SIAM J. MATH. ANAL.
Vol. 45, No. 3, pp. 1388–1421

SPATIAL DECAY OF TIME-DEPENDENT INCOMPRESSIBLE NAVIER–STOKES FLOWS WITH NONZERO VELOCITY AT INFINITY*

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Abstract. We consider the time-dependent Navier–Stokes system in a three-dimensional exterior domain with nonzero velocity at infinity. Under suitable assumptions on the data, it is shown that the velocity part of strong solutions, after subtraction of the far-field velocity, decays as $\big(|x| \cdot (1 + |x| - x_1)\big)^{-1}$, and its spatial gradient as $\big(|x| \cdot (1 + |x| - x_1)\big)^{-3/2}$, for $|x| \to \infty$. The solution class in question includes solutions obtained by L^2-variational methods and characterized by the fact that the velocity u and its spatial gradient $\nabla_x u$ are $L^\infty(L^2)$-functions, and $\nabla_x u$ is additionally L^2-integrable in time and in space.

Key words. exterior domain, time-dependent incompressible Navier–Stokes system, spatial decay, wake

AMS subject classifications. 35Q30, 65N30, 76D05

DOI. 10.1137/120872255

1. Introduction. Consider a rigid body moving with prescribed steady velocity $v_\infty \in \mathbb{R}^3 \backslash \{0\}$ and without rotation in a viscous incompressible fluid. Assume that the flow in a vicinity of the body is not influenced by some distant boundaries. Moreover suppose the flow in question is described with respect to a coordinate system attached to the body. Of course, although the rigid object moves with constant velocity, the flow near its boundary need not be stationary in general. A situation of this kind is usually modeled by the evolutionary Navier–Stokes system

$$\partial_t v - \nu \cdot \Delta_x v + (v \cdot \nabla_x) v + \varrho^{-1} \cdot \nabla_x \pi \;=\; \varrho^{-1} \cdot f, \quad \mathrm{div}_x v = 0,$$

in $Z_{T_0} := \overline{\Omega}^c \times (0, T_0)$, with a homogeneous Dirichlet boundary condition on $S_{T_0} := \partial\Omega \times (0, T_0)$ and a boundary condition at infinity,

$$v|S_{T_0} = 0, \quad v(x,t) \to v_\infty \; (|x| \to \infty) \;\text{ for }\; t \in (0, T_0),$$

supplemented by an initial condition,

$$v(x, 0) = v_0(x) \;\text{ for }\; x \in \overline{\Omega}^c,$$

where the open bounded set $\Omega \subset \mathbb{R}^3$ with connected Lipschitz boundary $\partial\Omega$ represents the rigid object, and the exterior domain $\overline{\Omega}^c := \mathbb{R}^3 \backslash \overline{\Omega}$ corresponds to the space occupied by the fluid. The given quantities of this problem are the viscosity $\nu \in (0, \infty)$, the density $\varrho \in (0, \infty)$, the far-field velocity v_∞, the initial velocity v_0, and the volume force f, whereas the velocity v and the pressure π are unknown. The role of the parameter T_0 depends on the type of solution under consideration: either T_0 cannot be determined in advance and thus is an unknown (solution locally in time), or it

*Received by the editors April 3, 2012; accepted for publication (in revised form) March 21, 2013; published electronically May 21, 2013.
http://www.siam.org/journals/sima/45-3/87225.html

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is a given quantity (solution globally in time). Without loss of generality, we may assume that $v_\infty = \beta \cdot (1,0,0)$ for some $\beta > 0$. Since a nonhomogeneous boundary condition at infinity is difficult to handle mathematically, we transform the velocity by a translation. In addition, we normalize the problem with respect to the size of the domain Ω and the magnitude β of the far-field velocity v_∞. In this way, we obtain the following initial-boundary value problem, denoting the transformed quantities in the same way as the original ones:

$$\partial_t v - \Delta_x v + \tau \cdot \partial_1 v + \tau \cdot (v \cdot \nabla_x) v + \nabla_x \pi \;=\; f, \quad \operatorname{div}_x v = 0 \quad \text{in } Z_{T_0}, \tag{1.1}$$

$$v|S_{T_0} = (-1,0,0), \quad v(x,t) \to 0 \ (|x| \to \infty) \ \text{ for } \ t \in (0,T_0), \tag{1.2}$$

$$v(x,0) = v_0(x) \ \text{ for } \ x \in \overline{\Omega}^c. \tag{1.3}$$

The parameter $\tau \in (0,\infty)$ is the Reynold's number. Note that the transformation of the velocity by a translation gives rise to the term $\tau \cdot \partial_1 v$ ("Oseen term") and to an inhomogeneous boundary condition on S_{T_0}. We are interested in a larger class of boundary values than just constant vectors like $(-1,0,0)$. Therefore we replace (1.2) by

$$v|S_{T_0} = v^*, \quad v(x,t) \to 0 \ (|x| \to \infty) \ \text{ for } \ t \in (0,T_0) \tag{1.4}$$

with a given function $v^* : S_{T_0} \mapsto \mathbb{R}^3$. A solution of (1.1), (1.3), (1.4) should reflect the main features of the physical flow in question. In particular, such a solution should exhibit a "wake." This means in our context that in a paraboloidal downstream region, the velocity converges slower than elsewhere to its constant boundary value at infinity. Such an asymptotic behavior is well established for solutions of the stationary version of (1.1)

$$-\Delta U + \tau \cdot \partial_1 U + \tau \cdot (U \cdot \nabla) U + \nabla \Pi = \Psi, \ \ \operatorname{div} U = 0 \quad \text{in } \ \overline{\Omega}^c. \tag{1.5}$$

In fact, if the velocity part U of a solution to this system belongs to $L^6(\overline{\Omega}^c)^3$, and ∇U to $L^2(\overline{\Omega}^c)^9$, and if the volume force Ψ decays sufficiently fast, then

$$|\partial_x^\alpha U(x)| = O\big(\big[\, |x| \cdot \nu(x) \,\big]^{-1-|\alpha|/2} \big) \quad \text{for } \ |x| \to \infty, \tag{1.6}$$

where $\alpha \in \mathbb{N}_0^3$ with $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3 \le 1$, and where the function $\nu : \mathbb{R}^3 \mapsto [0,\infty)$ is defined by

$$\nu(x) := 1 + |x| - x_1 \ \ (x \in \mathbb{R}^3). \tag{1.7}$$

The factor $\nu(x)$ in (1.6) may be considered as the mathematical manifestation of the above mentioned wake. The condition $|\alpha| \le 1$ means that (1.6) describes the asymptotic behavior of U and the gradient of U. The result in (1.7) has a long history. Key references are [24], [7], [23], [27, section IX]. In a note [10], we indicated how (1.7) follows from the theory in [27, section IX]. Active research is still ongoing in the context of (1.5), (1.6) [3], [4], [5].

In the work at hand, we want to show that strong solutions to the evolutionary problem (1.1), (1.3), (1.4) exhibit the spatial asymptotics of stationary flows, in the sense that velocity part v of these solutions decays like this:

$$|\partial_x^\alpha v(x,t)| = O\big(\big[\, |x| \cdot \nu(x) \,\big]^{-1-|\alpha|/2} \big) \ \text{ for } |x| \to \infty, \text{ uniformly in } t \in (0,T_0), \tag{1.8}$$

for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. The class of solutions we consider includes the "standard" strong solutions obtained by L^2-variational methods [29], [30]. Since in the stationary case, the decay rate $\big[\,|x| \cdot \nu(x)\,\big]^{-1-|\alpha|/2}$ appearing in (1.6) is the best possible in general, this rate should also be optimal in the evolutionary case. Therefore (1.8) seems to be the best results one can expect in general for the decay of v and ∇v if v is the velocity part of a solution to the initial-boundary value problem (1.1), (1.3), (1.4).

Let us state our results in more detail. To this end, we present a theorem on the stationary problem (1.5) with $\Psi = 0$. We need this theorem in order to be able to handle time-independent boundary data like the vector $(-1, 0, 0)$ in (1.2). Fix $S_0 > 0$ with $\overline{\Omega} \subset B_{S_0}$ and $\widetilde{b} \in H^{1/2}(\partial\Omega)^3$ with $\int_{\partial\Omega} \widetilde{b} \cdot n^{(\Omega)}\, do_x = 0$, where $n^{(\Omega)}$ denotes the outward unit normal to Ω. Then we have the following theorem.

THEOREM 1.1. *There is a function* $U \in L^6(\overline{\Omega}^c)^3$ *with* $\nabla U \in L^2(\overline{\Omega}^c)^9$ *and* $\operatorname{div} U = 0$, $U|\partial\Omega = \widetilde{b}$,

$$\int_{\overline{\Omega}^c} \big(\nabla U \cdot \nabla\vartheta + \tau \cdot \partial_1 U \cdot \vartheta + \tau \cdot (U \cdot \nabla)U \cdot \vartheta \big)\, dx = 0 \quad \textit{for } \vartheta \in V. \tag{1.9}$$

Moreover there is some $\mu_0 > 0$ *with*

$$|\partial_x^\alpha U(x)| \leq \mu_0 \cdot \big(|x| \cdot \nu(x) \big)^{-1-|\alpha|/2} \quad \textit{for } x \in B_{S_0}^c,\ \alpha \in \mathbb{N}_0^3 \ \textit{with } |\alpha| \leq 1. \tag{1.10}$$

For the first part of this theorem, we refer to [27, Theorem IX.4.1], [26, Theorem II.5.1]. As for inequality (1.10), it was proved in [7]; also see [27, section IX.7], [34], and [10].

Instead of (1.1), (1.3), (1.4), we consider a slightly more general problem, which may be stated as follows:

$$\begin{aligned} &\partial_t u(x,t) - \Delta_x u(x,t) + \tau \cdot \partial_1 u(x,t) + \tau \cdot \big(u(x,t) \cdot \nabla_x \big) u(x,t) \\ &\qquad + \tau \cdot \big(u(x,t) \cdot \nabla \big) U(x) + \tau \cdot \big(U(x) \cdot \nabla_x \big) u(x,t) + \nabla_x \pi(x,t) \\ &\quad = f(x,t),\ \operatorname{div}_x u(x,t) = 0 \quad \text{for } (x,t) \in Z_{T_0}, \end{aligned} \tag{1.11}$$

$$u|S_{T_0} = b, \quad u(x,t) \to 0\ (|x| \to \infty) \ \text{ for } t \in (0, T_0), \tag{1.12}$$

$$u(x,0) = a(x) \quad (x \in \overline{\Omega}^c). \tag{1.13}$$

Note that the pair of functions (u, π) solves (1.11) iff (v, π) with $v = u + U$ verifies (1.1). By considering problem (1.11)–(1.13) instead of (1.1), (1.3), (1.4), we may split the boundary data v^* from (1.4) into a stationary part $\widetilde{b}$, introduced in Theorem 1.1 as the boundary value of U, and into a part b supposed to decay for t tending to infinity. In fact, we require $b \in H_\infty$ with H_∞ being a certain subspace of $L^2(S_\infty)^3$ that will be defined in section 2. Thus a stationary function on S_∞ like the vector $(-1, 0, 0)$ from (1.2) cannot belong to H_∞. We further require that $a \in H_\sigma^{1/2+\epsilon_0}(\overline{\Omega}^c)$ for some $\epsilon_0 \in (0, 1/2]$ with $a|B_{S_0}^c \in W_{loc}^{1,1}(B_{S_0}^c)^3$ and

$$|\partial_y^\alpha a(y)| \leq \delta_0 \cdot \big(|y| \cdot \nu(y) \big)^{-1-|\alpha|/2-\kappa_0} \quad \text{for } y \in B_{S_0}^c, \tag{1.14}$$

where $\delta_0 \in (0, \infty)$ and $\kappa_0 \in (0, 1]$ are given constants. Since we want to show that the velocity u from (1.11)–(1.13) decays in the same way as v in (1.8), assumption (1.14) is the best possible, except perhaps for the fact that we exclude the case $\kappa_0 = 0$.

Turning to f, we require that $f : Z_\infty \mapsto \mathbb{R}^3$ is a measurable function such that $f|\Omega_{S_0} \times (0,\infty) \in L^2\big(\Omega_{S_0} \times (0,\infty)\big)^3$ and

$$|f(y,s)| \le \gamma(s) \cdot |y|^{-A} \cdot \nu(y)^{-B} \quad \text{for } y \in B^c_{S_0},\ s \in (0,\infty), \tag{1.15}$$

where $A \in (2,\infty)$ and $B \in [0,3/2]$ are constants with $A + \min\{1,B\} > 3$ and $A + B \ge 7/2$. The function γ belongs to $L^2\big((0,\infty)\big) \cap L^{p_0}\big((0,\infty)\big)$ for some $p_0 \in (2,\infty)$. We remark that the condition on spatial decay of f stated in (1.15) is again optimal in view of (1.8), because already in the case of the stationary Oseen system, a condition as in (1.15) has to be imposed on the right-hand side in order to obtain an asymptotic behavior of the (stationary) velocity as indicated in (1.6). We refer to [34, Theorems 3.1 and 3.2] for details.

We fix $T_0 \in (0,\infty]$, $s_0 \in [1,3)$, $r_0 \in (3,\infty)$, and we consider a function $u : (0,T_0) \mapsto W^{1,1}(\overline{\Omega}^c)^3$ with

$$u \in L^\infty\big(0,T_0,L^\kappa(\overline{\Omega}^c)^3\big) \quad \text{for } \kappa \in \{s_0,r_0\}, \tag{1.16}$$

$$\nabla_x u \in L^2\big(0,T_0,L^2(\overline{\Omega}^c)^9\big) \text{ and } (u \cdot \nabla_x)u \in L^2\big(0,T_0,L^{3/2}(\overline{\Omega}^c)^3\big). \tag{1.17}$$

Since $s_0 < 3$ and $r_0 > 3$, we may assume without loss of generality that $s_0 \ge 2$. This function u is supposed to be the velocity part of a solution to (1.11)–(1.13) in the sense that

$$u|S_{T_0} = b, \quad \operatorname{div}_x u = 0, \tag{1.18}$$

$$\begin{aligned}\int_0^T \int_{\overline{\Omega}^c} \Big[&-\big(u(x,t) \cdot \vartheta(x)\big) \cdot \varphi'(t) + \big(\nabla_x u(x,t) \cdot \nabla\vartheta(x)\big) \cdot \varphi(t) \\ &+ \tau \cdot \big(\partial_1 u(x,t) \cdot \vartheta(x)\big) \cdot \varphi(t) - \big(F(x,t) \cdot \vartheta(x)\big) \cdot \varphi(t)\Big]\, dx\, dt \\ = &\int_{\overline{\Omega}^c} a(x) \cdot \vartheta(x)\, dx \cdot \varphi(0)\end{aligned} \tag{1.19}$$

for $\varphi \in C_0^\infty\big([0,T_0)\big)$, $\vartheta \in C_0^\infty(\overline{\Omega}^c)^3$ with $\operatorname{div}\vartheta = 0$, where

$$\begin{aligned}&F(x,t) \\ &:= f(x,t) - \tau \cdot \big[\big(u(x,t) \cdot \nabla_x\big)u(x,t) + (U(x) \cdot \nabla_x)u(x,t) + \big(u(x,t) \cdot \nabla\big)U(x)\big]\end{aligned} \tag{1.20}$$

for $x \in \overline{\Omega}^c$, $t \in (0,T_0)$, with U from Theorem 1.1. It will become apparent below (Corollary 3.3) that the function F belongs to $L^2\big(0,T_0,L^{3/2}(\overline{\Omega}^c)^3\big)$, so that the integral of $\big(F(x,t) \cdot \vartheta(x)\big) \cdot \varphi(t)$ in (1.19) makes sense.

The conditions on u in (1.16)–(1.19) are satisfied, for example, by the solution to (1.11)–(1.13) constructed by Heywood [30] and characterized by the relations $u \in L^\infty\big(0,T_0,H^1(\overline{\Omega}^c)^3\big)$ and $\nabla_x u \in L^2\big(0,T_0,L^2(\overline{\Omega}^c)^9\big)$. In fact, in that case one may choose $s_0 = 2$, $r_0 = 6$ in (1.16). The second condition in (1.17) then becomes redundant; see the remark following Lemma 3.2. The same type of solution arises as a special case in the existence theory derived by Solonnikov [41] for a more general problem. (Choose $p = 2$ in [41, Theorem 10.1, Remark 10.1].) Heywood [30] and Solonnikov [41] obtained their existence results under a smallness condition on either T_0 or the data, as is usual for the three-dimensional nonstationary Navier–Stokes system. It is important to note that the decay results established in the work at hand hold without smallness conditions of any kind. In other words, once a solution of (1.11)–(1.13) satisfying (1.16)–(1.19) is available, it decays as in (1.8). More precisely, the following theorem—the main result of this article—is then valid.

THEOREM 1.2. *Let $R \in (S_0, \infty)$. Then*

$$|\partial_x^\alpha u(x,t)| \le \mathfrak{C}(R) \cdot \big(|x| \cdot \nu(x) \big)^{-1-|\alpha|/2} \quad \text{for } x \in B_R^c,\ t \in (0,T_0), \tag{1.21}$$

$\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$, *where the constant* $\mathfrak{C}(R)$ *depends on* Ω, *on the parameters* $\tau, S_0, \epsilon_0, \delta_0, \kappa_0, p_0, A, B, \mu_0, r_0, s_0, R$, *on upper bounds for the quantities* $\|a\|_{1/2+\epsilon_0,2}$, $\|\gamma\|_2$, $\|\gamma\|_{p_0}$, $\|f|\Omega_{S_0} \times (0,\infty)\|_2$, $\|U\|_6$, $\|\nabla U\|_2$, $\|u\|_{s_0,\infty;T_0}$, $\|u\|_{r_0,\infty;T_0}$, *as well as* $\|(u\cdot\nabla_x)u\|_{3/2,2;T_0}$ *and* $\|\nabla_x u\|_2$, *and on the function* $\varphi : [S_0,\infty) \mapsto [0,\infty)$ *defined by* $\varphi(r) := \|u|B_r^c \times (0,T_0)\|_{6,2;T_0}$ *for* $r \in [S_0, \infty)$.

As far as we know, up to now only Knightly [32] and Mizumachi [37] studied pointwise spatial decay of solutions to (1.11). Knightly detected the wake phenomenon, but he did not obtain the optimal decay rate $\big[\, |x| \cdot \nu(x) \,\big]^{-1-|\alpha|/2}$, and his results depend on various smallness assumptions. Mizumachi proved (1.21) for $\alpha = 0$ (decay of u but not of ∇u), under the assumptions $T_0 = \infty$, $U = 0$, $f = 0$, (u,π) a classical solution of (1.11)–(1.13), $u \in L^\infty\big(0,\infty, L^r(\overline{\Omega}^c)^3\big)$ for some $r \in [1,3)$, $\nabla_x u \in L^\infty\big(0,\infty, L^2(\overline{\Omega}^c)^3\big)$, $|u(x,t)| \to 0$ for $|x| \to \infty$ uniformly in $t \in [\widetilde{T}, \infty)$ for some $\widetilde{T} \in (0,\infty)$ [37, p. 515], and $|a(x) - u_0(x)| = O(|x|^{-2})$ ($|x| \to \infty$), where u_0 is the velocity part of some solution to the stationary problem (1.5). No conditions on the boundary data are specified in [37]. We think the pointwise decay condition $|u(x,t)| \to 0$ for $|x| \to \infty$ uniformly in t should be part of the conclusions, not of the assumptions. We further remark that the argument in [37] seems to have a gap: it is not clear why references [40] and [28] should imply [37, (2.42)]

Note that if $u \in L^\infty\big(0,\infty, L^r(\overline{\Omega}^c)^3\big)$ for some $r \in [1,3)$, and if $\nabla_x u$ is in $L^\infty\big(0,\infty, L^2(\overline{\Omega}^c)^9\big)$ and in $L^2\big(0,\infty, L^2(\overline{\Omega}^c)^9\big)$, then $u \in L^\infty\big(0,\infty, L^6(\overline{\Omega}^c)^9\big)$ by Lemma 2.3, and hence $(u\cdot\nabla_x)u \in L^2\big(0,\infty, L^{3/2}(\overline{\Omega}^c)^3\big)$ by Hölder's inequality. This means that the conditions on u in (1.16) and (1.17) are fulfilled if Mizumachi's assumptions on u (see above) hold, but with his pointwise decay condition replaced by the much weaker relation $\nabla_x u \in L^2\big(0,\infty, L^2(\overline{\Omega}^c)^9\big)$, which is valid even for standard weak solutions of the Navier–Stokes system.

The work at hand builds on previous papers [11, 12, 13, 14, 15, 16, 17, 18], where we studied Oseen flows, that is, the velocity part of solutions to the time-dependent Oseen system ((1.1) without the nonlinear term $\tau \cdot (v\cdot\nabla_x)v$). The decay result in (1.8) is established for such flows in [17] under the assumption that $\Omega = \emptyset$ (Cauchy problem) and $f = 0$. In [11], [14] we solved the integral equation (3.7) below in the Oseen case. As a consequence, in [15] and [16] we obtained a representation formula for Oseen flows, which allowed us to prove (1.8) for such flows in the case $\Omega \ne \emptyset$, first (see [15]) on the condition that v_0 and f have compact support, and then (see [18]) for v_0 as in (1.14) (with a replaced by v_0) and for f as in (1.15). Reference [12] deals with a criterion for f in view of solving the integral equation just mentioned, but this criterion will not play a role here. Regularity issues related to Oseen flows under the assumption $a = 0$, $f = 0$ are discussed in [13].

