[10] S. FUČIK - O. JOHN - A. KUFNER, *Function spaces*, Noordhoff, Leyden, 1977.

[11] M. G. GARONNI - V. A. SOLONNIKOV, *On parabolic oblique derivative problem with Hölder-continuous coefficients*, Comm. Partial Differential Equations, **9** (1984), 1323-1373.

[12] M. GIAQUINTA - G. MODICA, *Local existence for quasilinear parabolic systems under nonlinear boundary conditions*, Ann. Mat. Pura Appl., **149** (1987), 41-59.

[13] D. GILBARG - N. S. TRUDINGER, *Elliptic partial differential equations of second order*, Springer, Berlin e.a., 1977.

[14] L. I. KAMYNIN - V. N. MASLENNIKOVA, *Boundary estimates of the solution of the third boundary value problem for a parabolic equation* (Translation), Soviet Math. Dokl., **6** (1965), 128-131.

[15] L. I. KAMYNIN - V. N. MASLENNIKOVA, *Boundary estimates of the Schauder type for the solution of the problem with oblique derivatives for a parabolic equation in a noncylindrical region* (Translation), Siberian Math. J., **7** (1966), 65-103.

[16] L. I. KAMYNIN, *The smoothness of heat potentials, I* (Translation), Differential Equations, **1** (1967), 613-647.

[17] L. I. KAMYNIN, *On the smoothness of heat potentials, II* (Translation), Differential Equations, **2** (1969), 337-357.

[18] O. A. LADYZHENSKAYA - V. A. SOLONNIKOV - N. N. URAL'CEVA, *Linear and quasilinear equations of parabolic type* (Translation), Amer. Math. Soc., Trans. Math. Monographs, **23**, Providence, R.I., 1968.

[19] G. M. LIEBERMAN, *Hölder-continuity of the gradient of solutions of uniformly parabolic equations with conormal boundary conditions*, Ann. Mat. Pura Appl., **148** (1977), 77-99.

[20] W. POGORZELSKI, *Étude de la solution fondamentale de l'équation parabolique*, Ricerche Mat., **5** (1956), 25-57.

[21] W. POGORZELSKI, *Propriétés des intégrales de l'équation parabolique normale*, Ann. Polon. Math., **4** (1957), 61-92.

[22] W. POGORZELSKI, *Problèmes aux limites pour l'équation parabolique normale*, Ann. Polon. Mat., **4** (1957), 110-126.

[23] W. POGORZELSKI, *Propriétés de dérivées tangentielles d'une intégrale de l'équation parabolique*, Ricerche Mat., **6** (1957), 162-194.

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A Quasilinear Parabolic Initial-Value Problem with Conormal Boundary Conditions.

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Sunto. – *Si studia un problema quasilineare parabolico nel cilindro $\Omega \times (0, T)$, con condizione al contorno di tipo conormale. Si prova l'esistenza locale di una soluzione in spazi di funzioni Hölderiane. Si danno inoltre controesempi all'unicità e alla continuabilità delle soluzioni.*

1. – Introduction and main results.

We consider a quasilinear parabolic initial-value problem with conormal boundary conditions:

$$(1.1)\quad \partial/\partial t\, w(x,t) - \sum_{l,m=1}^{N} A_{lm}\big(x,t,(D_r w(x,t))_{0\le r\le N}\big)\cdot \partial^2/\partial x_l\, \partial x_m\, w(x,t) = F\big(x,t,(D_r w(x,t))_{0\le r\le N}\big) \quad \text{in } \Omega\times(0,T]\,;$$

$$(1.2)\quad \sum_{l,m=1}^{N} A_{l,m}\big(x,t,(D_r w(x,t))_{0\le r\le N}\big)\cdot n_m(x)\cdot D_l w(x,t) = G\big(x,t,w(x,t),(|D_r w(x,t)|^{\gamma})_{1\le r\le N}\big) \quad \text{on } M_T\,;$$

$$(1.3)\quad w(x,0) = \Psi(x) \quad \text{on } \bar{\Omega}\,,$$

where $\gamma \in (1, \infty)$, Ω bounded domain in R^N ($N \in N$), with C^2-boundary $\partial\Omega$. The parameters γ, N, and Ω will be kept fixed throughout. n denotes the outward unit normal to Ω. For brevity we use the abbreviations

$$M_T := \partial\Omega \times [0, T]\,, \quad Q_T := \Omega \times (0, T)\,, \qquad \text{for } T \in (0, \infty)\,.$$

Problems of the above type are motivated by applications, for example by the capillary problem; see [8; sections 1.2, 1.9, 1.10].

We impose the following conditions on the functions A_{lm}, F, Ψ, G, with the precise meaning of our notations explained in Section 2.

Let $\alpha \in (0, 1)$, $\tilde{C}, \tilde{T} \in (0, \infty)$. Then suppose

$$(1.4) \quad A_{lm} \in C^0(\bar{Q}_{\tilde{T}} \times (-\tilde{C}, \tilde{C})^{N+1}), \; A_{lm} = A_{ml},$$

$$A_{lm}(x, t, \cdot) \in C^1((-\tilde{C}, \tilde{C})^{N+1}) \quad \text{for } 1 \leqq l, \; m \leqq N, \; (x, t) \in Q_{\tilde{T}};$$

$$L_1 := 2 \cdot \sup \{|A_{lm}(\cdot, \cdot, P)|_\alpha : P \in (-\tilde{C}, \tilde{C})^{N+1}; 1 \leqq l, m \leqq N\} + \max \{|D_{N+2+r} A_{lm}|_0 : 1 \leqq l, m \leqq N, 0 \leqq r \leqq N\} < \infty;$$

$$E := \inf \Big\{ \sum_{l,m=1}^{N} A_{lm}(x, t, P) \cdot \xi_l \cdot \xi_m : (x, t, P) \in \bar{Q}_{\tilde{T}} \times (-\tilde{C}, \tilde{C})^{N+1}, \; \xi \in \mathbf{R}^N, |\xi| = 1 \Big\} > 0;$$

$$(1.5) \quad F \in C^0(\Omega \times (0, \tilde{T}] \times (-\tilde{C}, \tilde{C})^{N+1});$$

$$\sup \{|F(\cdot, t, \cdot)|_0 \cdot t^{1/2 - \alpha/2} : t \in (0, \tilde{T}]\} < \infty;$$

there is some $\beta \in (0, 1)$ such that

$$\sup \{|F(\cdot, t, P)|K|_{0,\beta} : t \in [S, \tilde{T}], P \in (-\tilde{C}, \tilde{C})^{N+1}\} < \infty$$

for $K \subset \mathbf{R}^N$ with $\bar{K} \subset \Omega$, $S \in (0, \tilde{T}]$;

for $0 \leqq r \leqq N$, the derivative $D_{N+2+r} F$ exists and is continuous;

$$\sup \{|D_{N+2+r} F(x, t, P)| : x \in \bar{K}, t \in [S, \tilde{T}], P \in (-\tilde{C}, \tilde{C})^{N+1}, 0 \leqq r \leqq N\} < \infty \quad \text{for } K \subset \mathbf{R}^N \text{ with } \bar{K} \subset \Omega, \; S \in (0, \tilde{T}];$$

$$(1.6) \quad G \in C^0(M_{\tilde{T}} \times (-\tilde{C}, \tilde{C}) \times [0, \tilde{C}^\gamma)^N);$$

$$G(x, t, \cdot) \in C^1((-\tilde{C}, \tilde{C}) \times [0, \tilde{C}^\gamma)^N) \quad \text{for } (x, t) \in M_{\tilde{T}},$$

$$L_2 := 2 \cdot \sup \{|G(\cdot, \cdot, P)|_\alpha : P \in (-\tilde{C}, \tilde{C}) \times [0, \tilde{C}^\gamma)^N\} + |D_{N+2} G|_0 + (\gamma - 1) \cdot \sum_{r=1}^{N} |D_{N+2+r} G|_0 < \infty;$$

$$(1.7) \quad \Psi \in C_{1,\alpha}(\bar{\Omega}) \quad \text{with } |D_r \Psi|_0 < \tilde{C} \text{ for } 0 \leqq r \leqq N;$$

$$(1.8) \quad D_l \Psi(x) = 0 = G\big(x, 0, \Psi(x), (|D_r \psi(x)|^\gamma)_{1 \leqslant r \leqslant N}\big),$$

for $x \in \partial\Omega$, $1 \leqslant l \leqslant N$.

Under these assumptions we shall prove

THEOREM 1.1. – *There is some* $T \in (0, \tilde{T}]$, *and a function* $w \in C_{1+\alpha}(\bar{Q}_T)$, *with* $|D_l w|_0 < \tilde{C}$ $(0 \leqq l \leqq N)$, *and with* $w|\Omega \times (0, T] \in C^{2,1}(\Omega \times (0, T])$, *such that* w *solves equations* (1.1)-(1.3).

Conditions (1.4)-(1.7) are fairly general. In particular we remark thar the right-hand side F in (1.1) may show a slight singularity at $t = 0$. However, compatibility condition (1.8) seems to be strange and unnatural. In its place, the following natural hypothesis should be expected:

$$(1.9) \quad \sum_{l,m=1}^{N} A_{lm}\big(x, 0, (D_r \Psi(x))_{0 \leqq r \leqq N}\big) \cdot n_m(x) \cdot D_l \Psi(x) = G\big(x, 0, \Psi(x), (|D_r \psi(x)|^\gamma)_{1 \leqslant r \leqslant N}\big) \quad \text{for } x \in \partial\Omega.$$

However, in Section 6 we shall give a counterexample which shows that condition (1.8) is reasonable: with (A_{lm}), Ψ, G only satisfying assumptions (1.4), (1.6), (1.7), (1.9), it cannot be expected that a number $T \in (0, \tilde{T}]$ exists such that equations (1.2) and (1.3) are solved by a function $w \in C^{1,0}(\bar{Q}_T)$. Note that a contradiction already arises from initial condition (1.3) and boundary condition (1.2); differential equation (1.1) need not be considered. Our counterexample further shows that this is true even for $G = 0$, and for functions A_{lm} and Ψ which are as smooth as possible.

Of course, compatibility condition (1.8) does not allow local continuation of solutions. But independent of any compatibility condition, it may happen that a solution of (1.1)-(1.3), given as in Theorem 1.1, cannot be continued beyond its compact interval of existence—at least not as a $C^{1,0}$-function satisfying (1.2) and (1.3). In Section 6 we shall illustrate this situation by a counterexample.

As for the question of uniqueness, another counterexample in Section 6 proves that our solutions are not uniquely determined, and that uniqueness cannot be obtained by choosing a smaller interval of existence.

Concerning other papers which relate to quasilinear parabolic equations with conormal boundary conditions, we have to mention the works of Amann; see [3-6]. Using an abstract approach, Amann proves local existence of solutions to quasilinear parabolic systems with nonlinear boundary conditions of various types. However, his results do not cover problem (1.1)-(1.3) because he considers coefficients only depending on the solution w of the problem involved, but not on ∇w. The same remark applies to [12, 19], where problems similar to (1.1)-(1.3) are treated. Acquistapace and Terreni [1] consider a parabolic system where the equations on the cylinder $\Omega \times (0, T]$ show the same type of nonlinearity as equation (1.1). Furthermore they consider a boundary-value problem with directional derivatives (see [18; (I.2.10)]). Thus they deal with a situation which is more general than ours. In spite of that, their results cannot be applied to our problem, since the coefficients in their boundary condition do not depend on ∇w.

Thus it seems there are no previous results on (1.1)-(1.3).

