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A Representation Formula for the Velocity Part of 3D Time-Dependent Oseen Flows

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Abstract. We prove a representation formula for the solution of an initial-boundary value problem associated with the 3D time-dependent Oseen system. This formula involves the solution of an integral equation on the lateral boundary of the space-time cylinder. The key point of this article is that the assumptions on the data are chosen in such a way that the formula in question may be used in a theory of the asymptotic behaviour of solutions to the nonstationary 3D Navier–Stokes system with Oseen term (incompressible viscous flow in an exterior domain with nonzero velocity at infinity).

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1. Introduction

We consider the time-dependent Oseen system

$$\begin{aligned}\partial_t u(x,t) - \Delta_x u(x,t) + \tau \cdot \partial_1 u(x,t) + \nabla_x \pi(x,t) &= f(x,t),\\ \operatorname{div}_x u(x,t) = 0 \quad \text{for} \quad (x,t) &\in \overline{\Omega}^c \times (0,\infty),\end{aligned} \tag{1.1}$$

where $\overline{\Omega}^c := \mathbb{R}^3 \backslash \overline{\Omega}$ for some bounded open set $\Omega \subset \mathbb{R}^3$ with connected Lipschitz boundary $\partial\Omega$. These assumptions on Ω mean in particular that $\overline{\Omega}^c$ is an exterior domain. We require a Dirichlet boundary condition on $S_\infty := \partial\Omega \times (0,\infty)$ and a boundary condition at infinity,

$$u|S_\infty = b, \quad u(x,t) \to 0 \quad (|x| \to \infty) \quad \text{for} \quad t \in (0,\infty), \tag{1.2}$$

as well as an initial condition,

$$u(x,0) = a(x) \quad \text{for} \quad x \in \overline{\Omega}^c. \tag{1.3}$$

Here the Reynolds number $\tau \in (0,\infty)$, the boundary data $b : S_\infty \mapsto \mathbb{R}^3$, the volume force $f : \overline{\Omega}^c \times (0,\infty) \mapsto \mathbb{R}^3$ and the initial data $a : \Omega \mapsto \mathbb{R}^3$ are given, whereas the velocity $u : \overline{\Omega}^c \times (0,\infty) \mapsto \mathbb{R}^3$ and the pressure $\pi : \overline{\Omega}^c \times (0,\infty) \mapsto \mathbb{R}$ are unknown. Problem (1.1)–(1.3) arises as a linearization of a mathematical model for the flow around a rigid body moving at a constant velocity and without rotation in a viscous incompressible fluid.

In the work at hand, we establish a representation formula for the velocity part u of a solution (u,π) to problem (1.1)–(1.3). In order to formulate this result in a precise way, let us introduce some potential functions. To this end, we write $\mathfrak{H}$ for the usual heat kernel in $\mathbb{R}^3$, that is,

$$\begin{aligned}&\mathfrak{H}(z,t) := (4\cdot\pi\cdot t)^{-3/2} \cdot e^{-|z|^2/(4\cdot t)} \quad \text{for} \quad z \in \mathbb{R}^3,\ t \in (0,\infty),\\ &\mathfrak{H}(z,t) := 0 \quad \text{for} \ \ (z,t) \in (\mathbb{R}^3 \backslash \{0\}) \times \{0\},\end{aligned} \tag{1.4}$$

we introduce the velocity part Γ of a fundamental solution of the time-dependent Stokes system by setting

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$$\Gamma_{jk}(z,t) := \delta_{jk} \cdot \mathfrak{H}(z,t) + \int\limits_t^\infty \partial_j \partial_k \mathfrak{H}(z,s)\, ds$$
$$\text{for } \ (z,t) \in \mathfrak{B} := (\mathbb{R}^3 \times [0,\infty))\backslash\{0\},\ 1 \le j,k \le 3, \tag{1.5}$$

and we define the velocity part Λ of a fundamental solution of the Oseen system (1.1) by

$$\Lambda_{jk}(z,t,\tau) := \Gamma_{jk}(z - \tau \cdot t \cdot e_1, t) \quad \text{for } \ (z,t) \in \mathfrak{B},\ \ j,k \in \{1,\, 2,\, 3\}. \tag{1.6}$$

Then we introduce volume potentials $\mathfrak{R}^{(\tau)}(f)$, $\mathfrak{I}^{(\tau)}(a) : \mathbb{R}^3 \times (0,\infty) \mapsto \mathbb{R}^3$ by

$$\mathfrak{R}^{(\tau)}(f)(x,t) := \int\limits_0^t \int\limits_{\mathbb{R}^3} \Lambda(x-y, t-\sigma,\, \tau) \cdot \widetilde{f}(y,\sigma)\, dy\, d\sigma, \tag{1.7}$$
$$\mathfrak{I}^{(\tau)}(a)(x,t) := \int\limits_{\mathbb{R}^3} \mathfrak{H}(x-y-\tau\cdot t\cdot e_1, t) \cdot \widetilde{a}(y)\, dy \tag{1.8}$$

for $x \in \mathbb{R}^3$, $t \in (0,\infty)$, $f \in L^s(0,T,\, L^q(A)^3)$, $a \in L^q(A)^3$, where $A \subset \mathbb{R}^3$ is measurable, $T \in (0,\infty]$, $q,s \in [1,\infty)$, and $\widetilde{f}$, $\widetilde{a}$ are the zero extension of f and a to $\mathbb{R}^3 \times (0,\infty)$ and $\mathbb{R}^3$, respectively. We further introduce a single-layer potential $\mathfrak{V}^{(\tau)}(\Phi) : \mathbb{R}^3 \times (0,\infty) \mapsto \mathbb{R}^3$ by setting

$$\mathfrak{V}^{(\tau)}(\Phi)(x,t) := \int\limits_0^t \int\limits_{\partial\Omega} \Lambda(x-y, t-\sigma, \tau) \cdot \widetilde{\Phi}(y,\sigma)\, d\Omega(y)\, d\sigma \tag{1.9}$$

for $x \in \mathbb{R}^3$, $t \in (0,\infty)$, $\Phi \in L^2(S_T)^3$ for some $T \in (0,\infty]$, with $S_T := \partial\Omega \times (0,T)$ and with $\widetilde{\Phi}$ denoting the zero extension of Φ to S_∞. In Sect. 5, we will give some explanations and references concerning the question as to why the preceding integrals make sense. Let the space H_T on S_T be defined via the norm

$$\|\varphi\|_{H_T} := \left(\int\limits_0^T \left(\|\varphi(\,\cdot\,,\, t) \,|\, \partial\Omega\|_{1,\,2}^2 + \|\partial_4^{1/2} \varphi(\,\cdot\,,\, t) | \partial\Omega\|_2^2 + \|\partial_4 \varphi(\,\cdot\,, t) \cdot n^{(\Omega)}\|_{H^1(\partial\Omega)'}^2 \right) dt \right)^{1/2} \tag{1.10}$$

and the condition $\varphi \in L^2_n(S_T)$, where $n^{(\Omega)}$ denotes the outward unit normal to Ω, the space $L^2_n(S_T)$ is given by

$$L^2_n(S_T) := \left\{ \mu \in L^2(S_T)^3 \ : \ \int\limits_{\partial\Omega} \mu(\,\cdot\,, t) \cdot n^{(\Omega)}\, d\Omega = 0 \ \ (t \in (0,T)) \right\}, \tag{1.11}$$

and the fractional derivative $\partial_4^{1/2}\varphi$ is defined by

$$\partial_4^{1/2} \varphi(x,t) := \pi^{-1/2} \cdot \partial_t \left(\int\limits_0^t (t-r)^{-1/2} \cdot \varphi(x,r)\, dr \right) \tag{1.12}$$

for $x \in \partial\Omega$, $t \in (0,T)$ (for more details we refer to Sect. 2). Moreover, for $s \in (0,1]$, we write $H^s_\sigma(\overline{\Omega}^c)$ for the completion in $H^s(\overline{\Omega}^c)^3$ of the set of all functions $\varphi \in C_0^\infty(\overline{\Omega}^c)^3$ with $div\, \varphi = 0$, where $H^s(\overline{\Omega}^c)$ is introduced in Sect. 2. For brevity, we write V instead of $H^1_\sigma(\overline{\Omega}^c)$. This space V is to be equipped with the norm of $H^1(\overline{\Omega}^c)^3$.

We will consider a function $u : (0,T) \mapsto H^1(\overline{\Omega}^c)^3$ as the velocity part of a weak solution to (1.1) – (1.3) if

$$u|\overline{\Omega}^c \times (0,T') \in L^2(0,T',\, H^1(\overline{\Omega}^c)^3) \quad \text{for } T' \in (0,T), \tag{1.13}$$

$$u|\partial\Omega \times (0,T) = b, \quad \operatorname{div}_x u = 0, \tag{1.14}$$

$$\int_0^T \Bigg(\int_{\overline{\Omega}^c} [-(u(x,t)\cdot\vartheta(x))\cdot\varphi'(t) + (\nabla_x u(x,t)\cdot\nabla\vartheta(x))\cdot\varphi(t) + \tau\cdot(\partial_1 u(x,t)\cdot\vartheta(x))\cdot\varphi(t)]\,dx - f(t)(\vartheta)\cdot\varphi(t)\Bigg)\,dt = \int_{\overline{\Omega}^c} a(x)\cdot\vartheta(x)\,dx\cdot\varphi(0) \tag{1.15}$$

for $\varphi \in C_0^\infty([0,T))$, $\vartheta \in C_0^\infty(\overline{\Omega}^c)^3$ with $\operatorname{div}\vartheta = 0$.

The main result of this article may now be stated as

Theorem 1.1. *Let $T \in (0,\infty]$. Suppose that $b \in H_T$ and $a \in H_\sigma^{1/2+\epsilon_0}(\overline{\Omega}^c)$ for some $\epsilon_0 \in (0,1/2]$. Further suppose that f belongs to $L^2(0,T,\, L^{3/2}(\overline{\Omega}^c)^3)$ as well as to $L^2(0,T,\, L^{q_0}(\overline{\Omega}^c)^3)$ for some $q_0 \in [1,3/2)$. In particular, $f \in L^2(0,T,V')$.*

Then there is a unique function $\Phi \in L^2_n(S_T)$ such that

$$\mathfrak{V}^{(\tau)}(\Phi)(x,t) = (-\mathfrak{R}^{(\tau)}(f) - \mathfrak{I}^{(\tau)}(a) + b)(x,t) \quad \textit{for} \quad (x,t)\in S_T. \tag{1.16}$$

Moreover, there is a unique function $u : (0,T) \mapsto H^1(\overline{\Omega}^c)^3$ such that (1.13)–(1.15) are valid. This function u is given by

$$u = \mathfrak{R}^{(\tau)}(f) + \mathfrak{I}^{(\tau)}(a) + \mathfrak{V}^{(\tau)}(\Phi)\,|\,\overline{\Omega}^c \times (0,T). \tag{1.17}$$

Equation (1.17) is the representation formula mentioned above. Note that the function Φ appearing in that formula solves an integral equation on S_T, that is, Eq. (1.16). In other words, we have to solve an integral equation in order to obtain (1.17). On the other hand, via the right-hand side of (1.17), we not only represent, but even construct the velocity part u of a solution to (1.1)–(1.3). In Theorem 1.1, this solution is characterized by the relations in (1.13)–(1.15), which are typical features of weak solutions. But these criteria were chosen only because they define a rather large uniqueness class. Actually our solution has better regularity properties, as should be expected by our assumptions on the data. Some of these properties are stated in [13, Corollary 2.17, 2.18, Theorem 2.22]. However, we will not pursue this point here because in the present context, Theorem 1.1 is motivated by the representation formula (1.17), and not by the existence result associated with it. But of course, this result might have interesting applications as well.

Some remarks are perhaps in order as to why Eq. (1.17) is of interest, what are the reasons for our assumptions of f, and how Theorem 1.1 is related to previous results, in particular to those in [13]. In this respect, consider the stationary Navier–Stokes system with Oseen term, that is,

$$-\Delta v + \tau\cdot\partial_1 v + \tau\cdot(v\cdot\nabla)v + \nabla p = F, \quad \operatorname{div} v = 0 \quad \text{in} \quad \overline{\Omega}^c. \tag{1.18}$$

If the velocity part v of a weak solution to this system belongs to $L^6(\overline{\Omega}^c)^3$, if $\nabla v \in L^2(\overline{\Omega}^c)^9$ and if F decays sufficiently fast, then

$$|\partial^\alpha v(x)| = O[(|x|\cdot\nu(x))^{-1-|\alpha|/2}] \quad \text{for } |x|\to\infty, \tag{1.19}$$

where $\alpha \in \mathbb{N}_0^3$ with $|\alpha| := \alpha_1+\alpha_2+\alpha_3 \le 1$ [3,8,20,21] [26, Section IX.8]. The function ν is defined by

$$\nu(x) := 1 + |x| - x_1 \quad \text{for} \quad x\in\mathbb{R}^3. \tag{1.20}$$

The factor $\nu(x)$ in (1.19) is a mathematical manifestation of the wake extending behind a rigid body moving steadily in a fluid described by (1.18). The decay result in (1.19) gives rise to the question as to whether an analogous result may be established in the nonstationary case. In this respect, Mizumachi [39] showed that the velocity part u of a solution to the instationary Navier–Stokes system with Oseen term,

$$\partial_t u - \Delta_x u + \tau \cdot \partial_1 u + \tau \cdot (u \cdot \nabla_x) u + \nabla_x \pi = f, \quad \mathrm{div}_x u = 0 \tag{1.21}$$

in $\overline{\Omega}^c \times (0, \infty)$ verifies the relation

$$|u(x,t)| = O[(|x| \cdot \nu(x))^{-1}] \quad \text{for} \quad |x| \to \infty, \quad \text{uniformly in} \quad t, \tag{1.22}$$

under assumptions whose key points may be stated as follows: $f = 0$, $u \in L^\infty(0, \infty,\, L^r(\overline{\Omega}^c)^3)$ for some $r \in [1,3)$, $\nabla_x u \in L^\infty(0, \infty,\, L^2(\overline{\Omega}^c)^9)$, and pointwise decay of u in the sense that $u(x,t) \to 0$ for $|x| \to \infty$, uniformly in $t \in (\widetilde{T}, \infty)$, for some $\widetilde{T} \in (0, \infty)$. The argument in [39] is based on an integral representation obtained in a standard way, that is, by applying Green's formula to a convolution integral involving a suitable fundamental solution—in the case of [39] a fundamental solution of the Oseen system (1.1), different from the one in (1.6), but satisfying the same estimates. But this approach gives rise to a term involving the boundary value of $\nabla_x u - \pi \cdot I$ on S_∞. The problem then consists in finding a bound for $\nabla_x u(x,t) - \pi(x,t) \cdot I$ $(x \in \partial\Omega,\ t \in (0,\infty))$ in a suitable norm. Mizumachi [39, p. 508] refers to [28] and [44] in this respect, but it is not clear to us how these references should lead to a bound as stated in [39, (2.42)]. In the case of the time-dependent Stokes system, Kozono [35] obtained a suitable bound by using L^p-L^q-estimates of the Stokes operator provided by Iwashita [31]. Although L^p-L^q-estimates are available for the Oseen operator as well [4,34], it seems to be an open question how they may be applied in order to estimate the term $(\nabla_x u - \pi \cdot I)\,|\,S_\infty$. This critical term does not arise in our representation formula (1.17) of Oseen flows; only the boundary value of the velocity u on S_T—a known quantity due to the boundary condition in (1.2)—is involved. But it takes some effort to derive (1.17) because, as mentioned above, the integral equation (1.16) has to be resolved in the process.

In [13], we exploited (1.17) in order to study the spatial asymptotics of the velocity part u of a solution to the linear problem (1.1)–(1.3). Under the condition that $u|\overline{\Omega}^c \times (0,T') \in L^2(0,T',\, H^1(\overline{\Omega})^3)$ for $T' \in (0,T)$, and if the support of f and a is compact, we obtained decay estimates for any space derivative and for the first time derivative of u. It should be noted that the assumption $u|\overline{\Omega}^c \times (0,T') \in L^2(0,T',\, H^1(\overline{\Omega})^3)$ in [13] (and in (1.13) above) means that the decay condition at infinity in (1.2) is satisfied only in the sense that $u(\cdot,t) \in H^1(\overline{\Omega}^c)^3$ for $t \in (0,T)$. No assumption on pointwise decay as in [39] is imposed. This is an important point because existence of a solution to (1.1)–(1.3) with $u|\overline{\Omega}^c \times (0,T') \in L^2(0,T',\, H^1(\overline{\Omega})^3)$ for $T' \in (0,T)$ may obtained rather easily, for example by a variational principle, whereas it is much less obvious how to construct a solution that decays pointwise if $|x| \to \infty$. Such a property should be part of the conclusions, and not of the assumptions.

The main difficulty of our argument in [13] consists in establishing the representation formula in (1.17). This is achieved in [13] under assumptions on f that are stronger than those in Theorem 1.1. Once Eq. (1.17) is available, decay results for the linear problem (1.1)–(1.3) follow rather easily ([13, Corollary 5.6]).

In this situation, we asked ourselves how to extend our representation formula (1.17) to solutions of the nonlinear problem (1.21), (1.2), (1.3). An obvious idea in this respect consists in considering (1.21) as an Oseen system with right-hand side $f - (u \cdot \nabla_x)u$. But of course, whether this approach works depends on the properties of u. We were interested in the solutions of (1.21), (1.2), (1.3) constructed by Heywood [29,30], and by Solonnikov [45, Theorem 10.1, Remark 10.1 with $q = 2$], and characterized by the relations

$$u \in L^\infty(0, T_0,\, L^2(\overline{\Omega}^c)^3),\ \nabla_x u \in L^p(0, T_0,\, L^2(\overline{\Omega}^c)^9) \text{ for } p \in \{2,\, \infty\}. \tag{1.23}$$

Actually we replaced the condition $u \in L^\infty(0, T_0,\, L^2(\overline{\Omega}^c)^3)$ by the requirement that u belongs to $L^\infty(0, T_0,\, L^r(\overline{\Omega}^c)^3)$ for some $r \in [1,3)$. This means that Mizumachi's assumptions are satisfied, except for his pointwise decay condition on u, which is replaced by the more natural relation $\nabla_x u \in L^2(0, T_0,$

$L^2(\overline{\Omega}^c)^9)$, verified even by weak solutions of the Navier–Stokes system. But even in the linear case [Oseen system (1.1)], the set of conditions

$$u \in L^\infty(0, T_0,\ L^r(\overline{\Omega}^c)^3),\ \nabla_x u \in L^p(0, T_0,\ L^2(\overline{\Omega}^c)^9) \text{ for } p \in \{2,\ \infty\} \tag{1.24}$$

with some $r \in [1, 3)$ gives rise to a problem with respect to uniqueness. In [15], we removed this problem, showing that if the velocity part u of a solution to (1.1)–(1.3) fulfills (1.24), and if $f \in L^2(0, T_0, V')$, $b \in H_{T_0}$ and $a \in L^2_\sigma(\overline{\Omega}^c)$, we get $u \in L^\infty(0, T_0, L^2(\overline{\Omega}^c)^3)$ [15, Theorem 4.4], and hence (1.23). This means such a function u lies within the framework of Theorem 1.1, so the representation formula (1.17) holds for this u [15, Corollary 4.5]. In [16], we exploited this result in order to deal with the nonlinear case, proving that if the velocity part u of a solution to (1.21), (1.2), (1.3) verifies (1.24), and if f is given as in Theorem 1.1, then the function $f - \tau \cdot (u \cdot \nabla_x)u$ fulfills the conditions imposed on f in that latter theorem. Thus we obtained an integral representation for this u as well. This opened up the way for the decay estimates of u we then derived in [16]. In fact, we not only showed (1.22), but additionally established that the spatial gradient $\nabla_x u(x, t)$ of u behaves like $O[(\,|x| \cdot \nu(x))^{-3/2}]$ for $|x| \to \infty$, uniformly in $t \in (0, T_0)$. In other words, we could derive the equivalent of (1.19) for the instationary case.