Concerning existence of solutions to (1.1), (1.3), (1.4) or to (1.11)–(1.13) or to the time-dependent Oseen system, Heywood's existence result in [30] with respect to (1.11)–(1.13) was already mentioned above; it is based on variational arguments and extends the theory he developed in [29]. Existence results in both Hölder and Sobolev spaces, for a generalized Stokes system and also for a nonlinear problem more general than (1.1), (1.3), (1.4), were established by Solonnikov in his article [41] mentioned above. Kobayashi and Shibata [33] constructed an Oseen semigroup in L^p-spaces. Such a semigroup implies existence results for the time-dependent Oseen system. Enomoto and Shibata [22] introduced an analogous semigroup in the case of space dimension $n \geq 3$. Mild solutions to the nonlinear problem (1.1), (1.3), (1.4) were constructed by Miyakawa [36] and by Shibata [39], with the first author considering local in time existence, and the second global in time existence for small data.

L^p-L^q-estimates of Oseen flows were established in [33]. These estimates were extended to the case of space dimension $n \geq 3$ in [22]. Local L^p-L^q-estimates for Oseen flows were shown in [33] ($n = 3$) and [21] ($n \geq 3$). Bae and Jin [8] proved weighted L^p-L^q-estimates for Oseen flows, using weight functions that are adapted to the wake appearing in such flows. Stability results for solutions to (1.1), (1.3), (1.4), in the sense of temporal decay estimates of spatial L^p-norms of the velocity, were proved by Masuda [35], Heywood [30, p. 675], Shibata [39], Enomoto and Shibata [22] (case $n \geq 3$), and Bae and Roh [9].

In the case $\Omega = \emptyset$ (Cauchy problem), certain aspects of the asymptotic behavior of a solution to (1.1) may be deduced from the decay properties exhibited by solutions of the Navier–Stokes system without Oseen term [31, p. 507], [42].

Let us indicate how our proof of Theorem 1.2 is structured. In section 3, we show that the function F from (1.20) belongs to $L^p\big(0, T_0, L^{3/2}(\overline{\Omega}^3)\big)$ for $p \in [q_0, 3/2]$ with some $q_0 \in [1, 3/2)$ (Corollary 3.3). This means that the velocity u enters into the framework of [18] if u is considered as the velocity part of a solution of the Oseen system with right-hand-side F. Thus the representation formula for Oseen flows given by [18, Corollary 4.5], which is a consequence of the theory in [16], yields an integral representation of u (Theorem 3.4). Our task then consists in estimating the integrals appearing in this representation. But all of them except one were already estimated in previous papers. The one exception consists of a volume potential involving the nonlinearity $\big((u+U)\cdot\nabla_x\big)(u+U)$. An estimate of this potential—the main difficulty of our proof—is achieved by a rather long and delicate iteration process presented in section 4, whereby every improved estimate of the potential in question implies a corresponding improved decay estimate of u or $\nabla_x u$, which in turn is used to revisit the critical potential. It is particularly challenging to advance to the relation $|u(x,t)| \to 0$ ($|x| \to \infty$) uniformly in $t \in (0, T_0)$ (Lemma 4.1 to 4.4), which in [37] is imposed as an assumption. Another difficulty consists in showing that $|\nabla_x u(x,t)|$ is bounded for large values of $|x|$ (Lemma 4.7). Once these results are available, we establish the decay rate $\big(|x| \cdot \nu(x)\big)^{-1}$ for $|u(x,t)|$ (Theorems 4.5, 4.6) and $\big(|x| \cdot \nu(x)\big)^{-3/2}$ for $|\nabla_x u(x,t)|$ (Theorem 4.8). In this context, the theory in [34] on the asymptotics of solutions of the stationary Oseen system in the whole space $\mathbb{R}^3$ turned out to be useful. Our argument is complicated by the presence of the terms $\big(u(t)\cdot\nabla\big)U$ and $(U\cdot\nabla_x)u(t)$, which transform into the matrix-valued function $\big(U_j \cdot u_k(t)\big)_{1\leq j,k\leq 3}$ after a partial integration. However, the technical difficulties related to these terms are reduced by Theorem 3.7, which states that $U_j \cdot u_k$ is an $L^\infty(L^1)$-function, a relation that is also valid for $u_j \cdot u_k$ ($1 \leq j, k \leq 3$) according to that theorem. Thus, in many estimates in section 4, we may handle $U_j \cdot u_k$ and $u_j \cdot u_k$ in the same way.

2. Notation. Some auxiliary results. We recall that the bounded Lipschitz domain Ω with connected boundary, the parameters $\tau, S_0, \epsilon_0, \delta_0, \kappa_0, A, B, p_0, \mu_0, r_0, s_0$ and the functions $\widetilde{b}$, U, a, f, γ, b, u, and F were introduced in section 1. We further recall that we denoted the length of a multiindex $\alpha \in \mathbb{N}_0^3$ by $|\alpha|$ (see the remark following (1.6)), defined a weight function ν describing the wake phenomenon (see (1.7)), introduced the notation $n^{(\Omega)}$ for the outward unit normal to Ω, and used the abbreviations $S_T := \partial\Omega \times (0,T)$ and $Z_T := \overline{\Omega}^c \times (0,T)$, for $T \in (0, \infty]$. Concerning the notation $\overline{\Omega}^c$ introduced at the beginning of section 1, we more generally write

$A^c := \mathbb{R}^3 \backslash A$ for any $A \subset \mathbb{R}^3$. We put $e_1 := (1,0,0)$. For $r > 0$, $x \in \mathbb{R}^3$, we set $B_r(x) := \{y \in \mathbb{R}^3 \,:\, |y-x| < r\}$, $B_r := B_r(0)$, $\Omega_r := \Omega \cap B_r$.

We write C for numerical constants, and $C(\gamma_1, \ldots, \gamma_n)$ for constants depending on parameters $\gamma_1, \ldots, \gamma_n$, for some $n \in \mathbb{N}$. These parameters typically belong to $(0, \infty)$, but the case $\gamma_1 = \Omega$ will also arise. In sections 3 and 4, the notation $\mathfrak{C}$ is used for constants that may depend on all the quantities listed at the end of Theorem 1.2, except on R and on the function φ. The exception with respect to φ is lifted from Lemma 4.4 onward. The symbol $\mathfrak{C}(\gamma_1, \ldots, \gamma_n)$ stands for constants that may additionally depend on $\gamma_1, \ldots, \gamma_n \in [0, \infty)$.

If $\mathfrak{H}$ is a vector space consisting of functions from an arbitrary nonempty set A into $\mathbb{R}$, and if $m \in \mathbb{N}$, we put $\mathfrak{H}^m := \{F : A \mapsto \mathbb{R}^m \,:\, F_1, \ldots, , F_m \in \mathfrak{H}\}$. If $\| \;\|$ is a norm on $\mathfrak{H}$, we will use the same notation $\| \;\|$ for the norm $(\sum_{j=1}^m \|F_i\|^2)^{1/2}$ $(F \in \mathfrak{H}^m)$ on $\mathfrak{H}^m$. For $n \in \mathbb{N}$, $A \subset \mathbb{R}^n$ measurable and $p \in [1, \infty]$, the usual norm of the Lebesgue space $L^p(A)$, defined with respect to Lebesgue measure in $\mathbb{R}^n$, is denoted by $\| \;\|_p$. The same notation is used for the norm of L^p-spaces on $\partial\Omega$ or on $\partial\Omega \times (0,T)$ for $T \in (0, \infty]$. Let A be an open set in $\mathbb{R}^3$. Given $m \in \mathbb{N}$, $p \in [1, \infty]$, we write $W^{m,p}(A)$ for the usual Sobolev space of order m on A with exponent p. The standard norm of this space is denoted by $\| \quad \|_{m,\,p}$. If $p = 2$, we write $H^m(A)$ instead of $W^{m,2}(A)$. The notation $W^{m,p}_{loc}(A)$ stands for the set of functions $w : A \mapsto \mathbb{R}$ such that $w|U \in W^{m,p}(U)$ for any open bounded set $U \subset \mathbb{R}^3$ with $\overline{U} \subset A$. If $p = 2$, we write $H^m_{loc}(A)$ instead of $W^{m,p}_{loc}(A)$. For $s \in (0,1)$, let $H^s(A)$ be the Sobolev space defined via the intrinsic norm with exponent $p = 2$ introduced in [1, Section 7.51]. This norm is denoted by $\| \;\|_{s,2}$. For $\epsilon \in (0,1]$, let $H^\epsilon_\sigma(\overline{\Omega}^c)$ denote the closure of the set $\big\{ v \in C^\infty_0(\overline{\Omega}^c)^3 \,:\, \operatorname{div} u = 0 \big\}$ with respect to the norm of $H^\epsilon(\overline{\Omega}^c)$. For brevity, put $V := H^1_\sigma(\overline{\Omega}^c)$. This means in particular that V is equiped with the norm $\| \;\|_{1,2}$ of $H^1(\overline{\Omega}^c)^3$. As usual, V' denotes the canonical dual space of V. The Sobolev space $H^1(\partial\Omega)$ is to be defined in the standard way (see [25, section III.6], for example). Let $\| \;\|_{1,\,2}$ denote the usual norm of this space with respect to some local coordinates of $\partial\Omega$ [25, section III.6.7]. The notation $\| \;\|_{H^1(\partial\Omega)'}$ stands for the canonical norm of the dual space $H^1(\partial\Omega)'$ of $H^1(\partial\Omega)$.

If $\mathfrak{B}$ is a Banach space, $T \in (0, \infty]$, and $p \in [1, \infty]$, then the norm of the space $L^p(0, T, \mathfrak{B})$ is denoted by $\| \;\|_{L^p(0,T,\mathfrak{B})}$. However, if $p, q \in [1, \infty]$, $\sigma \in \{0, 3\}$, and A is a measurable subset of $\mathbb{R}^3$, we write $\| \;\|_{q,p;T}$ for the norm of the space $L^p\big(0, T, L^q(A)^\sigma\big)$. If h is a function from $(0,T)$ into $L^q(A)^\sigma$ and $\widetilde{h}$ a function from $A \times (0,T)$ into $\mathbb{R}^\sigma$, and if $h(t)(x) = \widetilde{h}(x,t)$ for $x \in A$, $t \in (0,T)$, then we identify h and $\widetilde{h}$. This convention is justified because $h : (0,T) \mapsto L^q(A)^\sigma$ is measurable (or more precisely, strongly $L^q(A)^\sigma$-measurable in the sense of [43, p. 130]) iff $\widetilde{h} : A \times (0,T) \mapsto \mathbb{R}^\sigma$ is measurable in the usual Lebesgue sense; see [16, proof of Lemma 2.1] for some references on this well-known fact. In the case $q < \infty$, we may conclude with Fubini's theorem that the norm of the spaces $L^q\big(0, T, L^q(A)^\sigma\big)$ and $L^q\big(A \times (0,T)\big)^\sigma$ coincide. Therefore we may use the norm $\| \;\|_q$ instead of $\| \;\|_{q,q;T}$ for functions from $L^q\big(0, T, L^q(A)^\sigma\big)$, and we may identify the latter space with $L^q\big(A \times (0,T)\big)^\sigma$. Analogous remarks are valid with respect to the spaces $L^q\big(0, T, L^q(\partial\Omega)^\sigma\big)$ and $L^q\big(\partial\Omega \times (0,T)\big)^\sigma$.

Again take $T \in (0, \infty]$. Following [38], we introduce a space H_T of functions on S_T. To this end, we put

$$L^2_n(S_T) := \left\{ w \in L^2(S_T)^3 \,:\, \int_{\partial\Omega} w(x,t) \cdot n^{(\Omega)}(x)\, do_x = 0 \;\text{ for a.e. }\; t \in (0,T) \right\},$$

$$\widetilde{H}_T := \{\varrho | S_T \,:\, \varrho \in C^\infty_0(\mathbb{R}^4)^3,\; \varrho \,|\, \mathbb{R}^3 \times (-\infty, 0] = 0\}.$$

For $\varphi \in C^1\big((-\infty,T)\big)$ with $\varphi|(-\infty,0]=0$, and for $t\in(0,T)$, we set

$$\partial_t^{1/2}\varphi(t) := \pi^{-1/2}\cdot\partial_t\left(\int_0^t (t-r)^{-1/2}\cdot\varphi(r)\,dr\right)$$

("fractional derivative of φ"). We further put $\partial_4^{1/2}w(x,t) := \partial_t^{1/2}\big(w(x,\ \cdot\ )\big)(t)$ for $(x,t)\in\partial\Omega\times(0,T)$, $w\in\widetilde{H}_T$. For any $w\in L^2(S_T)$, we may define $F_w \in L^2\big(0,T,H^1(\partial\Omega)'\big)$ by setting

$$F_w(t)(\mu) := \int_{\partial\Omega} w(x,t)\cdot\mu(x)\,do_x \quad \text{for } \mu\in H^1(\partial\Omega) \text{ and for a.e. } t\in(0,T).$$

We will write w instead of F_w. For $w\in\widetilde{H}_T$, set

$$\|w\|_{H_T} := \left(\int_0^T \Big(\|w(\ \cdot\ ,t)\,|\,\partial\Omega\|_{1,2}^2 + \|\partial_4^{1/2}w(\ \cdot\ ,t)|\partial\Omega\|_2^2 + \|\partial_t w(\ \cdot\ ,t)\cdot n^{(\Omega)}\|_{H^1(\partial\Omega)'}^2\Big)\,dt\right)^{1/2}.$$

The mapping $\|\ \|_{H_T}$ is a norm on $\widetilde{H}_T$. Let the space H_T consist of all functions $w\in L_n^2(S_T)$ such that there exists a sequence (w_n) in $\widetilde{H}_T$ with the property that $\|w-w_n\|_2\to 0$ and such that (w_n) is a Cauchy sequence with respect to the norm $\|\ \|_{H_T}$. This means in particular that the sequence $\big(\|w_n\|_{H_T}\big)$ is convergent. Its limit value does not depend on the choice of the sequence (w_n) with the above properties. Thus, for $w\in H_T$, we may define the quantity $\|w\|_{H_T}$ in an obvious way. The mapping $\|\ \|_{H_T}$ is a norm on H_T, and the pair $\big(H_T,\|\ \|_{H_T}\big)$ is a Banach space.

It will be convenient to use the following abbreviations:

$$G(t) := \tau\cdot\big[\big(u(t)\cdot\nabla_x\big)u(t) + \big(u(t)\cdot\nabla\big)U + (U\cdot\nabla_x)u(t)\big], \tag{2.1}$$

$$H_{jk}(t) := \tau\cdot\big(u_j(t)\cdot u_k(t) + u_j(t)\cdot U_k + U_j\cdot u_k(t)\big), \tag{2.2}$$

$$g_k^{(b)}(t) := \tau\cdot\sum_{l=1}^3\big(b_l(t)\cdot b_k(t) + b_l(t)\cdot\widetilde{b}_k + \widetilde{b}_l\cdot b_k(t)\big)\cdot n_l^{(\Omega)} \tag{2.3}$$

for $1\le j,k\le 3,\ t\in(0,T_0)$. Next we present some estimates related to $\nu(x)$.

LEMMA 2.1 (see [23, Lemma 2.3]). *Let $\beta\in(1,\infty)$. Then $\int_{\partial B_r}\big(1+\nu(x)\big)^{-\beta}\,do_x \le C(\beta)\cdot r$ for $r\in(0,\infty)$.*

LEMMA 2.2 (see [19, Lemma 4.8]). *$\nu(x-y)^{-1} \le C\cdot(1+|y|)\cdot\nu(x)^{-1}$ for $x,y\in\mathbb{R}^3$.*

Let $S\in(0,\infty)$. Then $|y|^{-1}\le C(S)\cdot\nu(y)^{-1}$ for $y\in B_S^c$.

In [18], the following lemma is deduced from [26, Theorem II.5.1].

LEMMA 2.3 (see [18, Lemma 2.4]). *Let $v\in W_{loc}^{1,1}(\overline{\Omega}^c)$ with $v\in L^\kappa(\overline{\Omega}^c)$ for some $\kappa\in[1,\infty)$, and $\nabla v\in L^2(\overline{\Omega}^c)^3$. Then $v\in L^6(\overline{\Omega}^c)$ and $\|v\|_6\le C(\Omega)\cdot\|\nabla v\|_2$.*

In particular, $u\in L^2\big(0,T_0,L^6(\overline{\Omega}^c)\big)$ and $\|u\|_{6,2;T_0}\le C(\Omega)\cdot\|\nabla_x u\|_2$.

Next we introduce the fundamental solutions we will consider in the following. We write $\mathfrak{H}$ for the usual fundamental solution of the heat equation in $\mathbb{R}^3$, that is,

$$\mathfrak{H}(z,t) := (4\cdot\pi\cdot t)^{-3/2}\cdot e^{-|z|^2/(4\cdot t)} \quad \text{for } (z,t)\in\mathbb{R}^3\times(0,\infty).$$

Furthermore, we introduce a fundamental solution of the time-dependent Stokes system by setting as in [38]

$$\Gamma_{jk}(z,t) := \delta_{jk}\cdot\mathfrak{H}(z,t) + \int_t^\infty \partial_j\partial_k\mathfrak{H}(z,s)\,ds,\ \ E_k(x) := (4\cdot\pi)^{-1}\cdot x_k\cdot|x|^{-3} \tag{2.4}$$

for $(z,t) \in \mathbb{R}^3 \times (0,\infty),\ x \in \mathbb{R}^3 \backslash \{0\},\ 1 \le j,k \le 3$. (The function E_k is introduced for completeness; it will not play any role in the following.) Finally we define the velocity part of a fundamental solution of the time-dependent Oseen system by putting

$$\Lambda_{jk}(z,t,\tau) := \Gamma_{jk}(z - \tau \cdot t \cdot e_1, t) \quad \text{for } (z,t) \in \mathbb{R}^3 \times (0,\infty),\ j,k \in \{1, 2, 3\}.$$

We will need the following properties of Λ_{jk} and Γ_{jk}.

LEMMA 2.4 (see [38, Proposition 2.1.9]). *The functions Γ_{jk} and $\Lambda_{jk}(\cdot\,,\cdot\,,\tau)$ belong to $C^\infty\big(\mathbb{R}^3 \times (0,\infty)\big)$ for $1 \le j,k \le 3$. Moreover*

$$|\partial_z^\alpha \Gamma_{jk}(z,t)| \le C \cdot (|z|^2 + t)^{-3/2-|\alpha|/2},$$
$$|\partial_z^\alpha \Lambda_{jk}(z,t,\tau)| \le C \cdot (|z - \tau \cdot t \cdot e_1|^2 + t)^{-3/2-|\alpha|/2}$$

for $z \in \mathbb{R}^3,\ t \in (0,\infty),\ \alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$.

On the other hand, concerning the term $|z-\tau\cdot t\cdot e_1|^2+t$, we have the next lemma.

LEMMA 2.5 (see [11, Lemma 2]). *Let $K \in (0,\infty)$. For $z \in \mathbb{R}^3$, put $\gamma_K(z) := |z|^2$ if $|z| \le K$ and $\gamma_K(z) := |z| \cdot \nu(z)$ else.*

Then $|z - \tau \cdot t \cdot e_1|^2 + t \ge C(\tau,K) \cdot \big(\gamma_K(z) + t\big)$ for $z \in \mathbb{R}^3$, $t \in (0,\infty)$.

THEOREM 2.6. *Let $K \in (0,\infty),\ \mu \in (1,\infty)$. Then*

$$\int_0^\infty (|y - \tau \cdot t \cdot e_1|^2 + t)^{-\mu}\, dt \le C(\tau,K,\mu) \cdot \big(\,|y| \cdot \nu(y)\,\big)^{-\mu+1/2} \quad \textit{for } y \in B_K^c.$$

Proof. See [20, Theorem 2.19] with $z = 0,\ \delta = 1$, and S replaced by $K/2$. ◻

Lemma 2.4 and 2.5 yield the following.

COROLLARY 2.7. *Let $K \in (0,\infty)$. Then*

$$|\partial_z^\alpha \Lambda_{jk}(z,t,\tau)| \le C(\tau,K) \cdot \big(\,\gamma_K(z) + t\,\big)^{-3/2-|\alpha|/2} \quad \textit{for } z \in \mathbb{R}^3,\ t \in (0,\infty),$$

$\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$, where $\gamma_K(z)$ was defined in Lemma 2.5.

Another important tool in our proofs is the following estimate, which follows from Hölder's and Young's inequalities.