The proof of Theorem 1.1 is based on estimates of solutions of the following linear parabolic problem with conormal boundary conditions:

$$(1.10)\qquad \partial/\partial t\, v(x,t) - \sum_{l,m=1}^{N} a_{lm}(x,t)\cdot \partial^2/\partial x_l\, \partial x_m\, v(x,t) -$$

$$- \sum_{l=1}^{N} b_l(x,t)\cdot \partial/\partial x_l\, v(x,t) - c(x,t)\cdot v(x,t) = f(x,t) \quad \text{in } \Omega\times(0,T]\,;$$

$$(1.11)\qquad \sum_{l,m=1}^{N} a_{lm}(x,t)\cdot n_m(x)\cdot D_l v(x,t) = g(x,t) \quad \text{on } M_T\,;$$

$$(1.12)\qquad v(x,0) = \Psi(x) \quad \text{on } \bar{\Omega}\,.$$

The ensuing assumptions are imposed on (a_{lm}), (b_l), c, f, Ψ, g: Let $\alpha\in(0,1)$, $T\in(0,\infty)$. Then suppose

$$(1.13)\qquad a_{lm}\in C_\alpha(\bar{Q}_T)\,,\quad a_{lm}=a_{ml} \qquad \text{for } 1\leqq l, m\leqq N\,;$$

$$\inf\Big\{ \sum_{m=1}^{N} a_{lm}(x,t)\cdot\xi_l\cdot\xi_m\colon (x,t)\in\bar{Q}_T,\ \xi\in \mathbf{R}^N,\ |\xi|=1\Big\} > 0\,;$$

$$(1.14)\qquad f\in C^0(\Omega\times(0,T])\,;\ \sup\{|f(\cdot,t)|\cdot t^{1/2-\alpha/2}\colon t\in(0,T]\}<\infty\,;$$

there is some $\beta\in(0,1)$ such that

$$\sup\{|f(\cdot,t)|K|_{0,\beta}\colon t\in[S,T]\}<\infty$$

$$\text{for } K\subset\mathbf{R}^N \text{ with } \bar{K}\subset\Omega\,,\ S\in(0,T)\,;$$

$$(1.15)\qquad \Psi\in C_{1,\alpha}(\bar{\Omega})\,;$$

$$(1.16)\qquad g\in C_\alpha(M_T)\,;$$

$$(1.17)\qquad \sum_{l,m=1}^{N} a_{lm}(x,0)\cdot n_m(x)\cdot D_l\Psi(x) = g(x,0) \qquad \text{for } x\in\partial\Omega\,;$$

$$(1.18)\qquad b_l,\ c\in C_\alpha(\bar{Q}_T) \qquad \text{for } 1\leqq l\leqq N\,.$$

By a $C^{1,0}$-solution to (1.10)-(1.12), we understand a function $v\in C^{1,0}(\bar{Q}_T)$, with $v|\Omega\times(0,T]\in C^{2,1}(\Omega\times(0\ T])$, which of course satisfies (1.10)-(1.12).

The following theorem states in particular that under assumptions (1.13)-(1.18), any $C^{1,0}$-solution of (1.10)-(1.12) is uniquely determined, and even belongs to $C_{1+\alpha}(\bar{Q}_T)$:

THEOREM 1.2. – *For T, α, (a_{lm}), (b_l), c, f, Ψ, g as in (1.13)-(1.18), there is a uniquely determined $C^{1,0}$-solution of (1.10)-(1.12).*

There are functions $R_1, \ldots, R_4$, with R_i mapping $(0,\infty)^i$ into $(0,\infty)$ $(i=1,\ldots,4)$, and R_4 monotone increasing in its first variable, such that the following results hold true:

Take α, T, (a_{lm}), (b_l), c, f, Ψ, g as in (1.13)-(1.18). Let $\tilde{K}_1$, K_1, K_2, $M\in(0,\infty)$ be such that

$$(1.19)\qquad \sum_{l,m=1}^{N} a_{lm}(x,t)\cdot\xi_l\cdot\xi_m \geqq K_2\cdot|\xi|^2 \qquad \text{for } \xi\in\mathbf{R}^N,\ (x,t)\in\bar{Q}_T\,;$$

$$|a_{lm}|_0\leqq K_1\,,\ |a_{lm}|_\alpha,\ |b_l|_\alpha,\ |c|_\alpha\leqq \tilde{K}_1 \qquad \text{for } 1\leqq l, m\leqq N\,;$$

$$|f(\cdot,t)|_0\leqq M\cdot t^{-1/2+\alpha/2} \qquad \text{for } t\in(0,T]\,.$$

Then the ensuing inequalities hold for the corresponding $C^{1,0}$-solution v of (1.10)-(1.12):

$$(1.20)\qquad |v|_\alpha\leqq R_4(T,\tilde{K}_1,K_2,\alpha)\cdot T^{\alpha/2}\cdot(M+|g|_\alpha+|\Psi|_{1,\alpha})+R_1(\alpha)\cdot|\Psi|_{0,\alpha};$$

$$(1.21)\qquad |v|_{1+\alpha}\leqq [R_4(T,\tilde{K}_1,K_2,\alpha)\cdot T^{\alpha/8\wedge(1/2-\alpha/2)}+R_3(K_1,K_2,\alpha)]\cdot$$

$$\cdot\Big[M+|\Psi|_{1,\alpha}+|g|_\alpha+R_2(\tilde{K}_1,\alpha)\cdot\sum_{m=1}^{N}|D_m\Psi|\partial\Omega|_0\Big]\,.$$

We shall also need a comparison result:

THEOREM 1.3. – *There are functions $R_5\colon(0,\infty)^3\to(0,\infty)$, $R_6\colon(0,\infty)^4\to(0,\infty)$ with the following properties:*

Let T, α, (a_{lm}), f, Ψ, g, $\tilde{K}_1$, K_2, M be given as in (1.13)-(1.18) and Theorem 1.2. (The parameter K_1 is not needed here.) Suppose in addition that

$$(1.22)\qquad T\leqq R_5(\tilde{K}_1,K_2,\alpha)\,.$$

The functions $(\tilde{a}_{lm})$, $\tilde{f}$, $\tilde{g}$ may satisfy (1.13)-(1.17) in an analogous way. Furthermore we suppose that ellipticity condition (1.19) is valid *for* $(\tilde{a}_{lm})$, and

$$|\tilde{a}_{lm}|_\alpha \leqq \tilde{K}_1 \ (1 \leqq l, m \leqq N), \qquad |\tilde{f}(\cdot, t)|_0 \cdot t^{1/2-\alpha/2} \leqq M \ (t \in (0, T]).$$

Let $v, \tilde{v}$ *be the* $C^{1,0}$*-solution of* (1.10)-(1.12) *which corresponds to* $(a_{lm}), f, g, \Psi$, *and* $(\tilde{a}_{lm}), \tilde{f}, \tilde{g}, \Psi$ *respectively, with* $b_l = 0 \ (1 \leqq l \leqq N)$, $c = 0$ (*see Theorem* 1.2). *Then the ensuing inequality holds true*:

$$|v - \tilde{v}|_{1,0} \leqq R_6(T, \tilde{K}_1, K_2, \alpha) \cdot (1 + M + |\Psi|_{1,\alpha} + |g|_\alpha + |\tilde{g}|_\alpha)^2 \cdot$$
$$\cdot (|A - \tilde{A}|_0^{\alpha/32} + |g - \tilde{g}|_0^{\alpha/4} + \mathcal{K}^{\alpha/4}),$$

where $\mathcal{K}$ *is an abbreviation for* $\sup\{|(f - \tilde{f})(\cdot, t)|_0 \cdot t^{1/2-\alpha/2} : t \in (0, T]\}$.

Finally we need a rather weak regularity result for the second derivative in x and the first derivative in t of $C^{1,0}$-solutions of (1.10)-(1.12):

THEOREM 1.4. – *Let* $\alpha, T, (a_{lm}), (b_l), c, f, \beta, \Psi, g, \tilde{K}_1, K_2, M$ *be given as in* (1.13)-(1.18) *and Theorem* 1.2.

Take $S \in (0, T)$, $K \subset \boldsymbol{R}^N$ *with* K *open*, $\bar{K} \subset \Omega$.
Set $\delta := \operatorname{dist}(\bar{K}, \partial\Omega)$, $\Omega_\delta := \{y \in \boldsymbol{R}^N : \operatorname{dist}(y, \bar{K}) < \delta/2\}$.
Let $\mathcal{F} \in (0, \infty)$ *with* $\sup\{|f(\cdot, t)|\Omega_\delta|_{0,\beta} : t \in [S/2, T]\} \leqq \mathcal{F}$.
Finally take $a \in \boldsymbol{N}_0^N$, $l \in \boldsymbol{N}_0$ *with* $2 \cdot l + |a|_* = 2$. *Then for the* $C^{1,0}$*-solution of* (1.10)-(1.12), *corresponding to* $(a_{lm}), (b_l), c, f, \Psi, g$, *the following estimate holds true*:

$$|D_{N+1}^l D_a(v|K \times [S, T])|_{(\alpha \wedge \beta)/4} \leqq \mathcal{C},$$

with a constant $\mathcal{C}$ *which depends on* $\Omega, T, S, \alpha, \beta, \tilde{K}_1, K_2, M, K, \Psi, g, \mathcal{F}$.

We note that in Theorem 1.2-1.4, the coefficients a_{lm} need only be Hölder-continuous: $a_{lm} \in C_\alpha(\bar{Q}_T)$. There are many papers which deal with problem (1.10)-(1.12) under this (rather low) regularity condition. Probably the first author who studied (1.10)-(1.12) under this assumption was Pogorzelski. In [20-23] he constructs a solution v by applying the method of integral equations. Furthermore he estimates $|v|_\alpha$, and proves continuity of $\partial/\partial x_i\, v(x, t)$ in $\bar{\Omega} \times (0, T]$. Pogorzelski's results may be found in Friedman [9]. Another description of the method of integral equations, applied to problem (1.10)-(1.12), may be found in [18]. For the case of the heat equation on noncylindrical domains, Kamynin [16, 17] studies the potentials which will appear in our proof of Theorem 1.2-1.4. However, Kamynin considers problem (1.10)-(1.12) under assumptions on a_{lm}

which are stronger than Hölder-continuity; see [14, 15] and [17; Theorem 13].

Garonni and Solonnikov [11] study (1.10), (1.12) under more general boundary conditions (directional derivatives; see [18; (I.2.10)]). Their assumptions on the coefficients in the boundary condition are weak enough to permit an application of their results to problem (1.10)-(1.12) with $a_{lm} \in C_\alpha(\bar{Q}_T)$. To be precise, we mention that the conditions on f and Ψ in [11] are somewhat different from (1.14), (1.15). Among other results, the authors construct a $C^{1,0}$-solution v of their problem, and they estimate $|v|_{1+\alpha}$ against a constant times the sum of certain Hölder norms of f, Ψ, g. The constant appearing in this estimate is not further specified. The authors prove this result by reducing their problem to an initial-boundary value problem for the heat equation in half-space.

We prove Theorem 1.2-1.4 via the method of integral equations, which gives a rather explicit representation of solutions of (1.10)-(1.12). Thus we are able to specify how the parameters $\tilde{K}_1$ (upper bound for $|a_{lm}|_\alpha$), K_1 (upper bound for $|a_{lm}|_0$), and T (height of the cylinder Q_T) enter into the constants appearing on the right-hand side of the C_α- and $C_{1+\alpha}$-estimates (1.20) and (1.21) respectively. Furthermore our method yields the comparison result in Theorem 1.3, where the difference of the gradient of two solutions, corresponding to different coefficients, is estimated for small T. These results, which seem to be new, will play an essential role when we shall consider the quasilinear problem (Section 5).

We mention that the functions $R_1, \ldots, R_6$ can be constructed explicitly. This construction only involves elementary functions, as well as some parameters related to Ω. We shall refer to a description of $\partial\Omega$ as constructed in [7; § 2]. Thus we may concretely enumerate the just mentioned parameters. They are the dimension N of $\boldsymbol{R}^N$, the diameter of Ω, and numbers $k_\Omega, \alpha_\Omega, \varepsilon_{\mathcal{B}}, K_{\mathcal{B}}$ as given in [7; Lemma 2.1, 2.2].

Since the method of integral equations has been refined over a long time—see our references above—we may afford to be synoptic in the proof of Theorems 1.2-1.4, giving only outlines most of the time, and concentrating on those points where the existing theory is improved.

Let us indicate how Theorems 1.2-1.4 are used for proving an existence result for the quasilinear problem (1.1)-(1.3) (see Section 5). We shall first consider (1.1)-(1.3) under an additional hypothesis on F, namely

$$(1.23) \qquad \sup\{|D_{N+2+r} F(\cdot, t, \cdot)|_0 \cdot t^{1/2-\alpha/2} : t \in (0, T], 0 \leqq r \leqq N\} < \infty.$$

Then equations (1.1)-(1.3) may be solved by a fixed-point argument in $C_{1+\alpha}(\bar{Q}_T)$, for small T. It is at this point where Theorems 1.2 and 1.3 are applied. Afterwards we shall eliminate condition (1.23) by using an approximation argument. Here we shall need Theorem 1.4. Note that for the first and decisive step—the fixed-point argument—it is not necessary to estimate the second derivatives in x or the first derivative in t of solutions of the linear problem.