The transition to the nonlinear case would not have been possible on the basis of the linear theory in [13]. Indeed, in that latter reference, we had to require f to belong to $L^2(0, T_0, L^p(\overline{\Omega}^c)^3)$ for $p = 1$ and for some $p > 3/2$, whereas in Theorem 1.1, the condition $f \in L^2(0, T_0, L^p(\overline{\Omega}^c)^3)$ is imposed for $p = 3/2$ and some $p \in [1, 3/2)$. This apparently small difference is important because (1.23) implies the relation $(u \cdot \nabla_x)u \in L^2(0, T_0, L^{3/2}(\overline{\Omega}^c)^3)$, but we cannot see why $(u \cdot \nabla_x)u$ should belong to $L^2(0, T_0, L^p(\overline{\Omega}^c)^3)$ for some $p > 3/2$. Unfortunately the case $p = 3/2$ seems to be a limit case of the assumptions on f, leading to a much more complicated proof of (1.17) in the work at hand compared to our argument in [13].

These complications are related to the resolution theory for the integral equation (1.16). Such a theory requires a suitable choice of function spaces. Following Shen [41], who dealt with the Stokes case, we used the spaces $L^2(S_T)$ and H_T introduced above, with $T \in (0, \infty]$. For $\Phi \in L^2(S_T)^3$, let the single-layer potential $\mathfrak{V}(\Phi)$ be defined in the same way as $\mathfrak{V}^{(\tau)}(\Phi)$ [see (1.9)], but with the Oseen fundamental solution Λ replaced by the Stokes fundamental solution Γ. Then Shen [41] could show that for any $d \in H_T$, there is a unique function $\psi \in L^2_n(S_T)$ such that $\mathfrak{V}(\psi)|S_T = d$. In [9,12], we adapted this result to the Oseen case, in the sense that we showed an analogous relation to be valid for $\mathfrak{V}^{(\tau)}(\psi)$ in the place of $\mathfrak{V}(\psi)$. Thus the problem of solving (1.16) reduces to the question as to whether the right-hand side of (1.16) belongs to H_T. Among the terms b, $\mathfrak{I}^{(\tau)}(a)|S_T$ and $\mathfrak{R}^{(\tau)}(f)|S_T$ appearing in the sum on that right-hand side, only $\mathfrak{R}^{(\tau)}(f)|S_T$ needs further consideration. In fact, by assumption we have $b \in H_T$. Moreover, the relation $\mathfrak{I}^{(\tau)}(a)|S_T \in H_T$ was established in [13] under suitable conditions on a [13, Theorem 3.1]. Also in [13], it was shown that $\mathfrak{R}^{(\tau)}(f)|S_T$ belongs to H_T if $f \in L^2(0, T, L^p(\overline{\Omega}^c)^3)$ for any $p \in [1, p_0]$, with some $p_0 > 3/2$. But in view of our indications above, it should not be surprising we will have to establish the relation $\mathfrak{R}^{(\tau)}(f)|S_T \in H_T$ for the case that $f \in L^2(0, T, L^p(\overline{\Omega}^c)^3)$ for any $p \in [q_0, 3/2]$, where q_0 is an arbitrary but fixed number from $[1, 3/2)$. The greatest part of the work at hand is taken up by a proof of this claim.

After technical preparations, we start this proof by showing that for a given function $f \in C_0^\infty(\mathbb{R}^3 \times (0, \infty))^3$, the quantity $\|\mathfrak{R}^{(\tau)}(f)|S_T\|_{H_T}$ is bounded by a sum of terms having the form $\|G\|_{L^2(0,T,L^{3/2}(\Omega)^9)}$, for certain functions $G : \Omega \times (0, T) \mapsto \mathbb{R}^9$ (Theorem 5.5). This result is based on a trace estimate and a regularity result for solutions of Laplace's equation (Theorem 3.3), derived from the theory in [50]. Among the terms obtained in this way, the one that is probably the most difficult to handle may be written as $\|\nabla_x \partial_t \mathfrak{D}^{(\tau)}(f)\,|\,\Omega \times (0, \infty)\|_{3/2,2;\infty}$, with

$$\mathfrak{D}^{(\tau)}(f)(x, t) := \int_0^t \int_{\mathbb{R}^3} V^{(\tau)}(x - y, t - s) \cdot f(y, s)\, dy\, ds \tag{1.25}$$

$(x \in \mathbb{R}^3,\ t > 0)$, where

$$V_{jk}^{(\tau)}(z,r) := \int\limits_0^r (r-\sigma)^{-1/2} \cdot \Lambda_{jk}(z,\sigma,\tau)\, d\sigma \tag{1.26}$$

($z \in \mathbb{R}^3\backslash\{0\}$, $r \in (0,\infty)$, $1 \le j,k \le 3$). The symbol $\|\cdot\|_{q,p;T}$ is to denote the norm of the space $L^p(0,T,L^q(A)^\sigma)$, for $T \in (0,\infty]$, $A \subset \mathbb{R}^3$ open, $p,q \in [1,\infty]$, $\sigma \in \{1,3\}$. Let us briefly indicate how we estimate the term in question. We proceed in two steps. In the first one, we show that

$$\|\nabla_x \partial_t (\mathfrak{D}^{(\tau)}(f) - \mathfrak{D}(f))\,|\,\Omega \times (0,\infty)\|_{3/2,2;\infty} \le \mathfrak{C} \cdot \|f\|_{3/2,2;\infty}$$

(Lemma 6.3), where $\mathfrak{D}(f)$ is defined in the same way as $\mathfrak{D}^{(\tau)}(f)$, but with Λ replaced by the Stokes fundamental solution Γ from (1.5). In the second step, we observe that $\mathfrak{D}(f) = \mathfrak{M}(\mathfrak{F}_{3/2}(f))$, where $\mathfrak{F}_{3/2}(f)$ denotes the solenoidal part in $L^{3/2}(\mathbb{R}^3)^3$ of the zero extension of f to $\mathbb{R}^3 \times (0,\infty)$ [see (3.1)], and where

$$\mathfrak{M}(g)(x,t) := \int\limits_0^t \int\limits_{\mathbb{R}^3} L(x-y,t-s) \cdot g(y,s)\, dy\, ds \quad (x \in \mathbb{R}^3,\ t > 0), \tag{1.27}$$

with $g \in L^s(0,\infty,\, L^p(\mathbb{R}^3)^3)$ for some $s,p \in (1,\infty)$ (Lemma 5.11), and

$$L(z,r) := \int\limits_0^r (r-\sigma)^{-1/2} \cdot \mathfrak{H}(z,\sigma)\, d\sigma \quad \text{for} \quad z \in \mathbb{R}^3\backslash\{0\}, \quad r \in (0,\infty). \tag{1.28}$$

Thus we still have to estimate the term $\|\nabla_x \partial_t \mathfrak{M}(\mathfrak{F}_{3/2}(f)\,|\,\Omega \times (0,\infty)\|_{3/2,2;\infty}$. But for a function $g \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3$, the term $\nabla_x \partial_t \mathfrak{M}(g)$ may be written as a singular integral (Lemma 7.4). In order to deal with this integral, we apply a theorem by Benedek et al. [6] pertaining to singular integrals in Hilbert spaces. We thus obtain that

$$\|\nabla_x \partial_t \mathfrak{M}(g)\|_{2,p;\infty} \le \mathfrak{C}(p) \cdot \|g\|_{2,p;\infty} \quad \text{for} \quad p \in (1,\infty),$$

$g \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3$ (Corollary 7.5). Finally we estimate the L^p-norm in space and time of $\nabla_x \partial_t \mathfrak{M}(g)$ by the L^p-norm in space and time of g (Theorem 7.2), then use an interpolation theorem (Theorem 3.5) in order to conclude that $\partial_m \partial_t \mathfrak{M}(g)$ $(1 \le m \le 3)$ in the norm of $L^2(0,T,L^{3/2}(\mathbb{R}^3)^3)$ is bounded by g in the same norm (Corollary 7.6). A density argument then yields an analogous estimate of $\mathfrak{M}(\mathfrak{F}_{3/2}(f))$ (Theorem 7.3). This finishes our estimate of $\|\nabla_x \partial_t \mathfrak{D}^{(\tau)}(f)\|_{3/2,2;\infty}$. All the other terms appearing in the upper bound of $\|\mathfrak{R}^{(\tau)}(f)|S_T\|_{H_T}$ mentioned above may be handled in a similar way, or by simpler arguments, or were already treated in the literature. Some of these terms are shown to have upper bounds involving the quantity $\|f\|_{q_0,2;\infty}$ (Lemmas 6.1, 6.2), where q_0 is an arbitrary but fixed exponent smaller than 3/2, as indicated above. All this holds, as we may recall, if $f \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3$. The looked-for relation $\mathfrak{R}^{(\tau)}(f)|S_T \in H_T$ for $f \in L^2(0,T,L^p(\overline{\Omega}^c)^3)$ with $p \in [q_0, 3/2]$ then follows by a density argument (Theorem 8.1).

Our theory make use of a certain number of results which might not be new, but are not well known, and for which we cannot give a reference. For the convenience of the reader, a proof of these results is indicated in an appendix at the end of this article.

Let us make some further remarks on existing literature. In [14], we studied the asymptotic behaviour of the potential $\mathfrak{I}^{(\tau)}(a)$, which solves (1.1), (1.3) if $\Omega = \emptyset$ (Cauchy problem) and $f = 0$. Reference [10] deals with another criterion for f to satisfy the relation $f|S_{T_0} \in H_{T_0}$, but this criterion is not taken up in either [13,15,16]. Regularity properties of the potential $\mathfrak{V}^{(\tau)}(\Phi)$, which solves (1.1)–(1.3) with $a = 0$, $f = 0$, are discussed in [11].

In his article [45] mentioned above, Solonnikov considered a linear problem more general than (1.1)–(1.3), showing existence of strong solutions in Sobolev spaces and of classical solutions in Hölder spaces. Also, for a nonlinear problem more general than (1.21), (1.2), (1.3), he established local in time existence in the same type of spaces. Kobayashi, Shibata [34] constructed an Oseen semigroup in L^p-spaces. Such a semigroup yields existence results for problem (1.1)–(1.3). Enomoto and Shibata [19] introduced an analogous semigroup in the case of space dimension $n \geq 3$. Mild solutions to the nonlinear problem (1.21), (1.2), (1.3) were constructed by Miyakawa [38] and by Shibata [42], with the first author considering local in time existence, and the second global in time existence for small data. We already mentioned existence results for (1.21), (1.2), (1.3) due to Heywood [29,30], who used variational methods.

L^p-L^q-estimates of solutions to (1.1)–(1.3) with $f = 0$, $b = 0$ ("Oseen flows") were established in [34]. These estimates were extended to the case of space dimension $n \geq 3$ in [19]. Local L^p-L^q-estimates for Oseen flows were shown in [34] ($n = 3$) and [18] ($n \geq 3$). Bae and Jin [4] proved weighted L^p-L^q-estimates for Oseen flows, using weight functions that are adapted to the wake appearing in such flows. Stability results for solutions to (1.21), (1.2), (1.3), in the sense of temporal decay estimates of spatial L^p-norms of the velocity, were proved by Masuda [37], Heywood [30, p. 675], Shibata [42], Enomoto and Shibata [19] (case $n \geq 3$), and Bae and Roh [5]. An eigenvalue criterion for stability of solutions to (1.18) with Dirichlet boundary conditions is established in [17]. Knightly [33] showed that solutions to (1.21), (1.2), (1.3) exhibit a wake, but he required various smallness conditions.

A special situation arises when Eq. (1.1) or (1.21) is considered in the whole space $\mathbb{R}^3$, under an initial condition (Cauchy problem). Then certain aspects of the asymptotic behaviour of a solution to (1.1) or (1.21) may be deduced from the decay properties exhibited by solutions of respectively the Stokes system and the Navier–Stokes system without Oseen term; see [32, p. 507], [48].

2. Notation

Recall that in Sect. sec1, we introduced an open bounded set Ω with connected Lipschitz boundary $\partial\Omega$ and outward unit normal $n^{(\Omega)}$. Also we already introduced the notation $S_T := \partial\Omega \times (0, T)$ for $T \in (0, \infty]$, and the constant $\tau \in (0, \infty)$. Generalizing the notation $\overline{\Omega}^c$ defined in the context of (1.1), we put $A^c := \mathbb{R}^3 \backslash A$ for $A \subset \mathbb{R}^3$. The symbol $|\ |$ is to denote not only the Euclidean norm in $\mathbb{R}^n$ for $n \in \mathbb{N}$, but also the length of a multiindex $\alpha \in \mathbb{N}^3$, as was already indicated in the line following (1.19). Recall the definition of the weight function ν in (1.20). Put $B_r(x) := \{y \in \mathbb{R}^3 \ : \ |y - x| < r\}$ for $x \in \mathbb{R}^3$, $r > 0$, and $B_r := B_r(0)$, $\Omega_r := B_r \cap \Omega$. For $m \in \{1, 2, 3\}$, set $e_m := (\delta_{jm})_{1 \leq j \leq 3}$.

Numerical constants or constants only depending on Ω and τ will be denoted by $\mathfrak{C}$. We write $\mathfrak{C}(\gamma_1, \ldots, \gamma_n)$ for constants that may depend on Ω, τ and parameters $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$, for $n \in \mathbb{N}$.

Following [24, p. 269–270, 305–306], we choose $k(\Omega) \in \mathbb{N}$, $\alpha(\Omega) \in (0, \infty)$, orthonormal matrices $A_1^{(\Omega)}, \ldots, A_{k(\Omega)}^{(\Omega)} \in \mathbb{R}^{3\times 3}$, vectors $C_1^{(\Omega)}, \ldots, C_{k(\Omega)}^{(\Omega)} \in \mathbb{R}^3$, and Lipschitz continuous functions $a_1^{(\Omega)}, \ldots, a_{k(\Omega)}^{(\Omega)} : [-\alpha(\Omega),\, \alpha(\Omega)]^2 \mapsto \mathbb{R}$ such that the following properties hold: Defining

$$\begin{gathered}
\Delta^{\Omega} := (-\alpha(\Omega),\ \alpha(\Omega))^2, \quad H^{(i)}(\eta, r) := \ A_i^{(\Omega)} \cdot (\eta,\, a_i^{(\Omega)}(\eta) + r) \ + \ C_i^{(\Omega)} \\
\text{for } \ \eta \in \Delta^{\Omega},\, r \in (-\alpha(\Omega), \alpha(\Omega)), \\
U_i := \{H^{(i)}(\eta, r) \ : \ \eta \in \Delta^{\Omega}, \ \ r \in (-\alpha(\Omega)\,,\ \alpha(\Omega))\}, \\
h^{(i)}(\eta) := H^{(i)}(\eta, 0) \quad \text{for } \ \eta \in \Delta^{(\Omega)}, \ 1 \leq i \leq k(\Omega),
\end{gathered}$$

we have

$$\begin{gathered}
\overline{\Omega}^c \cap U_i = H^{(i)}[\Delta^{\Omega} \times (-\alpha(\Omega), 0)], \quad \Omega \cap U_i = H^{(i)}[\Delta^{\Omega} \times (0, \alpha(\Omega))], \\
\partial\Omega \cap U_i = H^{(i)}(\Delta^{\Omega} \times \{0\}) \quad (i \in \{1, \ldots, k(\Omega)\}), \\
\partial\Omega = \bigcup_{i=1}^{k(\Omega)} h^{(i)}[(-\alpha(\Omega)/4,\ \alpha(\Omega)/4)^2].
\end{gathered}$$

This means in particular that $h^{(i)}$ is a local map of $\partial\Omega$. Further note that

$$\int\limits_{U_i} g(x)\,dx = \int\limits_{-\alpha(\Omega)}^{\alpha(\Omega)} \int\limits_{\Delta^{(\Omega)}} (g \circ H^{(i)})(\eta,\, s)\,d\eta\,ds \quad \text{for} \quad g \in L^1(U_i), \tag{2.1}$$

$1 \le i \le k(\Omega)$. We further fix a nontangential vector field $m^{(\Omega)} \in C_0^\infty(\mathbb{R}^3)^3$ to Ω [40, p. 246]. This means that the relation $|m^{(\Omega)}(x)| = 1$ holds for x from a neighbourhood of $\partial\Omega$ in $\mathbb{R}^3$, and there are constants $\mathcal{D}_1, \mathcal{D}_2 \in (0,\infty)$ such that

$$|x + \delta \cdot m^{(\Omega)}(x) - x' - \delta' \cdot m^{(\Omega)}(x')| \ge \mathcal{D}_1 \cdot (|x - x'| + |\delta - \delta|'|) \tag{2.2}$$

for $x, x' \in \partial\Omega,\ \delta,\, \delta' \in [-\mathcal{D}_2, \mathcal{D}_2]$, and

$$x + \delta \cdot m^{(\Omega)}(x) \in \overline{\Omega}^c, \quad x - \delta \cdot m^{(\Omega)}(x) \in \Omega \quad \text{for} \ \ x \in \partial\Omega,\ \delta \in (0, \mathcal{D}_2]. \tag{2.3}$$

If $\mathfrak{H}$ is a space consisting of functions from a set B into $\mathbb{R}$, then let $\mathfrak{H}^3$ stand for the set $\{F : B \mapsto \mathbb{R}^3 : F_1, F_2, F_3 \in \mathfrak{H}\}$. If $\|\ \|$ is a norm on $\mathfrak{H}$, we will use the same notation $\|\ \|$ for the norm $(\sum_{j=1}^3 \|F_i\|^2)^{1/2}$ on $\mathfrak{H}^3$.

Let $n \in \mathbb{N}$, and suppose that $A \subset \mathbb{R}^n$ is measurable. Then the usual norm of the Lebesgue space $L^p(A)$ is denoted by $\|\ \|_p$ ($p \in [1,\infty]$). For $p \in (1,\infty),\ A \subset \mathbb{R}^3$ open, define $L^p_\sigma(A)$ as the closure of the set $\{v \in C_0^\infty(A)^3 \ :\ \operatorname{div} v = 0\}$ with respect to the norm of $L^p(A)^3$ ("space of solenoidal functions in $L^p(A)^3$"). Put $L^p_\sigma := L^p_\sigma(\mathbb{R}^3)$. For $G \in L^1(\mathbb{R}^4)$, we introduce the Fourier transform $\widehat{G}$ of G by setting $\widehat{G}(\xi,\, \varrho) := (2\cdot\pi)^{-2} \cdot \int_{\mathbb{R}} \int_{\mathbb{R}^3} e^{-i\cdot(\xi\cdot y + \varrho\cdot s)} \cdot G(y,s)\,dy\,ds$ ($\xi \in \mathbb{R}^3,\ \ \varrho \in \mathbb{R}$). The inverse Fourier transform $G^\vee$ of G is to be defined correspondingly.

Let $A \subset \mathbb{R}^3$ be open, $m \in \mathbb{N},\ p \in [1,\infty)$. Then we write $W^{m,p}(A)$ for the usual Sobolev space of order m on A, with exponent p. The standard norm of this space is denoted by $\|\ \ \|_{m,\,p}$. If $p = 2$, we write $H^m(A)$ instead of $W^{m,2}(A)$. For $s \in (0,1)$, let $H^s(A)$ be the Sobolev space defined via the intrinsic norm with exponent $p = 2$ introduced in [1, 7.51]. We write $\|\ \|_{s,2}$ for this norm. Let $H^s_\sigma(U)$ for $s \in (0,1]$ be defined as in the passage preceding Theorem 1.1, where we additionally abbreviated $V := H^1_\sigma(\overline{\Omega}^c)$. We write V' for the canonical dual space of V. The Sobolev space $H^1(\partial\Omega)$ is to be defined in the standard way (see [24, Section III.6], for example). Let $\|\ \ \|_{1,\,2}$ denote the usual norm of this space with respect to some local coordinates of $\partial\Omega$ [24, Section III.6.7]. The notation $\|\ \|_{H^1(\partial\Omega)'}$ stands for the canonical norm of the dual space $H^1(\partial\Omega)'$ of $H^1(\partial\Omega)$.