THEOREM 2.8. *Let $p,q \in [1,\infty],\ r \in (1,\infty],\ s \in [1,\infty]$ with $q < p,\ s \le r$.*

Then, for $h \in L^s\big(0,\infty, L^q(\mathbb{R}^3)\big),\ M \in (0,\infty),\ j,k \in \{1, 2, 3\},\ \alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$, the ensuing inequality holds: for $W = (0,M)$ if $1-|\alpha|/2+3\cdot(1/p-1/q)/2 > 1/s-1/r$, and for $W = (M,\infty)$ if $1-|\alpha|/2+3\cdot(1/p-1/q)/2 < 1/s-1/r$,

$$\begin{aligned}\Bigg(\int_0^\infty \Bigg[\int_{\mathbb{R}^3} \Bigg(\int_0^\infty \int_{\mathbb{R}^3} \chi_W(t-\sigma) \cdot |\partial_x^\alpha \Lambda_{jk}(x-y,t-\sigma,\tau)| \\ \cdot |h(y,\sigma)|\, dy\, d\sigma\Bigg)^p dx\Bigg]^{r/p} dt\Bigg)^{1/r} \\ \le C(\tau,p,q,r,s) \cdot M^{3\cdot(1/p-1/q)/2+1-|\alpha|/2-1/s+1/r} \cdot \|h\|_{q,s;\infty}\end{aligned}$$

if $p < \infty,\ r < \infty$. If $p = \infty$ and/or $r = \infty$, the preceding estimate has to be modified in an obvious way. For example, if $p = r = s = \infty$ and $1 - |\alpha|/2 - 3/(2\cdot q) > 0$, we have

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^3} \chi_{(0,M)}(t-\sigma) \cdot |\partial_x^\alpha \Lambda_{jk}(x-y,t-\sigma,\tau)| \cdot |h(y,\sigma)|\, dy\, d\sigma \\ \le C(\tau,q) \cdot M^{-3/(2\cdot q)+1-|\alpha|/2} \cdot \|h\|_{q,\infty;\infty}\end{aligned}$$

for $h \in L^\infty\big(0,\infty, L^q(\mathbb{R}^3)\big),\ M,j,k,\alpha$ as above, $x \in \mathbb{R}^3,\ t \in (0,\infty)$.

Proof. In the case $s < \infty$, we refer to [15, Lemma 2.7]. If $s = \infty$, the arguments from the preceding reference remain valid. As an example, take the case $p = r = s = \infty,\ q > 1$. For h, M, j, k, α as in the lemma, $W \in \{(0, M),\ (M, \infty)\}$, and for $x \in \mathbb{R}^3,\ t \in (0, \infty)$, we find by Lemma 2.4 and Hölder's inequality that

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^3} \chi_W(t-\sigma) \cdot |\partial_x^\alpha \Lambda_{jk}(x-y, t-\sigma, \tau)| \cdot |h(y,\sigma)|\, dy\, d\sigma \\
&\quad \le C(\tau) \cdot \int_0^\infty \int_{\mathbb{R}^3} \chi_W(t-\sigma) \cdot \Big(|x - y - \tau \cdot (t-\sigma) \cdot e_1| + (t-\sigma)^{1/2} \Big)^{-(3+|\alpha|)} \\
&\qquad \cdot |h(y,\sigma)|\, dy\, d\sigma \\
&\quad \le C(\tau) \cdot \int_0^\infty \chi_W(t-\sigma) \cdot \bigg(\int_{\mathbb{R}^3} \Big(|x - y - \tau \cdot (t-\sigma) \cdot e_1| + (t-\sigma)^{1/2} \Big)^{-(3+|\alpha|)\cdot q'} dy \bigg)^{1/q'} \\
&\qquad \cdot \|h(\,\cdot\,,\sigma)\|_q\, d\sigma \\
&\quad = C(\tau) \cdot \int_0^\infty \chi_W(t-\sigma) \cdot \bigg(\int_{\mathbb{R}^3} \Big(|z| + (t-\sigma)^{1/2} \Big)^{-(3+|\alpha|)\cdot q'} dz \bigg)^{1/q'} \cdot \|h(\,\cdot\,,\sigma)\|_q\, d\sigma \\
&\quad \le C(\tau, q) \cdot \int_0^\infty \chi_W(t-\sigma) \cdot (t-\sigma)^{-|\alpha|/2 - 3/(2\cdot q)} \cdot \|h(\,\cdot\,,\sigma)\|_q\, d\sigma \\
&\quad \le C(\tau, q) \cdot \|h\|_{q,\infty;\infty} \cdot \int_0^\infty \chi_W(t-\sigma) \cdot (t-\sigma)^{-|\alpha|/2 - 3/(2\cdot q)}\, d\sigma.
\end{aligned}$$

Now, distinguishing the case $W = (0, M),\ -|\alpha|/2 - 3/(2 \cdot q) > -1$ on the one hand and $W = (M, \infty),\ -|\alpha|/2 - 3/(2 \cdot q) < -1$ on the other, we may integrate with respect to σ and obtain the looked-for inequality. ∎

A first consequence of Theorem 2.8 is the following.

Corollary 2.9. *Let $s \in [1, \infty],\ q \in [1, \infty),\ h \in L^s\big(0, \infty, L^q(\mathbb{R}^3)^3\big)$. Then, for a.e. $(x,t) \in \mathbb{R}^3 \times (0, \infty),\ \alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$, we have $\int_0^t \int_{\mathbb{R}^3} \sum_{k=1}^3 |\partial_x^\alpha \Lambda_{jk}(x-y, t-\sigma, \tau) \cdot h_k(y,\sigma)|\, dy\, d\sigma < \infty$.*

Proof. Theorem 2.8 shows that the corollary holds for $(x,t) \in \mathbb{R}^3 \times (0, M)$ for any $M > 0$; compare the remarks in [15, p. 898]. ∎

Due to the preceding corollary, we may define a function $\mathfrak{R}^{(\tau)}(h) : \mathbb{R}^3 \times (0, \infty) \mapsto \mathbb{R}^3$ for any $h \in L^s\big(0, T, L^q(A)^3\big)$ with $s \in [1, \infty],\ q \in [1, \infty),\ A \subset \mathbb{R}^3$ measurable, $T \in (0, \infty]$, by setting

$$\mathfrak{R}_j^{(\tau)}(h)(x,t) := \int_0^t \int_{\mathbb{R}^3} \sum_{k=1}^3 \Lambda_{jk}(x-y, t-\sigma, \tau) \cdot \widetilde{h}_k(y,\sigma)\, dy\, d\sigma \tag{2.5}$$

for a.e. $x \in \mathbb{R}^3,\ t \in (0, \infty)$, where the function $\widetilde{h}$ in (2.5) denotes the zero extension of h to $\mathbb{R}^3 \times (0, \infty)$.

Lemma 2.10. *Let $s \in [1, \infty],\ q \in [1, \infty)$, and $h \in L^s\big(0, \infty,\ L^q(\mathbb{R}^3)^3\big)$. Then the weak derivative $\partial_l \mathfrak{R}_j^{(\tau)}(h)$ exists for $1 \le j, l \le 3$, in particular, $\mathfrak{R}^{(\tau)}(h)(\,\cdot\,,t) \in W^{1,1}_{loc}(\mathbb{R}^3)^3$ for $t \in (0, \infty)$. For a.e. $x \in \mathbb{R}^3,\ t \in (0, \infty),\ 1 \le j, l \le 3$,*

$$\partial_l \mathfrak{R}_j^{(\tau)}(h)(x,t) = \int_0^t \int_{\mathbb{R}^3} \sum_{k=1}^3 \partial_l \Lambda_{jk}(x-y, t-\sigma, \tau) \cdot h_k(y,\sigma)\, dy\, d\sigma.$$

In particular, the trace of $\mathfrak{R}^{(\tau)}(h)(\,\cdot\,,t)$ on $\partial\Omega$ is well defined for $t \in (0, \infty)$, so $\mathfrak{R}^{(\tau)}(h)$ has a boundary value on S_∞.

Proof. The lemma follows from Theorem 2.8. □

LEMMA 2.11 (see [17, Lemma 2.3]). *Let* $q \in [1,\infty],\ c \in L^q(\mathbb{R}^3)$. *Then the integral* $\int_{\mathbb{R}^3} |\mathfrak{H}(x-y-\tau\cdot t\cdot e_1,t)\cdot c(y)|\,dy$ *is finite for* $x \in \mathbb{R}^3,\ t \in (0,\infty)$. *Thus we may define*

$$\mathfrak{I}_j^{(\tau)}(c)(x,t) := \int_{\mathbb{R}^3} \mathfrak{H}(x-y-\tau\cdot t\cdot e_1,t)\cdot c_j(y)\,dy$$

for $x \in \mathbb{R}^3,\ t \in (0,\infty),\ 1 \le j \le 3$. *The function* $\mathfrak{I}^{(\tau)}(c)$ *belongs to* $C^1\big(\mathbb{R}^3\times(0,\infty)\big)^3$.

The function $\mathfrak{I}^{(\tau)}(c)$ even belongs to $C^\infty\big(\mathbb{R}^3\times(0,\infty)\big)^3$, but we will not need this fact. Next we consider convolutions of the Oseen fundamental solution on $\partial\Omega$.

LEMMA 2.12. *Let* $s,q \in [1,\infty],\ \varphi \in L^s\big(0,\infty,L^q(\partial\Omega)^3\big),\ 1 \le j,k \le 3$. *For* $x \in \mathbb{R}^3\backslash\partial\Omega,\ t \in (0,\infty),\ \alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$, *the integral* $\int_0^t \int_{\partial\Omega} |\partial_x^\alpha \Lambda_{jk}(x-y,t-\sigma,\tau)\cdot\varphi_k(y,\sigma)|\,d\Omega(y)\,d\sigma$ *is finite. Moreover, if* $\varphi \in L^2(S_\infty)^3$ *and* $\alpha = 0$, *the preceding integral is finite also for a.e.* $(x,t) \in S_\infty$.

Proof. The first claim of the lemma is obvious by Corollary 2.7. As for the second conclusion, we refer to [15, Lemma 2.19]. □

In view of Lemma 2.12, for $s,q \in [1,\infty],\ T \in (0,\infty],\ \varphi \in L^s\big(0,T,L^q(\partial\Omega)^3\big)$, we may define

$$\mathfrak{V}_j^{(\tau)}(\varphi)(x,t) := \int_0^t \int_{\partial\Omega} \sum_{k=1}^3 \Lambda_{jk}(x-y,t-\sigma,\tau)\cdot\widetilde{\varphi}_k(y,\sigma)\,d\Omega(y)\,d\sigma \tag{2.6}$$

for $1 \le j \le 3,\ x \in \mathbb{R}^3\backslash\partial\Omega,\ t \in (0,\infty)$, where $\widetilde{\varphi}$ denotes the zero extension of φ to S_∞. If $\varphi \in L^2(S_T)^3$ for some $T \in (0,\infty]$, we define $\mathfrak{V}_j^{(\tau)}(\varphi)(x,t)$ as in (2.6) for a.e. $x \in \partial\Omega,\ t \in (0,\infty)$. As a consequence of Corollary 2.7 and Lebesgue's theorem, we note the following.

COROLLARY 2.13. *Let* $s,q \in [1,\infty],\ \varphi \in L^s\big(0,\infty,L^q(\partial\Omega)^3\big)$. *Then for* $x \in \mathbb{R}^3\backslash\partial\Omega,\ t \in (0,\infty)$, *and* $j,l \in \{1,\,2,\,3\}$, *the derivative* $\partial_{x_l}\mathfrak{V}_j^{(\tau)}(\varphi)(x,t)$ *exists, and*

$$\partial_{x_l}\mathfrak{V}_j^{(\tau)}(\varphi)(x,t) = \int_0^t \int_{\partial\Omega} \partial_l\Lambda_{jk}(x-y,t-\sigma,\tau)\cdot\varphi_k(y,\sigma)\,d\Omega(y)\,d\sigma. \tag{2.7}$$

We note a consequence of (1.15).

LEMMA 2.14 (see [18, Lemma 2.27]). *Let* $\kappa \in [1,2]$. *Then* $f \in L^2\big(0,\infty,L^\kappa(\overline{\Omega}^c)^3\big)$ *and* $\|f\|_{\kappa,2;\infty} \le C(S_0,A,B,\kappa)\cdot\big(\|\gamma\|_2 + \|f|\Omega_{S_0}\times(0,\infty)\|_2\big)$.

In particular, $f \in L^2(0,\infty,V')$.

Concerning the spatial decay of $\mathfrak{I}^{(\tau)}(a)$, we will use a result from [17].

THEOREM 2.15 (see [17, Theorem 1.1]). *For* $S \in (S_0,\infty),\ x \in B_S^c,\ \alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$, *the inequality*

$$|\partial_x^\alpha\mathfrak{I}^{(\tau)}(a)(x,t)| \le C(\Omega,\tau,S_0,S,\kappa_0)\cdot(\delta_0 + \|a|\Omega_{S_0}\|_1)\cdot\big(|x|\cdot\nu(x)\big)^{-1-|\alpha|/2}$$

holds with δ_0 *and* κ_0 *from* (1.14).

In [18], we studied the spatial asymptotics of $\mathfrak{R}^{(\tau)}(f)$ and $\mathfrak{V}^{(\tau)}(\phi)$, obtaining the ensuing two results.

THEOREM 2.16 (see [18, Theorem 3.1]). *Let* $S \in (S_0,\infty)$. *Then*

$$\begin{aligned} &|\partial_x^\alpha\mathfrak{R}^{(\tau)}(f)(x,t)| \\ &\quad \le C(\tau,p_0,S_0,S,A,B)\cdot\big(\|\gamma\|_2 + \|\gamma\|_{p_0} + \|f\,|\,\Omega_{S_0}\times(0,\infty)\|_2\big)\cdot\big(|x|\cdot\nu(x)\big)^{-1-|\alpha|/2} \end{aligned} \tag{2.8}$$

for $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1,\ x \in B_S^c,\ t \in (0,\infty)$, *where* p_0, A, B, *and* γ *were introduced in* (1.15).

LEMMA 2.17 (see [18, Lemma 3.2]). *Let* $S \in (S_0,\infty),\ \kappa \in [1,\infty)$. *Then*

$$|\partial_x^\alpha \mathfrak{V}^{(\tau)}(\varphi)(x,t)| \le C(\Omega,\tau,S_0,S) \cdot \|\varphi\|_{L^2\left(0,\infty,L^\kappa(\partial\Omega)^3\right)} \cdot \big(|x| \cdot \nu(x)\big)^{-1-|\alpha|/2} \tag{2.9}$$

for $\varphi \in L^2\big(0,\infty,L^\kappa(\partial\Omega)^3\big),\ \alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1,\ x \in B_S^c,\ t \in (0,\infty)$.

If $\varphi \in L^1(S_\infty)^3$, *then for* $\alpha,\ x,\ t$ *as before,*

$$|\partial_x^\alpha \mathfrak{V}^{(\tau)}(\varphi)(x,t)| \le C(\Omega,\tau,S_0,S) \cdot \|\varphi\|_1 \cdot \big(|x| \cdot \nu(x)\big)^{-3/2-|\alpha|/2}.$$

In the following theorem, we cite a result from [34] involving convolution integrals of a type arising frequently in the context of Oseen flows.

THEOREM 2.18. *Let* $A \in [1,\infty),\ B \in [0,3/2]$, *and put*

$E := A - 1/2,\ F := B$ *if* $A \le 2,\ A < B+1,\ A + \min\{1,B\} < 3$;

$E := (A+B)/2,\ F := (A+B)/2 - 1/2$ *if* $A > B+1,\ A+B < 3$; *and*

$E := F := 3/2$ *if* $A + \min\{1,B\} > 3,\ A+B \ge 7/2$.

Then

$$\begin{aligned}\int_{\mathbb{R}^3} \Big((1+|x-y|)\cdot\nu(x-y)\Big)^{-3/2} \cdot (1+|y|)^{-A} \cdot \nu(y)^{-B}\,dy \\ \le C(A,B)\cdot(1+|x|)^{-E}\cdot\nu(x)^{-F}.\end{aligned} \tag{2.10}$$

Proof. See the proof of [34, Theorem 3.2]; in particular, see [34, p. 92, Figures 2 and 3]. □

In the ensuing corollary, we draw some conclusions from Theorem 2.8.

COROLLARY 2.19. *Let* $m \in [2,\infty),\ \mu \in (3/2,\infty)$ *with* $1/3 > 2/m - 1/\mu > 0$. *Moreover, let* $w \in L^\infty\big(0,\infty,L^2(\mathbb{R}^3)^3\big) \cap L^\infty\big(0,\infty,L^m(\mathbb{R}^3)^3\big)$ *and* $l \in \{1,\,2,\,3\}$. *Then* $w_l \cdot w \in L^\infty\big(0,\infty,L^1(\mathbb{R}^3)^3\big)$.

Let $t \in (0,\infty)$. *Then* $\partial_l \mathfrak{R}^{(\tau)}(w_l\cdot w)(\,\cdot\,,t) \in L^\mu(\mathbb{R}^3)^3$ *and*

$$\|\partial_l \mathfrak{R}^{(\tau)}(w_l\cdot w)(\,\cdot\,,t)\|_\mu \le C(\tau,m,\mu)\cdot\big[\,\|w\|_{2,\infty;\infty}^2 + \|w\|_{m,\infty;\infty}^2\,\big].$$

Proof. Let $j \in \{1,\,2,\,3\}$. For $x \in \mathbb{R}^3$,

$$\int_0^t \int_{\mathbb{R}^3} \sum_{k=1}^3 |\partial_l \Lambda_{jk}(x-y,t-\sigma,\tau)\cdot(w_l\cdot w_k)(y,\sigma)|\,dy\,d\sigma \le \mathfrak{A}^{(1)}(x) + \mathfrak{A}^{(2)}(x) \tag{2.11}$$

with

$$\mathfrak{A}^{(i)}(x) := \int_0^t \int_{\mathbb{R}^3} \chi_{W_i}(t-\sigma)\cdot\sum_{k=1}^3 |\partial_l \Lambda_{jk}(x-y,t-\sigma,\tau)\cdot(w_l\cdot w_k)(y,\sigma)|\,dy\,d\sigma$$

for $i \in \{1,\,2\}$ with $W_1 := (1,\infty),\ W_2 := (0,1]$. Noting that $\mu > 3/2$, hence $-1+3/(2\cdot\mu) < 0$, we may apply Theorem 2.8 with $r = s = \infty,\ p = \mu,\ q = 1$. It follows that

$$\|\mathfrak{A}^{(1)}\|_\mu \le C(\tau,\mu)\cdot\|w_l\cdot w\|_{1,\infty;\infty} \le C(\tau,\mu)\cdot\|w\|_{2,\infty;\infty}^2. \tag{2.12}$$

On the other hand, since $m \ge 2$ and $1/3 > 2/m - 1/\mu > 0$, we have $m/2 \ge 1,\ 1/2 + 3/(2\cdot\mu) - 3/m > 0$ and $m/2 < \mu$. Therefore Theorem 2.8 may be applied with $r = s = \infty,\ p = \mu,\ q = m/2$. This reference yields

$$\|\mathfrak{A}^{(2)}\|_\mu \le C(\tau,m,\mu)\cdot\|w_l\cdot w\|_{m/2,\infty;\infty} \le C(\tau,m,\mu)\cdot\|w\|_{m,\infty;\infty}^2. \tag{2.13}$$

The corollary follows from (2.11)–(2.13) and Lemma 2.10. □

We close this section by establishing various estimates of $\partial_l \mathfrak{R}^{(\tau)}(g)(x,t)$ for the case that $\operatorname{supp}\big(g(\,\cdot\,,t)\big)$ is bounded and the distance between x and $\operatorname{supp}\big(g(\,\cdot\,,t)\big)$ is larger than some given quantity.

Lemma 2.20. *Let $S,\, R_1,\, R_2 \in (0,\infty)$ with $R_1 < R_2$. Let $j,k,l \in \{1,\,2,\,3\}$. For measurable functions $g : B_{R_1} \times (0,\infty) \mapsto \mathbb{R}$, and for $x \in B^c_{R_2},\ t \in (0,\infty)$, define*

$$\mathfrak{A}(g)(x,t) := \int_0^t \int_{B_{R_1}} |\partial_l \Lambda_{jk}(x-y,t-\sigma,\tau)| \cdot |g(y,\sigma)|\, dy\, d\sigma.$$

Let $t \in (0,\infty)$. Then, for $p \in [1,\infty),\ q \in (2,\infty),\ g \in L^\infty\big(0,\infty,L^p(B_{R_1})\big)$,

$$\|\mathfrak{A}(g)(\,\cdot\,,t)\|_q \le C(\tau,R_1,R_2,q) \cdot \|g\|_{p,\infty;\infty}. \tag{2.14}$$

Moreover, for $g \in L^\infty\big(0,\infty,L^1(B_{R_1})\big),\ x \in B^c_{R_2}$,

$$\mathfrak{A}(g)(x,t) \le C(\tau,S) \cdot \|g\|_{1,\infty;\infty} \cdot \theta(x,R_1,R_2), \tag{2.15}$$

where $\theta(x,R_1,R_2) := \big(|x| \cdot (1-R_1/R_2)\big)^{-2}$ if $B_{R_1} \subset B_S(x)$, and $\theta(x,R_1,R_2) := \big[\,|x| \cdot (1-R_1/R_2) \cdot \max\{1,\, \nu(x)/(1+R_1)\}\,\big]^{-1}$ if $B_{R_1} \cap B_S(x) = \emptyset$.