2. – Notations.

For $a, b \in \boldsymbol{R}$, let $a \wedge b$, $a \vee b$ denote the minimum, maximum of $\{a, b\}$ respectively. If $T \in (0, \infty]$, and a_{lm} a function on $\bar{Q}_T$ for $1 \leqq l, m \leqq N$, we only write (a_{lm}) for the matrix-valued function $(a_{lm})_{1 \leqq l, m \leqq N}$. The abbreviation (b_l) is to be understood in a similar way.

For $n \in \boldsymbol{N}$, vectors in $\boldsymbol{R}^n$ are considered as matrices with one column. $\langle \beta, \gamma \rangle$ is the inner product of $\beta, \gamma \in \boldsymbol{R}^n$. $|\ |$ denotes the Euclidean norm in $\boldsymbol{R}^n$. We set $|a|_* := a_1 + \ldots + a_n$ for $a \in \boldsymbol{N}_0^n$.

Let U be a subset of $\boldsymbol{R}^n$, $n \in \boldsymbol{N}$, and let f be a function from U into $\boldsymbol{R}$. Then $|f|_0$ denotes the expression $\sup\{|f(p)|: p \in U\}$. We set for $\alpha \in (0, 1)$

$$|f|_{0,\alpha} := |f|_0 + \sup\{|f(p) - f(\tilde{p})|/|p - \tilde{p}|^\alpha: p, \tilde{p} \in U, p \neq \tilde{p}\}.$$

$C^0(U)$ is defined as usual. Let $i \in \{1, \ldots, n\}$, and suppose that the set $\{y_i: y \in U\}$ is open or an interval. If $\partial/\partial x_i f(x)$ exists for all $x \in U$, then $D_i f$ denotes the corresponding derivative. $D_{i(1)} \ldots D_{i(k)} f$ is defined by recursion ($k \in \boldsymbol{N}$, $i(1), \ldots, i(k) \in \{1, \ldots, n\}$). We set $D_0 f := f$. The meaning of $D_i^k f$ ($k \in \boldsymbol{N}_0, 1 \leqq i \leqq n$), and $D_a f$ ($a \in \boldsymbol{N}_0^n$) is obvious.

Take $B \subset \boldsymbol{R}^n$ open ($n \in \boldsymbol{N}$). Let $A \subset \boldsymbol{R}^n$ with $B \subset A \subset \bar{B}$, $g \in C^0(A)$, $k \in \boldsymbol{N}$, $i(1), \ldots, i(k) \in \{1, \ldots, n\}$. If $D_{i(1)\ldots i(k)}(g|B)$ exists and may be continuously extended to A, then $D_{i(1)\ldots i(k)} g$ denotes the corresponding extension. For $m \in \boldsymbol{N}_0 \cup \{\infty\}$, the sets $C^m(B)$, $C_0^m(B)$ are defined in the usual way. $C^m(A)$ denotes the set of all functions $f \in C^0(A)$ such that the derivative $D_a(f|B)$ exists and may be continuously extended to A, for $a \in \boldsymbol{N}^n$, with $|a|_* \leqq m$. We set

$$|\Psi|_{1,\alpha} := \sum_{l=0}^{n} |D_l \Psi|_{0,\alpha} \qquad \text{for } \psi \in C^1(A), \ \alpha \in (0, 1).$$

$C_{1,\alpha}(A)$ is the set of all functions $\Psi \in C^1(A)$ with $|\Psi|_{1,\alpha} < \infty$.

Take $T \in (0, \infty)$. $C_{1,0}(\bar{Q}_T)$ is the set of all functions $u \in C^0(\bar{Q}_T)$ such that the derivative $D_i(u|Q_T)$ exists, is continuous, and may be continuously extended to $\bar{Q}_T$, for $1 \leqq i \leqq N$. The set $C^{2,1}(\Omega \times (0, T])$ contains all functions $u \in C^0(\Omega \times (0, T])$ such that the derivatives $D_{N+1}^l D_a u$, for $l \in \boldsymbol{N}_0$, $a \in \boldsymbol{N}_0^N$ with $2 \cdot l + |a|_* \leqq 2$, exist and are continuous. The meaning of $C^{2,1}(\bar{Q}_T)$ is now clear. For $U \subset \boldsymbol{R}^{N+1}$, $\alpha \in (0, 1)$, $v \in C^0(U)$, we define

$$|v|_\alpha := |v|_0 + \sup\{|v(x, t) - v(\tilde{x}, \tilde{t})| \cdot (|x - \tilde{x}|^2 + |t - \tilde{t}|)^{-\alpha/2}: (x, t), (\tilde{x}, \tilde{t}) \in U, (x, t) \neq (\tilde{x}, \tilde{t})\}.$$

We set $C_\alpha(\bar{Q}_T) := \{v \in C^0(\bar{Q}_T): |v|_\alpha < \infty\}$. For $u \in C^{1,0}(\bar{Q}_T)$ we define

$$|u|_{1,0} := \sum_{l=0}^{N} |D_l u|_0 ; \qquad |u|_{1+\alpha} := \sum_{l=0}^{N} |D_l u|_\alpha . \quad \text{Set}$$

$$C_{1+\alpha}(\bar{Q}_T) := \{u \in C^{1,0}(\bar{Q}_T): |u|_{1+\alpha} < \infty\}.$$

Furthermore, for $u \in C^{1,0}(\bar{Q}_T)$, we write ∇u instead of $(D_1 u, \ldots, D_N u)$. Finally, for a matrix-valued function $C = (c_{lm})$ on $\bar{Q}_T$, we define

$$|C|_0 := \max\{|c_{lm}|_0: 1 \leqq l, m \leqq N\}.$$

3. – Proof of Theorem 1.2, 1.4.

Let us first note that the uniqueness result in Theorem 1.2 follows from [9; Lemma 5.3.2].

Let α, T, (a_{lm}), (b_l), c, f, Ψ, g, $\tilde{K}_1$, K_1, K_2, M be given as in (1.13)-(1.18) and Theorem 1.2.

Since we want to study the constants appearing in our estimates, we have to take special care in denoting these constants. For this purpose, we shall use the ensuing convention: The letter E is to denote various functions of the following type: they map $\boldsymbol{R}^k$ into $(0, \infty)$, for some $k \in \boldsymbol{N}_0$; they can be constructed *a priori*; their construction only involves elementary functions, as well as those parameters depending on Ω which were enumerated in Section 1. If we want to indicate that the function in question is monotone increasing in its first variable, then we shall write $\tilde{E}$ instead of E. The terms $E(K_1, K_2, \alpha)$, $\tilde{E}(T, \tilde{K}_1, K_2, \alpha)$, which will arise frequently, will be abbreviated by R, S respectively.

We now proceed to construct a $C^{1,0}$-solution to (1.10)-(1.12). We shall use the abbreviations $A := (a_{lm})$, $b := (b_l)$. The inverse of $A(y, s)$, for $(y, s) \in \bar{Q}_T$, is denoted by $A^{-1}(y, s)$. We define the function $Z := Z_A$—called parametrix—by setting for $z \in \boldsymbol{R}^N$, $r \in (0, \infty)$, $(y, s) \in \bar{Q}_T$:

$$(3.1)\qquad Z(z, r, y, s) := (4\cdot\pi\cdot r)^{-N/2}\cdot \det (A^{-1}(y, s))^{1/2}\cdot \exp (-\langle x, A^{-1}(y, s)\cdot x\rangle/(4\cdot r)) .$$

Note that

$$(3.2)\qquad \begin{cases} Z(\cdot, \cdot, y, s) \in C^\infty(\boldsymbol{R}^N\times(0, \infty)) & \text{for } (y, s) \in \bar{Q}_T ; \\ D_\sigma Z \in C_0(\boldsymbol{R}^N\times(0, \infty)\times\bar{Q}_T) & \text{for } \sigma \in \boldsymbol{N}_0^{N+1} . \end{cases}$$

For z, r, y, s as in (3.1), with $z \neq 0$, and for $a \in \boldsymbol{N}_0^N$, $l \in \boldsymbol{N}_0$ with $2\cdot l + |a|_* \leq 3$, $\mu \in (-\infty, (N + |a|_* + 2\cdot l)/2]$, we have

$$(3.3)\qquad |D^l_{N+1} D_a Z(z, r, y, s)| \leq P_1(K_1, K_2, \mu)\cdot r^{-\mu}\cdot |z|^{-N-|a|_*-2\cdot l+2\cdot\mu}\cdot \exp (-|z|^2/(8\cdot N^2\cdot K_1\cdot r)) ,$$

with a function P_1 from $\boldsymbol{R}^3$ into $(0, \infty)$. Departing from our convention above, we gave an individual name to this function. This was done in view of a later application (see Section 4).

For $z \in \boldsymbol{R}^N$, $z \neq 0$, $r \in (0, \infty)$, (y, s), $(\tilde{y}, \tilde{s}) \in \bar{Q}_T$, and for a, l, μ as in (3.3), the ensuing inequality holds:

$$(3.4)\qquad |D^l_{N+1} D_a Z(z, r, y, s) - D^l_{N+1} D_a Z(z, r, \tilde{y}, \tilde{s})| \leq E(\tilde{K}_1, K_2, \mu)\cdot r^{-\mu}\cdot |z|^{-N-|a|_*-2\cdot l+2\cdot\mu}\cdot (|y - \tilde{y}|^2 + |s - \tilde{s}|)^{\alpha/2}\cdot \exp (-|z|^2/(8\cdot N^2\cdot \tilde{K}_1\cdot r)) .$$

We define the function $LZ := LZ_{A,b,c}$ by setting for (x, t), $(y, s) \in \bar{Q}_T$ with $t > s$:

$$LZ(x, t, y, s) := \sum_{l,m=1}^{N} (a_{lm}(x, t) - a_{lm}(y, s))\cdot D_l D_m Z(x - y, t - s, y, s) + \sum_{l=1}^{N} b_l(x, t)\cdot D_l Z(x - y, t - s, y, s) + c(x, t)\cdot Z(x - y, t - s, y, s) .$$

It is proved in [9; Section 1.4] that there is a function $\Phi := \Phi_{A,b,c}$, defined on $C^0(\{(x, t, y, s) \in \bar{Q}_T^2 : t > s\})$, such that the ensuing integral equation is valid:

$$(3.5)\qquad \Phi(x, t, y, s) = LZ(x, t, y, s) + \int_s^t \int_\Omega LZ(x, t, \xi, \tau)\cdot \Phi(\xi, \tau, y, s)\, d\xi\, d\tau$$

for (x, t), $(y, s) \in \bar{Q}_T$ with $t > s$.