Let $T \in (0,\infty],\ p \in [1,\infty],\ \mathfrak{B}$ a Banach space. The usual norm of the space $L^p(0,T,\mathfrak{B})$ will be denoted by $\|\ \ \|_{L^p(0,T,\mathfrak{B})}$. In the case $\mathfrak{B} = L^q(A)^\sigma$ with $q \in [1,\infty],\ \sigma \in \{1,\,3\}$ and $A \subset \mathbb{R}^3$ open, it is convenient to write $\|\ \ \|_{q,p;T}$ instead of $\|\ \ \|_{L^p(0,T,L^q(A)^\sigma)}$, as we already did in Sect. 1. If J is an interval in $\mathbb{R}$, and if a function $h : J \mapsto \mathfrak{B}$ is measurable (see [51, p. 130], where the idiom "strongly $\mathfrak{B}$-measurable" is used), and if $\int_J \|h(r)\|_{\mathfrak{B}}\,dr < \infty$, then we write $\mathfrak{B} - \int_J h(r)\,dr$ for the Bochner integral of h on J [51, p. 132–134].

Again take $T \in (0,\infty]$. Recall the definition of the space $L^2_n(S_T)$ in (1.11). For a nonempty set $A \subset \mathbb{R}^3$ and a function $\varphi : A \times (0,T) \mapsto \mathbb{R}^3$ with $\varphi(x,\ \cdot) \in C^1((0,T))^3$ for $x \in A$, let $\partial_4^{1/2}\varphi(x,t)$ given by the right-hand side of (1.12). This defines the fractional derivative $\partial_4^{1/2}\varphi$ of φ. Moreover, for any $w \in L^2(S_T)$, we introduce a functional $F_w \in L^2(0,T,H^1(\partial\Omega)')$ by setting $F_w(t)(\mu) := \int_{\partial\Omega} w(x,t)\cdot\mu(x)\,dx$ for $\mu \in H^1(\partial\Omega)$ and $t \in (0,T)$. We will write w instead of F_w. Put $\widetilde{H}_T := \{\varphi|S_T \ :\ \varphi \in C_0^\infty(\mathbb{R}^4)^3,\ \varphi|\mathbb{R}^3 \times (-\infty,\,0] = 0\}$. Then, for $w \in \widetilde{H}_T$, we define $\|w\|_{H_T}$ as in (1.10). The mapping $\|\ \ \|_{H_T}$ is a norm on $\widetilde{H}_T$. Following [41], let the space H_T consist of all functions $w \in L^2_n(S_T)$ such that there exists a sequence (w_n) in $\widetilde{H}_T$ with the property that $\|w - w_n\|_2 \to 0$, and such that (w_n) is a Cauchy sequence with respect to the norm $\|\ \|_{H_T}$. This means in particular that the sequence $(\|w_n\|_{H_T})$ is convergent. Its limit value does not depend on the choice of the sequence (w_n) with the above properties. Thus, for $w \in H_T$, we may define the quantity $\|w\|_{H_T}$ in an obvious way. The mapping $\|\ \ \|_{H_T}$ is a norm on H_T, and the pair $(H_T,\ \|\ \ \|_{H_T})$ is a Banach space.

We end this section by some remarks that, although pedantic, may still be useful in avoiding ambiguities in the following.

Lemma 2.1. *Let* $n \in \mathbb{N}$, $A \subset \mathbb{R}^n$ *measurable,* $T \in (0, \infty]$, $p \in [1, \infty)$, $F : (0,T) \mapsto L^p(A)$, $f : A \times (0,T) \mapsto \mathbb{R}$ *with* $f(x,t) = F(t)(x)$ *for* $t \in (0,T)$, $x \in A$.

Then f *is Lebesgue-measurable as a function from* $A \times (0,T)$ *into* $\mathbb{R}$ *iff* F *is measurable as a function from* $(0,T)$ *into* $L^p(A)$ *(or more precisely: strongly* $L^p(A)$*-measurable in the sense of [51, p. 130]). If these conditions are satisfied, we have* $\|f\|_p = \|F\|_{p,p;T}$.

Proof. If F is measurable, the measurability of f follows by a reasoning as in the proof of [23, Lemma 10.1]. Suppose that f is measurable, and let $\mu \in L^{p'}(\mathbb{R}^n)$. Then by Fubini's theorem, the function $t \mapsto \int_{\mathbb{R}^n} f(x,t) \cdot \mu(x)\, dx$ $(t \in (0,T))$ is measurable on $(0,T)$. If follows by [51, p. 131-132] that F is measurable. The last claim of the lemma follows by Fubini's theorem. □

In the following, functions F and f as in Lemma 2.1 will be identified. In this sense, we have $L^p(0,T,L^p(A)^\sigma) = L^p(A \times (0,T))^\sigma$ (p, T, A as in Lemma 2.1, $\sigma \in \{1, 3\}$). Similarly, functions $f : S_T \mapsto \mathbb{R}$ and $F : (0,T) \mapsto L^p(\partial\Omega)$ with $F(t)(x) = f(x,t)$ $(x \in \partial\Omega,\ t \in (0,T))$ will be identified because Lemma 2.1 holds in an analogous way if A is replaced by $\partial\Omega$. In this sense, $L^2(0,T,L^2(\partial\Omega)^\sigma) = L^2(S_T)^\sigma$ for $\sigma \in \{1, 3\}$, $T \in (0, \infty]$.

Lemma 2.2. *Let* $S \in (0,\infty)$ *with* $\overline{\Omega} \subset B_S$, $T \in (0,\infty]$, $p, q, r \in [1,\infty)$. *Suppose there is* $C > 0$ *with* $\|v|\partial\Omega\|_r \le C \cdot \|v\|_{1,q}$ *for* $v \in W^{1,q}(\Omega_S)$. *Let* $w \in L^p(0,T,W^{1,q}(\Omega_S))$, *and define* $\mathfrak{T}(w) : (0,T) \mapsto L^r(\partial\Omega)$ *by* $\mathfrak{T}(w)(t) := w(\ \cdot\ ,t)|\partial\Omega$. *Then* $\mathfrak{T}(w) \in L^p(0,T,L^r(\partial\Omega))$. *In this sense, any function from* $L^p(0,T,W^{1,q}(\Omega_S))$ *has a measurable boundary value on* S_T.

Proof. The only point to check is whether $\mathfrak{T}(w)$ is a measurable function from $(0,T)$ into $L^r(\partial\Omega)$ (strongly $L^r(\partial\Omega)$-measurable in the sense of [51, p. 130]). But this follows from an argument involving simple functions from $(0,T)$ into $W^{1,q}(\Omega_S)$. □

Note that in view of Lemma 2.2 and the remark preceding it, functions from $L^2(0,T,H^1(\overline{\Omega}^c))$ admit a boundary value in $L^2(S_T)$.

Next we state a density result for the space $L^p(0,T,L^q(\mathbb{R}^3))$. We refer to the Appendix for a proof.

Lemma 2.3. *Let* $T \in (0,\infty]$, $n \in \mathbb{N}$, $p_1, p_2, q_1, q_2 \in [1,\infty)$, *and let* f *belong to* $L^{p_i}(0,T,L^{q_i}(\mathbb{R}^3))$ *for* $i \in \{1, 2\}$. *Then there is a sequence* φ_n *in* $C_0^\infty(\mathbb{R}^3 \times (0,T))$ *with* $\|f - \varphi_n\|_{q_i,p_i;T} \to 0$ *for* $i \in \{1, 2\}$. *In particular, the set* $C_0^\infty(\mathbb{R}^3 \times (0,T))$ *is dense in* $L^p(0,T,L^q(\mathbb{R}^3))$ *for any* $p, q \in [1,\infty)$.

Lemma 2.4. *Let* $T \in (0,\infty]$. *Then* $L^2(0,T,\ L^{3/2}(\overline{\Omega}^c)^3) \subset L^2(0,T,V')$.

Proof. Let $g \in L^{3/2}(\overline{\Omega}^c)^3$, $v \in V$. Then the integral $\int_{\overline{\Omega}^c} |g \cdot v| dx$ is bounded by $\|g\|_{3/2} \cdot \|v\|_3$, hence by $\|g\|_{3/2} \cdot \|v\|_2^{1/2} \cdot \|v\|_6^{1/2}$, and thus by $\mathfrak{C} \cdot \|g\|_{3/2} \cdot \|v\|_{1,2}$. □

3. Auxiliary Results

The ensuing four integral estimates are well known. Since they will be used frequently, we state them for the convenience of the reader.

Lemma 3.1. *Let* $\mu \in (-1,\infty)$, $t \in (0,\infty)$. *Then* $\int_0^t (t-s)^{-1/2} \cdot s^{-\mu}\, ds \le \mathfrak{C}(\mu) \cdot t^{1/2-\mu}$.

Proof. Use the change of variables $s = t \cdot r$. □

Lemma 3.2. [20, Lemma 2.3] $\int_{\partial B_r} \nu(x)^{-\beta}\, do_x \le \mathfrak{C}(\beta) \cdot r$ *for* $\beta \in (1,\infty)$, $r \in (0,\infty)$.

Lemma 3.2 is needed for the proof of

Lemma 3.3. *Let* $S \in (0,\infty)$. *Then* $\int_{B_S^c} (|z| \cdot \nu(z))^{-9/4} dz \le \mathfrak{C}(S)$.

Proof. $\int_{B_S^c} (|z| \cdot \nu(z))^{-9/4} dz = \int_S^\infty r^{-9/4} \cdot \int_{\partial B_r} \nu(z)^{-9/4} do_z\, dr \le \mathfrak{C} \cdot \int_S^\infty r^{-5/4} dr \le \mathfrak{C}(S)$. □

Next we state Minkowski's inequality for integrals (Theorem 3.1) and Young's inequality for convolutions (Theorem 3.2).

Theorem 3.1. [2, p. 26, Theorem 2.9] *Let* $m, n \in \mathbb{N},\ p \in [1,\infty),\ F : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$ *a measurable function. Then*

$$\left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |F(x,y)|\, dx \right)^p dy \right)^{1/p} \leq \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |F(x,y)|^p\, dy \right)^{1/p} dx.$$

Theorem 3.2. [2, p. 34, Corollary 2.25] *Suppose that* $n \in \mathbb{N},\ p,q,r \in [1,\infty]$ *with* $1/q = 1/p + 1/r - 1$. *Let* $f \in L^p(\mathbb{R}^n),\ g \in L^r(\mathbb{R}^n)$. *Then the convolution* $f * g$ *is well defined and belongs to* $L^q(\mathbb{R}^n)$, *with* $\|f * g\|_q \leq \|f\|_p \cdot \|g\|_r$.

Using the local maps $h^{(i)}$ of $\partial\Omega$ introduced at the beginning of Sect. 2, we prove an extension theorem based on the solution theory of Laplace's equation in Lipschitz domains, as presented in [50]. This theorem extends [13, Theorem 2.4] to the case $p = 3$.

Theorem 3.3. *For* $n \in \mathbb{N}$, *put* $\Omega^{(n)} := \{x \in \mathbb{R}^3 : dist(x,\Omega) < \mathfrak{D}_1 \cdot \mathfrak{D}_2/(2 \cdot n)\}$, *with* $\mathfrak{D}_1, \mathfrak{D}_2$ *from* (2.2) *and* (2.3), *respectively. Then* $\overline{\Omega} \subset \Omega^{(n)}$. *Moreover, for* $v \in H^1(\partial\Omega),\ n \in \mathbb{N}$, *there is a function* $E^{(n)}(v) \in C^\infty(\Omega^{(n)})$ *such that* $\|E^{(n)}(v)|\partial\Omega - v\|_2 \to 0\ (n \to \infty)$, *and such that there is a constant* $C \in (0,\infty)$ *with* $\|\nabla E^{(n)}(v)|\Omega\|_3 \leq C \cdot \|v\|_{1,2}$ *for* $n \in \mathbb{N},\ v \in H^1(\partial\Omega)$.

Proof. Put $\Delta^{\Omega}_{1/2} := (-\alpha(\Omega)/2,\, \alpha(\Omega)/2)^2$. Let $i \in \{1, \ldots, k(\Omega)\},\ \ \Phi \in L^2(\partial\Omega)$, and set

$$B_i(\Phi) := \left(\int_{U_i \cap \Omega} \left(\int_{h^{(i)}(\Delta^{\Omega}_{1/2})} |x-y|^{-2} \cdot |\Phi(y)|\, d\Omega(y) \right)^3 dx \right)^{1/3}.$$

Then, by (2.1) and because $h^{(i)}$ is a local map of $\partial\Omega$, the term $B_i(\Phi)$ is bounded by

$$\mathfrak{C} \cdot \left(\int_0^{\alpha(\Omega)} \int_{\Delta^{\Omega}} \left(\int_{\Delta^{\Omega}_{1/2}} \left| H^{(i)}(\varrho,\, r) - h^{(i)}(\eta) \right|^{-2} \cdot |\Phi \circ h^{(i)}(\eta)| d\eta \right)^3 d\varrho\, dr \right)^{1/3},$$

and thus by

$$\mathfrak{C} \cdot \left(\int_0^{\alpha(\Omega)} \int_{\Delta^{\Omega}} \left(\int_{\Delta^{\Omega}_{1/2}} (|\varrho - \eta| + r)^{-2} \cdot |\Phi \circ h^{(i)}(\eta)|\, d\eta \right)^3 d\varrho\, dr \right)^{1/3};$$

also see [13, p. 895]. Let $\widetilde{\Phi}_i$ denote the zero extension of $\Phi \circ h^{(i)}$ to $\mathbb{R}^2$. Then, by Minkowski's inequality (Theorem 3.1),

$$\begin{aligned} B_i(\Phi) &\leq \mathfrak{C} \cdot \left(\int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \left(\int_0^\infty (|\varrho - \eta| + r)^{-6} \cdot |\widetilde{\Phi}_i(\eta)|^3\, dr \right)^{1/3} d\eta \right]^3 d\varrho \right)^{1/3} \\ &\leq \mathfrak{C} \cdot \left(\int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} |\varrho - \eta|^{-5/3} \cdot |\widetilde{\Phi}_i(\eta)|\, d\eta \right]^3 d\varrho \right)^{1/3}, \end{aligned}$$

hence by the Hardy–Littlewood–Sobolev inequality [47, p. 119-120, Theorem 1]: $B_i(\Phi) \leq \mathfrak{C} \cdot \|\widetilde{\Phi}_i(\eta)\|_2$. With this inequality available, we may now establish Theorem 3.3 by repeating the arguments we used in an analogous situation in [13, p. 895-896], where we proved [13, Theorem 2.4] (the inequality $\|\nabla\mathfrak{E}^{(n)}(\Phi)\|_p \le \mathfrak{C}(p)\cdot\|\Phi\|_2$ on [13, p. 896] should read $\|\nabla\mathfrak{E}^{(n)}(\Phi)|\Omega\|_p \le \mathfrak{C}(p)\cdot\|\Phi\|_2$). □

Theorem 3.4. [25, Theorem II.5.1; p. 108–109] *For $p\in(1,\infty)$, there is $C=C(p)>0$ and for any $F\in L^p(\mathbb{R}^3)^3$ functions $P_p(F)\in L^p_\sigma$, $G_p(F)\in W^{1,p}_{loc}(\mathbb{R}^3)$ with $\nabla G_p(F)\in L^p(\mathbb{R}^3)^3$, $F=P_p(F)+\nabla G_p(F)$, $\|P_p(F)\|_p+\|\nabla G_p(F)\|_p\le C\cdot\|F\|_p$.*

If $p\in(1,3)$, $F\in L^p(\mathbb{R}^3)^3$, the function $G_p(F)$ may be chosen in such a way that $G_p(F)\in L^{3\cdot p/(3-p)}(\mathbb{R}^3)$. If $F\in C_0^\infty(\mathbb{R}^3)^3$, then $P_q(F)=P_r(F)$ for $q,r\in(1,\infty)$.

For $p\in(1,\infty)$, $f\in C_0^\infty(\mathbb{R}^3\times(0,\infty))^3$, we define $\mathfrak{F}_p(f):\mathbb{R}^3\times(0,\infty)\mapsto\mathbb{R}^3$ by

$$\mathfrak{F}_p(f)(x,t) := P_p\left(f(\,\cdot\,,t)\right)(x) \quad \text{for } x\in\mathbb{R}^3,\ t\in(0,\infty). \tag{3.1}$$

Theorem 3.4 yields

Lemma 3.4. *Let $p\in(1,\infty)$, $f\in C_0^\infty(\mathbb{R}^3\times(0,\infty))^3$. Then $supp(\mathfrak{F}_p(f))\subset\mathbb{R}^3\times(0,M)$ for some $M\in(0,\infty)$, and $\mathfrak{F}_p(f)\in L^\infty(0,\infty,\,L^p(\mathbb{R}^3)^3)$. In particular, $\mathfrak{F}_p(f)$ belongs to $L^s(0,\infty,\,L^p(\mathbb{R}^3)^3)$ for any $s\in[1,\infty)$. Moreover, $\|\mathfrak{F}_p(f)\|_{p,s;\infty}\le\mathfrak{C}(p)\cdot\|f\|_{p,s;\infty}$ for any $s\in[1,\infty)$.*

We will further need the ensuing interpolation result.

Theorem 3.5. *Let T be a linear operator whose domain includes $A_1:=L^6(0,\infty,\,L^2(\mathbb{R}^3)^3)$ and $A_2:=L^{6/5}(0,\infty,\,L^{6/5}(\mathbb{R}^3)^3)$. Suppose that $T(A_i)\subset A_i$, and that $T|A_i$ is bounded with respect to the norm of A_i ($i\in\{1,2\}$). Let $\|T|A_i\|^{(i)}$ denote the operator norm of $T|A_i:A_i\mapsto A_i$. Then $\|T(f)\|_{3/2,2;\infty}\le(\|T|A_1\|^{(1)})^{1/2}\cdot(\|T|A_2\|^{(2)})^{1/2}\cdot\|f\|_{3/2,2;\infty}$ for $f\in C_0^\infty(\mathbb{R}^3\times(0,\infty))^3$.*

Proof. The theorem holds by interpolation; see [7, Theorems 5.1.1, 5.1.2, 4.1.2]. □

Next we state [6, Theorem 3] in a form that will be convenient for our purposes.