If $R_2 - R_1 \ge S$, we have for $p \in (4/3,\infty),\ g \in L^2\big(0,\infty,L^1(B_{R_1})\big)$ that

$$\|\mathfrak{A}(g)(\,\cdot\,,t)\|_p \le C(\tau,S,R_1,R_2,p) \cdot \|g\|_{1,2;\infty}, \tag{2.16}$$

and for $x \in B^c_{R_2}$ and g as before

$$\mathfrak{A}(g)(x,t) \le C(\tau,S,R_1,R_2) \cdot \|g\|_{1,2;\infty} \cdot \big(\,|x| \cdot \nu(x)\,\big)^{-3/2}. \tag{2.17}$$

Proof. For $x \in B^c_{R_2},\ y \in B_{R_1}$, we have

$$\begin{aligned} |x-y| &\ge |x| \cdot (1-R_1/R_2) + |x| \cdot R_1/R_2 - |y| \\ &\ge |x| \cdot (1-R_1/R_2) + R_1 - |y| \ge |x| \cdot (1-R_1/R_2) \ge R_2 - R_1, \end{aligned} \tag{2.18}$$

and with Lemma 2.2,

$$\nu(x-y)^{-1} \le C \cdot (1+|y|) \cdot \nu(x)^{-1} \le C \cdot (1+R_1) \cdot \nu(x)^{-1}. \tag{2.19}$$

Take p,q,g as in (2.14). Referring to the preceding estimates and to Corollary 2.7 with $K = R_2 - R_1$, we find with the abbreviation $\widetilde{C} := C(\tau,R_1,R_2)$ that

(2.20)

$$\begin{aligned} &\|\mathfrak{A}(g)(\,\cdot\,,t)\|_q \\ &\quad\le \widetilde{C} \cdot \bigg(\int_{B^c_{R_2}} \Big(\int_0^t \int_{B_{R_1}} \big(\,|x-y| \cdot \nu(x-y) + t - \sigma\,\big)^{-2} \cdot |g(y,\sigma)|\, dy\, d\sigma \Big)^q dx \bigg)^{1/q} \\ &\quad\le \widetilde{C} \cdot \bigg(\int_{B^c_{R_2}} \Big(\int_0^t \big(\,|x| \cdot \nu(x) + t - \sigma\,\big)^{-2} \cdot \int_{B_{R_1}} |g(y,\sigma)|\, dy\, d\sigma \Big)^q dx \bigg)^{1/q} \\ &\quad\le \widetilde{C} \cdot \int_0^t \Big(\int_{B_{R_1}} |g(y,\sigma)|\, dy \Big) \cdot \Big(\int_{B^c_{R_2}} \big(\,|x| \cdot \nu(x) + t - \sigma\,\big)^{-2 \cdot q}\, dx \Big)^{1/q} d\sigma, \end{aligned}$$

where the last inequality follows by Minkowski's inequality for integrals [2, Theorem 2.9]. Put $\epsilon := (q-2)/(q+2)$. Since $q > 2$, we have $\epsilon > 0$ and $q > (q+2)/2$. It follows that

$$\epsilon \in (0,1), \quad (-1+\epsilon)\cdot q < (-1+\epsilon)\cdot (q+2)/2 = -2. \tag{2.21}$$

On the other hand, observing that $\int_{B_{R_1}} |g(y,\sigma)|\,dy \le C(R_1)\cdot \|g(\,\cdot\,,\sigma)\|_p$ for $\sigma \in (0,t)$, we deduce from (2.20) that

$$\begin{aligned} &\|\mathfrak{A}(g)(\,\cdot\,,t)\|_q \\ &\quad\le \widetilde{C}\cdot \|g\|_{p,\infty;\infty}\cdot \int_0^t \Big(\int_{B^c_{R_2}} \big(|x|\cdot\nu(x)+t-\sigma\big)^{(-1+\epsilon-1-\epsilon)\cdot q} dx\Big)^{1/q}\, d\sigma \\ &\quad\le \widetilde{C}\cdot \|g\|_{p,\infty;\infty}\cdot \int_0^t (R_2+t-\sigma)^{-1-\epsilon}\,d\sigma\cdot \left(\int_{B^c_{R_2}} \big(|x|\cdot\nu(x)\big)^{(-1+\epsilon)\cdot q} dx\right)^{1/q}. \end{aligned} \tag{2.22}$$

But with (2.21) and Lemma 2.1,

$$\begin{aligned} &\int_{B^c_{R_2}} \big(|x|\cdot\nu(x)\big)^{(-1+\epsilon)\cdot q} dx = \int_{R_2}^\infty r^{(-1+\epsilon)\cdot q}\int_{\partial B_r} \nu(x)^{(-1+\epsilon)\cdot q}\, do_x\, dr \\ &\quad\le \int_{R_2}^\infty r^{(-1+\epsilon)\cdot q}\int_{\partial B_r} \nu(x)^{-2}\, do_x\, dr \le C\cdot \int_{R_2}^\infty r^{(-1+\epsilon)\cdot q+1}\,dr \le C(q,R_2). \end{aligned}$$

Now inequality (2.14) follows from (2.22). Next take g, x as in (2.15). In the case $B_{R_1}\subset B_S(x)$, we have $|y-x|\le S$ for $y\in B_{R_1}$, so Corollary 2.7 with $K=S$ yields

$$\mathfrak{A}(g)(x,t) \le C(\tau,S)\cdot \int_0^t\int_{B_{R_1}} (|x-y|^2+t-\sigma)^{-2}\cdot |g(y,\sigma)|\,dy\,d\sigma;$$

hence with (2.18),

$$\begin{aligned} \mathfrak{A}(g)(x,t) &\le C(\tau,S)\cdot \int_0^t \Big[\big(|x|\cdot(1-R_1/R_2)\big)^2+t-\sigma\Big]^{-2}\cdot \int_{B_{R_1}} |g(y,\sigma)|\,dy\,d\sigma \\ &\le C(\tau,S)\cdot \|g\|_{1,\infty;\infty}\cdot \int_0^t \Big[\big(|x|\cdot(1-R_1/R_2)\big)^2+t-\sigma\Big]^{-2} d\sigma \\ &\le C(\tau,S)\cdot \|g\|_{1,\infty;\infty}\cdot \theta(x,R_1,R_2) \end{aligned}$$

with the term $\theta(x,R_1,R_2)$ defined in the lemma. Now suppose that $B_{R_1}\cap B_S(x)=\emptyset$, that is, $|x-y|\ge S$ for $y\in B_{R_1}$. Then Corollary 2.7 with $K=S$ implies

$$\mathfrak{A}(g)(x,t) \le C(\tau,S)\cdot \int_0^t\int_{B_{R_1}} (|x-y|\cdot\nu(x-y)+t-\sigma)^{-2}\cdot |g(y,\sigma)|\,dy\,d\sigma.$$

Using (2.18) and (2.19), and writing $\overline{C}$ instead of $C(\tau,S)$, we may conclude that

$$\begin{aligned} &\mathfrak{A}(g)(x,t) \\ &\quad\le \overline{C}\cdot \int_0^t \Big[|x|\cdot(1-R_1/R_2)\cdot \max\{1,\,\nu(x)/(1+R_1)\}+t-\sigma\Big]^{-2} \\ &\qquad\cdot \int_{B_{R_1}} |g(y,\sigma)|\,dy\,d\sigma \\ &\quad\le \overline{C}\cdot \|g\|_{1,\infty;\infty}\cdot \int_0^t \Big[|x|\cdot(1-R_1/R_2)\cdot \max\{1,\,\nu(x)/(1+R_1)\}+t-\sigma\Big]^{-2} d\sigma \\ &\quad\le \overline{C}\cdot \|g\|_{1,\infty;\infty}\cdot \theta(x,R_1,R_2). \end{aligned}$$

Thus we have proved (2.15). Next suppose that $R_2 - R_1 \geq S$, and let p, g be given as in (2.16). Due to (2.18), it follows that $|x-y| \geq S$ for $x \in B^c_{R_2},\ y \in B_{R_1}$, so Corollary 2.7 with $K = S$ implies

$$\|\mathfrak{A}(g)(\,\cdot\,,t)\|_p$$
$$\leq C(\tau,S)\cdot\Big(\int_{B^c_{R_2}}\Big(\int_0^t\int_{B_{R_1}}\big(\,|x-y|\cdot\nu(x-y)+t-\sigma\,\big)^{-2}\cdot|g(y,\sigma)|\,dy\,d\sigma\Big)^p\,dx\Big)^{1/p};$$

hence by (2.19) and again by (2.18),

$$\|\mathfrak{A}(g)(\,\cdot\,,t)\|_p$$
$$\leq C(\tau,S,R_1,R_2)\cdot\Big(\int_{B^c_{R_2}}\Big(\int_0^t\Big[|x|\cdot\nu(x)+t-\sigma\Big]^{-2}\cdot\int_{B_{R_1}}|g(y,\sigma)|\,dy\,d\sigma\Big)^p\,dx\Big)^{1/p}.$$

It follows by Hölder's inequality that

(2.23)

$$\|\mathfrak{A}(g)(\,\cdot\,,t)\|_p$$
$$\leq\ C(\tau,S,R_1,R_2)\cdot\bigg(\int_{B^c_{R_2}}\bigg[\Big(\int_0^t\Big[|x|\cdot\nu(x)+t-\sigma\Big]^{-4}d\sigma\Big)^{1/2}\cdot\|g\|_{1,2;\infty}\bigg]^p\,dx\bigg)^{1/p}$$
$$\leq C(\tau,S,R_1,R_2)\cdot\|g\|_{1,2;\infty}\cdot\bigg(\int_{B^c_{R_2}}\Big[|x|\cdot\nu(x)\Big]^{-3\cdot p/2}\,dx\bigg)^{1/p}.$$

But with Lemma 2.1 and the assumption $p > 4/3$,

$$\int_{B^c_{R_2}}\big(\,|x|\cdot\nu(x)\,\big)^{-3\cdot p/2}=\int_{R_2}^\infty r^{-3\cdot p/2}\cdot\int_{\partial B_r}\nu(x)^{-3\cdot p/2}\,do_x\,dr$$
$$\leq\int_{R_2}^\infty r^{-3\cdot p/2}\cdot\int_{\partial B_r}\nu(x)^{-3/2}\,do_x\,dr\leq C\cdot\int_{R_2}^\infty r^{-3\cdot p/2+1}\,dr\leq C(p)\cdot R_2^{-3\cdot p/2+2}.$$

Inserting this inequality into (2.23), we obtain (2.16). Again using Corollary 2.7 with $K = S$, as well as (2.18) and (2.19), we get for $x \in B^c_{R_2}$ that

$$\mathfrak{A}(g)(x,t)\leq C(\tau,S)\cdot\int_0^t\int_{B_{R_1}}(|x-y|\cdot\nu(x-y)+t-\sigma)^{-2}\cdot|g(y,\sigma)|\,dy\,d\sigma$$
$$\leq C(\tau,S)\cdot\int_0^t\Big[|x|\cdot(1-R_1/R_2)\cdot\nu(x)/(1+R_1)+t-\sigma\Big]^{-2}\cdot\int_{B_{R_1}}|g(y,\sigma)|\,dy\,d\sigma$$
$$\leq C(\tau,S)\cdot\|g\|_{1,2;\infty}\cdot\Big(\int_0^t\Big[|x|\cdot(1-R_1/R_2)\cdot\nu(x)/(1+R_1)+t-\sigma\Big]^{-4}d\sigma\Big)^{1/2}$$
$$\leq C(\tau,S)\cdot\|g\|_{1,2;\infty}\cdot\Big[|x|\cdot(1-R_1/R_2)\cdot\nu(x)/(1+R_1)\Big]^{-3/2},$$

so we have proved (2.17). □

3. Some properties of *U* and *u*. We begin with some observations with respect to the function U from Theorem 1.1.

LEMMA 3.1. *We have*

$$\|U\|_p \le C(S_0,p)\cdot(\|U\|_6+\mu_0) \le \mathfrak{C}(p) \quad \textit{for } p\in(2,6],$$
$$\|\nabla U\|_p \le C(S_0,p)\cdot(\|\nabla U\|_2+\mu_0) \le \mathfrak{C}(p) \quad \textit{for } p\in(4/3,2],$$
$$\|U|B^c_{S_0}\|_p \le C(S_0,p)\cdot\mu_0 \le \mathfrak{C}(p) \quad \textit{for } p\in(2,\infty],$$
$$\|\nabla U|B^c_{S_0}\|_p \le C(S_0,p)\cdot\mu_0 \le \mathfrak{C}(p) \quad \textit{for } p\in(4/3,\infty].$$

Proof. Let $\alpha\in\mathbb{N}_0^3$ with $|\alpha|\le 1$. With Lemma 2.1 we obtain that the integral $\int_{B^c_{S_0}}\big(|y|\cdot\nu(y)\big)^{(-1-|\alpha|/2)\cdot p}\,dy$ is bounded by a constant $C(p,S_0)$ for $p\in(2,\infty)$ if $\alpha=0$ and for $p\in(4/3,\infty)$ if $|\alpha|=1$. Since $U\in L^6(\overline{\Omega}^c)^3$ and $\nabla U\in L^2(\overline{\Omega}^c)^9$, and because of (1.10), the lemma follows. □

We are going to show that the function F from (1.20) is in $L^2\big(0,\infty,L^p(\overline{\Omega}^c)^3\big)$ for a range of p including the interval $[6/5,3/2]$. To this end, we draw some conclusions from (1.16), (1.17), and Theorem 1.1.

LEMMA 3.2. *We have* $1\le 2/(1+2/s_0)<6/5<2/(1+2/r_0)<2$. *Let* $p\in[s_0,r_0]$, *and put* $\theta:=(1/p-1/r_0)/(1/s_0-1/r_0)^{-1}$. *Then*

$$\begin{aligned}(3.1)\quad &\big\|\,|u|\cdot|\nabla_x u|\,\big\|_{2/(1+2/p),2;T_0}\\ &\quad\le C\cdot\big(\theta\cdot\|u\|_{s_0,\infty;T_0}+(1-\theta)\cdot\|u\|_{r_0,\infty;T_0}\big)\cdot\|\nabla_x u\|_2\le\mathfrak{C},\end{aligned}$$

$$(3.2)\quad \big\|\,|U|\cdot|\nabla_x u|\,\big\|_{11/10,2;T_0}\le C(S_0)\cdot(\|U\|_6+\mu_0)\cdot\|\nabla_x u\|_2\le\mathfrak{C},$$

$$(3.3)\quad \big\|\,|U|\cdot|\nabla_x u|\,\big\|_{3/2,2;T_0}\le\|U\|_6\cdot\|\nabla_x u\|_2\le\mathfrak{C},$$

$$(3.4)\quad \big\|\,|u|\cdot|\nabla U|\,\big\|_{11/10,2;T_0}\le C(S_0)\cdot\|u\|_{2,6;T_0}\cdot(\|\nabla U\|_2+\mu_0)\le\mathfrak{C},$$

$$(3.5)\quad \big\|\,|u|\cdot|\nabla U|\,\big\|_{3/2,2;T_0}\le\|u\|_{6,2;T_0}\cdot\|\nabla U\|_2\le\mathfrak{C}.$$

Proof. The inequalities in the first line of the lemma follow from the relations $s_0\in[2,3)$ and $r_0\in(3,\infty)$. For $\beta\in[2,\infty)$, we have $2/(1+2/\beta)\ge 1$, so Hölder's inequality yields

$$(3.6)\quad \big\|\,|V|\cdot|\nabla_x u|\,\big\|_{2/(1+2/\beta),2;T_0}\le\|V\|_\beta\cdot\|\nabla u\|_2 \quad \text{for } V\in L^\beta(\overline{\Omega}^c).$$

These observations and the inequality

$$\|u(t)\|_p\le\|u(t)\|_{s_0}^\theta\cdot\|u(t)\|_{r_0}^{1-\theta}\le\theta\cdot\|u(t)\|_{s_0}+(1-\theta)\cdot\|u(t)\|_{r_0}$$

for $t\in(0,T_0)$ prove (3.1). Inequality (3.2) follows from (3.6) with $\beta=22/9$ and Lemma 3.1, whereas (3.3) may be obtained by the same references, but with $\beta=6$ in (3.6). We further remark that for $\beta\in(4/3,2]$, Hölder's inequality implies that $\big\|\,|u|\cdot|\nabla U|\,\big\|_{6/(1+6/\beta),2;T_0}\le\|u\|_{2,6;T_0}\cdot\|\nabla U\|_\beta$. Inequality (3.4) is a consequence of this estimate with $\beta=66/49$ and Lemmas 2.3 and 3.1. As concerns (3.5), we may use the preceding inequality with $\beta=2$ and then apply Lemmas 2.3 and 3.1 again. □

If $r_0\ge 6$, we may use inequality (3.1) with $p=6$ to obtain $(u\cdot\nabla_x u)\in L^2\big(0,T_0,L^{3/2}(\overline{\Omega}^c)^3\big)$. Therefore the preceding relation is redundant as an assumption in (1.17) if $r_0\ge 6$.

COROLLARY 3.3. *Put* $\sigma_0:=\max\{11/10,\,2/(1+2/s_0)\}$. *Then* $\sigma_0\in(1,6/5)$ *and* $\|F\|_{p,2;T_0}\le\mathfrak{C}$, $\|G\|_{p,2;T_0}\le\mathfrak{C}$ *for* $p\in[\sigma_0,3/2]$, *where* F *and* G *were introduced in* (1.20) *and* (2.1), *respectively.*

Proof. See (1.16), (1.17), and Lemmas 3.2 and 2.14. ◻

Now we are in a position to derive the representation formula of u mentioned at the end of section 1, applying a result from [18].

THEOREM 3.4. *There is a unique function* $\psi \in L^2_n(S_{T_0})$ *with*

$$\mathfrak{V}^{(\tau)}(\psi)|S_{T_0} = \big(-\mathfrak{R}^{(\tau)}(F) - \mathfrak{I}^{(\tau)}(a) + b\big)\,|\,S_{T_0}. \tag{3.7}$$

This function verifies the estimate $\|\psi\|_2 \le \mathfrak{C}$. *The function* u *from* (1.16)–(1.19) *is given by* $u = \big(\mathfrak{R}^{(\tau)}(F) + \mathfrak{I}^{(\tau)}(a) + \mathfrak{V}^{(\tau)}(\psi)\big)\,|\,Z_{T_0}$.

Proof. Theorem 3.4 follows from [18, Corollary 4.5] with h replaced by F, Corollary 3.3, the relation $b \in H_\infty$, and the assumptions on a and u. We note that [18, Corollary 4.5] is a consequence of the representation formula for Oseen flows established in [16]. ◻

Since $F = f - G$, hence $\mathfrak{R}^{(\tau)}(F) = \mathfrak{R}^{(\tau)}(f) - \mathfrak{R}^{(\tau)}(G)$, and because the asymptotic behavior of $\mathfrak{R}^{(\tau)}(f)$, $\mathfrak{I}^{(\tau)}(a)$, and $\mathfrak{V}^{(\tau)}(\psi)$ has already been studied, we may now reduce the proof of Theorem 1.2 to an estimate of $\mathfrak{R}^{(\tau)}(G)$.

COROLLARY 3.5. *Abbreviate* $\mathfrak{J} := \mathfrak{R}^{(\tau)}(f) + \mathfrak{I}^{(\tau)}(a) + \mathfrak{V}^{(\tau)}(\psi)\,|\,Z_{T_0}$ *with* ψ *from Theorem* 3.4. *Then*

$$u = \mathfrak{J} - \mathfrak{R}^{(\tau)}(G). \tag{3.8}$$

Let $S \in (S_0, \infty),\ x \in B_S^c,\ t \in (0, \infty),\ \alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$. *Then*

$$|\partial_x^\alpha \mathfrak{J}(x,t)| \le \mathfrak{C}(S) \cdot \big(|x| \cdot \nu(x)\big)^{-1-|\alpha|/2}. \tag{3.9}$$

Proof. Equation (3.8) follows from Theorem 3.4 and the equation $F = f - G$ (see (1.20) and (2.1)). Inequality (3.9) is a consequence of (3.8), Theorems 2.15 and 2.16, and Lemma 2.17. ◻

In the next step we derive two estimates related to the product $U_k \cdot u_l$.