Φ satisfies the following inequality, for (x, t), $(y, s) \in \bar{Q}_T$ with $x \neq y$, $t > s$, $\mu \in (-\infty, (N + 2 - \alpha)/2]$:

$$(3.6)\qquad |\Phi(x, t, y, s)| \leq \tilde{E}(T, \tilde{K}_1, K_2, \alpha, \mu)\cdot (t - s)^{-\mu}\cdot |x - y|^{-N-2+2\cdot\mu+\alpha}\cdot \exp\big(-|x - y|^2/(8\cdot N^2\cdot \tilde{K}_1\cdot (t - s))\big) .$$

Inequality (3.6) is obtained by tracing constants in [9; Section 1.4]. We now introduce the functions $Z_0 := Z_{0A,b,c}$, $\Gamma := \Gamma_{A,b,c}$ by defining for (x, t), $(y, s) \in \bar{Q}_T$ with $t > s$:

$$Z_0(x, t, y, s) := \int_{\Omega\times(0,T)} Z(x - \xi, t - \tau, \xi, \tau)\cdot \Phi(\xi, \tau, y, s)\, d(\xi, \tau) ,$$

$$\Gamma(x, t, y, s) := Z(x - y, t - s, y, s) + Z_0(x, t, y, s) .$$

Γ is the fundamental solution of differential equation (1.10). From (3.3), (3.6), and [9; Lemma 4.3], it follows that the derivatives $D_i Z_0$ are defined on $\{(x, t, y, s) \in \bar{Q}_T^2 : t > s\}$ by continuous extension $(1 \leq i \leq N)$. Because of (3.2), this property is inherited to Γ. We have the following inequalities for Z_0:

$$(3.7)\qquad |D_a Z_0(x, t, y, s)| \leq \tilde{E}(T, \tilde{K}_1, K_2, \alpha, \mu)\cdot (t - s)^{-\mu}\cdot |x - y|^{-N-|a|_*+2\cdot\mu+\alpha}$$

for (x, t), $(y, s) \in \bar{Q}_T$, with $x \neq y$, $t > s$, for $a \in \boldsymbol{N}_0^N$, with $|a|_* \leq 1$, and for $\mu \in (-\infty, (N + |a|_* - \alpha)/2]$;

$$(3.8)\qquad |D_a Z_0(x, t, y, s) - D_a Z_0(\tilde{x}, \tilde{t}, y, s)| \leq \tilde{E}(T, \tilde{K}_1, K_2, \alpha, \mu)\cdot ((t - \tilde{t})^{(1+\alpha)/4} + |x - \tilde{x}|^{(1+\alpha)/2})\cdot (\tilde{t} - s)^{-\mu}\cdot |x - y|^{-N-|a|_*+2\cdot\mu-1/2+\alpha/2}$$

for $t, \tilde{t}, s \in [0, T]$ with $t \geq \tilde{t} > s$, $x, \tilde{x}, y \in \bar{\Omega}$ with $|x - y| \geq 2\cdot|x - \tilde{x}|$, $a \in \boldsymbol{N}_0^N$ with $|a|_* \leq 1$, $\mu \in (-\infty, (N + |a|_* + 1 - \alpha)/2]$.

These inequalities may be deduced from (3.3), (3.6), and [9; Lemma 1.4.3]. This approach, when applied to the proof of (3.8) in the case $|a|_* = 1$, leads to calculations which are not totally obvious. To give a hint, consider the case $x \neq \tilde{x}$, $t = \tilde{t}$. Then, referring to (3.5), the main trick consists in breaking the domain of integration (s, t) into the parts $\big(s, s \vee (t - |x - \tilde{x}|^2)\big)$, and its complement.

For $q \in C^0(\Omega \times (0, T])$ with

$$(3.9) \qquad \tilde{M} := \sup\{|q(\cdot, t)|_0 \cdot t^{1/2-\alpha/2} : t \in (0, T]\} < \infty ,$$

and for $(x, t) \in \bar{Q}_T$, we define

$$W_{A,b,c,q}(x, t) := 0 , \qquad \text{if } t = 0 ;$$

$$W_{A,b,c,q}(x, t) := \int_0^t \int_\Omega \Gamma(x, t, y, s) \cdot q(y, s)\, dy\, ds , \qquad \text{if } t > 0 .$$

With the method from [9; pp. 193-194], we obtain from (3.3), for q as above:

$$(3.10) \qquad |W_{A,b,c,q}|_{1+\alpha} \leqq (R + S \cdot T^{\alpha/2}) \cdot \tilde{M} ,$$

where $\tilde{M}$ is defined as in (3.9).

Next we introduce some notations concerning the description of $\partial\Omega$ by local parameters. As we mentioned in Section 1, we use the description from [7; § 2], which was inspired by [10; 6.2]. Thus, according to [7; Lemma 2.1, 2.2], we may choose $k(\Omega) \in N$, a cube $\Delta \subset R^{N-1}$, C^2-functions $\overset{1}{u}, \ldots, \overset{k(\Omega)}{u}$ from Δ into $\partial\Omega$, and functions $B_1, \ldots, B_{k(\Omega)} \in C_0^1(\Delta)$, with the properties to follow:

(3.11) For $1 \leqq i \leqq k(\Omega)$, B_i is given by $B_i = \overset{i}{\omega} \circ \overset{i}{u} \cdot J_i$,

for some $\overset{i}{\omega} \in C_0^\infty(R^N)$, $J_i \in C^1(\Delta)$.

$$(3.12) \qquad \int_\Omega f\, d\Omega = \sum_{i=1}^{k(\Omega)} \int_\Delta f \circ \overset{i}{u}(\eta) \cdot B_i(\eta)\, d\eta$$

for any integrable function f on $\partial\Omega$.

As an easy consequence of [7; Lemma 2.2] we note for $\varrho \in \Delta$,

$t \in [0, T]$, $1 \leqq i \leqq k(\Omega)$:

$$(3.13) \qquad \det\Big(D_1 \overset{i}{u}(\varrho), \ldots, D_{N-1} \overset{i}{u}(\varrho), A\big(\overset{i}{u}(\varrho), t\big) \cdot n \circ \overset{i}{u}(\varrho)\Big) \geqq K_2 .$$

Furthermore, according to [7; Lemma 2.2], we may choose $\varepsilon(\Omega)$, $K(\Omega) \in (0, \infty)$ with the following properties:

(3.14) For $x, \tilde{x} \in \bar{\Omega}$, and for $i \in \{1, \ldots, k(\Omega)\}$, one of these three cases holds:

1) $x = \overset{i}{u}(\varrho) - \varkappa \cdot n \circ \overset{i}{u}(\varrho)$, $\tilde{x} = \overset{i}{u}(\tilde{\varrho}) - \tilde{\varkappa} \cdot n \circ \overset{i}{u}(\tilde{\varrho})$

for some $\varrho, \tilde{\varrho} \in \Delta$, $\varkappa, \tilde{\varkappa} \in [0, \varepsilon(\Omega)]$;

2) $|\tilde{x} + \vartheta \cdot (x - \tilde{x}) - \overset{i}{u}(\eta)| \geqq \varepsilon(\Omega)/2$

for $\vartheta \in [0, 1]$, $\eta \in \operatorname{supp}(B_i)$;

3) $|x - \tilde{x}| \geqq \varepsilon(\Omega)/2$.

$$(3.15) \qquad |x - \varkappa \cdot n(x) - \tilde{x}| \geqq \varepsilon(\Omega) \cdot |x - \tilde{x}| , \qquad |x - \varkappa \cdot n(x) - \tilde{x}| \geqq \varkappa$$

for $x, \tilde{x} \in \partial\Omega$, $\varkappa \in [0, \varepsilon(\Omega)]$.

$$(3.16) \qquad |\overset{i}{u}(\varrho) - \overset{i}{u}(\eta)| \leqq K(\Omega) \cdot |\varrho - \eta| \qquad \text{for } \varrho, \eta \in \Delta ,\ 1 \leqq i \leqq k(\Omega) ;$$

$$(3.17) \qquad |\langle x - \tilde{x}, n(x) \rangle| \leqq K(\Omega) \cdot |x - \tilde{x}|^2 \qquad \text{for } x, \tilde{x} \in \partial\Omega .$$

For $h \in C^0(M_T)$, we shall introduce the single-layer potentials $U_h := U_{A,h}$, $U_{0h} := U_{0A,b,c,h}$, corresponding to the kernels Z, Z_0, by setting for $(x, t) \in \bar{Q}_T$:

$$U_h(x, t) := U_{0h}(x, t) := 0 , \qquad \text{if } t = 0 ;$$

$$U_h(x, t) := \int_0^t \int_{\partial\Omega} Z(x - y, t - s, y, s) \cdot h(y, s)\, d\Omega(y)\, ds ,$$

$$U_{0h}(x, t) := \int_0^t \int_{\partial\Omega} Z_0(x, t, y, s) \cdot h(y, s)\, d\Omega(y)\, ds , \qquad \text{if } t > 0 .$$

The ensuing inequality is an easy consequence of (3.3):

$$(3.18) \qquad |U_h|_\alpha \leqq R \cdot |h|_0 \qquad \text{for } h \in C^0(M_T) .$$

From (3.7), (3.8) we get that

$$|U_{\sigma h}|_{1+\alpha} \leq S \cdot |h|_0 \quad \text{for } h \in C^0(M_T)\,. \tag{3.19}$$

Now we introduce an abbreviation for the conormal derivative: For $g \in C^{1,0}(\bar{Q}_T)$, we define a function $D(A)g\colon M_T \to \mathbf{R}$ by setting for $(x, t) \in M_T$:

$$D(A)g(x, t) := \sum_{l,m=1}^{N} a_{lm}(x, t)\cdot n_m(x)\cdot D_l g(x, t)\,.$$

For $H \in \{Z_0, \Gamma\}$, $(x, t) \in M_T$, $(y, s) \in \bar{Q}_T$, the term $D(A)_{x,t}H(x, t, y, s)$ is an abbreviation for

$$\sum_{l,m=1}^{N} a_{lm}(x, t)\cdot n_m(x)\cdot D_l H(x, t, y, s)\,.$$

The meaning of $D(A)_{x,t}Z(z, r, y, s)$, for $(x, t) \in M_T$, $z \in \mathbf{R}^N$, $r \in (0, \infty)$, $(y, s) \in \bar{Q}_T$ is now obvious. By using the equality

$$\begin{aligned} D(A)_{y,s}Z(x-y, t-s, y, s) = \\ = (-1/2)\cdot(t-s)^{-1}\cdot\langle n(y), x-y\rangle\cdot Z(x-y, t-s, y, s)\,, \end{aligned} \tag{3.20}$$

it follows from (3.3) and (3.17):

$$\begin{aligned} |D(A)_{y,s}Z(x-y, t-s, y, s)| \leq \\ \leq E(\tilde{K}_1, K_2, \alpha, \mu)\cdot(t-s)^{-\mu}\cdot|x-y|^{-N-1+2\cdot\mu+\alpha}\,; \end{aligned} \tag{3.21}$$

hence from (3.7):

$$\begin{aligned} |D(A)_{y,s}\Gamma(x, t, y, s)| \leq \\ \leq P_2(T, \tilde{K}_1, K_2, \alpha, \mu)\cdot(t-s)^{-\mu}\cdot|x-y|^{-N-1+2\cdot\mu+\alpha} \end{aligned} \tag{3.22}$$

for $(y, s) \in M_T$, $(x, t) \in \bar{Q}_T$ with $x \neq y$, $t > s$, $\mu \in (-\infty, (N+1-\alpha)/2]$, where $P_2\colon \mathbf{R}^5 \to (0, \infty)$ is monotone increasing in its first variable. Contrary to our convention at the beginning of this section, we gave an individual name to this function. This will be convenient at a later point (see Section 4). We now consider an integral equation involving a double-layer potential: For any $h \in C_0(M_T)$, there is a function $\varphi_h := \varphi_{A,b,c,h} \in C^0(M_T)$ such that the

equation

$$\begin{aligned} \varphi_h(x, t) = \\ = (-2)\cdot\int_0^t \int_{\partial\Omega} D(A)_{x,t}\Gamma(x, t, y, s)\cdot\varphi_h(y, s)\, d\Omega(y)\, ds + 2\cdot h(x, t) \end{aligned} \tag{3.23}$$

holds for $(x, t) \in M_T$. φ_h satisfies the following inequality, for $h \in C^0(M_T)$:

$$|\varphi_h(x, t)| \leq S\cdot\sup\{|h(\cdot, s)|_0\colon s \in [0, t]\}\,. \tag{3.24}$$

These results are established by using (3.22) and [9; Lemma 5.2.1]; see [9; p. 145] for some indications.

By the method described in [2; 4.26], we may construct an extension operator $\mathcal{E}$ from $C_{1,\alpha}(\bar{\Omega})$ into $C_{1,\alpha}(\mathbf{R}^N)$ such that

$$|\mathcal{E}(\gamma)|_{1,\alpha} \leq E(\alpha)\cdot|\gamma|_{1,\alpha} \quad \text{for } \gamma \in C_{1,\alpha}(\bar{\Omega})\,.$$

We now define for $(x, t) \in \bar{Q}_T$:

$Y_{\Psi,T}(x, t) := \Psi(x)$, if $t = 0$;

$Y_{\Psi,T}(x, t) := \int_{\mathbf{R}^N} (4\cdot\pi\cdot t)^{-N/2}\cdot\exp(-|x-y|^2/(4\cdot t))\cdot\mathcal{E}(\Psi)(y)\, dy\,,$ if $t > 0$.