Theorem 3.6. *Let H be a Hilbert space whose norm is denoted by $\|\ \ \|$. Let $\mathfrak{L}$ be the space of linear bounded operators from H into H. This space is to be equipped with the usual operator norm, here denoted by $\|\ \ \|_{op}$. Let $k:(0,\infty)\mapsto\mathfrak{L}$ be a measurable function which is integrable on compact subsets of $(0,\infty)$. Suppose that for any $u\in H$, the term $(\mathfrak{L}-\int_\epsilon^1 k(s)\,ds)(u)$ converges as $\epsilon\downarrow 0$, and that there is $c>0$ with*

$$\left\| \mathfrak{L}-\int_\epsilon^\delta k(s)\,ds \right\|_{op} \le c \ \ \textit{for}\ \ \epsilon,\,\delta\in(0,\infty)\ \ \textit{with}\ \ \epsilon<\delta,$$

$$\int_0^\varrho s\cdot\|k(s)(u)\|\,ds \le c\cdot\varrho\cdot\|u\| \ \ \textit{for}\ \ \varrho\in(0,\infty),\ u\in H,$$

$$\int_{4\cdot|t|}^\infty \|k(s-t)-k(s)\|_{op}\,ds \le c \ \ \textit{for}\ \ t\in\mathbb{R}\backslash\{0\}.$$

Then, for any bounded measurable function $f:(0,\infty)\mapsto H$ with compact support, and for any $\epsilon>0$, the function $s\mapsto\chi_{(\epsilon,\infty)}(t-s)\cdot k(t-s)(f(s))$ $(s\in(0,\infty))$ is integrable, and the function $A_\epsilon(f):(0,\infty)\mapsto H$ defined by

$$A_\epsilon(f)(t) := H-\int_0^\infty \chi_{(\epsilon,\infty)}(t-s)\cdot k(t-s)(f(s))\,ds \quad \textit{for}\ \ t\in(0,\infty)$$

belongs to $L^2(0,\infty,\,H)$. Moreover the relation $\|A(f)-A_\epsilon(f)\|_{L^2(0,\infty,\,H)}\to 0$ $(\epsilon\downarrow 0)$ holds for some function $A(f)\in L^2(0,\infty,\,H)$. This function verifies the inequality $\|A(f)\|_{L^p(0,\infty,\,H)}\le\mathfrak{C}(p,c)\cdot\|f\|_{L^p(0,\infty,\,H)}$ for $p\in(1,\infty)$ and for f as before.

Proof. All statements of this theorem except the last inequality hold according to [6, Theorem 3] (choose $k(t) = 0$ for $t \in (-\infty, 0)$ in that reference). As for the proof of the inequality at the end of Theorem 3.6, we start from another result in [6, Theorem 3], namely the estimate

$$\|A_\epsilon(f)\|_{L^p(0,\infty,\,H)} \le \mathfrak{C}(p,c) \cdot \|f\|_{L^p(0,\infty,\,H)}, \tag{3.2}$$

which holds for $\epsilon \in (0,\infty),\ p \in (1,\infty)$ and for f as in Theorem 3.6. On the other hand, for $\epsilon > 0$,

$$\left(\int_0^\infty |\,\|A_\epsilon(f)(t)\| - \|A(f)(t)\|\,|^2\,dt\right)^{1/2} \le \|A_\epsilon(f) - A(f)\|_{L^2(0,\infty,\,H)}.$$

Hence $\|A_\epsilon(f)(t)\| \to \|A(f)(t)\|$ $(\epsilon \downarrow 0)$ in $L^2(0,\infty)$. Therefore we may choose a sequence (δ_n) in $(0,\infty)$ with $\delta_n \downarrow 0$ and $\|A_{\delta_n}(f)(t)\| \to \|A(f)(t)\|$ $(n \to \infty)$ for a.e. $t \in (0,\infty)$. By Fatou's lemma, we may conclude that

$$\int_0^\infty \|A(f)(t)\|^p\,dt \le \liminf_{n\to\infty} \int_0^\infty \|A_{\delta_n}(f)(t)\|^p\,dt$$

for $p \in (1,\infty)$. Now the looked-for inequality follows from (3.2). □

The next theorem states a uniqueness result for weak solutions to (1.1)–(1.3). A proof of this theorem is indicated in the Appendix.

Theorem 3.7. *Let $T \in (0,\infty],\ b : S_T \mapsto \mathbb{R}^3,\ a \in W^{1,1}_{loc}(\overline{\Omega}^c)^3,\ f : (0,T) \mapsto V'$ with $f|(0,T') \in L^2(0,T',V')$ for $T' \in (0,T)$. Then there is at most one function $u : (0,T) \mapsto H^1(\overline{\Omega}^c)^3$ such that (1.13)–(1.15) hold.*

The ensuing lemma links Bochner integrals in $L^2(\mathbb{R}^3)$ with standard Lebesgue integrals. Once more we refer to the Appendix for a proof.

Lemma 3.5. *Let J be an interval in $\mathbb{R}$, and let $v \in L^1(J, L^2(\mathbb{R}^3))$. Then for a.e. $x \in \mathbb{R}^3$, we have $\int_J |v(t)(x)|^2\,dt < \infty$ and $\int_J v(t)(x)\,dt = (L^2(\mathbb{R}^3) - \int_J v(t)\,dt)(x)$.*

4. Kernel Functions

In this section, we present estimates of the kernel functions that will later enter into the definition of our volume potentials (Sect.5). Recall that $\mathfrak{H}$, Γ and Λ (fundamental solution of respectively the heat equation, the time-dependent Stokes system and the time-dependent Oseen system) were introduced in (1.4)–(1.6). It will be convenient to use the abbreviation $K_{jk}(z,t) := \Lambda_{jk}(z,t,\tau) - \Gamma_{jk}(z,t)$ for $(z,t) \in \mathfrak{B},\ j,k \in \{1,2,3\}$, with the set $\mathfrak{B}$ defined in (1.5). In a slight generalization of (1.6), we put $\Lambda_{jk}(z,t,\kappa) := \Gamma_{jk}(z - \kappa \cdot t \cdot e_1,\, t)$ for $\kappa \in (0,\infty),\ 1 \le j,k \le 3,\ (z,t) \in \mathfrak{B}$. The following result is well known:

Lemma 4.1. $\int_{\mathbb{R}^3} \mathfrak{H}(z,t)\,dt = 1$ *for $z \in \mathbb{R}^3,\ t \in (0,\infty)$.*

Next we state estimates of $\mathfrak{H}$, Γ and Λ that are basic for our theory.

Lemma 4.2. *$\mathfrak{H} \in C^\infty(\mathfrak{B})$ and $\Gamma_{jk} \in C^\infty(\mathfrak{B})$, for $1 \le j,k \le 3$. For $l \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^3$, there is $C = C(l,\alpha) > 0$ such that for $z \in \mathbb{R}^3,\ t \in (0,\infty),\ j,k \in \{1,2,3\}$,*

$$|\partial_t^l \partial_z^\alpha \mathfrak{H}(z,t)| \le C \cdot (|z|^2 + t)^{-3/2 - |\alpha|/2 - l} \cdot e^{-|z|^2/(8\cdot t)},$$
$$|\partial_t^l \partial_z^\alpha \Gamma_{jk}(z,t)| \le C \cdot (|z|^2 + t)^{-3/2 - |\alpha|/2 - l}.$$

Proof. See [43] as concerns $\mathfrak{H}$, and [41, Proposition 2.1.9] for an estimate of the function Γ_{jk}. □

Lemma 4.3. *Let $\kappa \in (0,\infty)$. Then $\Lambda_{jk}(\,\cdot\,,\,\cdot\,,\kappa) \in C^\infty(\mathfrak{B})$ for $1 \le j,k \le 3$. Moreover, for $K \in (0,\infty)$, $l \in \{0,1\}$, $\alpha \in \mathbb{N}_0^3$, there is $C(K) = C(K,l,\alpha) > 0$ such that*

$$|\partial_t^l \partial_z^\alpha \Lambda_{jk}(z,t,\kappa)| \le C(K) \cdot (\max\{1,\kappa\})^{3/2+|\alpha|/2+3\cdot l/2} \cdot \Big((\gamma_K(z)+t)^{-3/2-|\alpha|/2-l} + (\gamma_K(z)+t)^{-3/2-|\alpha|/2-l/2} \Big)$$

for $z \in \mathbb{R}^3$, $t \in (0,\infty)$, $j,k \in \{1,\,2,\,3\}$, where $\gamma_K(z) := |z|^2$ if $|z| \le K$, and $\gamma_K(z) := |z| \cdot \nu(z)$ else.

Proof. By [9, Lemma 2], we have $(|z-\kappa\cdot t\cdot e_1|^2+t)^{-1} \le \mathfrak{C}(K)\cdot\max\{1,\kappa\}\cdot\gamma_K(z,t)^{-1}$ for $z \in \mathbb{R}^3$, $t > 0$. Thus Lemma 4.3 follows from Lemma 4.2. □

We note that a term is missing in the version of Lemma 4.3 given in [9, Lemma 3].

Lemma 4.4. *We have $K_{jk} \in C^\infty(\mathfrak{B})$ for $1 \le j,k \le 3$. Moreover, $K_{jk}(z,0) = 0$ $(z \in \mathbb{R}^3\backslash\{0\})$. Let $S \in (0,\infty)$, $l \in \{0,1\}$, $\alpha \in \mathbb{N}_0^3$. Then*

$$|\partial_t^l \partial_z^\alpha K_{jk}(z,t)| \le \mathfrak{C}(S,|\alpha|) \cdot \Big((|z|^2+t)^{-1-|\alpha|/2-l/2} + (|z|^2+t)^{-1-|\alpha|/2-l} \Big) \tag{4.1}$$

for $z \in B_S$, $t \in (0,\infty)$, $j,k \in \{1,\,2,\,3\}$.

Proof. The equation $K_{jk}(z,0) = 0$ follows from Lemmas 4.2, 4.3 and Lebesgue's theorem on dominated convergence. Take z,t,j,k as in (4.1). Proceeding as in the proof of [9, Lemma 9] (where a term is missing in estimate [9, (17)]), we observe that $\partial_z^\alpha K_{jk}(z,t) = \int_0^1 \partial_1\partial_z^\alpha \Gamma_{jk}(z-\vartheta\cdot\tau\cdot t\cdot e_1,\,t)\cdot(-\tau\cdot t)\,d\vartheta$, so

$$\partial_t^l \partial_z^\alpha K_{jk}(z,t) = \int_0^1 \Big(\partial_t^l \partial_z^{\alpha+e_1} \big(\Gamma_{jk}(z-\vartheta\cdot\tau\cdot t\cdot e_1,t)\big)\cdot(-\tau\cdot t) + \delta_{l1}\cdot\partial_z^{\alpha+e_1}\Gamma_{jk}(z-\vartheta\cdot\tau\cdot t\cdot e_1,t)\cdot(-\tau)\Big)\,d\vartheta.$$

Now we use Lemma 4.3 with $K = S$, $\kappa = \vartheta\cdot\tau$ to obtain

$$|\partial_t \partial_z^\alpha K_{jk}(z,t)| \le \mathfrak{C}(S)\cdot\Big((|z|^2+t)^{-2-|\alpha|/2-l/2}\cdot t + (|z|^2+t)^{-2-|\alpha|/2-l}\cdot t + \delta_{l1}\cdot(|z|^2+t)^{-2-|\alpha|/2}\Big).$$

Inequality (4.1) follows from the preceding estimate. □

Lemma 4.5. *Let $X \in C^\infty((\mathbb{R}^3\backslash\{0\})\times\mathbb{R})$. Then, for $z \in \mathbb{R}^3\backslash\{0\}$, $s \in (0,\infty)$, $l \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^3$, the function $r \mapsto (s-r)^{-1/2}\cdot|\partial_r^l\partial_z^\alpha X(z,r)|$ with $r \in (0,s)$ is integrable. Thus we may define $\mathfrak{X}(z,s) := \int_0^s (s-r)^{-1/2}\cdot X(z,r)\,dr$ for z,s as before. Then $\mathfrak{X} \in C^\infty((\mathbb{R}^3\backslash\{0\})\times(0,\infty))$, and*

$$\partial_z^\alpha \mathfrak{X}(z,s) = \int_0^s (s-r)^{-1/2}\cdot\partial_z^\alpha X(z,r)\,dr \tag{4.2}$$

for $z \in \mathbb{R}^3\backslash\{0\}$, $s \in (0,\infty)$, $\alpha \in \mathbb{N}_0^3$. Moreover, for α, z, s as in (4.2), and for $l \in \{1,\,2\}$,

$$\begin{aligned}\partial_s^l \partial_z^\alpha \mathfrak{X}(z,s) &= \sqrt{2}\cdot(-1)^{l-1}\cdot s^{-1/2-l+1}\cdot\partial_z^\alpha X(z,s/2) \\ &\quad + \delta_{l2}\cdot\sqrt{2}\cdot s^{-1/2}\cdot\partial_4\partial_z^\alpha X(z,s/2) + \int_{s/2}^s (s-r)^{-1/2}\cdot\partial_r^l\partial_z^\alpha X(z,r)\,dr \\ &\quad - (-3/2)^{l-1}\cdot\int_0^{s/2}(s-r)^{-3/2-l+1}\cdot\partial_z^\alpha X(z,r)\,dr/2. \end{aligned} \tag{4.3}$$

A proof of Lemma 4.5 may be found in the Appendix. We note that by Lemmas 4.2 and 4.3, the functions $X = \mathfrak{H}$, $X = \Gamma_{jk}$ and $X = \Lambda_{jk}(\,\cdot\,,\,\cdot\,,\tau)$ satisfy the assumptions of Lemma 4.5. Therefore we may put

$$V_{jk}(z,s) := \int_0^s (s-r)^{-1/2}\cdot\Gamma_{jk}(z,r)\,dr$$

for $z \in \mathbb{R}^3\backslash\{0\}$, $s \in (0,\infty)$, $j,k \in \{1,2,3\}$, and the kernels $(V^{(\tau)}_{jk})_{1\le j,k\le 3}$ and L introduced in (1.26) and (1.28), respectively, are well defined. In the following lemma, we collect all the estimates of L, V and $V^{(\tau)}$ we will need later on.

Lemma 4.6. *Let $j,k \in \{1,2,3\}$. Then $L,\ V_{jk},\ V^{(\tau)}_{jk} \in C^\infty((\mathbb{R}^3\backslash\{0\})\times(0,\infty))$. Let $X \in \{L, V_{jk}\}$, $M \in (0,\infty)$, $\epsilon \in (0,1]$, $z \in \mathbb{R}^3\backslash\{0\}$, $s \in (0,\infty)$, $x \in B_M\backslash\{0\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 2$. Then, using the function γ_M from Lemma 4.3, we get*

$$|\partial_s^l \partial_z^\alpha X(z,s)| \le \mathfrak{C}(\epsilon)\cdot|z|^{-1-|\alpha|-2\cdot\epsilon}\cdot s^{-1/2-l+\epsilon} \quad \text{for } l\in\{0,1\}, \tag{4.4}$$

$$|\partial_z^\alpha V^{(\tau)}_{jk}(z,s)| \le \mathfrak{C}(M,\epsilon)\cdot\gamma_M(z)^{-1/2-|\alpha|/2-\epsilon}\cdot s^{-1/2+\epsilon}, \tag{4.5}$$

$$\begin{aligned}|\partial_s\partial_z^\alpha V^{(\tau)}_{jk}(z,s)| &\le \mathfrak{C}(M,\epsilon)\cdot\Big(\gamma_M(z)^{-1/2-|\alpha|/2-\epsilon}\cdot s^{-3/2+\epsilon}\\ &\quad+\min\Big\{\gamma_M(z)^{-1/2-|\alpha|/2-\epsilon}\cdot s^{-1+\epsilon},\ \gamma_M(z)^{-|\alpha|/2-\epsilon}\cdot s^{-3/2+\epsilon}\Big\}\Big),\end{aligned} \tag{4.6}$$

$$|\partial_s^l\partial_z^\alpha L(z,s)|\le\mathfrak{C}\cdot\left(s^{-1-l-|\alpha|/2}\cdot e^{-|z|^2/(8\cdot s)}+s^{-1/2-l}\cdot\int_0^{s/2} r^{-3/2-|\alpha|/2}\cdot e^{-|z|^2/(8\cdot r)}\,dr\right)\quad \text{for } l\in\{1,2\}, \tag{4.7}$$

$$|\partial_s\partial_x^\alpha(V^{(\tau)}_{jk}-V_{jk})(x,s)| \le \mathfrak{C}(M,\epsilon)\cdot(s^{-3/2+\epsilon}+s^{-1-|\alpha|/2+\epsilon})\cdot|x|^{-|\alpha|-2\cdot\epsilon}. \tag{4.8}$$

Proof. The first claim of the lemma follows from Lemmas 4.2, 4.3 and 4.5. Moreover, Lemma 4.3, 3.1 and Eq. (4.2) imply

$$\begin{aligned}|\partial_z^\alpha V^{(\tau)}_{jk}(z,s)| &\le \mathfrak{C}(M)\cdot\int_0^s (s-r)^{-1/2}\cdot(\gamma_M(z)+r)^{-3/2-|\alpha|/2}\,dr\\ &\le \mathfrak{C}(M)\cdot\gamma_M(z)^{-1/2-|\alpha|/2-\epsilon}\cdot\int_0^s (s-r)^{-1/2}\cdot r^{-1+\epsilon}\,dr\\ &\le \mathfrak{C}(M,\epsilon)\cdot\gamma_M(z)^{-1/2-|\alpha|/2-\epsilon}\cdot s^{-1/2+\epsilon},\end{aligned}$$

so (4.5) is proved. Lemma 4.3 and Eq. (4.3) yield

$$|\partial_s\partial_z^\alpha V^{(\tau)}_{jk}(z,s)| \le \mathfrak{C}(M)\cdot(s^{-1/2}\cdot(\gamma_M(z)+s)^{-3/2-|\alpha|/2}+\mathfrak{A}_1+\mathfrak{A}_2), \tag{4.9}$$

with

$$\mathfrak{A}_1 := \int_{s/2}^s (s-r)^{-1/2}\cdot((\gamma_M(z)+r)^{-5/2-|\alpha|/2}+(\gamma_M(z)+r)^{-2-|\alpha|/2})\,dr,$$

$$\mathfrak{A}_2 := \int_0^{s/2} (s-r)^{-3/2}\cdot(\gamma_M(z)+r)^{-3/2-|\alpha|/2}\,dr.$$

In the first term on the right-hand side of (4.9), we may estimate $(\gamma_M(z)+s)^{-3/2-|\alpha|/2}$ by $s^{-1+\epsilon}\cdot\gamma_M(z)^{-1/2-\epsilon-|\alpha|/2}$. Concerning $\mathfrak{A}_1$, we observe that $(\gamma_M(z)+r)^{-1}$ is bounded by $2\cdot(\gamma_M(z)+s)^{-1}$ for $r\in(s/2,s)$. Since $\int_{s/2}^s (s-r)^{-1/2}\,dr \le (2\cdot s)^{1/2}$, we thus see that

$$\mathfrak{A}_1 \le (\gamma_M(z)+s)^{-2-|\alpha|/2}+(\gamma_M(z)+s)^{-3/2-|\alpha|/2},$$

where $(\gamma_M(z)+s)^{-2-|\alpha|/2} \le s^{-3/2+\epsilon}\cdot\gamma_M(z)^{-1/2-\epsilon-|\alpha|/2}$. As for $\mathfrak{A}_2$, we use that $(s-r)^{-3/2}\le(s/2)^{-3/2}$ for $r\in(0,s/2)$, and that $(\gamma_M(z)+r)^{-3/2-|\alpha|/2}$ is bounded by $\gamma_M(z)^{-1/2-\epsilon-|\alpha|/2}\cdot r^{-1+\epsilon}$. After integrating with respect to r, we get $\mathfrak{A}_2\le\mathfrak{C}(M,\epsilon)\cdot s^{-3/2+\epsilon}\cdot\gamma_M(z)^{-1/2-\epsilon-|\alpha|/2}$. On combining the preceding estimates, we deduce from (4.9) that

$$|\partial_s \partial_z^\alpha V_{jk}^{(\tau)}(z,s)| \le \mathfrak{C}(M,\epsilon) \cdot (s^{-3/2+\epsilon} \cdot \gamma_M(z)^{-1/2-\epsilon-|\alpha|/2} + (\gamma_M(z)+s)^{-3/2-|\alpha|/2}). \tag{4.10}$$