COROLLARY 3.6. *Let* $S \in (S_0, \infty),\ \kappa \in (3, \infty],\ t \in (0, T_0),\ l \in \{1, 2, 3\}$. *Then the inequality* $\|\partial_l \mathfrak{R}^{(\tau)}(u_l \cdot U + U_l \cdot u)(\,\cdot\,, t)|B_S^c\|_\kappa \le \mathfrak{C}(S, \kappa)$ *holds.*

Proof. Suppose $\kappa < \infty,\ j, k \in \{1, 2, 3\}$. Put

$$\mathfrak{A} := \bigg(\int_{B_S^c} \bigg(\int_0^t \int_{\overline{\Omega}^c} H(x,y,t,s)\,dy\,ds\bigg)^\kappa dx\bigg)^{1/\kappa}$$

with $H(x,y,t,s) := |\partial_l \Lambda_{jk}(x-y, t-s, \tau)| \cdot |u(y,s)| \cdot |U(y)|$ for $x \in B_S^c,\ y \in \overline{\Omega}^c,\ s \in (0,t)$. Then $\mathfrak{A} \le \mathfrak{A}_1 + \mathfrak{A}_2$, where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are defined in the same way as $\mathfrak{A}$, except that the domain of integration $\overline{\Omega}^c$ is replaced by Ω_{S_0} in the case of $\mathfrak{A}_1$ and by $B_{S_0}^c$ else. Inequality (2.14) yields $\mathfrak{A}_1 \le C(\tau, S_0, S) \cdot \big\|\,|u| \cdot |U|\,|\,\Omega_{S_0} \times (0, T_0)\,\big\|_{1,\infty,T_0}$, so that

$$\mathfrak{A}_1 \le C(\tau, S_0, S) \cdot \|u\|_{s_0,\infty,T_0} \cdot \|U|\Omega_{S_0}\|_{s_0'} \le C(\tau, S_0, S) \cdot \|u\|_{s_0,\infty,T_0} \cdot \|U\|_6, \tag{3.10}$$

where the last inequality holds because $s_0 \in [2,3)$, so that $s_0' \in (3/2, 2]$. We further observe that $\mathfrak{A}_2 \le \mathfrak{B}_1 + \mathfrak{B}_2$ with $\mathfrak{B}_1, \mathfrak{B}_2$ obtained from $\mathfrak{A}_2$ by introducing the additional factor $\chi_{(0,1)}(t-s)$ in the case of $\mathfrak{B}_1$ and $\chi_{[1,\infty)}(t-s)$ in the case of $\mathfrak{B}_2$, inside the triple integral abbreviated by $\mathfrak{A}_2$. By Theorem 2.8 with $M = 1,\ s = r = \infty,\ p = \kappa,\ q = 3$, and Hölder's inequality we get

$$\begin{aligned} \mathfrak{B}_1 &\le C(\tau, \kappa) \cdot \big\|\,|u| \cdot |U|\,|\,B_{S_0}^c \times (0, T_0)\,\big\|_{3,\infty;T_0} \\ &\le C(\tau, \kappa) \cdot \|u\|_{r_0,\infty;T_0} \cdot \|U|B_{S_0}^c\|_{3\cdot(1-3/r_0)^{-1}}. \end{aligned} \tag{3.11}$$

Now Lemma 3.1 implies that $\mathfrak{B}_1 \le C(\tau, S_0, \kappa) \cdot \|u\|_{r_0,\infty;T_0} \cdot \mu_0$. Again referring to Theorem 2.8, this time with $M = 1,\ r = \infty,\ s = 2,\ p = \kappa,\ q = 2$, we obtain

$$\mathfrak{B}_2 \le C(\tau, \kappa) \cdot \big\| \, |u| \cdot |U| \, \big\|_2 \le C(\tau, \kappa) \cdot \|u\|_{6,2;T_0} \cdot \|U\|_3.$$

Hence by Lemmas 3.1 and 2.3, $\mathfrak{B}_2 \le C(\Omega, \tau, S_0, \kappa) \cdot \|\nabla_x u\|_2 \cdot (\|U\|_6 + \mu_0)$. Thus, if $\kappa < \infty$, the corollary follows from Lemma 2.10, (3.10), and the preceding estimates of $\mathfrak{B}_1$ and $\mathfrak{B}_2$.

In the case $\kappa = \infty$, we handle the term $\mathfrak{A}_1$ by using inequality (2.15) with S replaced by $S + S_0$, for example. Moreover, instead of (3.11), we observe that by Theorem 2.8 and Lemma 3.1,

$$\begin{aligned}\mathfrak{B}_1 &\le C(\tau, \kappa, r_0) \cdot \big\| \, |u| \cdot |U| \, | \, B^c_{S_0} \times (0, T_0) \, \big\|_{r_0,\infty;T_0} \\ &\le C(\tau, \kappa, r_0) \cdot \|u\|_{r_0,\infty;T_0} \cdot \|U|B^c_{S_0}\|_\infty \le C(\tau, \kappa, r_0, S_0) \cdot \|u\|_{r_0,\infty;T_0} \cdot \mu_0.\end{aligned}$$

The estimate of $\mathfrak{B}_2$ given above remains valid. ∎

THEOREM 3.7. *There is $\sigma_1 \in (1, 2)$ with $\|u\|_{p,\infty;T_0} \le \mathfrak{C}(p)$ for $p \in [\sigma_1, 2]$. Moreover the estimate $\big\| \, |u| \cdot |U| \, \big\|_{1,\infty;T_0} \le \mathfrak{C}$ holds.*

Proof. For $x \in B_{2\cdot S_0},\ y \in \partial\Omega$, we have $|x - y| \ge |x| - |y| \ge |x| - S_0 \ge |x|/2 \ge S_0$ and $\nu(x - y) \ge C(S_0) \cdot \nu(x)$, where the last inequality holds according to Lemma 2.2.

Recall the function ψ introduced in Theorem 3.4. For $x \in \mathbb{R}^3 \backslash \partial\Omega,\ t \in (0, \infty)$, let the functions $V^{(1)}(x, t)$, $V^{(2)}(x, t)$ be defined as $\mathfrak{V}^{(\tau)}(\psi)(x, t)$ (see (2.6)), but with an additional factor $\chi_{(0,1)}(t - \sigma)$ and $\chi_{[1,\infty)}(t - \sigma)$, respectively, inside the double integral in (2.6). Let $p \in (3/2, 2],\ t \in (0, \infty)$. By Corollary 2.7 with $K = S_0$ and the two estimates at the beginning of this proof, we obtain $|\Lambda_{jk}(x - y, t - \sigma, \tau)| \le C(\tau, S_0) \cdot \big(|x| \cdot \nu(x) \big)^{-3/2}$ for $x \in B^c_{2\cdot S_0},\ y \in \partial\Omega,\ \sigma \in (0, t),\ 1 \le j, k \le 3$. Therefore

$$\|V^{(1)}(\,\cdot\,, t)|B^c_{2\cdot S_0}\|_p \le C(\tau, S_0) \cdot \left(\int_{B^c_{2\cdot S_0}} \big(|x| \cdot \nu(x) \big)^{-3\cdot p/2} \, dx \right)^{1/p} \cdot \left(\int_0^t \int_{\partial\Omega} \chi_{(0,1)}(t - \sigma) \cdot |\psi(y, \sigma)| \, do_y \, d\sigma \right). \tag{3.12}$$

Since $p > 3/2$, we have $3 \cdot p/2 > 9/4$; hence the first integral on the right-hand side of (3.12) is bounded by $C(S_0, p) \cdot \int_{B^c_{2\cdot S_0}} \big(|x| \cdot \nu(x) \big)^{-9/4} dx$. But the preceding integral may be rewritten as $\int_{2\cdot S_0}^\infty r^{-9/4} \cdot \int_{\partial B_r} \nu(x)^{-9/4} \, do_x \, dr$. Thus, on applying Lemma 2.1, we see that the first integral on the right-hand side of (3.12) may be estimated by a constant depending on S_0 and p. Concerning the second integral, we apply may apply Hölder's inequality obtaining $\|V^{(1)}(\,\cdot\,, t)|B^c_{2\cdot S_0}\|_p \le C(\Omega, \tau, S_0, p) \cdot \|\psi\|_2$. In order to estimate $\|V^{(2)}(\,\cdot\,, t)\|_p$, we apply Minkowski's inequality to obtain

$$\|V^{(2)}(\,\cdot\,, t)\|_p \le \sum_{k=1}^3 \int_0^t \int_{\partial\Omega} \left(\int_{\mathbb{R}^3} |\Lambda_{jk}(x - y, t - \sigma, \tau)|^p \, dx \right)^{1/p} \cdot |\psi(y, \sigma)| \, do_y \, d\sigma.$$

Now we may proceed as in [13, inequality (25)], using Lemma 2.4, but with p in the place of the exponent 2 considered in [13]. We get

$$\|V^{(2)}(\,\cdot\,, t)\|_p \le C(\tau, p) \cdot \int_0^t \chi_{[1,\infty)}(t - \sigma) \cdot (t - \sigma)^{-3/(2\cdot p')} \cdot \|\psi(\,\cdot\,, \sigma)\|_1 \, d\sigma.$$

Since $p > 3/2$, we have $-3/p' < -1$, so $\int_0^t \chi_{[1,\infty)}(t-\sigma)\cdot(t-\sigma)^{-3/p'}\,d\sigma \le C(p)$. Thus the preceding estimate of $\|V^{(2)}(\,\cdot\,,t)\|_p$ and Hölder's inequality yield $\|V^{(2)}(\,\cdot\,,t)\|_p \le C(\tau,p)\cdot\|\psi\|_2$. Altogether we have shown that $\|\mathfrak{V}^{(\tau)}(\psi)(\,\cdot\,,t)\|_p \le C(\Omega,\tau,S_0,p)\cdot\|\psi\|_2$. Hence with the estimate of ψ in Theorem 3.4, $\|\mathfrak{V}^{(\tau)}(\psi)\|_{p,2;\infty} \le \mathfrak{C}(p)$ for any $p \in (3/2, 2]$.

For $x \in \mathbb{R}^3$, $t \in (0,\infty)$, let $R^{(1)}(x,t)$, $R^{(2)}(x,t)$ be defined as $\mathfrak{R}^{(\tau)}(F)(x,t)$ (see (2.5)), but with an additional factor $\chi_{(0,1)}(t-\sigma)$ and $\chi_{[1,\infty)}(t-\sigma)$, respectively, inside the double integral in (2.5). Let $\kappa \in (3/2, 3)$. By Theorem 2.8 with $M = 1$, $r = \infty$, $s = 2$, $p = \kappa$, $q = 3/2$, we get $\|R^{(1)}\|_{\kappa,\infty;\infty} \le C(\tau,\kappa)(\kappa)\cdot\|F\|_{3/2,2;\infty}$. Consider the parameter σ_0 from Corollary 3.3. Since $1 < \sigma_0 < 6/5$, we have $3/2 < (1/\sigma_0 - 1/3)^{-1} < 2$. Let $\kappa \in ((1/\sigma_0 - 1/3)^{-1}, 2]$. Then we have in particular $\kappa > 6/5 > \sigma_0$ and $1/\sigma_0 - 1/3 > 1/\kappa$ so that $1 + 3/(2\cdot\kappa) - 3/(2\cdot\sigma_0) < 1/2$. Therefore Theorem 2.8 with $M = 1$, $r = \infty$, $s = 2$, $p = \kappa$, $q = \sigma_0$ yields $\|R^{(2)}\|_{\kappa,\infty;\infty} \le C(\tau,\kappa,\sigma_0)\cdot\|F\|_{\sigma_0,2;\infty}$. Thus, with the above estimate of $\|R^{(1)}\|_{\kappa,\infty;\infty}$ and Corollary 3.3, we arrive at the inequality $\|\mathfrak{R}^{(\tau)}(F)\|_{\kappa,\infty;\infty} \le \mathfrak{C}(\kappa)$ for $\kappa \in ((1/\sigma_0 - 1/3)^{-1}, 2]$.

Let $p \in (2/(1+\kappa_0), 2]$, with κ_0 from (1.14). Then we obtain with the latter assumption that $\|a|B^c_{S_0}\|_p^p \le \delta_0\cdot\int_{B^c_{S_0}}\big(|y|\cdot\nu(y)\big)^{(-1-\kappa_0)\cdot p}\,dy$. The integral on the right-hand side of this estimate is bounded by $\int_{S_0}^\infty r^{(-1-\kappa_0)\cdot p}\cdot\int_{\partial B_r}\nu(y)^{(-1-\kappa_0)\cdot p}\,do_y\,dr$, so we get with Lemma 2.1 and because $(1+\kappa_0)\cdot p > 2$ that $\|a|B^c_{S_0}\|_p \le \delta_0\cdot C(p,\kappa_0)$. Obviously $\|a|\Omega_{S_0}\|_p \le C(S_0)\cdot\|a|\Omega_{S_0}\|_2 \le \mathfrak{C}$, so we have found that $\|a\|_p \le \mathfrak{C}(p)$. Since $\int_{\mathbb{R}^3}\mathfrak{H}(y,t)\;dy = 1$ for $t \in (0,\infty)$, it follows with Young's inequality that $\|\mathfrak{I}^{(\tau)}(a)(\,\cdot\,,t)\|_p \le \|a\|_p \le \mathfrak{C}(p)$ for $t \in (0,\infty)$.

Due to the preceding estimates of $\mathfrak{V}^{(\tau)}(\psi)$, $\mathfrak{R}^{(\tau)}(F)$ and $\mathfrak{I}^{(\tau)}(a)$, we may refer to Theorem 3.4, concluding that $\|u\|_{p,\infty;T_0} \le \mathfrak{C}(p)$ for any p belonging to the interval $\big(\max\{(1/\sigma_0 - 1/3)^{-1}, 2/(1+\kappa_0)\},\ 2\big]$. This proves the first claim of the theorem. The second follows from the first and Lemma 3.1. ∎

Finally we transform $\mathfrak{R}(G)$ by a partial integration in order to obtain a variant of (3.8) and (3.9), respectively, that will be useful in the following.

Lemma 3.8. *Let $j,k \in \{1, 2, 3\}$, $x \in \overline{\Omega}^c$, $t \in (0,T_0)$. Then*

$$\int_{\overline{\Omega}^c}\Lambda_{jk}(x-y,t-\sigma,\tau)\cdot G_k(y,\sigma)\,dy = -\int_{\partial\Omega}\Lambda_{jk}(x-y,t-\sigma,\tau)\cdot g_k^{(b)}(y,\sigma)\,d\Omega(y)$$
$$+\int_{\overline{\Omega}^c}\sum_{l=1}^3\partial_l\Lambda_{jk}(x-y,t-\sigma,\tau)\cdot H_{kl}(y,\sigma)\,dy$$

for $\sigma \in (0,t)$. (The functions $G, H, g^{(b)}$ were introduced in (2.1)–(2.3)*). In particular*

$$\mathfrak{R}_j^{(\tau)}(G)(x,t) = -\mathfrak{V}_j^{(\tau)}(g^{(b)})(x,t) + \sum_{l=1}^3\partial_l\mathfrak{R}_j^{(\tau)}\big((H_{ml})_{1\le m\le 3}\big)(x,t). \tag{3.13}$$

Proof. Take $\sigma \in (0,t)$. Put $h := |u(\,\cdot\,,\sigma)|^2 + |u(\,\cdot\,,\sigma)|\cdot|U|$. Since $U, u(\,\cdot\,,\sigma) \in L^3(\overline{\Omega}^c)^3$ (see (1.16) and Lemma 3.1), we obtain that $h \in L^{3/2}(\overline{\Omega}^c)$. We further note the equation $\int_{B^c_R} h(y)^{3/2}\,dy = \int_R^\infty\int_{\partial B_r} h(y)^{3/2}\,do_y\,dr$ for $R \in [S_0,\infty)$. Suppose for a contradiction that there is $R_0 \in [S_0,\infty)$ with $\int_{\partial B_r} h(y)^{3/2}\,do_y \ge 1/r$ for $r \in [R_0,\infty)$. Then $\int_{B_{R_0}} h(y)^{3/2}\,dy = \infty$ by the preceding equation, in contradiction to the relation $h \in L^{3/2}(\overline{\Omega}^c)$. Therefore we may choose a strictly increasing sequence (R_n) in $[S_0,\infty)$

with $R_n \to \infty$ and $\int_{\partial B_{R_n}} h(y)^{3/2}\, do_y \le 1/R_n$ for $n \in \mathbb{N}$. On the other hand, in view of (1.18), and by Theorem 1.1, we get with a partial integration that

$$\begin{aligned}(3.14)\qquad &\int_{\Omega_{R_n}} \Lambda_{jk}(x-y, t-\sigma, \tau)\cdot G_k(y,\sigma)\, dy \\ &= A_n - \int_{\partial\Omega} \Lambda_{jk}(x-y,t-\sigma,\tau)\cdot g_k^{(b)}(y,\sigma)\, d\Omega(y) \\ &\quad + \int_{\Omega_{R_n}} \sum_{l=1}^{3} \partial_l \Lambda_{jk}(x-y,t-\sigma,\tau)\cdot H_{lk}(y,\sigma)\, dy \quad \text{for } n\in\mathbb{N},\end{aligned}$$

where $A_n := \int_{\partial B_{R_n}} \sum_{l=1}^{3} \Lambda_{jk}(x-y,t-\sigma,\tau)\cdot H_{lk}(y,\sigma)\cdot y_l/R_n\, do_y$. Let $n_0 \in \mathbb{N}$ be so large that $|x| \le R_{n_0}/2$ and $S_0 \le R_{n_0}$. Then, for $n \in \mathbb{N}$ with $n \ge n_0$ and for $y \in \partial B_{R_n}$, we find

$$|x-y| \ge R_n - |x| \ge R_n/2 + R_{n_0}/2 - |x| \ge R_n/2 \ge R_{n_0}/2 \ge S_0/2.$$

Corollary 2.7 with $K = S_0/2$ now yields

$$|A_n| \le C(\tau, S_0)\cdot \int_{\partial B_{R_n}} \big(\, |x-y|\cdot \nu(x-y) + t - \sigma\,\big)^{-3/2} \cdot h(y)\, do_y.$$

We thus get $|A_n| \le C(\tau,S_0)\cdot R_n^{-3/2} \int_{\partial B_{R_n}} h(y)\ do_y$, so with Hölder's inequality, $|A_n| \le C(\tau,S_0)\cdot R_n^{-5/6}\cdot (\int_{\partial B_{R_n}} h(y)^{3/2}\, do_y)^{2/3}$ if $n \ge n_0$. Recalling the choice of the sequence (R_n), we may conclude that $|A_n| \le C(\tau,S_0)\cdot R_n^{-3/2}$ for $n \in \mathbb{N},\ n \ge n_0$. Thus we obtain that $A_n \to 0$ for $n \to \infty$. As concerns the two integrals over Ω_{R_n} in (3.14), they converge to the corresponding integrals over $\overline{\Omega}^c$ for $n \to \infty$. This follows from Corollary 2.9, the relations $G \in L^2\big(0, T_0, L^{3/2}(\overline{\Omega}^c)^3\big)$ (Corollary 3.3), $H_{kl} \in L^\infty\big(0,T_0,L^1(\overline{\Omega}^c)^3\big)$ (Theorem 3.7), and Lebesgue's theorem. Therefore, on letting n tend to infinity in (3.14), we arrive at the first equation in Lemma 3.8. Integrating this equation with respect to σ, summing up with respect to k, and applying Lemma 2.10 yields the second equation. □

The function $\mathfrak{V}^{(\tau)}(g^{(b)})$ may be estimated as follows.

Lemma 3.9. *The function $g^{(b)}$ belongs to $L^2\big(0,\infty,L^1(\partial\Omega)^3\big) + L^1(S_\infty)^3$, and*

$$\begin{aligned}(3.15)\qquad &\int_0^t \int_{\partial\Omega} \sum_{k=1}^{3} |\partial_x^\alpha \Lambda_{jk}(x-y,s-\sigma,\tau)\cdot g_k^{(b)}(y,\sigma)|\, d\Omega(y)\, d\sigma \\ &\le \mathfrak{C}(S)\cdot \big(\,|x|\cdot\nu(x)\,\big)^{-1-|\alpha|/2}\end{aligned}$$

for $S \in (S_0,\infty),\ x \in B_S^c,\ t\in(0,\infty),\ j\in\{1,2,3\},\ \alpha\in\mathbb{N}_0^3$ with $|\alpha|\le 1$. This means in particular that $|\partial_x^\alpha \mathfrak{V}_j^{(\tau)}(g^{(b)})(x,t)|$ is bounded by the right-hand side of (3.15).

Proof. Since $U|\partial\Omega = \widetilde{b}$ (Theorem 1.1), we get by a standard trace theorem that $\|\widetilde{b}\|_2 \le C(\Omega,S_0)\cdot\|U|\Omega_{S_0}\|_{1,2}$. Hence $\|\widetilde{b}\|_2 \le C(\Omega,S_0)\cdot(\|U|\Omega_{S_0}\|_6 + \|\nabla U|\Omega_{S_0}\|_2) \le \mathfrak{C}$, in particular, $|b|\cdot|\widetilde{b}| \in L^2\big(0,\infty,L^1(\partial\Omega)\big)$. Obviously $|b|^2 \in L^1(S_\infty)$. As a consequence, Lemma 2.17 with $\kappa=1$ yields that the left-hand side of (3.15) is bounded by $C(\Omega,\tau,S_0,S)\cdot(\|b\|_2\cdot\|\widetilde{b}\|_2 + \|b\|_2^2)\cdot\big(\,|x|\cdot\nu(x)\,\big)^{-1-|\alpha|/2}$. Recalling the estimate of $\|\widetilde{b}\|_2$ given above, we see that $\|b\|_2\cdot\|\widetilde{b}\|_2 + \|b\|_2^2 \le \mathfrak{C}$, so (3.15) is proved. The last claim of the lemma follows from that latter estimate and (2.7). □

At this point we may prove the variants of (3.8) and (3.9) mentioned above.