It is clear that $Y_{\Psi,T}|\Omega\times(0, T]$ belongs to $C^{2,1}(\Omega\times(0, T])$. The results in [18; pp. 278-279] imply that

$$|Y_{\Psi,T}|_{1+\alpha} \leq E(\alpha)\cdot|\Psi|_{1,\alpha}\,. \tag{3.25}$$

With the abbreviation $Y := Y_{\Psi,T}|\Omega\times(0, T]$, we define

$$G := G_{a,b,c,\Psi} := -D_{N+1}Y + \sum_{l,m=1}^{N} a_{lm}\cdot D_l D_m Y + \sum_{l=1}^{N} b_l\cdot D_l Y + c\cdot Y\,.$$

Again by adapting arguments from [18; pp. 278-279], we obtain for $t \in (0, T]$:

$$|G(\cdot, t)|_0 \leq E(K_1, \alpha)\cdot(1 + T^{1/2-\alpha/2})\cdot|\Psi|_{1,\alpha}\cdot t^{-1/2+\alpha/2}\,.$$

As a consequence of (3.10), it follows:

$$|W_{A,b,c,G+f}| \leq (R + S\cdot T^{\alpha/2\wedge(1/2-\alpha/2)})\cdot(M + |\Psi|_{1,\alpha})\,. \tag{3.26}$$

Define

$$H := H_{A,b,c,f,\Psi,g} := \varphi_{-D(A)(W_{A,b,c,f+G} + Y_{\Psi,T}) + g}\,. \tag{3.27}$$

Because of (3.25) and (1.17), we have for $t \in (0, T]$:

$$|D(A)\, Y_{\Psi,T}(\cdot, t) - g(\cdot, t)|_0 \leqq \tilde{E}(T, \tilde{K}_1, \alpha)\cdot(|\Psi|_{1,\alpha} + |g_\alpha|)\cdot t^{\alpha/2}\,. \tag{3.28}$$

From (3.24), (3.26), (3.28), we may now conclude, for $t \in [0, T]$:

$$|H(\cdot, t)|_0 \leqq S\cdot(M + |\Psi|_{1,\alpha} + |g|_\alpha)\cdot t^{\alpha/2}\,. \tag{3.29}$$

We are now in a position to define a $C^{1,0}$-solution of (1.10)-(1.12), namely

$$v := v_{A,b,c,f,\Psi,g} := W_{A,b,c,f+G} + Y_{\Psi,T} + U_H + U_{cH}\,. \tag{3.30}$$

From [18; § IV.13] we may conclude that $v|\Omega\times(0, T]$ belongs to $C^{2,1}(\Omega\times(0, T])$, and solves differential equation (1.10). By simple but tedious calculations, we obtain from the just cited reference that v satisfies the estimate in Theorem 1.4. In addition, from the definition of the potentials appearing in (3.30), it follows at once that v solves initial-condition (1.12). Finally, these potentials, with the exception of U_H, belong to $C_{1+\alpha}(\bar{Q}_T)$, and were already estimated in the norm of this space; see (3.19), (3.29), (3.25), (3.26). Concerning the potential U_H, we only evaluated $|U_H|_\alpha$; see (3.18). This means that we have at least established (1.20). This leaves us to estimate $|D_l(U_H|\Omega\times(0, T])|_\alpha$, for $1 \leqq l \leqq N$. More concretely, we shall establish that

$$|D_l(U_H|\Omega\times(0, T])|_\alpha \leqq (R + S\cdot T^{\alpha/8\wedge(1/2-\alpha/2)})\cdot \Big(M + |\Psi|_{1,\alpha} + |g|_\alpha + E(\tilde{K}_1, \alpha)\cdot \sum_{m=1}^{N} |D_m\Psi|\,\partial\Omega|_0\Big). \tag{3.31}$$

When this point is settled, we may conclude that inequality (1.21) holds. This means in particular that the function v, defined in (3.30), belongs to $C_{1+\alpha}(\bar{Q}_T)$. From [18; § IV.15] we then get that v solves boundary condition (1.11). Thus, when we have shown (3.31), Theorem 1.2 is completely proved.

Inequality (3.31) will be established in two steps. First we take $l \in \{1, \ldots, N\}$, $i \in \{1, \ldots, k(\Omega)\}$, $h \in C_\alpha(M_T)$ with $h(\cdot, 0) = 0$,

and consider the function $\mathcal{M}$, defined by

$$\mathcal{M}(x, t) := \int_0^t \int_\Delta D_l Z(x - \overset{i}{u}(\eta), t - s, \overset{i}{u}(\eta), s)\cdot h(\overset{i}{u}(\eta), s)\cdot B_i(\eta)\, d\eta\, ds \quad \text{for } (x, t) \in \Omega\times(0, T]\,. \tag{3.32}$$

We shall show that

$$|\mathcal{M}|_\alpha \leqq |h|_\alpha\cdot(R + S\cdot T^{\alpha/2})\,. \tag{3.33}$$

In the second step we shall prove for the function H defined in (3.27):

$$|H|_\alpha \leqq (R + S\cdot T^{\alpha/8\wedge(1/2-\alpha/2)})\cdot \Big(M + |\Psi|_{1,\alpha} + |g|_\alpha + E(\tilde{K}_1, \alpha)\cdot \sum_{m=1}^{N} |D_m\Psi|\,\partial\Omega|_0\Big)\,. \tag{3.34}$$

Inequality (3.31) then follows from (3.33), (3.34), and (3.12).

For the proof of (3.33), we first observe a consequence of (3.13) and (1.19), namely

$$|V_\varrho| \leqq E(K_1, K_2)\cdot\Big(\sum_{m=1}^{N-1} |\langle D_m \overset{i}{u}(\varrho), V\rangle| + |\langle A(\overset{i}{u}(\varrho), t)\cdot n_m(\varrho), V\rangle|\Big) \tag{3.35}$$

for $V \in \mathbf{R}^N$, $1 \leqq i \leqq k(\Omega)$, $\varrho \in \Delta$, $t \in (0, T]$.

Now take l, i, h as in (3.32). By means of (3.35), we shall show that

$$|\mathcal{M}(\overset{i}{u}(\varrho) - \varkappa\cdot n\circ\overset{i}{u}(\varrho), t) - \mathcal{M}(\overset{i}{u}(\varrho) - \tilde{\varkappa}\cdot n\circ\overset{i}{u}(\varrho), t)| \leqq |h|_\alpha\cdot(R + S\cdot T^{\alpha/2})\cdot|\varkappa - \tilde{\varkappa}|^\alpha, \tag{3.36}$$

for $\varrho \in \Delta$, $\varkappa, \tilde{\varkappa} \in (0, \varepsilon(\Omega)]$, $t \in (0, T]$;

$$|\mathcal{M}(\overset{i}{u}(\varrho) - \varkappa\cdot n\circ\overset{i}{u}(\varrho), t) - \mathcal{M}(\overset{i}{u}(\tilde{\varrho}) - \varkappa\cdot n\circ\overset{i}{u}(\tilde{\varrho}), t)| \leqq |h|_\alpha\cdot(R + S\cdot T^{\alpha/2})\cdot|\varrho - \tilde{\varrho}|^\alpha, \tag{3.37}$$

for $\varrho, \tilde{\varrho} \in \Delta$, $\varkappa \in (0, \varepsilon(\Omega)]$ with $|\overset{i}{u}(\varrho) - \overset{i}{u}(\tilde{\varrho})| \leqq \varkappa$, $t \in (0, T]$.

For the proof of (3.37), we first apply the mean-value theorem, relative to ϱ, $\tilde{\varrho}$. From that point onward, (3.37) may be shown by the same reasoning as used for proving (3.36). So we may restrict ourselves to considering (3.36). Thus take $\varrho, \varkappa, \tilde{\varkappa}, t$ as in (3.36). Abbreviate

$$z := \overset{i}{u}(\varrho) - \varkappa \cdot n \circ \overset{i}{u}(\varrho)\,, \qquad \tilde{z} := \overset{i}{u}(\varrho) - \tilde{\varkappa} \cdot n \circ \overset{i}{u}(\varrho)\,.$$

Then we conclude from (3.3) and (3.4):

$$|\mathcal{M}(z, t) - \mathcal{M}(\tilde{z}, t)| \leqq (S \cdot |h|_0 + R \cdot |h|_\alpha) \cdot |\varkappa - \tilde{\varkappa}|^\alpha + |I_i \cdot h(\overset{i}{u}(\varrho), t)|\,,$$

where I_p, for $1 \leqq p \leqq N$, is defined by

$$I_p := \int_0^t \int_\Delta \{D_p Z(z - \overset{i}{u}(\eta), t - s, \overset{i}{u}(\varrho), t) - D_p Z(\tilde{z} - \overset{i}{u}(\eta), t - s, \overset{i}{u}(\varrho), t)\} \cdot B_i(\eta)\, d\eta\, ds\,.$$

But for $1 \leqq r \leqq N - 1$, we have by (3.3):

$$\Big|\sum_{p=1}^N D_r \overset{i}{u}_p(\varrho) \cdot I_p\Big| \leqq R \cdot |\varkappa - \tilde{\varkappa}|^\alpha + \Big|\int_0^t \int_\Delta \Big\{\partial/\partial\eta_r Z\Big(z - \overset{i}{u}(\eta), t - s, \overset{i}{u}(\varrho), t\Big) - \partial/\partial\eta_r Z\Big(\tilde{z} - \overset{i}{u}(\eta), t - s, \overset{i}{u}(\varrho), t\Big)\Big\} \cdot B_i(\eta)\, d\eta\, ds\Big|\,.$$

By applying partial integration in the preceding integral, we may achieve that the derivative in η acts on B_i. Note that no boundary terms appear since $B_i \in C_0^1(\Delta)$. Now a simple application of (3.3) yields that

$$\Big|\sum_{p=1}^N D_r \overset{i}{u}_p(\varrho) \cdot I_p\Big| \leqq R \cdot |\varkappa - \tilde{\varkappa}|^\alpha \qquad (1 \leqq r \leqq N - 1)\,.$$

Next, by applying [9; (1.2.7)] and (3.3), we get:

$$\Big|\sum_{p,m=1}^N a_{pm}\Big(\overset{i}{u}(\varrho), t\Big) \cdot n_m \circ \overset{i}{u}(\varrho) \cdot I_p\Big| \leqq R \cdot |\varkappa - \tilde{\varkappa}|^\alpha + \Big|\lim_{\varepsilon \downarrow 0} \int_\Omega \Big\{Z\Big(z - y, \varepsilon, \overset{i}{u}(\varrho), t\Big) - Z\Big(\tilde{z} - y, \varepsilon, \overset{i}{u}(\varrho), t\Big) - Z\Big(z - y, t, \overset{i}{u}(\varrho), t\Big) + Z\Big(\tilde{z} - y, t, \overset{i}{u}(\varrho), t\Big)\Big\} \cdot \overset{i}{\omega}(y)\, dy\Big|\,,$$

where $\overset{i}{\omega}$ was introduced in (3.11). But the preceding integral is bounded by $R \cdot t^{-\alpha/2} \cdot |\varkappa - \tilde{\varkappa}|^\alpha$, as follows from (3.3) and [9; Theorem 1.1.1].

By combining the preceding estimates, and recalling that $h \in C_\alpha(M_T)$, $h(\cdot, 0) = 0$, we may derive (3.36) from (3.35).