But we observe that firstly, $(\gamma_M(z)+s)^{-3/2-|\alpha|/2} \le \gamma_M(z)^{-1/2-|\alpha|/2-\epsilon} \cdot s^{-1+\epsilon}$ and secondly, $(\gamma_M(z)+s)^{-3/2-|\alpha|/2} \le \gamma_M(z)^{-|\alpha|/2-\epsilon} \cdot s^{-3/2+\epsilon}$, so inequality (4.6) follows from (4.10). The same type of estimate, but based on Lemma 4.2 instead of 4.3, yields (4.4). Concerning (4.7), we use (4.3) with $l = 2$ and Lemma 4.2 to obtain that $|\partial_s^2 \partial_z^\alpha L(z,s)|$ is bounded by

$$\mathfrak{C} \cdot \Bigg(s^{-3-|\alpha|/2} \cdot e^{-|z|^2/(4\cdot s)} + \int_{s/2}^{s} (s-r)^{-1/2} \cdot r^{-7/2-|\alpha|/2} \cdot e^{-|z|^2/(8\cdot r)}\, dr + s^{-5/2} \cdot \int_0^{s/2} r^{-3/2-|\alpha|/2} \cdot e^{-|z|^2/(8\cdot r)}\, dr \Bigg).$$

But the first of the preceding integrals may be estimated by $\mathfrak{C} \cdot s^{-7/2-|\alpha|/2} \cdot e^{-|z|^2/(8\cdot s)} \cdot \int_{s/2}^{s} (s-r)^{-1/2}\, dr$, and hence by $\mathfrak{C} \cdot s^{-3-|\alpha|/2} \cdot e^{-|z|^2/(8\cdot s)}$. Thus inequality (4.7) with $l = 2$ follows. The case $l = 1$ may be dealt with in the same way. As for the proof of (4.8), we apply (4.1) and (4.3), proceeding in the same way as in the proof of (4.6). Recalling that $x \in B_M \backslash \{0\}$, we obtain

$$|\partial_s \partial_x^\alpha (V_{jk}^{(\tau)} - V_{jk})(x,s)| \le \mathfrak{C}(M) \cdot (s^{-1/2} \cdot (|x|^2+s)^{-1-|\alpha|/2} + \widetilde{\mathfrak{A}_1} + \widetilde{\mathfrak{A}_2}),$$

where the definition of $\widetilde{\mathfrak{A}_j}$ is analogous to that of $\mathfrak{A}_j$ above ($j \in \{1, 2\}$), with $|x|^2$ in the place of $\gamma_M(z)$, and with the exponent $-\varrho - |\alpha|/2$ for $\varrho \in \{5/2, 2, 3/2\}$ replaced by $-\varrho - 1/2 - |\alpha|/2$. Concerning $\widetilde{\mathfrak{A}_1}$, we estimate $(|x|^2+r)^{-1}$ by $2 \cdot (|x|^2+s)^{-1}$, for $r \in (s/2, s)$, and integrate the remaining term $(s-r)^{-1/2}$ with respect to r. In this way we arrive at the estimate $\widetilde{\mathfrak{A}_1} \le \mathfrak{C} \cdot ((|x|^2+s)^{-1-|\alpha|/2} + (|x|^2+s)^{-3/2-|\alpha|/2})$. As for the term $\widetilde{\mathfrak{A}_2}$, we observe that the integrand in its definition is bounded by $\mathfrak{C} \cdot s^{-3/2} \cdot |x|^{-|\alpha|-2\cdot\epsilon} \cdot r^{-1+\epsilon}$. Integrating with respect to $r \in (0, s/2)$, we obtain $\widetilde{\mathfrak{A}_2} \le \mathfrak{C} \cdot s^{-3/2+\epsilon} \cdot |x|^{-|\alpha|-2\cdot\epsilon}$. In view of the assumption $|x| \le M$, inequality (4.8) follows. □

In the proof of the ensuing lemma, we exploit the fact that the inner integral on the left-hand side of the estimate in Lemma 4.7 extends over $\mathbb{R}^3$. Due to this observation, we are able to pass from the Oseen to the Stokes fundamental solution, and thus obtain an upper bound that is sharper than any we could have obtained by applying Lemma 4.6.

Lemma 4.7. *Let $p \in (1,\infty)$, $g \in L^p(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$, $j,k \in \{1, 2, 3\}$, $t \in [1,\infty)$. Then*

$$\Bigg(\int_\Omega \Bigg(\int_{\mathbb{R}^3} |\partial_t V_{jk}^{(\tau)}(x-y,t) \cdot g(y)|\, dy \Bigg)^{3/2} dx \Bigg)^{2/3} \le \mathfrak{C}(p) \cdot (t^{-3/(2\cdot p)} + t^{-5/4}) \cdot (\|g\|_p + \|g\|_{3/2}).$$

Proof. Let $x \in \mathbb{R}^3$. Using (4.3) with $X = \Lambda_{jk}(\,\cdot\,,\ \cdot\,,\tau)|(\mathbb{R}^3\backslash\{0\}) \times (0,\infty)$, $l = 1$, $\alpha = 0$, we find

$$\int_{\mathbb{R}^3} |\partial_t V_{jk}^{(\tau)}(x-y,t) \cdot g(y)|\, dy \le \mathfrak{C} \cdot \sum_{m=1}^{4} \mathfrak{A}_m(x),$$

with $\mathfrak{A}_1(x) := t^{-1/2} \cdot \int_{\mathbb{R}^3} |\Lambda_{jk}(x-y, t/2, \tau) \cdot g(y)|\, dy$,

$$\mathfrak{A}_2(x) := \int_{t/2}^{t} (t-s)^{-1/2} \cdot \int_{\mathbb{R}^3} |\partial_s \Lambda_{jk}(x-y, s, \tau) \cdot g(y)|\, dy\, ds,$$

$$\mathfrak{A}_3(x) := \int_{0}^{t/2} (t-s)^{-3/2} \cdot \int_{B_1(x)^c} |\Lambda_{jk}(x-y, s, \tau) \cdot g(y)|\, dy\, ds.$$

The term $\mathfrak{A}_4(x)$ is defined in the same way as $\mathfrak{A}_3(x)$, but with the domain of integration $B_1(x)^c$ replaced by $B_1(x)$. For $l \in \{0,1\},\ s \in (0,\infty),$ we find with Hölder's inequality that

$$\int_{\mathbb{R}^3} |\partial_s^l \Lambda_{jk}(x-y, s, \tau) \cdot g(y)|\, dy \le \|g\|_p \cdot \Big(\int_{\mathbb{R}^3} |\partial_s^l \Lambda_{jk}(x-y, s, \tau)|^{p'}\, dy \Big)^{1/p'}$$
$$\le \mathfrak{C} \cdot \|g\|_p \cdot \sum_{m \in \{1,4\}} \Big(\int_{\mathbb{R}^3} |\partial_m^l \Gamma_{jk}(x-y-\tau \cdot s \cdot e_1,\, s)|^{p'}\, dy \Big)^{1/p'}.$$

But with Lemma 4.2,

$$\Big(\int_{\mathbb{R}^3} |\partial_1^l \Gamma_{jk}(x-y-\tau \cdot s \cdot e_1,\, s)|^{p'}\, dy \Big)^{1/p'}$$
$$= \Big(\int_{\mathbb{R}^3} |\partial_1^l \Gamma_{jk}(z,\, s)|^{p'}\, dz \Big)^{1/p'} \le \mathfrak{C} \cdot \Big(\int_{\mathbb{R}^3} (|z| + \sqrt{s})^{(-3-l) \cdot p'}\, dz \Big)^{1/p'}$$
$$\le \mathfrak{C}(p) \cdot s^{(-3-l)/2 + 3/(2 \cdot p')} \le \mathfrak{C}(p) \cdot s^{-3/(2 \cdot p) - l/2}. \tag{4.11}$$

If in the left-hand side of (4.11), the derivative ∂_1 is replaced by ∂_4, a similar computation yields the upper bound $\mathfrak{C}(p) \cdot s^{-3/(2 \cdot p) - l}$. Thus we may conclude that the left-hand side of the estimate preceding (4.11) may be estimated by $\mathfrak{C}(p) \cdot \|g\|_p \cdot (s^{-3/(2 \cdot p) - l/2} + s^{-3/(2 \cdot p) - l})$. It follows that

$$\|\mathfrak{A}_1 | \Omega\|_{3/2} \le \mathfrak{C} \cdot \|\mathfrak{A}_1\|_\infty \le \mathfrak{C}(p) \cdot t^{-3/(2 \cdot p) - 1/2} \cdot \|g\|_p \le \mathfrak{C}(p) \cdot t^{-3/(2 \cdot p)} \cdot \|g\|_p,$$

and

$$\|\mathfrak{A}_2 | \Omega\|_{3/2} \le \mathfrak{C} \cdot \|\mathfrak{A}_2\|_\infty$$
$$\le \mathfrak{C}(p) \cdot \|g\|_p \cdot \int_{t/2}^{t} (t-s)^{-1/2} \cdot (s^{-3/(2 \cdot p) - 1/2} + s^{-3/(2 \cdot p) - 1})\, ds$$
$$\le \mathfrak{C}(p) \cdot \|g\|_p \cdot (t^{-3/(2 \cdot p) - 1/2} + t^{-3/(2 \cdot p) - 1}) \cdot \int_{t/2}^{t} (t-s)^{-1/2}\, ds$$
$$\le \mathfrak{C}(p) \cdot \|g\|_p \cdot t^{-3/(2 \cdot p)},$$

where we used the assumption $t \ge 1$. We further find with Lemma 4.3 with $K = 1$ that for $x \in \mathbb{R}^3$,

$$\begin{aligned}\mathfrak{A}_3(x) &\le \mathfrak{C} \cdot t^{-3/2} \cdot \int_0^{t/2} \int_{B_1(x)^c} (|x-y| \cdot \nu(x-y) + s)^{-3/2} \cdot |g(y)| \, dy \, ds \\ &\le \mathfrak{C} \cdot t^{-3/2} \cdot \left(\int_0^{t/2} s^{-3/4} \, ds \right) \cdot \int_{B_1(x)^c} (|x-y| \cdot \nu(x-y))^{-3/4} \cdot |g(y)| \, dy \\ &\le \mathfrak{C} \cdot t^{-5/4} \cdot \left(\int_{B_1(x)^c} (|x-y| \cdot \nu(x-y))^{-9/4} \, dy \right)^{1/3} \cdot \|g\|_{3/2},\end{aligned}$$

hence $\mathfrak{A}_3(x) \le \mathfrak{C} \cdot t^{-5/4} \cdot \|g\|_{3/2}$ according to Lemma 3.3. Thus $\|\mathfrak{A}_3|\Omega\|_{3/2} \le \mathfrak{C} \cdot \|\mathfrak{A}_3\|_\infty \le \mathfrak{C} \cdot t^{-5/4} \cdot \|g\|_{3/2}$. Moreover, again with Lemma 4.3 with $K = 1$,

$$\mathfrak{A}_4(x) \le \mathfrak{C} \cdot t^{-3/2} \cdot \int_{B_1(x)} \int_0^{t/2} (|x-y|^2 + s)^{-3/2} \cdot |g(y)| \, ds \, dy.$$

for $x \in \mathbb{R}^3$. Hence by an integration with respect to s, and an application of Young's inequality (Theorem 3.2), we get

$$\|\mathfrak{A}_4\|_{3/2} \le \mathfrak{C} \cdot t^{-3/2} \cdot \left(\int_{\mathbb{R}^3} \chi_{(0,1)}(|z|) \cdot |z|^{-1} \, dz \right) \cdot \|g\|_{3/2} \le \mathfrak{C} \cdot t^{-3/2} \|g\|_{3/2}.$$

Lemma 4.7 now follows from the preceding estimates. □

5. Volume Potentials: Definition and Some Properties

In this section, we introduce a number of volume potentials, and also a single-layer potential, associated with the evolutionary Oseen or Stokes system, or with the heat equation. We start by stating a result from [13].

Lemma 5.1. [13, Lemma 2.7] *Let $p, q \in [1, \infty]$, $r \in (1, \infty]$, $s \in [1, \infty)$ with $q < p$, $s \le r$.*

Then, for $f \in L^s(0, \infty, L^q(\mathbb{R}^3))$, $M \in (0, \infty)$, $j, k \in \{1, 2, 3\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$, the following estimate holds for $W = (0, M)$ if $1 - |\alpha|/2 + 3 \cdot (1/p - 1/q)/2 > 1/s - 1/r$, and for $W = (M, \infty)$ if $1 - |\alpha|/2 + 3 \cdot (1/p - 1/q)/2 < 1/s - 1/r$:

$$\left(\int_0^\infty \left[\int_{\mathbb{R}^3} \left(\int_0^\infty \int_{\mathbb{R}^3} \chi_W(t-\sigma) \cdot |\partial_x^\alpha \Lambda_{jk}(x-y, t-\sigma, \tau)| \cdot |f(y,\sigma)| \, dy \, d\sigma \right)^p dx \right]^{r/p} dt \right)^{1/r}$$
$$\le \mathfrak{C}(p,q,r,s) \cdot M^{3 \cdot (1/p - 1/q)/2 + 1 - |\alpha|/2 - 1/s + 1/r} \cdot \|f\|_{q,s;\infty}.$$

(Of course, if $r = \infty$ and/or $p = \infty$, the preceding inequality has to be modified in an obvious way.)

Lemma 5.2. *Let $s, q \in [1, \infty)$, $A \subset \mathbb{R}^3$ measurable, $\gamma \in (0, \infty]$, $f \in L^s(0, \gamma, L^q(A)^3)$. Then the integral in the definition of $\mathfrak{R}^{(\tau)}(f)(x,t)$ in (1.7) exists for a.e. $x \in \mathbb{R}^3$, $t \in (0, \infty)$, so the function $\mathfrak{R}^{(\tau)}(f)$ is well defined.*

Suppose that $s \in [1, 2]$, $q \in [1, 2)$ with $7/4 > 1/s + 3/(2 \cdot q)$. Let $T \in (0, \infty)$. Then $\mathfrak{R}^{(\tau)}(f)|\mathbb{R}^3 \times (0,T) \in L^2(0, T, H^1(\mathbb{R}^3)^3)$. Suppose in addition that $\widetilde{f}|\overline{\Omega}^c \times (0,T) \in L^2(0,T,V')$, where $\widetilde{f}$ denotes the zero extension of f to $\mathbb{R}^3 \times (0, \infty)$. Then $\mathfrak{R}^{(\tau)}(f)|\overline{\Omega}^c \times (0,T) \in H^1(0,T,V')$.

Proof. Lemma 5.2 follows from Lemma 5.1; see the remark in [13, p. 898 below] and [13, Lemma 2.11, Corollary 2.17]. □

We note that in view of Lemmas 5.2 and 2.2, the boundary value of $\mathfrak{R}^{(\tau)}(f)$ on S_∞ is well defined if the assumptions in the second part of Lemma 5.2 hold.

Lemma 5.3. [13, Lemma 2.14, Corollary 2.18] *Let $p \in [1,\infty)$, $A \subset \mathbb{R}^3$ measurable, $a \in L^p(A)^3$. Then the integral in the definition of $\mathfrak{I}^{(\tau)}(a)(x,t)$ in (1.8) exists for $x \in \mathbb{R}^3$, $t > 0$, so the function $\mathfrak{I}^{(\tau)}(a)$ introduced by (1.8) is well defined. Moreover, $\mathfrak{I}^{(\tau)}(a) \in C^\infty(\mathbb{R}^3 \times (0,\infty))^3$.*

If $\epsilon \in (0,1]$, $a \in H^\epsilon(\overline{\Omega}^c)^3$ and $T \in (0,\infty)$, we have $\mathfrak{I}^{(\tau)}(a)|\mathbb{R}^3 \times (0,T) \in L^2(0,T,H^1(\mathbb{R}^3)^3)$ and $\mathfrak{I}^{(\tau)}(a)|\overline{\Omega}^c \times (0,T) \in H^1(0,T,V')$.

Theorem 5.1. [11, Corollary 2.1], [13, Lemma 2.19] *Let $T \in (0,\infty]$ and $\Phi \in L^2(S_T)^3$. Then the integral appearing in the definition of $\mathfrak{V}^{(\tau)}(\Phi)(x,t)$ in (1.9) exists for $(x,t) \in (\mathbb{R}^3\backslash\partial\Omega) \times (0,\infty)$ and for a.e. $(x,t) \in \partial\Omega \times (0,\infty)$. Thus the function $\mathfrak{V}^{(\tau)}(\Phi)$ introduced in (1.9) is well defined. Moreover $\mathfrak{V}^{(\tau)}(\Phi)|\overline{\Omega}^c \times (0,T') \in L^2(0,T',H^1(\overline{\Omega}^c)^3) \cap H^1(0,T',V')$ for $T' \in (0,\infty)$.*

Note that there is a mistake in [11, Corollary 2.1]: the condition $T < \infty$ is missing. In fact, the function $\mathfrak{V}^{(\tau)}(\Phi)|\overline{\Omega}^c \times (0,\infty)$ is in $L^\infty(0,\infty,L^2(\overline{\Omega}^c)^3)$, the gradient $\nabla_x(\mathfrak{V}^{(\tau)}(\Phi)|\overline{\Omega}^c \times (0,\infty))$ belongs to $L^2(0,\infty,L^2(\overline{\Omega}^c)^9)$, and the time derivative $\partial_t(\mathfrak{V}^{(\tau)}(\Phi)|\overline{\Omega}^c \times (0,\infty))$ to $L^2(0,\infty,V')$ [11, Theorem 2.3]. But of course, this does not imply that the last claim in Theorem 5.1 holds for $T' = \infty$.

The uniqueness result in Theorem 3.7 and the regularity results in Lemmas 5.2, 5.3 and Theorem 5.1 yield the representation formula (1.17), provided the integral equation in (1.16) can be solved:

Theorem 5.2. *Let $T \in (0,\infty]$, $\epsilon \in (0,1]$, $s \in [1,2]$, $\widetilde{q} \in [1,2)$ with $7/4 > 1/s + 3/(2\cdot\widetilde{q})$. Moreover, let b a function from S_T into $\mathbb{R}^3$, $a \in H^\epsilon_\sigma(\overline{\Omega}^c)$, $f : (0,T) \mapsto L^{\widetilde{q}}(\overline{\Omega}^c)^3$ such that $f|\overline{\Omega}^c \times (0,T')$ belongs to $L^s(0,T',L^{\widetilde{q}}(\overline{\Omega}^c)^3) \cap L^2(0,T',V')$ for $T' \in (0,T)$. Further suppose there is $\Phi \in L^2_n(S_T)$ such that (1.16) holds.*

Then there is unique function $u : (0,T) \mapsto H^1(\overline{\Omega})^3$ with (1.13)–(1.15). This function is given by (1.17).

Proof. Put $u := (\mathfrak{V}^{(\tau)}(\Phi) + \mathfrak{R}^{(\tau)}(f) + \mathfrak{I}^{(\tau)}(a))\,|\,\overline{\Omega}^c \times (0,\infty)$. As explained in the proof of [13, Corollary 5.1], the relations in (1.13)–(1.15) hold for u. Uniqueness of this function follows from Theorem 3.7. □

Note that in [13, Corollary 5.1], existence of a solution to (1.16) was a claim, and not an assumption as in Theorem 5.2. This explains why the latter theorem holds under conditions on d, a and f that are weaker than those in [13, Corollary 5.1]. We further remark that a typing error occurred in the list of assumptions in [13, Corollary 5.1]: the parameter q at the beginning of that reference need not coincide with the quantity q in [13, (5.3)]. That latter q corresponds to the parameter $\widetilde{q}$ in Theorem 5.2. As for solving (1.16), we will need the next two theorems.