Corollary 3.10. *Abbreviate* $\mathfrak{K} := \mathfrak{J} + \mathfrak{V}^{(\tau)}(g^{(b)})$ *with* $g^{(b)}$ *from* (2.3) *and* ψ *from Theorem* 3.4. *Then*

$$u = \mathfrak{K} - \sum_{l=1}^{3} \partial_l \mathfrak{R}^{(\tau)}\big((H_{ml})_{1 \le m \le 3} \big). \tag{3.16}$$

Let $S \in (S_0, \infty),\ x \in B_S^c,\ t \in (0, \infty),\ \alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$. *Then*

$$|\partial_x^\alpha \mathfrak{K}(x,t)| \le \mathfrak{C}(S) \cdot \big(|x| \cdot \nu(x) \big)^{-1-|\alpha|/2}. \tag{3.17}$$

Proof. Equation (3.16) follows from (3.8) and (3.13), whereas inequality (3.17) is a consequence of (3.9) and Lemma 3.9. □

Corollary 3.11. *Let* $S \in (S_0, \infty),\ \alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1,\ p \in [3, \infty)$ *if* $\alpha = 0$, *and* $p \in [2, \infty)$ *if* $|\alpha| = 1$. *Let* $t \in (0, T_0)$. *Then*

$$\|\partial_x^\alpha \mathfrak{J}(\,\cdot\,, t) \,|\, B_S^c\|_p + \|\partial_x^\alpha \mathfrak{K}(\,\cdot\,, t) \,|\, B_S^c\|_p \le \mathfrak{C}(S). \tag{3.18}$$

Proof. Let us estimate the term $(\int_{B_S^c} \big(|x| \cdot \nu(x) \big)^{-p-|\alpha| \cdot p/2}\, dx)^{1/p}$ in order to exploit (3.9) and (3.17). Consider the case $\alpha = 0$ (hence $p \ge 3$). Then, with Lemma 2.1,

$$\begin{aligned}\Big(\int_{B_S^c} \big(|x| \cdot \nu(x) \big)^{-p-|\alpha| \cdot p/2}\, dx \Big)^{1/p} &= \Big(\int_S^\infty r^{-p} \cdot \int_{\partial B_r} \nu(x)^{-p}\, do_x\, dr \Big)^{1/p} \\ &\le \Big(\int_S^\infty r^{-p} \cdot \int_{\partial B_r} \nu(x)^{-3}\, do_x\, dr \Big)^{1/p} \le C \cdot \Big(\int_S^\infty r^{-p+1}\, dr \Big)^{1/p} \\ &= C \cdot (p-2)^{-1/p} \cdot S^{-1+2/p}.\end{aligned} \tag{3.19}$$

Since $p \ge 3$, we have $(p-2)^{-1/p} \le 1,\ S^{-1+2/p} \le \max\{S^{-1}, 1\}$. Thus we may conclude from (3.19) that $(\int_{B_S^c} \big(|x| \cdot \nu(x) \big)^{-p}\, dx)^{1/p} \le C(S)$. Inequality (3.18) now follows from (3.9) and (3.17). An analogous argument may be used in the case $|\alpha| = 1$. □

As the reader may easily verify (compare Lemma 3.1), inequality (3.18) holds for $p \in (2, \infty)$ in the case $\alpha = 0$, and for $p \in (4/3, \infty)$ if $|\alpha| = 1$. But the range of parameters p considered in Corollary 3.11 is sufficient for our purposes. And since this range stays away from, respectively, 2 ($\alpha = 0$) and 4/3 ($|\alpha| = 1$), the constant in inequality (3.18) may be chosen independent of p, a fact that will be useful later on (proof of Lemma 4.1).

4. Proof of Theorem 1.2. We begin by showing that u belongs to the function space $L^\infty\big(0, T_0, L^{\nu_0}(B_S^c)^3\big)$ for some $\nu_0 \in (6, \infty)$.

Lemma 4.1. *There is some* $\nu_0 \in (6, \infty)$ *such that* $\|u | B_S^c \times (0, T_0)\|_{\nu_0, \infty; T_0} \le \mathfrak{C}(S)$ *for* $S \in (S_0, \infty)$.

Proof. If $r_0 > 6$, assumption (1.16) implies that we may take $\nu_0 = r_0$. Now suppose that $r_0 \le 6$. Then in particular $1/r_0 - (1/3 - 1/r_0)/2 = 3/(2 \cdot r_0) - 1/6 > 0$. Since $r_0 > 3$, we have $1/3 - 1/r_0 > 0$. Thus there is a unique $k_0 = k_0(r_0) \in \mathbb{N}$ with

$$1/r_0 - k_0 \cdot (1/3 - 1/r_0)/2 > 0, \quad 1/r_0 - (k_0 + 1) \cdot (1/3 - 1/r_0)/2 \le 0. \tag{4.1}$$

Due to the first relation in (4.1), we may define $r_i := \big(1/r_0 - i \cdot (1/3 - 1/r_0)/2 \big)^{-1}$ for $i \in \{1, \dots, k_0\}$. Since $1/3 - 1/r_0 > 0$, we have $r_i < r_{i+1}$ for $0 \le i \le k_0 - 1$. Moreover

$2/r_i - 1/r_{i+1} = 1/r_{i-1}$, hence $0 < 2/r_i - 1/r_{i+1} \le 1/r_0 < 1/3$ for $1 < i < k_0 - 1$. In addition, $0 < 2/r_0 - 1/r_1 < 1/3$. By the second relation in (4.1), we obtain $1/r_{k_0} \le (1/3 - 1/r_0)/2$. This inequality and the assumption $r_0 \le 6$ yield that $r_{k_0} > 6$. (We even get $r_{k_0} \ge 12$, although we do not need this.)

Let $S \in (S_0, \infty)$. For $i \in \{1, \ldots, k_0\}$, put $R_i := S_0 + i \cdot (S - S_0)/k_0$. Then $S_0 < R_1 < R_2 < \cdots < R_{k_0} = S$. By (3.16) and definition (2.2), we find for $t \in (0, T_0)$ that

$$\|u(\,\cdot\,,t)\,|\,B^c_{R_1}\|_{r_1} \le \|\mathfrak{K}(\,\cdot\,,t)\,|\,B^c_{R_1}\|_{r_1} + \tau \cdot \sum_{l=1}^{3} \|\partial_l \mathfrak{R}^{(\tau)}(u_l \cdot U + U_l \cdot u)(\,\cdot\,,t)\,|\,B^c_{R_1}\|_{r_1}$$
$$+ \tau \cdot \sum_{l=1}^{3} \|\partial_l \mathfrak{R}^{(\tau)}(u_l \cdot u)(\,\cdot\,,t)\,|\,B^c_{R_1}\|_{r_1}.$$

Recalling that $0 < 2/r_0 - 1/r_1 < 1/3$, we apply Corollary 2.19 with m, μ replaced by r_0, r_1, respectively. Moreover, we use (3.18) with R_1 in the place of S, Corollary 3.6, and the first inequality in Theorem 3.7 with $p = 2$. It follows that

$$\|u(\,\cdot\,,t)\,|\,B^c_{R_1}\|_{r_1} \le \mathfrak{C}(S) + C(\tau, r_0, r_1) \cdot (\|u\|^2_{2,\infty;T_0} + \|u\|^2_{r_0,\infty;T_0}) \le \mathfrak{C}(S). \tag{4.2}$$

Assume that $k_0 > 1$, let $i \in \{1, \ldots, k_0 - 1\}$, and suppose that $\|u|B_{R_i} \times (0,T_0)\|^2_{r_i,\infty;T_0} \le \mathfrak{C}(S)$. Then, noting that $0 < 2/r_i - 1/r_{i+1} < 1/3$ and $B^c_{R_{i+1}} \subset B^c_{R_i}$, we may use the same references as in the proof of (4.2) to obtain for $t \in (0, T_0)$ that

$$\begin{aligned}
&\|u(\,\cdot\,,t)\,|\,B^c_{R_{i+1}}\|_{r_{i+1}} \le \|\mathfrak{K}(\,\cdot\,,t)\,|\,B^c_{R_{i+1}}\|_{r_{i+1}} \\
&\qquad + \tau \cdot \sum_{l=1}^{3} \Big(\|\partial_l \mathfrak{R}^{(\tau)}(u_l \cdot U + U_l \cdot u)(\,\cdot\,,t)\,|\,B^c_{R_i+1}\|_{r_i+1} \\
&\qquad + \|\partial_l \mathfrak{R}^{(\tau)}\big(\,u_l \cdot u\,|\,B^c_{R_i} \times (0,T_0)\,\big)(\,\cdot\,,t)\,|\,B^c_{R_{i+1}}\|_{r_{i+1}} \\
&\qquad + \|\partial_l \mathfrak{R}^{(\tau)}\big(\,u_l \cdot u\,|\,\Omega_{R_i} \times (0,T_0)\,\big)(\,\cdot\,,t)\,|\,B^c_{R_{i+1}}\|_{r_{i+1}} \Big) \\
&\le \mathfrak{C}(S) + C(\tau, r_i, r_{i+1}) \cdot (\|u|B^c_{R_i} \times (0,T_0)\|^2_{2,\infty;T_0} + \|u|B^c_{R_i} \times (0,T_0)\|^2_{r_i,\infty;T_0}) \\
&\qquad + \tau \cdot \sum_{l=1}^{3} \|\partial_l \mathfrak{R}^{(\tau)}\big(\,u_l \cdot u\,|\,\Omega_{R_i} \times (0,T_0)\,\big)(\,\cdot\,,t)\,|\,B^c_{R_{i+1}}\|_{r_{i+1}}.
\end{aligned}$$

The last term on the right-hand side of the preceding inequality is estimated by (2.14), whereas the assumption $\|u|B^c_{R_i} \times (0,T_0)\|^2_{r_i,\infty;T_0} \le \mathfrak{C}(S)$ and Theorem 3.7 are applied to the second. We thus obtain for $t \in (0, T_0)$ that

$$\|u(\,\cdot\,,t)\,|\,B^c_{R_{i+1}}\|_{r_{i+1}} \le \mathfrak{C}(S) + C(\tau, r_{i+1}, R_i, R_{i+1}) \cdot \sum_{l=1}^{3} \|u_l \cdot u\,|\,\Omega_{R_i} \times (0,T_0)\|_{1,\infty;T_0}.$$

Hence $\|u(\,\cdot\,,t)\,|\,B^c_{R_{i+1}}\|_{r_{i+1}} \le \mathfrak{C}(S) + \mathfrak{C}(S) \cdot \|u\|^2_{2,\infty;T_0} \le \mathfrak{C}(S)$. Since $R_{k_0} = S$, we may conclude by induction that $\|u|B^c_S \times (0,T_0)\|_{r_{k_0},\infty;T_0} \le \mathfrak{C}(S)$. Therefore we may choose $\nu_0 = r_{k_0}$. ∎

Next we show that u is bounded on $B^c_S \times (0, T_0)$ for $S \in (S_0, \infty)$.

LEMMA 4.2. *Let $S \in (S_0, \infty)$. Then $|u(x,t)| \le \mathfrak{C}(S)$ for $x \in B^c_S$, $t \in (0, T_0)$.*

Proof. Abbreviate $R := (S_0 + S)/2$. Let $(x,t) \in B_S^c \times (0, T_0)$. By (3.16),

$$|u(x,t)| \le |\mathfrak{K}(x,t)| + \tau \cdot \sum_{l=1}^{3} \big(|\partial_l \mathfrak{R}^{(\tau)}(u_l \cdot U + U_l \cdot u)(x,t)| + |\partial_l \mathfrak{R}^{(\tau)}\big(u_l \cdot u \,|\, B_R^c \times (0,T_0) \big)(x,t)| + |\partial_l \mathfrak{R}^{(\tau)}\big(u_l \cdot u \,|\, \Omega_R \times (0,T_0) \big)(x,t)| \big). \tag{4.3}$$

If $|x| \le 2 \cdot R$, we have $B_R \subset B_{3 \cdot R}(x)$. On the other hand, if $|x| > 2 \cdot R$, it follows for $y \in B_R$ that $|y - x| \ge |x|/2 + |x|/2 - R \ge |x|/2 \ge R$, so that $B_R \cap B_R(x) = \emptyset$. Thus, using (2.15) with R_2, R_1, S replaced either by S, R, $3 \cdot R$, respectively, or by S, R, and R, respectively, we get

$$|\partial_l \mathfrak{R}^{(\tau)}\big(u_l \cdot u \,|\, \Omega_R \times (0,T_0) \big)(x,t)| \le C(\tau, S_0, S) \cdot \|u_l \cdot u\|_{1,\infty;T_0} \le C(\tau, S_0, S) \cdot \|u\|_{2,\infty;T_0}^2 \le \mathfrak{C}(S) \tag{4.4}$$

for $1 \le l \le 3$; see Theorem 3.7 for the last inequality. Moreover,

$$|\partial_l \mathfrak{R}_j^{(\tau)}\big(u_l \cdot u \,|\, B_R^c \times (0,T_0) \big)(x,t)| \le |\mathfrak{A}_{j,l}^{(1)}| + |\mathfrak{A}_{j,l}^{(2)}|, \tag{4.5}$$

where

$$\mathfrak{A}_{j,l}^{(i)} := \int_0^t \int_{B_R^c} \chi_{W_i}(t-\sigma) \cdot \sum_{k=1}^{3} \partial_l \Lambda_{jk}(x-y, t-\sigma, \tau) \cdot (u_l \cdot u_k)(y,\sigma)\, dy\, d\sigma \tag{4.6}$$

for $1 \le j, l \le 3,\ 1 \le i \le 2$, with $W_1 := (1, \infty),\ W_2 := (0, 1]$. By Theorem 2.8 with $p = r = s = \infty,\ q = 1$, and Theorem 3.7, we have

$$|\mathfrak{A}_{j,l}^{(1)}| \le C(\tau) \cdot \|u_l \cdot u\|_{1,\infty;T_0} \le C(\tau) \cdot \|u\|_{2,\infty;T_0}^2 \le \mathfrak{C}.$$

Choose $\nu_0 = \nu_0(r_0) \in (6, \infty)$ as in Lemma 4.1. Since $\nu_0 > 6$, Theorem 2.8 with $p = r = s = \infty,\ q = \nu_0/2$ yields

$$|\mathfrak{A}_{j,l}^{(2)}| \le C(\tau, \nu_0) \cdot \|u_l \cdot u | B_R^c \times (0,T_0)\|_{\nu_0/2,\infty;T_0} \le C(\tau, \nu_0) \cdot \|u | B_R^c \times (0,T_0)\|_{\nu_0,\infty;T_0}^2.$$

Collecting inequalities (4.3)–(4.5) and the preceding estimate of $|\mathfrak{A}_{j,l}^{(1)}|$ and $|\mathfrak{A}_{j,l}^{(2)}|$, using Lemma 4.1 with $S = R$ in order to evaluate the term $\|u | B_R^c \times (0,T_0)\|_{\nu_0,\infty;T_0}^2$, and applying (3.17) as well as Corollary 3.6 with $\kappa = \infty$, we obtain Lemma 4.2. ∎

In the ensuing step of our argument, we are going to show that $u(x,t)$ decays as $O(|x|^{-1})$ for $|x| \to \infty$. Our proof is based on an approach by Babenko [6, pp. 23–24], also taken up by Mizumachi [37, Lemma 6, p. 513]. We begin with two preparatory lemmas, before using Babenko's method in the proof of Theorem 4.5, where we present the result just mentioned with respect to the decay of $u(x,t)$.

Lemma 4.3. *For $R \in [2 \cdot S_0, \infty)$, define*

$$\psi(R) := \sup\{ |u(x,t)| + (|u(x,t) \cdot U(x)|)^{1/2} \ :\ x \in B_R^c,\ t \in (0, T_0) \}.$$

Then $\psi(R) \le \mathfrak{C} \cdot (R^{-1} + \psi(R/2)^{7/6})$ for $R \ge 4 \cdot S_0$.

Proof. Let $R \in [4 \cdot S_0, \infty),\ (x,t) \in B_R^c \times (0,T_0),\ j \in \{1, 2, 3\}$. By (3.16), (3.17) with $S = 2 \cdot S_0$, we have

$$|u_j(x,t)| \le \mathfrak{C} \cdot |x|^{-1} + \sum_{l=1}^{3} \Big(|\partial_l \mathfrak{R}_j^{(\tau)}\big((H_{ml})_{1 \le m \le 3} \,|\, B_{R/2}^c \times (0,T_0) \big)(x,t)| + |\partial_l \mathfrak{R}_j^{(\tau)}\big((H_{ml})_{1 \le m \le 3} \,|\, \Omega_{R/2} \times (0,T_0) \big)(x,t)| \Big). \tag{4.7}$$

Since $|x| \geq R \geq 4 \cdot S_0$, we find for $y \in B_{S_0}(x)$ that $|y| \geq |x| - |y-x| > R - S_0 \geq R/2$. Therefore $B_{R/2} \cap B_{S_0}(x) = \emptyset$. Thus, by (2.15) with R, $R/2$, S_0 in the place of R_2, R_1, S, and by Theorem 3.7, we get

$$\begin{aligned}(4.8)\qquad &\sum_{l=1}^{3} |\partial_l \mathfrak{R}_j^{(\tau)}\big((H_{ml})_{1\leq m\leq 3} \,|\, \Omega_{R/2} \times (0,T_0)\big)(x,t)| \\ &\quad \leq C(\tau, S_0) \cdot |x|^{-1} \cdot \sum_{l=1}^{3} \|(H_{ml})_{1\leq m \leq 3}\|_{1,\infty;T_0} \\ &\quad \leq C(\tau, S_0)\cdot |x|^{-1} \cdot \big(\|u\|^2_{2,\infty;T_0} + \big\| \, |u|\cdot|U| \, \big\|_{1,\infty;T_0}\big) \leq \mathfrak{C} \cdot |x|^{-1}.\end{aligned}$$

Further observe that

$$(4.9)\qquad \sum_{l=1}^{3} |\partial_l \mathfrak{R}_j^{(\tau)}\big((H_{ml})_{1\leq m\leq 3} \,|\, B^c_{R/2} \times (0,T_0)\big)(x,t)| \leq \sum_{l=1}^{3}\sum_{i=1}^{2} |\mathfrak{A}^{(i)}_{j,l}|$$

with $\mathfrak{A}^{(i)}_{j,l}$ defined as in (4.6) but with R replaced by $R/2$ $(1 \leq i \leq 2,\ 1 \leq l \leq 3)$ and $u_l \cdot u$ by $(H_{ml})_{1\leq m\leq 3}$. We get by Theorem 2.8 with $p = r = s = \infty$, $q = 12/5$, and by Theorem 3.7 that

$$\begin{aligned}(4.10)\quad |\mathfrak{A}^{(1)}_{j,l}| &\leq \psi(R/2)^{7/6} \cdot \int_0^t \int_{B^c_{R/2}} \chi_{(1,\infty)}(t-\sigma) \\ &\quad \cdot \sum_{k,l=1}^{3} |\partial_l \Lambda_{jk}(x-y, t-\sigma, \tau)| \cdot |H_{kl}(y,\sigma)|^{5/12} \, dy \, d\sigma \\ &\leq C(\tau) \cdot \psi(R/2)^{7/6} \cdot \big\| \, (|u|^2 + |u|\cdot|U|)^{5/12} \, \big\|_{12/5,\infty;T_0} \\ &\leq C(\tau)\cdot \psi(R/2)^{7/6} \cdot \big(\|u\|^{5/6}_{2,\infty;T_0} + \big\| \, |u|\cdot|U| \, \big\|^{5/12}_{1,\infty;T_0} \big) \leq \mathfrak{C}\cdot \psi(R/2)^{7/6}\end{aligned}$$

for $1 \leq l \leq 3$. Moreover, Theorem 2.8 with $p = r = s = \infty$, $q = 6$ yields for $1 \leq l \leq 3$ that

$$\begin{aligned}|\mathfrak{A}^{(2)}_{j,l}| &\leq C(\tau) \cdot \|(H_{ml})_{1\leq m\leq 3} | B^c_{R/2} \times (0,T_0)\|_{6,\infty;T_0} \\ &\leq C(\tau) \cdot \big(\|u|B^c_{R/2} \times (0,T_0)\|^2_{12,\infty;T_0} + \big\| \, |u|\cdot|U| \, | \, B^c_{R/2}\times(0,T_0) \, \big\|_{6,\infty;T_0} \big) \\ &\leq C(\tau)\cdot \psi(R/2)^{5/3} \\ &\quad \cdot \Big(\|u|B^c_{R/2}\times(0,T_0)\|^{1/3}_{2,\infty;T_0} + \big\| \, |u|\cdot|U| \, | \, B^c_{R/2}\times(0,T_0)\, \big\|^{1/6}_{1,\infty;T_0} \Big),\end{aligned}$$

so that we arrive at the upper bound $\mathfrak{C}\cdot\psi(R/2)^{5/3}$ for $|\mathfrak{A}^{(2)}_{j,l}|$. But since $R \geq 4\cdot S_0$, hence $R/2 \geq 2 \cdot S_0$, Lemma 4.2 with $S = 2\cdot S_0$ and Lemma 3.1 imply $\psi(R/2)^{1/2} \leq \mathfrak{C}$. Thus $|\mathfrak{A}^{(2)}_{j,l}| \leq \mathfrak{C}\cdot\psi(R/2)^{7/6}$. Combining (4.7)–(4.10) with the preceding estimate of $|\mathfrak{A}^{(2)}_{j,l}|$, we obtain the lemma. □