Similar arguments yield for $x \in \Omega$, $t, \tilde{t} \in (0, T]$ with $t \geqq \tilde{t}$, $q \in C_{\alpha/2}(M_T)$:

$$(3.38) \qquad \Big|\int_0^{\tilde{t}} \int_\Delta D_i Z(x - \overset{i}{u}(\eta), t - s, \overset{i}{u}(\eta), s) \cdot q(\overset{i}{u}(\eta), s) \cdot B_i(\eta)\, d\eta\, ds\Big| \leqq \leqq S \cdot (|q|_{\alpha/2} \cdot T^{\alpha/8} + |q|_0)\,.$$

In particular we get in the situation of (3.32):

$$(3.39) \qquad |\mathcal{M}(x, t)| \leqq S \cdot T^{\alpha/8} \cdot |h|_\alpha \qquad \text{for } x \in \Omega\,,\ t \in (0, T]\,.$$

Now we consider $|\mathcal{M}(x, t) - \mathcal{M}(\tilde{x}, t)|$, for $x, \tilde{x} \in \Omega$, $t \in (0, T]$. In view of (3.39) we assume

$$(3.40) \qquad |x - \tilde{x}| \leqq \varepsilon(\Omega) \wedge K(\Omega)\,, \text{ with } \varepsilon(\Omega)\,, K(\Omega) \text{ from (3.14)-(3.17)}\,.$$

We may further suppose that the first case in (3.14) holds. Thus we may choose $\varkappa, \tilde{\varkappa} \in (0, \varepsilon(\Omega)]$, $\varrho, \tilde{\varrho} \in \Delta$ as in (3.14) 1). From [7; (2.26)] we know that

$$|\varkappa - \tilde{\varkappa}| \leqq |x - \tilde{x}|\,, \qquad |\overset{i}{u}(\varrho) - \overset{i}{u}(\tilde{\varrho})| \leqq \sqrt{2} \cdot |x - \tilde{x}|\,.$$

Now it follows from (3.36), (3.37):

$$|\mathcal{M}(x, t) - \mathcal{M}(\tilde{x}, t)| \leqq |h|_\alpha \cdot (R + S \cdot T^{\alpha/2}) \cdot |x - \tilde{x}|^\alpha\,.$$

The proof of (3.33) is completed by the inequality

$$|\mathcal{M}(x, t) - \mathcal{M}(x, \tilde{t})| \leqq |h|_\alpha \cdot (R + S \cdot T^{\alpha/2}) \cdot |t - \tilde{t}|^{\alpha/2} \quad (x \in \Omega\,,\ t, \tilde{t} \in (0, T])\,,$$

which may be shown by arguments similar to those used in the proof of (3.36).

This leaves us to show (3.34). The main difficulty consists in proving that

$$(3.41) \qquad |\varphi_h - 2 \cdot h|_\alpha \leqq S \cdot (|h|_\alpha \cdot T^{\alpha/8} + |h|_0) \qquad \text{for } h \in C_\alpha(M_T)\,,$$

with φ_h from (3.23).

We note that $\varphi_h - 2\cdot h$ is given by a double-layer potential; see (3.23). Applying (3.12), (3.23), (3.21), (3.7), (3.24), we obtain by an easy calculation:

$$(3.42)\qquad |\varphi_h|_{\alpha/2} \leqq S\cdot|h|_\alpha\,;$$

$$|\varphi_h - 2\cdot h|_\alpha \leqq S\cdot|h|_0 +$$
$$+\sup\{L(x,\tilde{x},t,i)\cdot|x-x|^{-\alpha} + M(x,t,\tilde{t},i)\cdot|t-\tilde{t}|^{-\alpha/2}:$$
$$(x,t),(\tilde{x},\tilde{t})\in M_T,\ x\neq\tilde{x},\ t>\tilde{t},\ 1\leqq i\leqq k(\Omega)\}\,.$$

Here $L(x,\tilde{x},t,i)$, for $x,\tilde{x}\in\partial\Omega$, $t\in(0,T]$, $1\leqq i\leqq k(\Omega)$, is an abbreviation for

$$\Big|\int_0^t\int_\Delta\Big\{\big(D(A)_{x,t}Z - D(A)_{\overset{i}{u}(\eta),t}Z\big)\big(x-\overset{i}{u}(\eta),\,t-s,\,\overset{i}{u}(\eta),\,s\big) -$$
$$-\big(D(A)_{\tilde{x},t}Z - D(A)_{\overset{i}{u}(\eta),t}Z\big)\big(\tilde{x}-\overset{i}{u}(\eta),\,t-s,\,\overset{i}{u}(\eta),\,s\big)\Big\}\cdot$$
$$\cdot\varphi_h\big(\overset{i}{u}(\eta),s\big)\cdot B_i(\eta)\,d\eta\,ds\Big|\,,$$

and $M(x,t,\tilde{t},i)$, for $x\in\partial\Omega$, $t,\tilde{t}\in(0,T]$, $1\leqq i\leqq k(\Omega)$, is given by

$$\Big|\sum_{1,m=1}^{N}\big(a_{lm}(x,t)-a_{lm}(x,\tilde{t})\big)\cdot n_m(x)\cdot$$
$$\cdot\int_0^{0\vee(2\cdot\tilde{t}-t)}\int_\Delta D_l Z\Big(x-\overset{i}{u}(\eta),\,t-s,\,\overset{i}{u}(\eta),\,s\Big)\cdot\varphi_h\Big(\overset{i}{u}(\eta),s\Big)\cdot B_i(\eta)\,d\eta\,ds\Big|\,.$$

As an easy consequence of (3.42), (3.3), (3.38) we find that

$$M(x,t,\tilde{t},i)\leqq S\cdot\big(|h|_\alpha\cdot T^{\alpha/8}+|h|_0\big)\cdot|t-\tilde{t}|^{\alpha/2}$$
$$\text{for } x\in\partial\Omega\,,\ \tilde{t},t\in(0,T] \text{ with } t>\tilde{t}\,,\ 1\leqq i\leqq k(\Omega)\,.$$

This leaves us to estimate $L(x,\tilde{x},t,i)$, for $x,\tilde{x}\in\partial\Omega$, $t\in(0,T]$, $1\leqq i\leqq k(\Omega)$. Because of (3.14), we may restrict ourselves to the case that $x=\overset{i}{u}(\varrho)$, $\tilde{x}=\overset{i}{u}(\tilde{\varrho})$ for some $\varrho,\tilde{\varrho}\in\Delta$. By the triangle inequality, $L(x,\tilde{x},t,i)$ is bounded by a sum $J_1+J_2+J_3$, with

$$J_j := \Big|\int_0^t\int_\Delta\sum_{l,m=1}^{N}\mathcal{K}_{lmj}(\eta,s)\cdot\varphi_h\Big(\overset{i}{u}(\eta),s\Big)\cdot B_i(\eta)\,d\eta\,ds\Big|\qquad(j=1,2,3)\,,$$

where we have set for $\eta\in\Delta$, $s\in(0,t)$, $1\leqq l$, $m\leqq N$:

$$\mathcal{K}_{lm1}(\eta,s) := \sum_{v\in\{x,\tilde{x}\}}\big[a_{lm}(v,t)\cdot n_m(v) - a_{lm}(\overset{i}{u}(\eta),t)\cdot n_m\circ\overset{i}{u}(\eta)\big]\cdot$$
$$\cdot\big[D_l Z(v-\overset{i}{u}(\eta),\,t-s,\,\overset{i}{u}(\eta),\,s) -$$
$$- D_l Z(v-|x-\tilde{x}|\cdot n(v)-\overset{i}{u}(\eta),\,t-s,\,\overset{i}{u}(\eta),\,s)\big]\,,$$

$$\mathcal{K}_{lm2}(\eta,s) := \{a_{lm}(x,t)\cdot n_m(x) - a_{lm}(\tilde{x},t)\cdot n_m(\tilde{x})\}\cdot$$
$$\cdot D_l Z(x-|x-\tilde{x}|\cdot n(x)-\overset{i}{u}(\eta),\,t-s,\,\overset{i}{u}(\eta),\,s)\,,$$

$$\mathcal{K}_{lm3}(\eta,s) := \{a_{lm}(\tilde{x},t)\cdot n_m(\tilde{x}) - a_{lm}(\overset{i}{u}(\eta),t)\cdot n_m\circ\overset{i}{u}(\eta)\}\cdot$$
$$\cdot\{D_l Z(x-|x-\tilde{x}|\cdot n(x)-\overset{i}{u}(\eta),\,t-s,\,\overset{i}{u}(\eta),\,s) -$$
$$- D_l Z(\tilde{x}-|x-\tilde{x}|\cdot n(\tilde{x})-\overset{i}{u}(\eta),\,t-s,\,\overset{i}{u}(\eta),\,s)\}\,.$$

From (3.2) and (3.3) we easily derive that $J_1\leqq S\cdot|h|_0\cdot|x-\tilde{x}|^\alpha$. By (3.38), (3.42), and (3.24) we get

$$J_2\leqq S\cdot\big(|h|_\alpha\cdot T^{\alpha/2}+|h|_0\big)\cdot|x-\tilde{x}|^\alpha\,.$$

As for J_3, we first apply the mean-value theorem. The resulting expression is then estimated by using (3.3) and the following inequality:

$$|\overset{i}{u}(\tilde{\varrho}+\vartheta\cdot(\varrho-\tilde{\varrho})) - |x-\tilde{x}|\cdot n\circ\overset{i}{u}(\tilde{\varrho}+\vartheta\cdot(\varrho-\tilde{\varrho})) - \overset{i}{u}(\eta)|\geqq$$
$$\geqq E\cdot|\tilde{x}-\overset{i}{u}(\eta)|\qquad\text{for } \vartheta\in[0,1]\,,\ \eta\in\Delta\,.$$

This result may be derived from (3.15), (3.16). Now we obtain that $J_3\leqq S\cdot|h|_0\cdot|x-\tilde{x}|^\alpha$.

By combining the preceding estimates, we arrive at inequality (3.41). For the proof of (3.34), we need a last auxiliary result, namely:

$$(3.43)\qquad |D(A)\,Y_{\Psi,T}-g|_\alpha\leqq\{E(K_1,\alpha)+E(\tilde{K}_1,\alpha)\cdot T^{\alpha/2}\}\cdot|\Psi|_{1,\alpha}+$$
$$+|g|_\alpha+E(\tilde{K}_1,\alpha)\cdot\sum_{m=1}^{N}|D_m\Psi|\,\partial\Omega|_0\,.$$

(3.34) is now a consequence of (3.41), (3.43), and (3.26).

4. - Proof of Theorem 1.3.

We begin by defining the function R_5. To this purpose, set for $\alpha \in (0, 1)$, $\tilde{K}_1 \in (0, \infty)$:

$$P_3(\tilde{K}_1, \alpha) := (\pi \cdot 8 \cdot N^2 \cdot \tilde{K}_1)^{N/2} \cdot B(\alpha/2, \alpha/4)\,, \tag{4.1}$$

where B is the Bernoulli-function;

$$P_4(\alpha) := \sup\Big\{ \int_{\partial\Omega} |x - z|^{-N+1+\alpha/2}\, d\Omega(z) : x \in \partial\Omega \Big\}\,.$$

Now, for $\tilde{K}_1, K_2 \in (0, \infty)$, $\alpha \in (0, 1)$, we define $R_5(\tilde{K}_1, K_2, \alpha)$ as

$$\min\big\{1, [2 \cdot \tilde{K}_1 \cdot N^2 \cdot P_1(\tilde{K}_1, K_2, (N+2-\alpha)/2) \cdot P_3(\tilde{K}_1, \alpha)]^{-2/\alpha},\ [P_2(1, \tilde{K}_1, K_2, \alpha, 1-\alpha/4) \cdot (8/\alpha) \cdot P_4(\alpha)]^{-4/\alpha}\big\}\,,$$

where P_1, P_2 were introduced in (3.3), (3.22) respectively.

Now consider the situation of Theorem 1.3. For brevity we set $A := (a_{lm})$, $\tilde{A} := (\tilde{a}_{lm})$, and for $(B, q, h) \in \{(A, f, g), (\tilde{A}, \tilde{f}, \tilde{g})\}$:

$$LZ_B := (LZ)_{B,0,0}\,, \quad \Phi_B := \Phi_{B,0,0}\,, \quad Z_{\circ B} := Z_{\circ B,0,0}\,, \quad \Gamma_B := \Gamma_{B,0,0}\,,$$

$$W_B := W_{B,0,0,q+G_{B,0,0,\Psi}}\,, \quad H_B := H_{B,0,0,q,\Psi,h}\,.$$

These functions were introduced in Section 3. Concerning the constants which will appear in our estimates, we adopt the same convention as in Section 3.

The proof of Theorem 1.3 is based on the ensuing estimate of $Z_A - Z_{\tilde{A}}$:

$$|D^l_{N+1} D_a (Z_A - Z_{\tilde{A}})(z, r, y, s)| \le E(\tilde{K}_1, K_2, \mu) \cdot |A - \tilde{A}|_0 \cdot r^{-\mu} \cdot |z|^{-N-|a|_*-2\cdot l+2\cdot\mu} \cdot \exp(-|z|^2/(8 \cdot N^2 \cdot \tilde{K}_1 \cdot r)) \tag{4.2}$$

for $l \in N_0$, $a \in N_0^N$ with $2 \cdot l + |a|_* \le 3$, $z \in R^N \setminus \{0\}$, $r \in (0, \infty)$, $(y, s) \in \bar{Q}_T$.