Theorem 5.3. [12, Corollary 3] *Let $T \in (0,\infty]$, $d \in H_T$. Then there is a unique function $\Phi \in L^2_n(S_T)$ such that $\mathfrak{V}^{(\tau)}(\Phi)|S_T = d$.*

Theorem 5.4. [13, Theorem 3.1] *Let $\kappa \in (0,1/2]$, $c \in H^{1/2+\kappa}_\sigma(\overline{\Omega}^c)$, $T \in (0,\infty]$. Then $\mathfrak{I}^{(\tau)}(c)|S_T \in H_T$.*

Next we introduce volume potentials related to the kernels L, V and $V^{(\tau)}$.

Lemma 5.4. *Let $p \in (1,\infty)$, $T \in (0,\infty)$, $g \in L^1(0,T,L^p(\mathbb{R}^3)^3)$, $t \in (0,T]$. Then*

$$\left(\int_{\mathbb{R}^3} \left(\int_0^t (t-\sigma)^{-1/2} \cdot \int_0^\sigma \int_{\mathbb{R}^3} \mathfrak{H}(x-y,\sigma-r) \cdot |g(y,r)|\,dy\,dr\,d\sigma \right)^p dx \right)^{1/p} \le \mathfrak{C} \cdot t^{1/2} \cdot \|g\|_{p,1;T}.$$

Proof. Let $\mathfrak{A}(t)$ denote the left-hand side of the preceding inequality. By Minkowski's inequality (Theorem 3.1), $\mathfrak{A}(t)$ is bounded by

$$\int_0^t (t-\sigma)^{-1/2} \cdot \int_0^\sigma \Bigg(\int_{\mathbb{R}^3} \Bigg(\int_{\mathbb{R}^3} \mathfrak{H}(x-y,\sigma-r)\cdot|g(y,r)|\,dy \Bigg)^p dx \Bigg)^{1/p} dr\,d\sigma.$$

Applying Young's inequality (Theorem 3.2) to the two innermost integrals in the preceding term, and using Lemma 4.1, we conclude that

$$\mathfrak{A}(t) \le \int_0^t (t-\sigma)^{-1/2} \cdot \int_0^\sigma \|g(\,\cdot\,,r)\|_p\,dr\,d\sigma \le \mathfrak{C}\cdot t^{1/2}\cdot\|g\|_{p,1;T}.$$

□

Lemma 5.4 means that the definition of the function $\mathfrak{M}(g):\mathbb{R}^3\times(0,\infty)\mapsto\mathbb{R}^3$ in (1.27) makes sense if $g\in L^s(0,\infty,\,L^p(\mathbb{R}^3)^3)$ for some $s,p\in(1,\infty)$. By proceeding as in the preceding proof, but applying Hölder's inequality and Lemma 3.1 at the end, we get

Corollary 5.1. *Let $s,p\in(1,\infty),\ f\in L^s(0,\infty,\,L^p(\mathbb{R}^3)^3),\ T\in(0,\infty)$. Then the inequality $\|\mathfrak{M}(f)|\mathbb{R}^3\times(0,T)\|_{p,s;T}\le\mathfrak{C}\cdot T^{3/2-1/s}\cdot\|f\|_{p,s;\infty}$ holds.*

Proceeding as in the proof of Lemma 5.4 and Corollary 5.1, we obtain

Lemma 5.5. *Let $p\in(1,\infty),\ T\in(0,\infty),\ g\in L^1(0,T,L^p(\mathbb{R}^3)^3),\ t\in(0,T]$. Then*

$$\Bigg(\int_{\mathbb{R}^3} \Bigg(\int_0^t \int_{\mathbb{R}^3} \mathfrak{H}(x-y,t-\sigma)\cdot|g(y,\sigma)|\,dy\,d\sigma \Bigg)^p dx \Bigg)^{1/p} \le \mathfrak{C}\cdot\|g\|_{p,1;T}.$$

Thus, for $s,p\in(1,\infty),\ f\in L^s(0,\infty,\,L^p(\mathbb{R}^3)^3)$, we may define the mapping $\mathfrak{K}(f):\mathbb{R}^3\times(0,\infty)\mapsto\mathbb{R}^3$ by

$$\mathfrak{K}(f)(x,t) := \int_0^t \int_{\mathbb{R}^3} \mathfrak{H}(x-y,t-\sigma)\cdot f(y,\sigma)\,dy\,d\sigma\ (x\in\mathbb{R}^3,\ t\in(0,\infty)),$$

and we have $\|\mathfrak{K}(f)\|_{p,s;T}\le\mathfrak{C}(T,s)\cdot\|f\|_{p,s;\infty}$ for $T\in(0,\infty)$.

A proof of the next lemma is given in the Appendix.

Lemma 5.6. *Let $U\in C^0((\mathbb{R}^3\backslash\{0\})\times(0,\infty))$ with $\int_0^T\int_{B_S}|U(z,r)|\,dz\,dr<\infty$ for $S,T\in(0,\infty)$. Let $f\in C_0^\infty(\mathbb{R}^3\times(0,\infty))$. Then, for $\alpha\in\mathbb{N}_0^3,\ l\in\mathbb{N}_0,\ x\in\mathbb{R}^3,\ t\in[0,\infty)$,*

$$\int_0^t \int_{\mathbb{R}^3} |U(x-y,t-s)\cdot\partial_s^l\partial_y^\alpha f(y,s)|\,dy\,ds<\infty, \tag{5.1}$$

the function $F(x,t):=\int_0^t\int_{\mathbb{R}^3}U(x-y,t-s)\cdot f(y,s)\,dy\,ds\ (x\in\mathbb{R}^3,\ t\in[0,\infty))$ belongs to $C^\infty(\mathbb{R}^3\times[0,\infty))$, and for $\alpha,\,l,x,t$ as in (5.1),

$$\partial_t^l\partial_x^\alpha F(x,t) = \int_0^t \int_{\mathbb{R}^3} U(x-y,t-s)\cdot\partial_s^l\partial_y^\alpha f(y,s)\,dy\,ds. \tag{5.2}$$

In particular $\partial_t^l\partial_x^\alpha F(x,t)_{|t=0}=0$ for $\alpha,\,l,x$ as in (5.1).

We consider the volume potentials we will work with in the following. Some of them were already defined before.

Lemma 5.7. *For $j, k \in \{1, 2, 3\}$, $U \in \{\mathfrak{H}, \Gamma_{jk}, \Lambda_{jk}(\cdot , \cdot , \tau), L, V_{jk}, V^{(\tau)}_{jk}\}$, and for $g \in C^\infty_0(\mathbb{R}^3 \times (0,\infty))$, $x \in \mathbb{R}^3$, $t \in [0,\infty)$, the relation $\int_0^t \int_{\mathbb{R}^3} |U(x-y, t-\sigma) \cdot g(y,\sigma)| \, dy \, d\sigma < \infty$ holds.*

Let $f \in C^\infty_0(\mathbb{R}^3 \times (0,\infty))^3$. Recall the definition of $\mathfrak{D}^{(\tau)}(f)$ in (1.25). As already mentioned in Sect. 1, we define $\mathfrak{D}(f)$ in the same way as $\mathfrak{D}^{(\tau)}(f)$, but with the kernel $V^{(\tau)}$ replaced by V. Moreover, let $\mathfrak{R}(f)$ be defined as $\mathfrak{R}^{(\tau)}(f)$ [see (1.7)], but with $\Lambda(\cdot , \cdot , \tau)$ replaced by Γ. Recall the definition of $\mathfrak{M}(f)$ in (1.27), and of $\mathfrak{K}(f)$ in Lemma 5.5. Then the integrals appearing in these definitions exist for any $x \in \mathbb{R}^3$ and $t \in [0,\infty)$, the functions $\mathfrak{K}(f)$, $\mathfrak{R}(f)$, $\mathfrak{R}^{(\tau)}(f)$, $\mathfrak{M}(f)$, $\mathfrak{D}(f)$ and $\mathfrak{D}^{(\tau)}(f)$ belong to $C^\infty_0(\mathbb{R}^3 \times [0,\infty))^3$, and

$$\partial_4^l \partial^\alpha \mathfrak{K}(f) = \mathfrak{K}(\partial_4^l \partial^\alpha f) \ \ (l \in \mathbb{N}_0, \ \alpha \in \mathbb{N}_0^3), \quad \mathfrak{K}(f)(x,0) = 0 \ \ (x \in \mathbb{R}^3), \tag{5.3}$$

with analogous formulas being valid for the other preceding functions. Moreover, $div_x \mathfrak{R}^{(\tau)}(f)(x,t) = 0$ for $x \in \mathbb{R}^3$, $t \in [0,\infty)$.

Proof. For $S \in (0,\infty)$, all the kernels U listed at the beginning of the lemma satisfy the estimate $|U(z,t)| \le \mathfrak{C}(S) \cdot |z|^{-2} \cdot t^{-1/2}$ for $z \in B_S$, $t \in (0,\infty)$; see Lemmas 4.2, 4.3 and 4.6. Therefore all the statements of Lemma 5.7 except the equation in its last sentence follow from Lemma 5.6. As for that latter equation, it may be deduced from (5.3) and the relation $\sum_{k=1}^3 \partial_k \Lambda_{jk}(\cdot , \cdot , \tau) = 0$, via an integration by parts. □

In the ensuing lemma, we rewrite the time derivative of $\mathfrak{D}^{(\tau)}(f)$, $\mathfrak{D}(f)$ and $\mathfrak{M}(f)$.

Lemma 5.8. *Let $j, k \in \{1, 2, 3\}$, $U \in \{L, V^{(\tau)}_{jk}, V_{jk}\}$, $f \in C^\infty_0(\mathbb{R}^3 \times (0,\infty))$. Define $F(x,t) := \int_0^t \int_{\mathbb{R}^3} U(x-y, t-s) \cdot f(y,s) \, dy \, ds$ for $x \in \mathbb{R}^3$, $t \in [0,\infty)$.*

Then $\int_0^t \int_{\mathbb{R}^3} |\partial_t U(x-y, t-s) \cdot f(y,s)| \, dy \, ds < \infty$ for $x \in \mathbb{R}^3$, $t > 0$, and

$$\partial_t F(x,t) = \int_0^t \int_{\mathbb{R}^3} \partial_t U(x-y, t-s) \cdot f(y,s) \, dy \, ds. \tag{5.4}$$

Proof. Let x, t be given as in (5.4). By (5.3), the first claim in Lemma 5.7 with g replaced by $\partial_4 f$, Fubini's and Lebesgue's theorem, we obtain $\partial_t F(x,t) = \lim_{\delta \downarrow 0} \int_{\mathbb{R}^3} \int_0^{t-\delta} U(x-y, t-s) \cdot \partial_s f(y,s) \, ds \, dx$, so that $\partial_t F(x,t)$ equals

$$\lim_{\delta \downarrow 0} \int_{\mathbb{R}^3} \left(\int_0^{t-\delta} \partial_t U(x-y, t-s) \cdot f(y,s) \, ds + U(x-y, \delta) \cdot f(y, t-\delta) \right) dy.$$

There are numbers $K, S \in (0,\infty)$ with $|f(y,r)| \le K \cdot \chi_{B_S}(y)$ for $y \in \mathbb{R}^3$, $r \in (0,\infty)$. Moreover, by (4.4), (4.5) with $\epsilon = 3/4$, $M = S + |x|$, we get $|U(x-y, \delta)| \le \mathfrak{C}(S+|x|) \cdot |x-y|^{-5/2} \cdot \delta^{1/4}$ for $y \in B_S$, $\delta \in (0,1]$, so

$$\int_{\mathbb{R}^3} |U(x-y,\delta) \cdot f(y, t-\delta)| \, dy \le \mathfrak{C}(S+|x|) \cdot \delta^{1/4} \cdot K \cdot \int_{B_{S+|x|}(x)} |x-y|^{-5/2} \, dy \le \mathfrak{C}(S+|x|) \cdot \delta^{1/4} \cdot K.$$

On the other hand, by (4.4), (4.6) with $l = 1$, $\alpha = 0$, $\epsilon = 3/4$, we have $|\partial_t U(z,r)| \le \mathfrak{C}(R) \cdot |z|^{-5/2} \cdot r^{-3/4}$ for $R, r \in (0,\infty)$, $z \in B_R$. Hence by (5.1), the first claim in Lemma 5.8 holds, so Eq. (5.4) follows with Lebesgue's and Fubini's theorem. □

In a series of lemmas, we are going to show that $\mathfrak{D}(f) = \mathfrak{M}(\mathfrak{F}_p(f))$ for smooth f.

Lemma 5.9. *Let $s \in (0,\infty)$, $j \in \{1, 2, 3\}$. Then $\partial^\alpha \Gamma_{jk}(\cdot , s) \in L^q(\mathbb{R}^3)$ for $1 \le j \le 3$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$, $q \in (1,\infty]$.*

Let $p \in (1,3)$, $G \in W^{1,1}_{loc}(\mathbb{R}^3) \cap L^{3 \cdot p/(3-p)}(\mathbb{R}^3)$ with $\nabla G \in L^p(\mathbb{R}^3)^3$. Then, for $x \in \mathbb{R}^3$, the equation $\int_{\mathbb{R}^3} \sum_{k=1}^3 \Gamma_{jk}(x-y, s) \cdot \partial_k G(y) \, dy = 0$ holds.

Proof. According to Lemma 4.2, we have $|\partial^\alpha \Gamma_{jk}(z,s)| \le \mathfrak{C}\cdot(|z|+\sqrt{s})^{-3-|\alpha|}$ for $z \in \mathbb{R}^3,\ 1 \le k \le 3,\ \alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. This implies the first claim in Lemma 5.9. By [25, Theorem II.6.1, II.5.1], there is a sequence (φ_n) in $C_0^\infty(\mathbb{R}^3)$ such that $\|\nabla G - \nabla\varphi_n\|_p \to 0$. But by an integration by parts and the properties of the functions Γ_{jk} $(1 \le j,k \le 3)$, we get $\int_{\mathbb{R}^3} \sum_{k=1}^3 \Gamma_{jk}(x-y,s)\cdot \partial_k\varphi_n(y)\,dy = 0$ for $n \in \mathbb{N}$. Now the second part of the lemma follows from the first. □

Lemma 5.10. *Let $p \in (1,\infty),\ F \in L^p_\sigma,\ x \in \mathbb{R}^3,\ t \in (0,\infty),\ j \in \{1, 2, 3\}$. Then we have $\int_{\mathbb{R}^3}\int_t^\infty \sum_{k=1}^3 |\partial_j\partial_k \mathfrak{H}(x-y,s)\cdot F_k(y)|\,ds\,dy < \infty$, and*

$$\int_{\mathbb{R}^3}\int_t^\infty \sum_{k=1}^3 \partial_j\partial_k \mathfrak{H}(x-y,s)\cdot F_k(y)\,ds\,dy = 0.$$

Proof. Let $k \in \{1, 2, 3\},\ \psi \in L^p(\mathbb{R}^3)$. Then we obtain with Lemma 4.2 and Hölder's inequality that

$$\int_{\mathbb{R}^3}\int_t^\infty |\partial_j\partial_k\mathfrak{H}(x-y,s)\cdot\psi(y)|\,ds\,dy \le \int_{\mathbb{R}^3}(|x-y|^2+t)^{-3/2}\cdot|\psi(y)|\,dy \le \mathfrak{C}\cdot t^{-3/(2\cdot p)}\cdot\|\psi\|_p. \tag{5.5}$$

On the other hand, by the definition of L^p_σ, there is a sequence (φ_n) in $C_0^\infty(\mathbb{R}^3)^3$ with $\operatorname{div}\varphi_n = 0$ for $n \in \mathbb{N}$ and $\|F-\varphi_n\|_p \to 0$ $(n \to \infty)$. The first of these relations implies by an integration by parts that the equation at the end of Lemma 5.10 holds with $\varphi_{n,k}$ in the place of F_k, for $n \in \mathbb{N},\ 1 \le k \le 3$. Therefore this latter equation follows from (5.5) and the second property of the sequence (φ_n). □

Lemma 5.11. *Let $f \in C_0^\infty(\mathbb{R}^3\times(0,\infty))^3$. Then $\partial_4^{1/2}\mathfrak{R}^{(\tau)}(f) = \partial_4\mathfrak{D}^{(\tau)}(f)$ and $\partial_4^{1/2}\mathfrak{R}^{(\tau)}(f) \in C^\infty(\mathbb{R}^3\times[0,\infty))^3$.*

Let $p \in (1,3)$. Then $\mathfrak{D}(f) = \mathfrak{M}(\mathfrak{F}_p(f))$ and $\mathfrak{M}(\mathfrak{F}_p(f)) \in C^\infty(\mathbb{R}^3\times[0,\infty))^3$ [the function $\mathfrak{F}_p(f)$ was introduced in (3.1)].

Proof. Let $j \in \{1, 2, 3\},\ x \in \mathbb{R}^3,\ t \in (0,\infty)$. There is $S > 0$ with $supp(f) \subset B_S\times(0,\infty)$. Thus, by Lemma 4.3 with $K = S+|x|$,

$$|\Lambda_{jk}(x-y,s-\sigma)\cdot f_k(y,\sigma)| \le \|f\|_\infty\cdot\mathfrak{C}(S+|x|)\cdot(|x-y|^2+s-\sigma)^{-3/2}\cdot\chi_{B_{S+|x|}}(x-y)$$

for $s \in (0,t),\ \sigma \in (0,s),\ y \in \mathbb{R}^3,\ 1 \le k \le 3$. Since the right-hand side of the preceding inequality multiplied by $(t-s)^{-1/2}$ may be integrated with respect to such s,σ and y, we may apply Fubini's theorem to the triple integral appearing when the definition of $\mathfrak{R}_j^{(\tau)}(f)$ [see (1.7)] is inserted into the term $\int_0^t (t-s)^{-1/2}\cdot\mathfrak{R}_j^{(\tau)}(f)(x,s)\,ds$. With a suitable change of variables, we then obtain that the latter term equals $\mathfrak{D}_j^{(\tau)}(f)(x,t)$. But $\mathfrak{D}_j^{(\tau)}(f) \in C^\infty(\mathbb{R}^3\times[0,\infty))^3$ by Lemma 5.7, so it follows by a differentiation with respect to t that $\partial_4^{1/2}\mathfrak{R}^{(\tau)}(f) = \partial_4\mathfrak{D}^{(\tau)}(f)$ and $\partial_4^{1/2}\mathfrak{R}^{(\tau)}(f) \in C^\infty(\mathbb{R}^3\times[0,\infty))^3$.

Take j,x,t as before. By the preceding arguments, we have $\mathfrak{D}_j(f)(x,t) = \int_0^t (t-s)^{-1/2}\cdot\mathfrak{R}_j(f)(x,s)\,ds$. On the other hand, for $s \in (0,t),\ \sigma \in (0,s)$,

$$\begin{aligned}\int_{\mathbb{R}^3}\sum_{k=1}^3 \Gamma_{jk}(x-y,s-\sigma)\cdot f_k(y,\sigma)\,dy &= \int_{\mathbb{R}^3}\sum_{k=1}^3 \Gamma_{jk}(x-y,s-\sigma)\cdot P_p(f(\,\cdot\,,\sigma))_k(y)\,dy\\ &= \int_{\mathbb{R}^3}\mathfrak{H}(x-y,s-\sigma)\cdot\mathfrak{F}_p(f)_j(y,\sigma)\,dy,\end{aligned} \tag{5.6}$$

where the first equation holds by Theorem 3.4 and Lemma 5.9. The second is a consequence of (3.1), the definition of Γ_{jk} [see (1.5)] and Lemma 5.10. Since $\mathfrak{F}_p(f) = \mathfrak{F}_2(f)$ according to the last claim of Theorem 3.4, we may deduce from Lemma 4.2 and Hölder's inequality that

$$\int_{\mathbb{R}^3}\mathfrak{H}(x-y,s-\sigma)\cdot|\mathfrak{F}_p(f)_j(y,\sigma)|\,dy \le \mathfrak{C}\cdot(s-\sigma)^{-3/4}\cdot\|\mathfrak{F}_2(f)_j(\,\cdot\,,\sigma)\|_2$$

for $s \in (0,t),\ \sigma \in (0,s)$. Therefore, in view of Lemma 3.4, the triple integral $\int_0^t \int_0^s \int_{\mathbb{R}^3} (t-s)^{-1/2} \cdot \mathfrak{H}(x-y, s-\sigma) \cdot |\mathfrak{F}_p(f)_j(y,\sigma)|\ dy\ d\sigma\ ds$ is finite, so we may apply Fubini's theorem in the triple integral arising when we write out the definition of $\mathfrak{M}(\mathfrak{F}_p(f))$ and L. Combining this observation and the above equation for $\mathfrak{D}_j(f)(x,t)$ with (5.6), we obtain that $\mathfrak{M}_j(\mathfrak{F}_p(f))(x,t) = \mathfrak{D}_j(f)(x,t)$. By Lemma 5.7, we have $\mathfrak{D}(f) \in C^\infty(\mathbb{R}^3 \times [0,\infty))^3$, so $\mathfrak{M}(\mathfrak{F}_p(f))$ also belongs to the latter function space. □

The ensuing lemma is a consequence of Eq. (5.6) and Lemma 5.7.