LEMMA 4.4. *Let $\kappa \in (0,\infty)$. Then there is $R_\kappa \in [4\cdot S_0, \infty)$ depending on the quantities listed in Theorem 1.2 (except R) and on κ such that $|u(x,t)| \leq \kappa$ for $x \in B^c_{R_\kappa}$, $t \in (0,T_0)$.*

Proof. Let $j \in \{1, 2, 3\}$, $R \in [4\cdot S_0, \infty)$, $M \in (0,\infty)$, $(x,t) \in B^c_R \times (0,T_0)$. Then by (3.16) and Lemma 2.10,

$$(4.11)\qquad u_j(x,t) = \mathfrak{K}_j(x,t) + \sum_{m=1}^{4}\sum_{k,l=1}^{3} \mathfrak{B}^{(m)}_{k,l},$$

where for $k, l \in \{1,\, 2,\, 3\},\ m \in \{1,\, \dots,\, 4\}$, we used the abbreviation

$$\mathfrak{B}^{(m)}_{k,l} := \int_0^t \int_{A_m} \chi_{W_m}(t-\sigma) \cdot \partial_l \Lambda_{jk}(x-y, t-\sigma, \tau) \cdot H_{kl}(y,\sigma)\, dy\, d\sigma$$

with $A_1 := B^c_{2\cdot S_0},\ W_1 := (0, M],\ A_2 := \Omega_{2\cdot S_0},\ W_2 := (0, M],\ A_3 := \Omega_{R/2},\ W_3 := (M, \infty),\ A_4 := B^c_{R/2},\ W_4 := (M, \infty)$. Choose $\nu_0 = \nu_0(r_0) \in (6, \infty)$ as in Lemma 4.1. By Theorem 2.8 with $p = r = s = \infty,\ q = \nu_0/2$,

$$\begin{aligned}|\mathfrak{B}^{(1)}_{k,l}| &\le C(\tau, \nu_0) \cdot M^{1/2-3/\nu_0} \cdot \|H_{kl}|B^c_{2\cdot S_0} \times (0, T_0)\|_{\nu_0/2,\infty;T_0}\\ &\le C(\tau, \nu_0) \cdot M^{1/2-3/\nu_0}\\ &\quad \cdot \big(\, \|u|B^c_{2\cdot S_0} \times (0, T_0)\|^2_{\nu_0,\infty;T_0} + \|u|B^c_{2\cdot S_0} \times (0, T_0)\|_{\nu_0,\infty;T_0} \cdot \|U|B^c_{2\cdot S_0}\|_{\nu_0} \,\big)\end{aligned}$$

for $1 \le k, l \le 3$. It follows with Lemmas 4.1 and 3.1 that

$$|\mathfrak{B}^{(1)}_{k,l}| \le \mathfrak{C} \cdot M^{1/2-3/\nu_0} \quad (1 \le k, l \le 3), \tag{4.12}$$

where $1/2 - 3/\nu_0 > 0$ because $\nu_0 > 6$. Next we estimate $\mathfrak{B}^{(2)}_{k,l}$ by applying (2.15) with $R_1 = 2 \cdot S_0,\ R_2 = R,\ S = S_0$. Note that for $y \in B_{2\cdot S_0}$, we have $|x - y| \ge |x| - |y| \ge R - 2 \cdot S_0 \ge 2 \cdot S_0$, so that $B_{2\cdot S_0} \cap B_{S_0}(x) = \emptyset$. We obtain

$$\begin{aligned}|\mathfrak{B}^{(2)}_{k,l}| &\le C(\tau, S_0) \cdot \|H_{kl}|\Omega_{S_0} \times (0, T_0)\|_{1,\infty;T_0} \cdot |x|^{-1}\\ &\le C(\tau, S_0) \cdot \big(\, \|u\|^2_{2,\infty;T_0} + \big\|\, |u| \cdot |U|\, \big\|_{1,\infty;T_0} \big) \cdot R^{-1} \le \mathfrak{C} \cdot R^{-1},\end{aligned} \tag{4.13}$$

where the last inequality follows from Theorem 3.7. By inequality (2.15) with $R_1 = R/2,\ R_2 = R,\ S = 2 \cdot S_0$ (so that $B_{R_1} \cap B_S(x) = \emptyset,\ R_1/R_2 = 1/2$) and by Theorem 3.7, we see that (4.13) also holds for $\mathfrak{B}^{(3)}_{kl}$. Turning to $\mathfrak{B}^{(4)}_{kl}$, we apply Theorem 2.8 with $p = r = \infty,\ q = 3,\ s = 1$, and with $p = r = \infty,\ q = 3,\ s = 2$, to obtain

$$\begin{aligned}|\mathfrak{B}^{(4)}_{kl}| &\le C(\tau) \cdot (M^{-1} \cdot \big\||u|^2 \,\big|\, B^c_{R/2} \times (0, T_0)\big\|_{3,1;T_0}\\ &\qquad + M^{-1/2} \cdot \big\||u| \cdot |U| \,\big|\, B^c_{R/2} \times (0, T_0)\big\|_{3,2;T_0})\\ &\le C(\tau) \cdot \|u|B^c_{R/2} \times (0, T_0)\|_{6,2;T_0} \cdot (M^{-1} \cdot \|u\|_{6,2;T_0} + M^{-1/2} \cdot \|U\|_6).\end{aligned}$$

It follows with Lemmas 2.3 and 3.1 that

$$|\mathfrak{B}^{(4)}_{kl}| \le \mathfrak{C} \cdot (M^{-1} + M^{-1/2}) \cdot \varphi(R/2)$$

with the function φ defined in Theorem 1.2. We may conclude, referring to the preceding inequality and to (4.11)–(4.13), and by observing that $|\mathfrak{K}_j(x, t)| \le \mathfrak{C} \cdot R^{-1}$ by inequality (3.17) with $S = 4 \cdot S_0$,

$$|u(x, t)| \le \mathfrak{C} \cdot \big(\, M^{1/2-1/\nu_0} + R^{-1} + (M^{-1} + M^{-1/2}) \cdot \varphi(R/2) \big). \tag{4.14}$$

According to Lemma 2.3, we have $u \in L^2\big(0, T_0, L^6(\overline{\Omega}^c)^3\big)$, so that $\varphi(R/2) \to 0$ ($R \to \infty$) by Lebesgue's theorem. Thus, recalling that $1/2 - 3/\nu_0 > 0$, we see that $|u(x, t)| \le \kappa$ for $(x, t) \in B^c_R \times (0, T_0)$ if we first choose M sufficiently small and then R sufficiently large in (4.14). $\square$

THEOREM 4.5. *Let $S \in (S_0, \infty)$. Then $|u(x, t)| \le \mathfrak{C}(S) \cdot |x|^{-1}$ for $x \in B^c_S,\ t \in (0, T_0)$.*

Proof. Let the function $\psi : [2 \cdot S_0, \infty) \mapsto [0, \infty)$ be defined as in Lemma 4.3. Note that ψ is monotone decreasing. By referring to Lemma 4.3, we see there is a constant $C_0 > 0$ depending on the quantities listed in Theorem 1.2 (except R and φ) such that

$$\psi(R) \leq C_0 \cdot (R^{-1} + \psi(R/2)^{7/6}) \quad \text{for} \ \ R \in [4 \cdot S_0, \infty). \tag{4.15}$$

Due to Lemmas 4.4 and 3.1, we may choose some $\widetilde{R} \in [4 \cdot S_0, \infty)$, depending on the same quantities as C_0 as well as on the function φ such that $\psi(\widetilde{R}) \leq (4 \cdot C_0)^{-6}$. Put $R_0 := \max\{\widetilde{R}, (4 \cdot C_0)^7/2\}$. Note that $R_0 \geq \max\{4 \cdot S_0, (4 \cdot C_0)^7/2\}$. Since ψ is decreasing and $R_0 \geq \widetilde{R}$, we further have $\psi(R_0) \leq (4 \cdot C_0)^{-6}$. Due to (4.15), $\psi(2 \cdot R_0) \leq C_0 \cdot \big((2 \cdot R_0)^{-1} + \psi(R_0)^{7/6} \big)$, hence $\psi(2 \cdot R_0) \leq C_0 \cdot 2 \cdot (4 \cdot C_0)^{-7} = (4 \cdot C_0)^{-6}/2$.

Let $n \in \mathbb{N}$, and suppose that $\psi(2^n \cdot R_0) \leq (4 \cdot C_0)^{-6} \cdot 2^{-n}$. Then with (4.15),

$$\begin{aligned}\psi(2^{n+1} \cdot R_0) &\leq C_0 \cdot \big((2^{n+1} \cdot R_0)^{-1} + \psi(2^n \cdot R_0)^{7/6} \big) \\ &\leq C_0 \cdot \big(2^{-n} \cdot (4 \cdot C_0)^{-7} + (4 \cdot C_0)^{-7} \cdot 2^{-7 \cdot n/6} \big) \leq 2^{-n-1} \cdot (4 \cdot C_0)^{-6}.\end{aligned}$$

Thus we have shown by induction that $\psi(2^n \cdot R_0) \leq 2^{-n} \cdot (4 \cdot C_0)^{-6}$ for $n \in \mathbb{N}$. Now let $R \in [2 \cdot R_0, \infty)$. Then there is $n \in \mathbb{N}$ with $2^n \cdot R_0 \leq R < 2^{n+1} \cdot R_0$, so that

$$\psi(R) \leq \psi(2^n \cdot R_0) \leq 2^{-n} \cdot (4 \cdot C_0)^{-6} \leq R^{-1} \cdot 2 \cdot R_0 \cdot (4 \cdot C_0)^{-6}. \tag{4.16}$$

For $x \in B^c_{4 \cdot R_0}$, $t \in (0, T_0)$, inequality (4.16) with $R = |x|/2$ yields

$$|u(x,t)| \leq \psi(R) \leq 4 \cdot R_0 \cdot (4 \cdot C_0)^{-6} \cdot |x|^{-1}. \tag{4.17}$$

On the other hand, we have $\min\{4 \cdot R_0, S\} > S_0$, so Lemma 4.2 yields that $|u(x,t)| \leq \mathfrak{C}(S)$ for $x \in B^c_{\min\{4 \cdot R_0, S\}}$, $t \in (0, T_0)$. Hence $|u(x,t)| \leq \mathfrak{C}(S) \cdot 4 \cdot R_0 \cdot |x|^{-1}$ for $x \in B_{4 \cdot R_0} \backslash B_{\min\{4 \cdot R_0, S\}}$, $t \in (0, T_0)$. Theorem 4.5 follows from (4.17) and the preceding inequality. ∎

At this point, we may prove one of the main results of this article—a pointwise decay estimate for u.

THEOREM 4.6. *Let $S \in (S_0, \infty)$. Then $|u(x,t)| \leq \mathfrak{C}(S) \cdot \big(|x| \cdot \nu(x) \big)^{-1}$ for $x \in B^c_S$, $t \in (0, T_0)$.*

Proof. Let $j \in \{1, 2, 3\}$, $(x,t) \in B^c_{2 \cdot S} \times (0, T)$. Then, with (3.16), (3.17), and Lemma 2.10, we find

$$|u_j(x,t| \leq \mathfrak{C}(S) \cdot \big(|x| \cdot \nu(x) \big)^{-1} + \sum_{i=1}^{3} \sum_{k,l=1}^{3} \mathfrak{B}^{(i)}_{k,l}, \tag{4.18}$$

where

$$\mathfrak{B}^{(i)}_{k,l} := \int_0^t \int_{\mathfrak{D}_i} |\partial_l \Lambda_{jk}(x-y, t-\sigma, \tau) \cdot H_{kl}(y,\sigma)| \, dy \, d\sigma \quad \text{for} \ \ i,k,l \in \{1, 2, 3\}$$

with $\mathfrak{D}_1 := B^c_S \cap B_S(x)$, $\mathfrak{D}_2 := \Omega_S$, $\mathfrak{D}_3 := B^c_S \backslash B_S(x)$. We observe that for $y \in B_S(x)$

$$|y| \geq |x| - |x-y| \geq |x| - S = |x|/2 + |x|/2 - S \geq |x|/2. \tag{4.19}$$

Let $k, l \in \{1, 2, 3\}$. By Theorem 4.5 and (1.10), we have

$$|H_{kl}(y,\sigma)| \leq \mathfrak{C}(S) \cdot |y|^{-2} \quad \text{for} \ \ y \in B^c_S, \ \sigma \in (0, T_0). \tag{4.20}$$

Due to (4.19) and the preceding inequality, and by Corollary 2.7 with $K = S$, we find

(4.21)

$$\begin{aligned}|\mathfrak{B}^{(1)}_{k,l}| &\le \mathfrak{C}(S) \cdot \int_0^t \int_{B_S^c \cap B_S(x)} (|x-y|^2 + t - \sigma)^{-2} \cdot |y|^{-2}\, dy\, d\sigma \\ &\le \mathfrak{C}(S) \cdot |x|^{-2} \int_{B_S^c \cap B_S(x)} \cdot \int_0^t (|x-y|^2 + t - \sigma)^{-2}\, d\sigma\, dy \\ &\le \mathfrak{C}(S) \cdot |x|^{-2} \cdot \int_{B_S(x)} |x-y|^{-2}\, dy \le \mathfrak{C}(S) \cdot |x|^{-2} \le \mathfrak{C}(S) \cdot \big(\,|x| \cdot \nu(x)\,\big)^{-1}\end{aligned}$$

with the last inequality following from Lemma 2.2. For $y \in B_S$, we have $|x-y| \ge |x| - |y| > S$, so that $B_S \cap B_S(x) = \emptyset$. Therefore inequality (2.15) with R_2, R_1 replaced by $2 \cdot S,\ S$, respectively, yields

$$\begin{aligned}|\mathfrak{B}^{(2)}_{k,l}| &\le C(\tau, S) \cdot \|H_{kl}|\Omega_S \times (0, T_0)\|_{1,\infty;T_0} \cdot \big(\,|x| \cdot \nu(x)\,\big)^{-1} \\ &\le \mathfrak{C}(S) \cdot \big(\,|x| \cdot \nu(x)\,\big)^{-1},\end{aligned} \tag{4.22}$$

where the last inequality follows from Theorem 3.7. We further find with (4.20) and Lemma 2.4 that

$$|\mathfrak{B}^{(3)}_{k,l}| \le \mathfrak{C}(S) \cdot \int_0^t \int_A \big(\,|x - y - \tau \cdot (t-\sigma) \cdot e_1|^2 + t - \sigma\,\big)^{-2} \cdot |y|^{-2}\, dy\, d\sigma \tag{4.23}$$

with $A := B_S^c \backslash B_S(x)$. Next we use Theorem 2.6 with $K = S$ to obtain

$$|\mathfrak{B}^{(3)}_{k,l}| \le \mathfrak{C}(S) \cdot \int_{B_S^c \backslash B_S(x)} \big(\,|x-y| \cdot \nu(x-y)\,\big)^{-3/2} \cdot |y|^{-2}\, dy. \tag{4.24}$$

But for $y \in B_S^c \backslash B_S(x) = \mathbb{R}^3 \backslash \big(\,B_S \cup B_S(x)\,\big)$, we have

$$|y| = |y|/2 + |y|/2 \ge S/2 + |y|/2 \ge \min\{S/2,\ 1/2\} \cdot (1 + |y|), \tag{4.25}$$

$$|x-y| \ge S/2 + |x-y|/2 \ge \min\{S/2,\ 1/2\} \cdot (1 + |x-y|). \tag{4.26}$$

Thus, from (4.24),

$$|\mathfrak{B}^{(3)}_{k,l}| \le \mathfrak{C}(S) \cdot \int_{\mathbb{R}^3} \big(\,(1 + |x-y|) \cdot \nu(x-y)\,\big)^{-3/2} \cdot (1 + |y|)^{-2}\, dy. \tag{4.27}$$

Now we refer to Theorem 2.18, with $A = 2,\ B = 0,\ E = 1,\ F = 1/2$, to obtain from (4.27) that $|\mathfrak{B}^{(3)}_{k,l}| \le \mathfrak{C}(S) \cdot (1 + |x|)^{-1} \cdot \nu(x)^{-1/2}$. This implies

$$|\mathfrak{B}^{(3)}_{k,l}| \le \mathfrak{C}(S) \cdot (1 + |x|)^{-3/4} \cdot \nu(x)^{-1/2}.$$

We may conclude from (4.18), (4.21), (4.22), and the preceding estimate that

$$|u(x,t)| \le \mathfrak{C}(S) \cdot |x|^{-3/4} \cdot \nu(x)^{-1/2} \quad \text{for } x \in B^c_{2\cdot S},\ t \in (0, T_0). \tag{4.28}$$

But by Lemma 2.2, we have $1 \le 4 \cdot S^2 \cdot |x|^{-2} \le C(S) \cdot \big(\,|x| \cdot \nu(x)\,\big)^{-1}$ for $x \in B_{2\cdot S} \backslash B_S$, so that with Lemma 4.2,

$$|u(x,t)| \le \mathfrak{C}(S) \cdot \big(\,|x| \cdot \nu(x)\,\big)^{-1} \quad \text{for } x \in B_{2\cdot S} \backslash B_S,\ t \in (0, T_0). \tag{4.29}$$

Now we may conclude that inequality (4.28) extends to $x \in B_S^c$. It follows with (1.10) that

$$|H_{kl}(y,s)| \le \mathfrak{C}(S) \cdot |y|^{-3/2} \cdot \nu(y)^{-1} \quad \text{for } y \in B_S^c,\ s \in (0,T_0),\ 1 \le k,l \le 3. \tag{4.30}$$

Again take $j \in \{1, 2, 3\},\ (x,t) \in B_{2\cdot S}^c \times (0,T_0)$ Once more starting from (4.18), we improve our estimate of $\mathfrak{B}_{k,l}^{(3)}$. To this end, we proceed as in (4.23), (4.24), (4.27) but use inequality (4.30) instead of (4.20). In this way, we arrive at the estimate

$$|\mathfrak{B}_{k,l}^{(3)}| \le \mathfrak{C}(S) \cdot \int_{\mathbb{R}^3} \big((1+|x-y|) \cdot \nu(x-y) \big)^{-3/2} \cdot (1+|y|)^{-3/2} \cdot \nu(y)^{-1}\, dy$$

$(1 \le k,l \le 3)$. Again referring to Theorem 2.18, this time with $A = 3/2,\ B = E = F = 1$, we may conclude that $|\mathfrak{B}_{k,l}^{(3)}| \le \mathfrak{C}(S) \cdot \big((1+|x|) \cdot \nu(x) \big)^{-1}$ for $1 \le k,l \le 3$. This estimate and (4.18), (4.21), and (4.22) imply $|u(x,t)| \le \mathfrak{C}(S) \cdot \big(|x| \cdot \nu(x) \big)^{-1}$ for $x \in B_{2\cdot S}^c,\ t \in (0,T_0)$. Inequality (4.29) states that the preceding estimate is valid also for $x \in B_{2\cdot S} \backslash B_S$. Thus Theorem 4.6 is proved. ▯

We finally turn to pointwise decay estimates for $\nabla_x u$. In a first step, we show that $\nabla_x u$ is bounded on $B_S^c \times (0,T_0)$, for any $S \in (S_0,\infty)$.

Lemma 4.7. *Let $S \in (S_0,\infty)$. Then $|\nabla_x u(x,t)| \le \mathfrak{C}(S)$ for $x \in B_S^c,\ t \in (0,T_0)$.*

Proof. Put $R_i := S_0 + i \cdot (S-S_0)/4$ for $i \in \{1, 2, 3\}$. Let $j,l \in \{1, 2, 3\}$. For $(x,t) \in Z_{T_0}$, (3.8) and Lemma 2.10 yield

$$\partial_l u_j(x,t) = \partial_l \mathfrak{I}_j(x,t) - \sum_{i=1}^{3} \mathfrak{G}_{jl}^{(i)}(x,t) \tag{4.31}$$

with

$$\mathfrak{G}_{jl}^{(i)}(x,t) := \sum_{k=1}^{3} \int_0^t \int_{A_i} \chi_{W_i}(t-\sigma) \cdot \partial_l \Lambda_{jk}(x-y,t-\sigma,\tau) \cdot G_k(y,\sigma)\, dy\, d\sigma$$

for $j,k \in \{1, 2, 3\}$, where $A_1 := B_{R_1}^c,\ A_2 := \Omega_{R_1},\ A_3 := \overline{\Omega}^c,\ W_1 := W_2 := (0,1],\ W_3 := (1,\infty)$. For $p \in (6/5,\infty]$, we deduce from Theorem 2.8 with $q = 6/5,\ r = \infty,\ s = 2$ that

$$\|\mathfrak{G}_{jl}^{(3)}\|_{p,\infty;T_0} \le C(\tau,p) \cdot \|G\|_{6/5,2;T_0} \le \mathfrak{C}(p) \tag{4.32}$$

with the last inequality following from Corollary 3.3. Moreover, inequality (2.16) with S replaced by $(S-S_0)/4$ yields

$$\|\mathfrak{G}_{jl}^{(2)} | B_{R_2}^c \times (0,T_0)\|_{2,\infty;T_0} \le C(\tau,S_0,S) \cdot \|G|\Omega_{R_1} \times (0,T_0)\|_{1,2;T_0},$$

so that we obtain $\|\mathfrak{G}_{jl}^{(2)} | B_{R_2}^c \times (0,T_0)\|_{2,\infty;T_0} \le C(\tau,S_0,S) \cdot \|G\|_{6/5,2;T_0} \le \mathfrak{C}(S)$, where the last inequality is a consequence of Corollary 3.3. By this estimate, (4.31), (4.32), and (3.18), we obtain

$$\|(\partial_l u_j - \mathfrak{G}_{jl}^{(1)}) | B_{R_2}^c \times (0,T_0)\|_{2,\infty;T_0} \le \mathfrak{C}(S) \quad \text{for } 1 \le j,l \le 3. \tag{4.33}$$

Theorem 2.8 with $p = r = 3,\ q = s = 2$ implies $\|\mathfrak{G}_{jl}^{(1)}\|_3 \le C(\tau) \cdot \|G|B_{R_1}^c \times (0,T_0)\|_2$. But by (2.1), we see that

$$\begin{aligned}&\|G|B_{R_1}^c \times (0,T_0)\|_2 \\ &\quad \le C(\tau) \cdot \big((\|u|B_{R_1}^c \times (0,T_0)\|_\infty + \|U|B_{R_1}^c\|_\infty) \cdot \|\nabla_x u\|_2 + \|u\|_{6,2;T_0} \cdot \|\nabla U|B_{R_1}^c\|_3 \big).\end{aligned}$$

Thus we may conclude by Lemma 4.2 with S replaced by R_1 and Lemmas 2.3 and 3.1 that $\|\mathfrak{G}^{(1)}_{jl}\|_3 \le \mathfrak{C}(S)$.