Setting

$$\mathcal{T}_\Phi := \sup\Big\{ |(\Phi_A - \Phi_{\tilde{A}})(x, t, y, s)| \cdot (t-s)^{(N+2-\alpha/2)/2} \cdot \exp\big(|x-y|^2/(8 \cdot N^2 \cdot \tilde{K}_1 \cdot (t-s))\big) : (x, t), (y, s) \in \bar{Q}_T \text{ with } t > s \Big\}\,, \tag{4.3}$$

we shall deduce from (4.2) that the inequality

$$\mathcal{T}_\Phi \le S \cdot |A - \tilde{A}|_0^{\alpha/4} \tag{4.4}$$

is valid. For the proof of (4.4), we first note that

$$|(LZ_A - LZ_{\tilde{A}})(x, t, y, s)| \le S \cdot |A - \tilde{A}|_0^{\alpha/4} \cdot (t-s)^{-(N+2-\alpha/2)/2} \cdot \exp\big(-|x-y|^2/(8 \cdot N^2 \cdot \tilde{K}_1 \cdot (t-s))\big) \tag{4.5}$$

for $(x, t), (y, s) \in \bar{Q}_T$ with $t > s$.

This inequality follows from (3.3), (4.2) by distinguishing between the cases $t > s + |A - \tilde{A}|_0$, and $t \le s + |A - \tilde{A}|_0$. Now take (x, t), $(y, s) \in \bar{Q}_T$ with $t > s$. By referring to (4.5), (3.6), and [9; Lemma 1.4.3], we arrive at the estimate

$$|(\Phi_A - \Phi_{\tilde{A}})(x, t, y, s)| \le J + S \cdot |A - \tilde{A}|_0^{\alpha/4} \cdot (t-s)^{-(N+2-\alpha/2)/2} \cdot \exp\big(-|x-y|^2/(8 \cdot N^2 \cdot \tilde{K}_1 \cdot (t-s))\big)\,, \tag{4.6}$$

where J is an abbreviation for the integral

$$\int_s^t \int_\Omega |LZ_A(x, t, \xi, \tau) \cdot (\Phi_A - \Phi_{\tilde{A}})(\xi, \tau, y, s)|\, d\xi\, d\tau\,.$$

From [9; Lemma 1.4.3] and (3.3) we obtain:

$$J \le N^2 \cdot \tilde{K}_1 \cdot P_1(\tilde{K}_1, K_2, (N+2-\alpha)/2) \cdot P_3(\tilde{K}_1, \alpha) \cdot (t-s)^{-(N+2-3\alpha/2)/2} \cdot \exp\big(-|x-y|^2/(8 \cdot N^2 \cdot \tilde{K}_1 \cdot (t-s))\big) \cdot \mathcal{T}_\Phi\,,$$

with $\mathcal{T}_\Phi$, P_3 as in (4.3), (4.1) respectively. Observing that

$$(t-s)^{\alpha/2} \le T^{\alpha/2} \le R_5(\tilde{K}_1, K_2, \alpha)^{\alpha/2} \qquad \text{(see (1.22))}\,,$$

it follows from the definition of R_5 that

$$J \le (1/2) \cdot \mathcal{T}_\Phi \cdot (t-s)^{-(N+2-\alpha/2)/2} \cdot \exp\big(-|x-y|^2/(8 \cdot N^2 \cdot \tilde{K}_1 \cdot (t-s))\big)\,. \tag{4.7}$$

Now (4.4) may be deduced from (4.6) and (4.7). From (4.2) and (4.4) it follows for $(x, t), (y, s) \in \bar{Q}_T$ with $t > s$, $x \neq y$, $a \in N_0^N$ with $|a|_* \leqq 1$, $\mu \in \big(-\infty, (N + |a|_*)/2\big]$:

$$(4.8) \qquad |(D_a Z_{\circ A} - D_a Z_{\tilde{A}})(x, t, y, s)| + |(D_a \Gamma_A - D_a \Gamma_{\tilde{A}})(x, t, y, s)| \leqq \leqq S \cdot |A - \tilde{A}|_0^{\alpha/4} \cdot (t - s)^{-\mu} \cdot |x - y|^{-N - |a|_* + 2 \cdot \mu}.$$

By some easy calculations we may conclude from (4.8) that

$$(4.9) \qquad |W_A - W_{\tilde{A}}|_{1,0} \leqq S \cdot \{|A - \tilde{A}|_0^{\alpha/4} \cdot (M + |\Psi|_{1,\alpha}) + \mathcal{K}\},$$

with $\mathcal{K}$ as in Theorem 1.3.

Besides in the proof of (4.4), the smallness condition (1.22) on T is also needed for showing the following estimate:

$$(4.11) \qquad |\varphi_{A,0,0,h} - \varphi_{\tilde{A},0,0,\tilde{h}}|_0 \leqq S \cdot [|A - \tilde{A}|_0^{\alpha/8} \cdot (|h|_0 + |\tilde{h}|_0) + |h - \tilde{h}|_0] \quad \text{for } h, \tilde{h} \in C^0(M_T).$$

This result follows from integral equation (3.23). There we split the domain of integration $(0, t)$ into the parts $\big(0 \vee (t - |A - \tilde{A}|_0), t\big)$, and its complement, and then apply (3.22), (4.8), the definition of R_5, and (1.22).

From (4.11), (4.9), (3.25), and (3.26) we obtain

$$(4.12) \qquad |H_A - H_{\tilde{A}}|_0 \leqq S \cdot (1 + M + |\psi|_{1,\alpha} + |g|_\alpha + |\tilde{g}|_\alpha) \cdot \cdot (|A - \tilde{A}|_0^{\alpha/8} + |g - \tilde{g}|_0 + \mathcal{K}),$$

with $\mathcal{K}$ as in Theorem 1.3. We observe that for $h, \tilde{h} \in C_\alpha(M_T)$, with $h(\cdot, 0) = \tilde{h}(\cdot, 0) = 0$, we have the following inequality:

$$(4.13) \qquad |U_{A,h} - U_{\tilde{A},\tilde{h}}|_{1,0} + |U_{\circ A,0,0,h} - U_{\circ \tilde{A},0,0,\tilde{h}}|_{1,0} \leqq \leqq S \cdot (1 + |h|_\alpha + |\tilde{h}|_\alpha) \cdot (|A - \tilde{A}|_0^{\alpha/8} + |h - \tilde{h}|_0^{\alpha/4}).$$

The second summand on the left-hand side of (4.13) is evaluated by means of (3.7) and (4.8). As for the first summand, we refer to the arguments in the proof of (3.36).

Due to uniqueness, any $C^{1,0}$-solution to (1.10)-(1.12) is equal to the solution introduced in (3.30). Thus Theorem 1.3 follows from (4.9), (4.12), (4.13), and (3.29).

5. – Proof of Theorem 1.1.

Let $\tilde{T}$, $\tilde{C}$, α, $(A_{lm})_{1 \leqq l, m \leqq N}$, F, Ψ, G be given as in (1.4)-(1.8). The constants L_1, E, L_2 are to be defined as in (1.4), (1.6). Take $L_3 \in (0, \infty)$ such that

$$(5.1) \qquad \sup\{|F(\cdot, t, \cdot)|_0 \cdot t^{1/2 - \alpha/2} : t \in (0, T]\} \leqq L_3.$$

Now we set

$$K_1 := \max\{|A_{lm}|_0 : 1 \leqq l, m \leqq N\};$$
$$\varepsilon_\Psi := (1/2) \cdot \max\{\tilde{C} - |D_l \Psi|_0 : 0 \leqq l \leqq N\};$$
$$I := L_3 + |\Psi|_{1,\alpha} + 2 \cdot L_2 + L_2 \cdot R_1(\alpha) \cdot |\Psi|_{1,\alpha} + 1;$$
$$\tilde{K}_1 := L_1 \cdot [2 + R_3(K_1, E, \alpha) \cdot I].$$

Take $T \in (0, \tilde{T} \wedge 1]$ such that T is smaller than

$$\min\{[R_4(1, \tilde{K}_1, E, \alpha) \cdot I]^{-2/\alpha},$$
$$[N \cdot (1 + R_3(K_1, E, \alpha) \cdot I)^\gamma \cdot L_2]^{-2/((\gamma - 1) \cdot \alpha)},$$
$$\varepsilon_\Psi^{2/\alpha} \cdot [R_4(1, \tilde{K}_1, E, \alpha) + R_3(K_1, E, \alpha)]^{-2/\alpha} \cdot I^{-2/\alpha},$$
$$[R_4(1, \tilde{K}_1, E, \alpha) \cdot I]^{-1/[\alpha/8 \wedge (1/2 - \alpha/2)]}, \quad R_5(\tilde{K}_1, E, \alpha)\}.$$

Define $\mathfrak{m}$ as the set of all functions $u \in C_{1+\alpha}(\bar{Q}_T)$ such that

$$u(x, 0) = \Psi(x) \quad \text{for } x \in \bar{\Omega}; \qquad |D_l u|_0 < \tilde{C} \quad \text{for } 0 \leqq l \leqq N;$$
$$|u|_\alpha \leqq 1 + R_1(\alpha) \cdot |\Psi|_{1,\alpha}; \qquad |u|_{1+\alpha} \leqq 1 + R_3(K_1, K_2, \alpha) \cdot I.$$

For $u \in \mathfrak{m}$, let $\mathfrak{L}u$ be the $C^{1,0}$-solution to the linear problem (1.10)-(1.12), with (a_{lm}), (b_l), c, f, g given by

$$a_{lm}(x, t) := A_{lm}\big(x, t, (D_r u(x, t))_{0 \leqq r \leqq N}\big) \quad \text{for } (x, t) \in \bar{Q}_T, \ 1 \leqq l, m \leqq N;$$
$$b_l := c := 0 \quad \text{for } 1 \leqq l \leqq N;$$
$$g(x, t) := G\big(x, t, u(x, t), (|D_r u(x, t)|^\gamma)_{1 \leqq r \leqq N}\big) \quad \text{for } (x, t) \in M_T;$$
$$f(x, t) := F\big(x, t, (D_r u(x, t))_{0 \leqq r \leqq N}\big) \quad \text{for } (x, t) \in \Omega \times (0, T];$$

and with the initial value Ψ which was fixed at the beginning of this section. We obtain from Theorem 1.2 that $\mathfrak{L}$ maps $\mathfrak{m}$ into $\mathfrak{m}$.

Now we require in addition that F satisfy (1.23). Then Theorem 1.3 yields that $\mathfrak{L}$ is continuous in the norm $|\ |_{1,0}$ of $C^{1,0}(\bar{Q}_T)$. Thus we may apply Schauder's fixed point theorem in the form of [13; Theorem 10.1]. It follows that there is some $w \in \mathfrak{m}$ with $\mathfrak{L}w = w$. This function w is a solution of the nonlinear problem (1.1)-(1.3).

Now we suppose that F does not necessarily satisfy (1.23). Then we choose a sequence (ζ_k) in $C_0^\infty(\boldsymbol{R}^{N+1})$ such that $0 \leqq \zeta_k \leqq 1$, $\operatorname{supp}(\zeta_k) \subset Q_{T+1}$, $\zeta_k(x, t) = 1$ for $t \in [1/k, T]$, $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) \geqq 1/k$ $(k \in \boldsymbol{N})$. For $k \in \boldsymbol{N}$, $(x, t) \in \Omega \times (0, T]$, $P \in (-\bar{C}, \bar{C})^{N+1}$, we set

$$F_k(x, t, P) := \zeta_k(x, t) \cdot F(x, t, P).$$

Then conditions (1.5), (1.23), and inequality (5.1) are valid for F_k $(k \in \boldsymbol{N})$. Thus the previous arguments yield a solution $w_k \in \mathfrak{m}$ of (1.1)-(1.3), with F replaced by F_k in (1.1). Since the functions w_k belong to $\mathfrak{m}$ for $k \in \boldsymbol{N}$, they are uniformly bounded in the norm $|\ |_{1+\alpha}$. From this we may conclude that a subsequence $(w_{k(i)})$ converges in the norm of $C^{1,0}(\bar{Q}_T)$. Let w be the limit of this subsequence. Then w belongs to $C_{1+\alpha}(\bar{Q}_T)$, and solves (1.2), (1.3). By combining Theorem 1.4 with a convergence argument on $K \times [S, T]$, for $S \in (0, T)$, $K \subset \boldsymbol{R}^N$ with $\bar{K} \subset \Omega$, we obtain that $w|\Omega \times (0, T]$ is an element of $C^{2,1}(\Omega \times (0, T])$, and solves (1.1).