Lemma 5.12. *Let $p \in (1,3),\ f \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3$. Then $\mathfrak{R}(f) = \mathfrak{K}(\mathfrak{F}_p(f))$ and $\mathfrak{K}(\mathfrak{F}_p(f)) \in C^\infty(\mathbb{R}^3 \times [0,\infty))^3$ (see Lemma 5.5 for the definition of the function $\mathfrak{K}(\mathfrak{F}_p(f))$).*

Lemma 5.13. *Let $f \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3$. Then $\mathfrak{R}^{(\tau)}(f)|S_\infty \in H_\infty$ and $\|\mathfrak{R}^{(\tau)}(f)|S_\infty\|_{H_\infty}$ equals the right-hand side of (1.10) with $\varphi = \mathfrak{R}^{(\tau)}(f)|S_\infty$.*

Proof. See Appendix. □

We end this section by the first step of our estimates of $\|\mathfrak{R}^{(\tau)}(f)|S_\infty\|_{H_\infty}$.

Theorem 5.5. *Let $f \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3$. Then*

$$\begin{aligned} \|\mathfrak{R}^{(\tau)}(f)|S_\infty\|_{H_\infty} &\le \mathfrak{C} \cdot (\|\partial_4 \mathfrak{R}^{(\tau)}(f)|\Omega \times (0,\infty)\|_{3/2,2;\infty} \\ &\quad + \|\mathfrak{R}^{(\tau)}(f)|\Omega \times (0,\infty)\|_{L^2(0,\infty,\, W^{2,3/2}(\Omega)^3)} \\ &\quad + \|\partial_4 \mathfrak{D}^{(\tau)}(f)|\Omega \times (0,\infty)\|_{L^2(0,\infty,\, W^{1,3/2}(\Omega)^3)}). \end{aligned} \tag{5.7}$$

Proof. Let $t \in (0,\infty)$. By a trace theorem (see [24, Theorem 6.4.1] for example), we see that $\|\mathfrak{R}^{(\tau)}(f)(\,\cdot\,,t)|\partial\Omega\|_{1,2} \le \mathfrak{C} \cdot \|\mathfrak{R}^{(\tau)}(f)(\,\cdot\,,t)|\Omega\|_{2,3/2}$, and $\|\partial_4^{1/2}\mathfrak{R}^{(\tau)}(f)(\,\cdot\,,t)|\partial\Omega\|_2 \le \mathfrak{C} \cdot \|\partial_4^{1/2}\mathfrak{R}^{(\tau)}(f)(\,\cdot\,,t)|\Omega\|_{1,3/2}$. Abbreviate $K := \partial_4 \mathfrak{R}^{(\tau)}(f)(\,\cdot\,,t)$. Let $v \in H^1(\partial\Omega)$. For $n \in \mathbb{N}$, choose $E^{(n)}(v)$ as in Theorem 3.3. Then

$$\begin{aligned} \Big| \int_{\partial\Omega} (n^{(\Omega)} \cdot K) \cdot v\ do_x \Big| &= \lim_{n\to\infty} \Big| \int_{\partial\Omega} (n^{(\Omega)} \cdot K) \cdot E^{(n)}(v)\ do_x \Big| \\ &= \lim_{n\to\infty} \Big| \int_{\Omega} (\operatorname{div} K) \cdot E^{(n)}(v) + K \cdot \nabla E^{(n)}(v))\ dx \Big|. \end{aligned}$$

Since $\operatorname{div} K = 0$ (see Lemma 5.7), it follows with Theorem 3.3 that the left-hand side of the preceding estimate is bounded by $\mathfrak{C} \cdot \|K\|_{3/2} \cdot \|v\|_{1,2}$. This shows that $\|n^{(\Omega)} \cdot K|\partial\Omega\|_{H^1(\partial\Omega)'} \le \mathfrak{C} \cdot \|K|\Omega\|_{3/2}$. The theorem follows from the preceding inequalities, Lemmas 5.13 and 5.11. □

6. Reduction to Potential Functions Involving the Heat Kernel

We will show that the right-hand side of (5.7) is bounded by a constant times a sum of firstly, $\|f\|_{3/2,2;\infty} + \|f\|_{q_0,2;\infty}$ with q_0 arbitrary but fixed in $[1,3/2)$, and secondly, the $\|\ \ \|_{3/2,2;\infty}$-norm of some convolutions involving the heat kernel.

Lemma 6.1. *Let $f \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3$ and $q_0 \in [1,3/2)$. Then*

$$\|\mathfrak{R}^{(\tau)}(f)|\Omega \times (0,\infty)\|_{L^2(0,\infty,\, W^{1,3/2}(\Omega)^3)} \le \mathfrak{C}(q_0) \cdot (\|f\|_{q_0,2;\infty} + \|f\|_{3/2,2;\infty}).$$

Proof. Let $j \in \{1,2,3\},\ \alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. By Lemma 5.7 and an integration by parts, we get for $x \in \mathbb{R}^3,\ t \in (0,\infty)$ that

$$|\partial_x^\alpha \mathfrak{R}_j^{(\tau)}(f)(x,t)| = |\mathfrak{R}_j^{(\tau)}(\partial^\alpha f)(x,t)| \le \mathfrak{A}_1(x,t) + \mathfrak{A}_2(x,t), \tag{6.1}$$

where $\mathfrak{A}_1(x,t)$ is an abbreviation for the term

$$\int_0^\infty \int_{\mathbb{R}^3} \chi_{[1,\infty)}(t-\sigma)\cdot \sum_{k=1}^3 |\partial_x^\alpha \Lambda_{jk}(x-y,t-\sigma,\,\tau)|\cdot|f_k(y,\sigma)|\,dy\,d\sigma,$$

and where $\mathfrak{A}_2(x,t)$ is defined in the same way as $\mathfrak{A}_1(x,t)$, except that the term $\chi_{[1,\infty)}(t-\sigma)$ is replaced by $\chi_{(0,1)}(t-\sigma)$. In order to estimate $\mathfrak{A}_1(x,t)$, we apply Lemma 5.1 with $r=s=2,\ p=\infty,\ q=q_0,\ M=1$. This is possible because $q_0<3/2$. It follows

$$\|\mathfrak{A}_1|\Omega\times(0,\infty)\|_{3/2,2;\infty} \le \mathfrak{C}\cdot\|\mathfrak{A}_1\|_{\infty,\,2;\infty} \le \mathfrak{C}(q_0)\cdot\|f\|_{q_0,2;\infty}. \tag{6.2}$$

As for $\mathfrak{A}_2$, we use Lemma 5.1 with $r=s=2,\ p=2,\ q=3/2$ and $M=1$, to obtain

$$\|\mathfrak{A}_2|\Omega\times(0,\infty)\|_{3/2,2;\infty} \le \mathfrak{C}\cdot\|\mathfrak{A}_2\|_2 \le \mathfrak{C}(q_0)\cdot\|f\|_{3/2,2;\infty}. \tag{6.3}$$

Lemma 6.1 follows from (6.1)–(6.3). □

Lemma 6.2. *Let $f\in C_0^\infty(\mathbb{R}^3\times(0,\infty))^3$ and $q_0\in[1,3/2)$. Then*

$$\|\partial_4\mathfrak{D}^{(\tau)}(f)|\Omega\times(0,\infty)\|_{3/2,2;\infty} \le \mathfrak{C}(q_0)\cdot(\|f\|_{q_0,2;\infty}+\|f\|_{3/2,2;\infty}).$$

Proof. Let $j\in\{1,\,2,\,3\}$. Then, by Lemma 5.8 and Minkowski's inequality (Theorem 3.1), the quantity $\|\partial_4\mathfrak{D}_j^{(\tau)}(f)|\Omega\times(0,\infty)\|_{3/2,2;\infty}$ may be estimated by

$$\left(\int_0^\infty\left(\int_0^\infty \chi_{(0,\infty)}(t-\sigma)\cdot\mathfrak{A}(t,\sigma)\,d\sigma\right)^2 dt\right)^{1/2},$$

with $\mathfrak{A}(t,\sigma):=(\int_\Omega(\int_{\mathbb{R}^3}\sum_{k=1}^3|\partial_t V_{jk}^{(\tau)}(x-y,t-\sigma)\cdot f_k(y,\sigma)|\,dy)^{3/2}\,dx)^{2/3}$. But for $t\in(0,\infty),\ \sigma\in(0,t)$ with $t-\sigma\ge1$, by Lemma 4.7 with $g=f(\,\cdot\,,\sigma)$, we get that $\mathfrak{A}(t,\sigma)$ is bounded by

$$\mathfrak{C}(q_0)\cdot((t-\sigma)^{-3/(2\cdot q_0)}+(t-\sigma)^{-5/4})\cdot(\|f(\,\cdot\,,\sigma)\|_{q_0}+\|f(\,\cdot\,,\sigma)\|_{3/2}).$$

Now consider $t,\sigma\in(0,\infty)$ with $0<t-\sigma\le1$. We observe that $\mathfrak{A}(t,\sigma)\le\mathfrak{A}_1(t,\sigma)+\mathfrak{A}_2(t,\sigma)$, with $\mathfrak{A}_\mu(t,\sigma)$ for $\mu\in\{1,\,2\}$ defined in the same way as $\mathfrak{A}(t,\sigma)$, except that the domain of integration $\mathbb{R}^3$ is to be replaced by $B_1(x)^c$ in the case $\mu=1$, and by $B_1(x)$ if $\mu=2$. By (4.6) with $\epsilon=3/4,\ M=1,\ \alpha=0$, we obtain in view of the assumption $t-\sigma\le1$ that

$$\mathfrak{A}_1(t,\sigma)\le\mathfrak{C}\cdot(t-\sigma)^{-3/4}\cdot\left(\int_\Omega\left(\int_{B_1(x)^c}(|x-y|\cdot\nu(x-y))^{-5/4}\cdot|f(y,\sigma)|\,dy\right)^{3/2}dx\right)^{2/3}. \tag{6.4}$$

By Hölder's inequality, the innermost integral of the preceding term may be evaluated by $(\int_{B_1(x)^c}|x-y|^{-15/4}\,dy)^{1/3}\cdot\|f(\,\cdot\,,\sigma)\|_{3/2}$, so we get $\mathfrak{A}_1(t,\sigma)\le\mathfrak{C}\cdot(t-\sigma)^{-3/4}\cdot\|f(\,\cdot\,,\sigma)\|_{3/2}$. Inequality (6.4) also holds for $\mathfrak{A}_2(t,\sigma)$, except that the quantity $(|x-y|\cdot\nu(x-y))^{-5/4}$ has to be replaced by $|x-y|^{-5/2}$. It follows with Young's inequality (Theorem 3.2) that

$$\mathfrak{A}_2(t,\sigma)\le\mathfrak{C}\cdot(t-\sigma)^{-3/4}\cdot\left(\int_{\mathbb{R}^3}\chi_{(0,1)}(|z|)\cdot|z|^{-5/2}\,dz\right)\cdot\|f(\,\cdot\,,\sigma)\|_{3/2},$$

so that $\mathfrak{A}_2(t,\sigma)\le\mathfrak{C}\cdot(t-\sigma)^{-3/4}\cdot\|f(\,\cdot\,,\sigma)\|_{3/2}$. By combining the preceding estimates, we see that $\|\partial_4\mathfrak{D}^{(\tau)}(f)|\Omega\times(0,\infty)\|_{3/2,2;\infty}$ is bounded by

$$\mathfrak{C}(q_0)\cdot\left(\int_0^\infty\left(\int_0^\infty\varrho(t-\sigma)\cdot\left(\|f(\,\cdot\,,\sigma)\|_{q_0}+\|f(\,\cdot\,,\sigma)\|_{3/2}\right)\,d\sigma\right)^2 dt\right)^{1/2},$$

with $\varrho(r) := \chi_{[1,\infty)}(r) \cdot (r^{-3/(2\cdot q_0)} + r^{-5/4}) + \chi_{(0,1)}(r) \cdot r^{-3/4}$ for $r \in \mathbb{R}$. Since $q_0 < 3/2$, we have $-3/(2 \cdot q_0) < -1$, so $\int_0^\infty \varrho(r)\, dr < \infty$. Thus Lemma 6.2 follows with Young's inequality. □

Lemma 6.3. *Let $m \in \{1, 2, 3\}$, $f \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3$. Then*

$$\|\partial_m \partial_4 (\mathfrak{D}^{(\tau)}(f) - \mathfrak{D}(f)) \,|\, \Omega \times (0,\infty)\|_{3/2,2;\infty} \le \mathfrak{C} \cdot \|f\|_{3/2,2;\infty}.$$

Proof. Let $j \in \{1, 2, 3\}$, and abbreviate $H_k := \partial_4 (V_{jk}^{(\tau)} - V_{jk})$ for $1 \le k \le 3$. Let $x \in \mathbb{R}^3$, $t > 0$. By Lemmas 5.7, 5.8 and Lebesgue's theorem, we see that $\partial_m \partial_4 (\mathfrak{D}_j^{(\tau)}(f) - \mathfrak{D}_j(f))(x,t)$ equals

$$\lim_{\delta \downarrow 0} \int_0^t \int_{\mathbb{R}^3 \setminus B_\delta(x)} \sum_{k=1}^3 H_k(x-y, t-\sigma) \cdot \partial_m f_k(y,\sigma)\, dy\, d\sigma.$$

But $H_k \in C^\infty((\mathbb{R}^3 \setminus \{0\}) \times (0,\infty))$ by Lemma 4.6, so the preceding integral may be transformed into

$$\lim_{\delta \downarrow 0} \Bigg(\int_0^t \int_{\mathbb{R}^3 \setminus B_\delta(x)} \sum_{k=1}^3 \partial_{x_m} H_k(x-y, t-\sigma) \cdot f_k(y,\sigma)\, dy\, d\sigma + \int_0^t \int_{\partial B_\delta(x)} \sum_{k=1}^3 H_k(x-y, t-\sigma) \cdot f_k(y,\sigma) \cdot (x-y)_m/\delta \, do_y\, d\sigma \Bigg). \tag{6.5}$$

On the other hand, referring to (4.8) with $\epsilon = 3/4$, $\alpha = 0$, $M = 1$, we see that for $\delta \in (0,1]$, the absolute value of the second of the two preceding integrals is bounded by $\mathfrak{C} \cdot \|f\|_\infty \cdot \int_0^t \int_{\partial B_\delta(x)} ((t-\sigma)^{-3/4} + (t-\sigma)^{-1/4}) \cdot \delta^{-3/2}\, do_y\, d\sigma$, hence by $\mathfrak{C} \cdot \|f\|_\infty \cdot (t^{1/4} + t^{3/4}) \cdot \delta^{1/2}$. So the integral in question tends to zero as δ tends to zero. Hence we may estimate $|\partial_m \partial_4 (\mathfrak{D}_j^{(\tau)}(f) - \mathfrak{D}_j(f))(x,t)|$ by

$$\int_0^t \int_{\mathbb{R}^3} \sum_{k=1}^3 |\partial_{x_m} H_k(x-y, t-\sigma) \cdot f_k(y,\sigma)|\, dy\, d\sigma.$$

It follows by Minkowski's inequality that

$$\|\partial_m \partial_4 (\mathfrak{D}_j^{(\tau)}(f) - \mathfrak{D}_j(f)) | \Omega \times (0,\infty)\|_{3/2,2;\infty} \le \left(\int_0^\infty \left(\int_0^t \mathfrak{A}(t,\sigma) d\sigma \right)^2 dt \right)^{1/2},$$

with $\mathfrak{A}(t,\sigma) := (\int_\Omega (\int_{\mathbb{R}^3} \sum_{k=1}^3 |\partial_{x_m} H_k(x-y, t-\sigma) \cdot f_k(y,\sigma)|\, dy)^{3/2}\, dx)^{2/3}$ for $t, \sigma \in (0,\infty)$ with $t > \sigma$. But $\mathfrak{A}(t,\sigma) \le \mathfrak{A}_1(t,\sigma) + \mathfrak{A}_2(t,\sigma)$, where $\mathfrak{A}_1(t,\sigma)$, $\mathfrak{A}_2(t,\sigma)$ are defined in the same way as $\mathfrak{A}(t,\sigma)$, except that the domain of integration $\mathbb{R}^3$ is replaced by $B_1(x)^c$ and $B_1(x)$, respectively. Let $z \in \mathbb{R}^3$ with $|z| \ge 1$, $r \in (0,\infty)$, $1 \le k \le 3$. We use (4.4), (4.6) with $\epsilon = 3/4$, $l = 1$, $\alpha = e_m$, $M = 1$ to obtain in the case $r \le 1$

$$\begin{aligned} |\partial_{z_m} H_k(z,r)| &\le \mathfrak{C} \cdot \chi_{[1,\infty)}(|z|) \cdot ((r^{-3/4} + r^{-1/4}) \cdot |z|^{-7/4} + r^{-3/4} \cdot |z|^{-7/2}) \\ &\le \mathfrak{C} \cdot \chi_{[1,\infty)}(|z|) \cdot r^{-3/4} \cdot |z|^{-7/4}. \end{aligned}$$

Moreover, inequalities (4.4), (4.6) with $\epsilon = 1/4$, $l = 1$, $\alpha = e_m$, $M = 1$ yield in the case $r \ge 1$ that

$$\begin{aligned} |\partial_{z_m} H_k(z,r)| &\le \mathfrak{C} \cdot \chi_{[1,\infty)}(|z|) \cdot r^{-5/4} \cdot (|z|^{-5/4} + [|z| \cdot \nu(z)]^{-3/4} + |z|^{-5/2}) \\ &\le \mathfrak{C} \cdot \chi_{[1,\infty)}(|z|) \cdot r^{-5/4} \cdot (|z|^{-5/4} + [|z| \cdot \nu(z)]^{-3/4}). \end{aligned}$$

Setting $\varrho(r) := \chi_{(0,1]}(r) \cdot r^{-3/4} + \chi_{[1,\infty)}(r) \cdot r^{-5/4}$, we thus obtain

$$|\partial_{z_m} H_k(z,r)| \le \mathfrak{C} \cdot \varrho(r) \cdot \chi_{[1,\infty)}(|z|) \cdot (\,|z|^{-5/4} + [|z| \cdot \nu(z)]^{-3/4}).$$