Again take $j, l \in \{1, 2, 3\}$. For $i \in \{1, 2\}$, $(x,t) \in Z_{T_0}$, define $\widetilde{\mathfrak{G}}^{(i)}_{jl}(x,t)$ in the same way as $\mathfrak{G}^{(i)}_{jl}(x,t)$, but with $A_1 := B^c_{R_2}$, $A_2 := \Omega_{R_2}$. Put $\widetilde{\mathfrak{G}}^{(3)}_{jl} := \mathfrak{G}^{(3)}_{jl}$. Then from (3.8) and Lemma 2.10,

$$\partial_l u_j(x,t) = \partial_l \mathfrak{I}_j(x,t) - \sum_{i=1}^{3} \widetilde{\mathfrak{G}}^{(i)}_{jl}(x,t) \quad \big((x,t) \in Z_{T_0} \big). \tag{4.34}$$

We further introduce the splitting $\widetilde{\mathfrak{G}}^{(1)}_{jl}(x,t) = \mathfrak{B}^{(1)}_{jl}(x,t) + \mathfrak{B}^{(2)}_{jl}(x,t)$ for $(x,t) \in Z_{T_0}$ with

(4.35)

$$\mathfrak{B}^{(1)}_{jl}(x,t) := \tau \cdot \sum_{m,k=1}^{3} \int_0^t \int_{B^c_{R_2}} \chi_{(0,1]}(t-\sigma) \cdot \partial_l \Lambda_{jk}(x-y, t-\sigma, \tau)$$
$$\cdot \big[\big(u_m(y,\sigma) + U_m(y) \big) \cdot (\partial_m u_k - \mathfrak{G}^{(1)}_{km})(y,\sigma) \; + \; u_m(y,\sigma) \cdot \partial_m U_k(y) \big] \, dy \, d\sigma$$

and with $\mathfrak{B}^{(2)}_{jl}(x,t)$ defined in the same way as $\mathfrak{B}^{(1)}_{jl}(x,t)$, except that the expression in brackets in (4.35) is to be replaced by $\big(u_m(y,\sigma) + U_m(y) \big) \cdot \mathfrak{G}^{(1)}_{km}(y,\sigma)$. Theorem 2.8 with $r = s = \infty$, $p = 4$, $q = 2$ shows that $\|\mathfrak{B}^{(1)}_{jl}\|_{4,\infty;T_0}$ is bounded by

$$C(\tau) \cdot \big[(\|u|B^c_{R_2} \times (0,T_0)\|_\infty + \|U|B^c_{R_2}\|_\infty)$$
$$\cdot \sum_{k,m=1}^{3} \|(\partial_m u_k - \mathfrak{G}^{(1)}_{km})|B^c_{R_2} \times (0,T_0)\|_{2,\infty;T_0} \; + \; \|u|B^c_{R_2} \times (0,T_0)\|_\infty \cdot \|\nabla U\|_2 \big].$$

Due to this estimate, Lemma 4.2 with S replaced by R_2, Lemma 3.1, and (4.33), we thus arrive at the inequality $\|\mathfrak{B}^{(1)}_{jl}\|_{4,\infty;T_0} \le \mathfrak{C}(S)$. On the other hand, again by Theorem 2.8, this time with $r = \infty$, $p = 4$, $q = s = 3$,

$$\|\mathfrak{B}^{(2)}_{jl}\|_{4,\infty;T_0} \le C(\tau) \cdot (\|u|B^c_{R_2} \times (0,T_0)\|_\infty + \|U|B^c_{R_2}\|_\infty) \cdot \sum_{k,m=1}^{3} \|\mathfrak{G}^{(1)}_{km}\|_3 \le \mathfrak{C}(S),$$

with the last inequality being a consequence of Lemmas 4.2 and 3.1 and the estimate $\|\mathfrak{G}^{(1)}_{km}\|_3 \le \mathfrak{C}(S)$ shown above for $1 \le k, m \le 3$. Thus we get $\|\widetilde{\mathfrak{G}}^{(1)}_{jl}\|_{4,\infty;T_0} \le \mathfrak{C}(S)$. From (2.16) with $(S - S_0)/4$, R_2, R_3 in the place of, respectively, S, R_1, R_2, and from Corollary 3.3, we may deduce

$$\begin{aligned} \|\widetilde{\mathfrak{G}}^{(2)}_{jl}|B^c_{R_3} \times (0,T_0)\|_{4,\infty;T_0} &\le C(\tau, S_0, S) \cdot \|G|\Omega_{R_2} \times (0,T_0)\|_{1,2;T_0} \\ &\le C(\tau, S_0, S) \cdot \|G\|_{6/5,2;T_0} \le \mathfrak{C}(S). \end{aligned}$$

Thus we get by (3.18), (4.32) with $p = 4$ and (4.34) that $\|\partial_l u_j|B^c_{R_3} \times (0,T_0)\|_{4,\infty;T_0} \le \mathfrak{C}(S)$. It follows with Lemma 4.2 with $S = R_3$ and Lemma 3.1 that

$$\begin{aligned} &\|G|B^c_{R_3} \times (0,T_0)\|_{4,\infty;T_0} \\ &\quad \le C(\tau) \cdot \big[\|U|B^c_{R_3}\|_\infty \cdot \|\nabla_x u|B^c_{R_3} \times (0,T_0)\|_{4,\infty;T_0} \\ &\qquad + \|u|B^c_{R_3} \times (0,T_0)\|_\infty \cdot (\|\nabla_x u|B^c_{R_3} \times (0,T_0)\|_{4,\infty;T_0} + \|\nabla U|B^c_{R_3}\|_4) \big] \\ &\quad \le \mathfrak{C}(S). \end{aligned} \tag{4.36}$$

Once more take $j, l \in \{1, 2, 3\}$, and let $(x,t) \in B_S^c \times (0, T_0)$. By (3.8) and Lemma 2.10,

$$\partial_l u_j(x,t) = \partial_l \mathfrak{J}_j(x,t) - \sum_{i=1}^{3} \overline{\mathfrak{G}}_{j,l}^{(i)}(x,t), \tag{4.37}$$

where $\overline{\mathfrak{G}}_{j,l}^{(3)}(x,t) := \mathfrak{G}_{j,l}^{(3)}(x,t)$, and $\overline{\mathfrak{G}}_{j,l}^{(i)}(x,t)$ for $i \in \{1, 2\}$ is defined in the same way as $\mathfrak{G}_{j,l}^{(i)}(x,t)$, but with $A_1 := B_{R_3}^c,\ A_2 := \Omega_{R_3}$. According to (4.32), we have $|\overline{\mathfrak{G}}_{j,l}^{(3)}(x,t)| \le \mathfrak{C}(S)$. In addition, we use inequality (2.17) with R_2, R_1, S replaced by S, R_3, $(S - S_0)/4$, respectively. It follows with Corollary 3.3 that

$$\begin{aligned} |\overline{\mathfrak{G}}_{j,l}^{(2)}(x,t)| &\le C(\tau, S_0, S) \cdot \big(\,|x| \cdot \nu(x)\,\big)^{-3/2} \cdot \|G|\Omega_{R_3} \times (0, T_0)\|_{1,2;T_0} \\ &\le C(\tau, S_0, S) \cdot \|G\|_{6/5,2;T_0} \le \mathfrak{C}(S). \end{aligned}$$

Finally, applying Theorem 2.8 again, this time with $p = r = s = \infty,\ q = 4$, we obtain $|\overline{\mathfrak{G}}_{j,l}^{(1)}(x,t)| \le C(\tau) \cdot \|G|B_{R_3}^c \times (0, T_0)\|_{4,\infty;T_0} \le \mathfrak{C}(S)$, where the last inequality holds according to (4.36). Thus we conclude with (4.37) and (3.9) that $|\nabla_x u(x,t)| \le \mathfrak{C}(S)$. This proves the lemma. □

Now we arrive at our main result on the pointwise decay of $\nabla_x u$.

THEOREM 4.8. *Let $S \in (S_0, \infty)$. Then $|\nabla_x u(x,t)| \le \mathfrak{C}(S) \cdot \big(\,|x| \cdot \nu(x)\,\big)^{-3/2}$ for $x \in B_S^c,\ t \in (0, T_0)$.*

Proof. Let $j, l \in \{1, 2, 3\},\ (x,t) \in B_{2\cdot S}^c \times (0, T_0)$. Then by (3.8) and Lemma 2.10, we find

$$\partial_l u_j(x,t) = \partial_l \mathfrak{J}_j(x,t) - \sum_{i=1}^{3} \sum_{k=1}^{3} \mathfrak{A}_k^{(i)} \tag{4.38}$$

with $\mathfrak{A}_k^{(i)} := \int_0^t \int_{A_i} \partial_l \Lambda_{jk}(x-y, t-\sigma, \tau) \cdot G_k(y,\sigma)\ dy\ d\sigma$ for $1 \le i, k \le 3$, where $A_1 := \Omega_S,\ A_2 := B_S^c \cap B_S(x),\ A_3 := B_S^c \backslash B_S(x)$. By (2.17) with $R_2 = 2 \cdot S,\ R_1 = S$, and by Corollary 3.3, we get

$$\begin{aligned} |\mathfrak{A}_k^{(1)}| &\le C(\tau, S) \cdot \big(\,|x| \cdot \nu(x)\,\big)^{-3/2} \cdot \|G|\Omega_S \times (0, T_0)\|_{1,2;T_0} \\ &\le C(\tau, S) \cdot \big(\,|x| \cdot \nu(x)\,\big)^{-3/2} \cdot \|G\|_{6/5,2;T_0} \le \mathfrak{C}(S) \cdot \big(\,|x| \cdot \nu(x)\,\big)^{-3/2}. \end{aligned} \tag{4.39}$$

Next we observe that by (1.10), Lemma 4.7, and Theorem 4.6,

$$|G(y,\sigma)| \le \mathfrak{C}(S) \cdot \big(\,|y| \cdot \nu(y)\,\big)^{-1} \quad \big(\,y \in B_S^c,\ \sigma \in (0, T_0)\,\big). \tag{4.40}$$

Thus, by Corollary 2.7 with $K = S$,

$$\begin{aligned} |\mathfrak{A}_k^{(2)}| &\le \mathfrak{C}(S) \cdot \int_0^t \int_{B_S^c \cap B_S(x)} (|x-y|^2 + t - \sigma)^{-2} \cdot \big(\,|y| \cdot \nu(y)\,\big)^{-1}\ dy\ d\sigma \\ &\le \mathfrak{C}(S) \cdot \int_{B_S^c \cap B_S(x)} |x-y|^{-2} \cdot \big(\,|y| \cdot \nu(y)\,\big)^{-1}\ dy \quad \text{for } 1 \le k \le 3. \end{aligned} \tag{4.41}$$

But Lemma 2.2 yields for $y \in B_S(x)$ that

$$\nu(y)^{-1} \le C \cdot (1 + |x-y|) \cdot \nu(x)^{-1} \le C(S) \cdot \nu(x)^{-1}.$$

Thus we may conclude with (4.19) that for $1 \leq k \leq 3$,

$$|\mathfrak{A}_k^{(2)}| \leq \mathfrak{C}(S) \cdot \big(|x| \cdot \nu(x) \big)^{-1} \cdot \int_{B_S(x)} |x-y|^{-2} \, dy \leq \mathfrak{C}(S) \cdot \big(|x| \cdot \nu(x) \big)^{-1}. \tag{4.42}$$

Next we apply Lemma 2.4, inequality (4.40), as well as Theorem 2.6 with $K = S$ to obtain with the abbreviation $h(x,y,t,\sigma) := |x - y - \tau \cdot (t-\sigma) \cdot e_1|^2 + t - \sigma$ that

$$\begin{aligned} |\mathfrak{A}_k^{(3)}| &\leq C(\tau, S) \cdot \int_0^t \int_{B_S^c \backslash B_S(x)} h(x,y,t,\sigma)^{-2} \cdot |G(y,\sigma)| \, dy \, d\sigma \\ &\leq \mathfrak{C}(S) \cdot \int_{B_S^c \backslash B_S(x)} \big(|x-y| \cdot \nu(x-y) \big)^{-3/2} \cdot \big(|y| \cdot \nu(y) \big)^{-1} \, dy \quad (1 \leq k \leq 3). \end{aligned} \tag{4.43}$$

In the next step, we refer to (4.25) and (4.26). It follows that

$$|\mathfrak{A}_k^{(3)}| \leq \mathfrak{C}(S) \cdot \int_{\mathbb{R}^3} \big((1+|x-y|) \cdot \nu(x-y) \big)^{-3/2} \cdot \big((1+|y|) \cdot \nu(y) \big)^{-1} \, dy. \tag{4.44}$$

At this point, we may apply Theorem 2.18 with $A = B = F = 1, \ E = 1/2$, to obtain

$$|\mathfrak{A}_k^{(3)}| \leq \mathfrak{C}(S) \cdot (1+|x|)^{-1/2} \cdot \nu(x)^{-1}. \tag{4.45}$$

Combining (4.38), (3.9), (4.39), (4.42), and (4.45) yields

$$|\nabla_x u(x,t)| \leq \mathfrak{C}(S) \cdot |x|^{-1/2} \cdot \nu(x)^{-1} \quad \text{for } \ x \in B_{2 \cdot S}^c, \ t \in (0, T_0). \tag{4.46}$$

We have $1 \leq (2 \cdot S)^3 \cdot |x|^{-3} \leq C(S) \cdot \big(|x| \cdot \nu(x) \big)^{-3/2}$ for $x \in B_{2 \cdot S} \backslash B_S$ (see Lemma 2.2); hence with Lemma 4.7,

$$|\nabla_x u(x,t)| \leq \mathfrak{C}(S) \cdot \big(|x| \cdot \nu(x) \big)^{-3/2} \quad \text{for } \ x \in B_{2 \cdot S} \backslash B_S. \tag{4.47}$$

From (1.10), (4.46), (4.47), and Theorem 4.6,

$$|G(y,\sigma)| \leq \mathfrak{C}(S) \cdot |y|^{-3/2} \cdot \nu(y)^{-2} \leq \mathfrak{C}(S) \cdot \big(|y| \cdot \nu(y) \big)^{-3/2} \tag{4.48}$$

for $y \in B_S^c, \ \sigma \in (0, T_0)$. Now again take $j, l \in \{1, 2, 3\}, \ (x,t) \in B_{2 \cdot S}^c \times (0, T_0)$. Once more starting from (4.38), we revisit the estimates of $\mathfrak{A}_k^{(2)}$ and $\mathfrak{A}_k^{(3)}$, using (4.48) instead of (4.40). Then it follows as in (4.41) and (4.42) that

$$|\mathfrak{A}_k^{(2)}| \leq \mathfrak{C}(S) \cdot \big(|x| \cdot \nu(x) \big)^{-3/2} \quad (1 \leq k \leq 3). \tag{4.49}$$

Moreover, proceeding as in (4.43) and (4.44), we obtain

$$|\mathfrak{A}_k^{(3)}| \leq \mathfrak{C}(S) \cdot \int_{\mathbb{R}^3} \big((1+|x-y|) \cdot \nu(x-y) \big)^{-3/2} \cdot \big((1+|y|) \cdot \nu(y) \big)^{-3/2} \, dy$$

for $1 \leq k \leq 3$. Again we refer to Theorem 2.18, choosing $A = B = F = 3/2, \ E = 1$. We get

$$|\mathfrak{A}_k^{(3)}| \leq \mathfrak{C}(S) \cdot |x|^{-1} \cdot \nu(x)^{-3/2} \quad (1 \leq k \leq 3). \tag{4.50}$$

Bringing together (4.38), (3.9), (4.39), (4.49), and (4.50), we get

$$|\nabla_x u(x,t)| \le \mathfrak{C}(S) \cdot |x|^{-1} \cdot \nu(x)^{-3/2} \quad \text{for } x \in B^c_{2\cdot S},\ t \in (0,T_0). \tag{4.51}$$

By (4.47), we may conclude that inequality (4.51) even holds for $x \in B^c_S,\ t \in (0,T_0)$. As a consequence, with Theorem 4.6 and (1.10),

$$|G(y,\sigma)| \le \mathfrak{C}(S) \cdot |y|^{-2} \cdot \nu(y)^{-5/2} \le \mathfrak{C}(S) \cdot |y|^{-7/4} \cdot \nu(y)^{-3/2} \tag{4.52}$$

for $y \in B^c_S,\ \sigma \in (0,T_0)$. Once more take $j,l \in \{1,\,2,\,3\},\ (x,t) \in B^c_{2\cdot S} \times (0,T_0)$, and start from (4.38) a third time. But now we only need to improve our estimate of $\mathfrak{A}^{(3)}_k$, using (4.52) instead of (4.48). It follows as in (4.43) and (4.44) that

$$|\mathfrak{A}^{(3)}_k| \le \mathfrak{C}(S) \cdot \int_{\mathbb{R}^3} \big((1+|x-y|) \cdot \nu(x-y) \big)^{-3/2} \cdot (1+|y|)^{-7/4} \cdot \nu(y)^{-3/2}\, dy$$

for $1 \le k \le 3$. Turning to Theorem 2.18 with $A = 7/4,\ B = 3/2,\ E = 5/4,\ F = 3/2$, we may conclude that

$$|\mathfrak{A}^{(3)}_k| \le \mathfrak{C}(S) \cdot (1+|x|)^{-5/4} \cdot \nu(x)^{-3/2} \le \mathfrak{C}(S) \cdot |x|^{-5/4} \cdot \nu(x)^{-3/2} \quad (1 \le k \le 3).$$

It follows with (4.38), (3.9), (4.39), and (4.49) that

$$|\nabla_x u(x,t)| \le \mathfrak{C}(S) \cdot |x|^{-5/4} \cdot \nu(x)^{-3/2} \quad \text{for } x \in B^c_{2\cdot S},\ t \in (0,T_0). \tag{4.53}$$

As before, we may refer to (4.47) to see that (4.53) even holds if $x \in B^c_S$. Thus with Theorem 4.6 and (1.10),

$$|G(y,\sigma)| \le \mathfrak{C}(S) \cdot |y|^{-9/4} \cdot \nu(y)^{-5/2} \le \mathfrak{C}(S) \cdot |y|^{-9/4} \cdot \nu(y)^{-3/2} \tag{4.54}$$

for $y \in B^c_S,\ \sigma \in (0,T_0)$. A last time take $j,l \in \{1,\,2,\,3\},\ (x,t) \in B^c_{2\cdot S} \times (0,T_0)$, and return to (4.38). As in (4.43) and (4.44), but with inequality (4.54) in the place of (4.52), we get

$$|\mathfrak{A}^{(3)}_k| \le \mathfrak{C}(S) \cdot \int_{\mathbb{R}^3} \big((1+|x-y|) \cdot \nu(x-y) \big)^{-3/2} \cdot (1+|y|)^{-9/4} \cdot \nu(y)^{-3/2}\, dy$$

for $1 \le k \le 3$. In this situation, we apply Theorem 2.18 with $A = 9/4,\ B = 3/2,\ E = F = 3/2$. It follows that

$$|\mathfrak{A}^{(3)}_k| \le \mathfrak{C}(S) \cdot \big((1+|x|) \cdot \nu(x) \big)^{-3/2} \le \mathfrak{C}(S) \cdot \big(|x| \cdot \nu(x) \big)^{-3/2} \quad (1 \le k \le 3),$$

so by (4.38), (3.9), (4.39), and (4.49),

$$|\nabla_x u(x,t)| \le \mathfrak{C}(S) \cdot \big(|x| \cdot \nu(x) \big)^{-3/2} \quad \text{for } x \in B^c_{2\cdot S},\ t \in (0,T_0). \tag{4.55}$$

We know by (4.47) that inequality (4.55) also holds if $x \in B_{2\cdot S} \backslash B_S$, so Theorem 4.8 is proved. ∎

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