6. – Counterexamples.

Our first counterexample concerns the question whether the natural compatibility condition (1.9) is really natural in the context of problem (1.1)-(1.3). Consider the case that Ω is a domain in $\boldsymbol{R}^2$, with C^∞-boundary, and

$$[-2, 2] \times (-2, 2) \subset \Omega \subset (-4, 4) \times (-2, 2).$$

Furthermore choose a function $\Psi \in C^\infty(\bar{\Omega})$ such that

$$D_1\Psi(x) = 1 \quad \text{for } x \in [-1, 1] \times \{2\},$$
$$D_2\Psi(x) = 0 \quad \text{for } x \in \partial\Omega,$$
$$D_1\Psi(x) = 0 \quad \text{for } x \in \partial\Omega \setminus [-3/2, 3/2] \times \{2\}.$$

Set $\bar{C} := |\Psi|_1 + 1$, $a := 16 \cdot (1 + 3 \cdot \bar{C})^2 + 1$. Take $\varphi \in C^\infty(\boldsymbol{R})$ such that

$$\varphi(s) = 1/s \quad \text{for } s \geqq 1/2, \qquad \varphi(s) = 0 \quad \text{for } s \leqq 1/4, \ 0 \leqq \varphi \leqq 4.$$

For $(x, t, p_0, p_1, p_2) \in \bar{Q}_1 \times [-\bar{C}, \bar{C}]^3$, let $A_{lm}(x, t, p_0, p_1, p_2)$ be given by

$$\begin{cases} a, & \text{if } (l, m) = (1, 1); \\ 1, & \text{if } (l, m) = (2, 2); \\ [(1 + p_0 - \Psi(x)) \cdot t - p_2] \cdot \varphi(p_1), & \text{if } (l, m) \in \{(2, 1), (1, 2)\}. \end{cases}$$

Then from our choice of $\bar{C}$ and a, it follows that

$$\det((A_{lm})_{1 \leqq l, m \leqq 2}) \geqq 1.$$

Furthermore we have $A_{lm} \in C^\infty(\bar{Q}_1 \times [-\bar{C}, \bar{C}]^3)$ $(1 \leqq l, m \leqq 2)$. In particular $(A_{lm})_{1 \leqq l, m \leqq 2}$ satisfies the assumptions in (1.4). We assume $G = 0$. Then Ψ, G satisfy (1.9), as follows by an easy calculation; note that $n(x) = (0, 1)$ for $x \in [-2, 2] \times \{2\}$. Now let us suppose there is some $T \in (0, 1]$ and a function $w \in C^{1,0}(\bar{Q}_T)$ such that (1.2) and (1.3) hold, and $|D_l w|_0 < \bar{C}$ $(0 \leqq l \leqq 2)$. Then we may choose $S \in (0, T]$ such that

$$(6.1) \qquad |D_1 w(x, t) - D_1\Psi(x)| \leqq 1/4 \quad \text{for } (x, t) \in \bar{Q}_S.$$

Take $x \in [-1, 1] \times \{2\}$. Since $D_1\Psi(x) = 1$, it follows from (6.1):

$$\sum_{l,m=1}^{2} A_{lm}\big(x, t, (D_r w(x, t))_{0 \leqq r \leqq 2}\big) \cdot n_m(x) \cdot D_l w(x, t) = $$
$$= [1 + w(x, t) - \Psi(x)] \cdot t \quad \text{for } t \in [0, S];$$

hence from (1.9): $0 = 1 + w(x, t) - \Psi(x)$ for $t \in (0, S]$. But this is a contradiction, since $w(x, t)$ converges to $\Psi(x)$ for t tending to 0.

In our second counterexample, we consider the question whether our solutions of (1.1)-(1.3) may be continued locally. Take Ω, Ψ, $\bar{C}$, a, φ as above. For

$$(x, t, p_0, p_1, p_2) \in \bar{Q}_2 \times [-\bar{C}, \bar{C}]^3, \qquad (l, m) \in \{(1, 2), (2, 1)\},$$

we define $A_{lm}(x, t, p_0, p_1, p_2)$ by

$$\begin{cases} -p_2 \cdot \varphi(p_1), & \text{if } t \in [0, 1]; \\ [(1 + p_0 - \Psi(x)) \cdot (t - 1) - p_2] \cdot \varphi(p_1), & \text{if } t \geqq 1. \end{cases}$$

We further set $A_{11} := a$, $A_{22} := 1$. Choose $\zeta \in C^\infty(\boldsymbol{R})$ such that $0 \leqq \zeta \leqq 1$, $\zeta(t) = 0$ for $t \in (-\infty, 1/2]$, $\zeta(t) = 1$ for $t \in [3/4, \infty)$. Set $w(x, t) := \zeta(t) \cdot \Psi(x)$ for $(x, t) \in \bar{Q}_1$,

$$F(x, t, p_0, p_1, p_2) := \Psi(x) \cdot \zeta'(t) -$$
$$- \sum_{l,m=1}^{2} A_{lm}\big(x, t, (\zeta(t) \cdot D_r \Psi(x))_{0 \leqq r \leqq 2}\big) \cdot \zeta(t) \cdot D_l D_m \Psi(x)$$

for $x \in \Omega$, $t \in (0, 2]$, $p_0, p_1, p_2 \in [-\tilde{C}, \tilde{C}]^3$. This means that F is independent of p_0, p_1, p_2. The functions $(A_{lm})_{1 \leqq l, m \leqq 2}$, F satisfy (1.4), (1.5). The function w belongs to $C^\infty(\bar{Q}_1)$, and satisfies (1.1), with $T = 1$, and with $(A_{lm})_{1 \leqq l, m \leqq 2}$, F as introduced above. Moreover, w solves (1.2), (1.3) for $T = 1$, $\Psi = 0$, $G = 0$. But there is no $T' \in (1, 2)$ such that a function $v \in C^{1,0}(Q_{T'})$ exists satisfying (1.2), (1.3), as well as the equation $w = v | \bar{Q}_1$. This is a consequence of the previous counterexample.

Our last counterexample shows that the solutions from Theorem 1.1 are not uniquely determined, and uniqueness cannot be achieved by choosing a smaller value of T. In order to see this, consider the case $\Omega = (1, 2) \subset \boldsymbol{R}$, and set

$$w(x, t) := \exp\,(-x/t - 1/(x-1) - 1/(2-x)) \quad \text{for } (x, t) \in \Omega \times (0, \infty)\,;$$
$$w(x, t) := 0 \qquad \text{for } (x, t) \in \bar{\Omega} \times \{0\} \cup \partial\Omega \times [0, \infty)\,.$$

Then we have $w \in C^{2,1}(\bar{\Omega} \times [0, \infty))$, $D_1 w(x, t) = 0$ for $(x, t) \in \partial\Omega \times [0, \infty)$; $w(x, 0) = 0$ for $x \in \bar{\Omega}$. We set

$$\tilde{C} := 12 \cdot \big[\sup \big\{\exp\,(r \cdot \ln(r) - r) : r \in (0, 8]\big\}\big]^2 + 1\,.$$

Note that

$$12 \cdot [s^r \cdot \exp(-s)]^2 \leqq \tilde{C} \qquad \text{for } r \in (0, 8]\,,\ s \in [0, \infty)\,;$$
$$|D_l w|_0 \leqq \tilde{C} - 1 \qquad \text{for } 0 \leqq l \leqq 1\,.$$

Next we choose $\varphi \in C^\infty(\boldsymbol{R})$ such that

$$0 \leqq \varphi \leqq 1\,, \quad \varphi | [-\tilde{C} + 1, \tilde{C} - 1] = 1\,, \quad \varphi | \boldsymbol{R} \setminus (-\tilde{C}, \tilde{C}) = 0\,.$$

For $\delta, \tilde{\delta} \in [0, 4]$, $k \in \{1, 2\}$, $(x, t, p_0, p_1) \in \Omega \times (0, \infty) \times \boldsymbol{R}^2$, we define

$$f_1^{\delta,\tilde{\delta}}(x, t, p_0, p_1) := \varphi[(x-1)^{-\delta} \cdot (2-x)^{-\tilde{\delta}} \cdot p_0] \cdot (x-1)^{-\delta} \cdot (2-x)^{-\tilde{\delta}} \cdot p_0\,;$$
$$f_2^{\delta,\tilde{\delta},k}(x, t, p_0, p_1) := \varphi[(x-1)^{-\delta} \cdot (2-x)^{-\tilde{\delta}} \cdot p_1^{k+1} \cdot w(x, t)^{-k}] \cdot$$
$$\cdot (x-1)^{-\delta} \cdot (2-x)^{-\tilde{\delta}} \cdot p_1^{k+1} \cdot w(x, t)^{-k}\,.$$

For $\delta, \tilde{\delta} \in [0, 2]$, we set

$$f_3^{\delta,\tilde{\delta}} := -f_2^{\delta,\tilde{\delta},0} + f_1^{2+\delta,\tilde{\delta}} - f_1^{\delta,2+\tilde{\delta}}\,,$$
$$f_4 := f_2^{0,0,1} + 2 \cdot f_3^{2,0} - 2 \cdot f_3^{0,2} - f_1^{4,0} + 2 \cdot f_1^{2,2} - f_1^{0,4}$$

Finally we define for $x \in (1, 2)$, $t \in (0, 1]$, $p_0, p_1 \in (-\tilde{C}, \tilde{C})$:

$$F(x, t, p_0, p_1) := x \cdot f_4(x, t, p_0, p_1) +$$
$$+ (2 \cdot f_1^{3,0} - 2 \cdot f_1^{0,3} - f_4 + 2 \cdot f_3^{2,0} - 2 \cdot f_3^{0,2} - f_1^{4,0} + 2 \cdot f_1^{2,2} - f_1^{0,4})(x, t, p_0, p_1)\,.$$

Then F satisfies (1.5). For any $T \in (0, 1]$, equations (1.1)-(1.3), with F as just defined, and with $\Psi = G = 0$, $A_{11} = 1$, are solved by two different functions, namely by the zero function on $\bar{Q}_T$, and by $w | \bar{Q}_T$, with w as defined above.

Note added in proof.

The author learned of another paper which, though not dealing with equations (1.1)-(1.3), gives very general results on related problems: A. LUNARDI, *Maximal space regularity innonhomogeneous initial boundary value problem*, preprint.

REFERENCES

[1] P. ACQUISTAPACE - B. TERRENI, *On quasilinear parabolic systems*, Math. Ann., **282** (1988), 315-335.

[2] R. A. ADAMS, *Sobolev spaces*, Academic Press, New York e.a., 1975.

[3] H. AMANN, *Quasilinear parabolic systems under nonlinear boundary conditions*, Arch. Rational Mech. Anal., **92** (1986), 153-192.

[4] H. AMANN, *Semigroups and nonlinear evolution equations*, Linear Algebra Appl., **84** (1986), 47-84.

[5] H. AMANN, *Dynamic theory of quasilinear parabolic equations. - I: Abstract evolution equations*, Nonlinear Anal., **12** (1988), 895-919.

[6] H. AMANN, *Dynamic theory of quasilinear parabolic equations. - II: Reaction-diffusion systems*, Preprint.

[7] P. DEURING - W. VON WAHL - P. WEIDEMAIER, *Das Stokes-Problem in $\boldsymbol{R}^3$. - I: Vorlesung über das Innenraumproblem*, Bayreuth. Math. Schr., **27** (1988), 1-252.

[8] R. FINN, *Equilibrium capillary surfaces*, Springer, New York e.a., 1986.

[9] A. FRIEDMAN, *Partial differential equations of parabolic type*, Prentice-Hall, Englewood Cliffs, N.J., 1964.