Thus, with Hölder's inequality,

$$\mathfrak{A}_1(t,\sigma) \le \mathfrak{C} \cdot \max_{x\in\mathbb{R}^3} \int\limits_{B_1(x)^c} \sum_{k=1}^{3} |\partial_{x_m} H_k(x-y,t-\sigma) \cdot f_k(y,\sigma)|\, dy \tag{6.6}$$

$$\le \mathfrak{C} \cdot \varrho(t-\sigma) \cdot \Big(\int\limits_{\mathbb{R}^3} \chi_{[1,\infty)}(|z|) \cdot \Big(|z|^{-15/4} + (|z|\cdot\nu(z))^{-9/4} \Big)\, dz \Big)^{1/3}$$

$$\cdot \|f(\,\cdot\,,\sigma)\|_{3/2} \quad \le \quad \mathfrak{C} \cdot \varrho(t-\sigma) \cdot \|f(\,\cdot\,,\sigma)\|_{3/2} \tag{6.7}$$

$(t \in (0,\infty),\ \sigma \in (0,t))$, where we used Lemma 3.3 in the last estimate. Inequality (4.8) with $M=1,\ \alpha = e_m,\ \epsilon = 3/4$ and $\epsilon = 1/4$, respectively, yields that $\mathfrak{A}_2(t,\sigma)$ is bounded by $\mathfrak{C}\cdot\varrho(t-\sigma)$ times the term

$$\Bigg(\int\limits_{\mathbb{R}^3} \Big(\int\limits_{\mathbb{R}^3} \chi_{(0,1]}(|x-y|) \cdot (|x-y|^{-5/2} + |x-y|^{-3/2}) \cdot |f(y,\sigma)|\, dy \Big)^{3/2} dx \Bigg)^{2/3}.$$

Hence by Young's inequality (Theorem 3.2),

$$\mathfrak{A}_2(t,\sigma) \le \mathfrak{C}\cdot\varrho(t-\sigma) \cdot \Big(\int\limits_{\mathbb{R}^3} \chi_{(0,1]}(|z|)\cdot |z|^{-5/2}\, dz \Big) \cdot \|f(\,\cdot\,,\sigma)\|_{3/2}$$

$$\le \mathfrak{C}\cdot\varrho(t-\sigma)\cdot \|f(\,\cdot\,,\sigma)\|_{3/2} \quad (t\in(0,\infty),\ \sigma\in(0,t)). \tag{6.8}$$

By the relations $\mathfrak{A}(t,\sigma) \le \mathfrak{A}_1(t,\sigma) + \mathfrak{A}_2(t,\sigma)$, (6.6) and (6.8) we see that the term $(\int_0^\infty (\int_0^t \mathfrak{A}(t,\sigma)\, d\sigma)^2\, dt)^{1/2}$ is bounded by

$$\mathfrak{C}\cdot \Bigg(\int\limits_0^\infty \Big(\int\limits_0^\infty \varrho(t-\sigma)\cdot\|f(\,\cdot\,,\sigma)\|_{3/2}\, d\sigma \Big)^2 dt \Bigg)^{1/2},$$

hence with Young's inequality by $\int_0^\infty \varrho(r)\ dr \cdot \|f\|_{3/2,2;\infty}$, and thus by $\mathfrak{C}\cdot\|f\|_{3/2,2;\infty}$. Now the lemma follows. □

Lemma 6.4. *Let* $f \in C_0^\infty(\mathbb{R}^3\times(0,\infty))^3,\ m,n\in\{1,2,3\}$. *Then*

$$\|\partial_4(\mathfrak{R}^{(\tau)}(f) - \mathfrak{R}(f))|\Omega\times(0,\infty)\|_{3/2,2;\infty} + \|\partial_m\partial_n(\mathfrak{R}^{(\tau)}(f)-\mathfrak{R}(f))|\Omega\times(0,\infty)\|_{3/2,2;\infty} \le \mathfrak{C}\cdot\|f\|_{3/2,2;\infty}.$$

Proof. Let $j\in\{1,2,3\}$. Recall the definition of the function K_{jk} $(1\le k\le 3)$ at the beginning of Sect. 4. Let $x\in\mathbb{R}^3,\ t\in(0,\infty)$. By Lemma 5.7, we know that

$$\partial_t^l\partial_x^\alpha(\mathfrak{R}^{(\tau)}(f)-\mathfrak{R}(f))(x,t) = \int\limits_0^t \int\limits_{\mathbb{R}^3} \sum_{k=1}^3 K_{jk}(x-y,t-\sigma)\cdot\partial_\sigma^l\partial_y^\alpha f(y,\sigma)\, dy\, d\sigma \tag{6.9}$$

$(\alpha\in\mathbb{N}_0^3,\ l\in\mathbb{N}_0)$. For any $\sigma\in(0,t),\ k\in\{1,2,3\}$, the function $K_{jk}(\,\cdot\,,t-\sigma)$ belongs to $C^\infty(\mathbb{R}^3)$ (Lemma 4.2, 4.3), so in the case $\alpha = e_m+e_n$, we may integrate by parts with respect to y_m and y_n in the preceding integral. Moreover, if $\alpha=0,\ l=1$, by Lemma 5.7 we may apply Fubini's theorem to that integral. We further observe that for $z\in\mathbb{R}^3\backslash\{0\}$, the function $K_{jk}(z,\,\cdot\,)$ belongs to $C^\infty(\mathbb{R})$ (Lemmas 4.2, 4.3), and the equation $K_{jk}(z,0)=0$ holds (Lemma 4.4). Thus, if $\alpha=0,\ l=1$, we may integrate by parts with respect to σ in the preceding integral, and this integration does not give rise to a boundary term with respect to the interval $[0,t]$. Now let $l\in\{0,1\},\ \alpha\in\mathbb{N}_0^3$ with $|\alpha|+2\cdot l=2$. Then the preceding transformations of the right-hand side of (6.9) imply

$$|\partial_t^l\partial_x^\alpha(\mathfrak{R}^{(\tau)}(f)-\mathfrak{R}(f))(x,t)| \le G(x,t), \tag{6.10}$$

with $G(x,t) := \int_0^t \int_{\mathbb{R}^3} \sum_{k=1}^3 |\partial_t^l \partial_x^\alpha K_{jk}(x-y,t-\sigma)\cdot f(y,\sigma)|\,dy\,d\sigma$. Our aim now is to estimate the term $\|G|\Omega\times(0,\infty)\|_{3/2,2;\infty}$. By Minkowski's inequality we get

$$\|G|\Omega\times(0,\infty)\|_{3/2,2;\infty} \le \left(\int_0^\infty \left(\int_0^\infty \chi_{(0,\infty)}(t-\sigma)\cdot \mathfrak{A}(t,\sigma)\,d\sigma\right)^2 dt\right)^{1/2},$$

with $\mathfrak{A}(t,\sigma) := (\int_\Omega (\int_{\mathbb{R}^3} \sum_{k=1}^3 |\partial_t^l \partial_x^\alpha K_{jk}(x-y,t-\sigma)\cdot f_k(y,\sigma)|\,dy)^{3/2}\,dx)^{2/3}$ for $t,\sigma\in(0,\infty)$ with $t>\sigma$. But $\mathfrak{A}(t,\sigma)\le \mathfrak{A}_1(t,\sigma)+\mathfrak{A}_2(t,\sigma)$, with $\mathfrak{A}_1(t,\sigma)$ and $\mathfrak{A}_2(t,\sigma)$ defined in the same way as $\mathfrak{A}(t,\sigma)$, except that the domain of integration $\mathbb{R}^3$ is replaced by $B_1(x)^c$ in the case of $\mathfrak{A}_1(t,\sigma)$, and by $B_1(x)$ else. For $z\in\mathbb{R}^3$ with $|z|\ge 1$, $r\in(0,\infty)$, $1\le k\le 3$, we find with Lemmas 4.2 and 4.3 that

$$\begin{aligned}|\partial_r^l \partial_z^\alpha K_{jk}(z,r)| &\le \mathfrak{C}\cdot\chi_{[1,\infty)}(|z|)\cdot([|z|\cdot\nu(z)+r]^{-2}+[|z|\cdot\nu(z)+r]^{-5/2}\\ &\quad +(|z|^2+r)^{-5/2}) \;\le\; \mathfrak{C}\cdot\chi_{[1,\infty)}(|z|)\cdot\varrho_1(r)\cdot([|z|\cdot\nu(z)]^{-3/4}+|z|^{-5/4}),\end{aligned}$$

with $\varrho(r) := \chi_{(0,1]}(r)+\chi_{(1,\infty)}(r)\cdot r^{-5/4}$. Therefore, by Hölder's inequality and Lemma 3.3, we find for $t\in(0,\infty)$, $\sigma\in(0,t)$, $x\in\mathbb{R}^3$ that

$$\begin{aligned}\mathfrak{A}_1(t,\sigma) &\le \mathfrak{C}\cdot\max_{x\in\mathbb{R}^3}\int_{B_1(x)^c}\sum_{k=1}^3 |\partial_t^l\partial_x^\alpha K_{jk}(x-y,t-\sigma)\cdot f_k(y,\sigma)|\,dy\\ &\le \mathfrak{C}\cdot\varrho_1(t-\sigma)\cdot\left(\int_{\mathbb{R}^3}\chi_{[1,\infty)}(|z|)\cdot((\,|z|\cdot\nu(z))^{-9/4}+|z|^{-15/4})\,dz\right)^{1/3}\cdot\|f(\,\cdot\,,\sigma)\|_{3/2}\\ &\le \quad \mathfrak{C}\cdot\varrho_1(t-\sigma)\cdot\|f(\,\cdot\,,\sigma)\|_{3/2}.\end{aligned}$$

In order to estimate $\mathfrak{A}_2(t,\sigma)$, we use (4.1) instead of Lemmas 4.2 and 4.3. We find for $r\in(0,\infty)$, $z\in B_1\backslash\{0\}$ that $|\partial_r^l\partial_z^\alpha K_{jk}(z,r)|$ is bounded by

$$\mathfrak{C}\cdot\chi_{(0,1)}(|z|)\cdot(\chi_{(0,1)}(r)\cdot(|z|^2+r)^{-2}+\chi_{[1,\infty)}(r)\cdot(|z|^2+r)^{-3/2})$$

and hence by $\mathfrak{C}\cdot\chi_{(0,1)}(|z|)\cdot(\chi_{(0,1)}(r)\cdot r^{-3/4}\cdot|z|^{-5/2}+\chi_{[1,\infty)}(r)\cdot r^{-3/2})$, so that

$$|\partial_r^l\partial_z^\alpha K_{jk}(z,r)| \le \mathfrak{C}\cdot\varrho_2(r)\cdot\chi_{(0,1)}(|z|)\cdot|z|^{-5/2},$$

with $\varrho_2(r) := \chi_{(0,1)}(r)\cdot r^{-3/4}+\chi_{[1,\infty)}(r)\cdot r^{-3/2}$. Now we obtain with Young's inequality (Theorem 3.2) that $\mathfrak{A}_2(t,\sigma)$ may be estimated by

$$\mathfrak{C}\cdot\varrho_2(t-\sigma)\cdot\left(\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\chi_{(0,1)}(|x-y|)\cdot|x-y|^{-5/2}\cdot|f(y,\sigma)|\,dy\right)^{3/2}dx\right)^{2/3},$$

and thus by

$$\mathfrak{C}\cdot\varrho_2(t-\sigma)\cdot\left(\int_{\mathbb{R}^3}\chi_{(0,1)}(|z|)\cdot|z|^{-5/2}\,dz\right)\cdot\|f(\,\cdot\,,\sigma)\|_{3/2},$$

so that $\mathfrak{A}_2(t,\sigma) \le \mathfrak{C} \cdot \varrho_2(t-\sigma) \cdot \|f(\,\cdot\,,\sigma)\|_{3/2}$. We may conclude from the preceding estimates, again using Young's inequality,

$$\|G\|_{3/2,2;\infty} \le \mathfrak{C} \cdot \left(\int\limits_0^\infty \left(\int\limits_0^\infty (\varrho_1 + \varrho_2)(t-\sigma) \cdot \|f(\,\cdot\,,\sigma)\|_{3/2} \, d\sigma \right)^2 dt \right)^{1/2}$$

$$\le \mathfrak{C} \cdot \left(\int\limits_0^\infty (\varrho_1 + \varrho_2)(r) \, dr \right) \cdot \|f\|_{3/2,2;\infty} \le \mathfrak{C} \cdot \|f\|_{3/2,2;\infty}.$$

This completes the proof of Lemma 6.4. □

Theorem 5.5, Lemmas 6.1–6.4, 5.11 and 5.12 imply

Theorem 6.1. *Let $f \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3$, $q_0 \in [1, 3/2)$. Then*

$$\|\mathfrak{R}^{(\tau)}(f)|S_\infty\|_{H_\infty} \le \mathfrak{C}(q_0) \cdot (\|f\|_{q_0,2;\infty} + \|f\|_{3/2,2;\infty}) + \mathfrak{C} \cdot \Bigg(\|\partial_4 \mathfrak{K}\left(\mathfrak{F}_{3/2}(f)\right)\|_{3/2,2;\infty}$$

$$+ \sum_{m,n=1}^{3} \|\partial_n \partial_m \mathfrak{K}(\mathfrak{F}_{3/2}(f))\|_{3/2,2;\infty} + \|\nabla_x \partial_4 \mathfrak{M}(\mathfrak{F}_{3/2}(f))\|_{3/2,2;\infty} \Bigg).$$

7. Estimate of $\mathfrak{M}(\mathfrak{F}_{3/2}(f))$ and $\mathfrak{K}(\mathfrak{F}_{3/2}(f))$

Theorem 6.1 leaves the task of estimating the terms $\|\partial_4 \mathfrak{K}(\mathfrak{F}_{3/2}(f))\|_{3/2,2;\infty}$, $\|\partial_m \partial_n \mathfrak{K}(\mathfrak{F}_{3/2}(f))\|_{3/2,2;\infty}$ and $\|\partial_m \partial_4 \mathfrak{M}(\mathfrak{F}_{3/2}(f))\|_{3/2,2;\infty}$, for $1 \le m, n \le 3$. We will consider the last one of these three quantities. The first two may be handled in a similar, but somewhat simpler way, or may be estimated by applying a result from literature on maximal regularity. We will give more specific indications in this respect at the end of this section.

Let $m \in \{1, 2, 3\}$ be fixed throughout this section.

Lemma 7.1. *Let $t \in (0,\infty)$, $l \in \{0, 1, 2\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. Then $\int_{\mathbb{R}^3} |\partial_t^l \partial_z^\alpha L(z,t)| \, dz \le \mathfrak{C} \cdot t^{-l+1/2-|\alpha|/2}$. In particular, for $g \in C_0^\infty(\mathbb{R}^3)$, $x \in \mathbb{R}^3$, the integral $\int_{\mathbb{R}^3} |\partial_t^l \partial_x^\alpha L(x-y,t) \cdot g(y)| \, dy$ is finite*

Proof. We have $\int_{\mathbb{R}^3} |\partial_z^\alpha L(z,t)| \, dz \le \int_0^t (t-s)^{-1/2} \cdot \int_{\mathbb{R}^3} |\partial_z^\alpha \mathfrak{H}(z,s)| \, dz \, ds$ by (4.2), so with Lemmas 4.2, 4.1 and 3.1,

$$\int\limits_{\mathbb{R}^3} |\partial_z^\alpha L(z,t)| \, dz \le \mathfrak{C} \cdot \int\limits_0^t (t-s)^{-1/2} \cdot s^{-3/2-|\alpha|/2} \cdot \int\limits_{\mathbb{R}^3} e^{-|z|^2/(8\cdot s)} \, dz \, ds$$

$$\le \mathfrak{C} \cdot \int\limits_0^t (t-s)^{-1/2} \cdot s^{-|\alpha|/2} \, ds \le \mathfrak{C} \cdot t^{1/2-|\alpha|/2}.$$

Let $l \in \{1, 2\}$. Then by (4.7), $\int_{\mathbb{R}^3} |\partial_t^l \partial_z^\alpha L(z,t)| \, dz \le \mathfrak{C} \cdot (\mathfrak{A}_1 + \mathfrak{A}_2)$, with

$$\mathfrak{A}_1 := t^{-1-|\alpha|/2-l} \cdot \int\limits_{\mathbb{R}^3} e^{-|z|^2/(8\cdot t)} \, dz,$$

$$\mathfrak{A}_2 := t^{-1/2-l} \cdot \int\limits_0^{t/2} r^{-3/2-|\alpha|/2} \cdot \int\limits_{\mathbb{R}^3} e^{-|z|^2/(8\cdot r)} \, dz \, dr.$$

Lemma 4.1 yields $\mathfrak{A}_1 \le \mathfrak{C} \cdot t^{1/2-|\alpha|/2-l}$ and $\mathfrak{A}_2 \le \mathfrak{C} \cdot t^{-1/2-l} \cdot \int_0^{t/2} r^{-|\alpha|/2} \, dr \le \mathfrak{C} \cdot t^{1/2-|\alpha|/2-l}$. □

Lemma 7.2. *Let $g \in C_0^\infty(\mathbb{R}^3)$, and put $\mathfrak{J}(x,t) := \int_{\mathbb{R}^3} \partial_m L(x-y,t) \cdot g(y)\, dy$ for $x \in \mathbb{R}^3,\ t \in (0,\infty)$. Then the function $\mathfrak{J}$ is continuously differentiable with respect to t, and the equation $\partial_t \mathfrak{J}(x,t) = \int_{\mathbb{R}^3} \partial_4 \partial_m L(x-y,t) \cdot g(y)\, dy$ holds for x, t as above.*

Proof. There is $S > 0$ with $supp(g) \subset B_S$. Let $R, \delta \in (0,\infty)$. Using (4.4) with $\epsilon = 1/4$, we obtain for $y \in \mathbb{R}^3,\ x \in B_R,\ t \in [\delta, \infty),\ l \in \{0,1\}$ that

$$|\partial_4^l \partial_m L(x-y,t) \cdot g(y)| \le \mathfrak{C} \cdot \|g\|_\infty \cdot |x-y|^{-5/2} \cdot \delta^{-1/4-l} \cdot \chi_{B_S}(y).$$

On taking into account that $\chi_{B_S}(y) \le \chi_{B_{S+R}}(x-y)$ for $y \in \mathbb{R}^3,\ x \in B_R$, we obtain with Lebesgue's theorem that $\mathfrak{J}|B_R \times (\delta, \infty)$ is continuously differentiable with respect to t, and the equation at the end of Lemma 7.2 holds for $x \in B_R,\ t \in (\delta, \infty)$. This proves the lemma. □

Lemma 7.3. *Let $g \in C_0^\infty(\mathbb{R}^3),\ x \in \mathbb{R}^3,\ t \in (0,\infty)$. Then $\int_{\mathbb{R}^3} \partial_4 L(x-y,t) \cdot \partial_m g(y)\, dy = \int_{\mathbb{R}^3} \partial_m \partial_4 L(x-y,t) \cdot g(y)\, dy$.*

Proof. By Lemma 7.1 and Lebesgue's theorem, we see that the left-hand side of the equation in Lemma 7.3 equals $\lim_{\delta \downarrow 0} \int_{\mathbb{R}^3 \setminus B_\delta(x)} \partial_4 L(x-y,t) \cdot \partial_m g(y)\, dy$. Recall that $\partial_4 L(\,\cdot\,,t) \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ (Lemma 4.6), so

$$\int_{\mathbb{R}^3} \partial_4 L(x-y,t) \cdot \partial_m g(y)\, dy = \lim_{\delta \downarrow 0} (\mathfrak{A}_1(\delta) + \mathfrak{A}_2(\delta)),$$

with $\mathfrak{A}_1(\delta) := \int_{\mathbb{R}^3 \setminus B_\delta(x)} \partial_m \partial_4 L(x-y,t) \cdot g(y)\, dy, \mathfrak{A}_2(\delta) := \int_{\partial B_\delta(x)} \partial_4 L(x-y,t) \cdot g(y) \cdot (x-y)_m/\delta\, do_y$, for $\delta \in (0,\infty)$.

But $\mathfrak{A}_1(\delta) \to \int_{\mathbb{R}^3} \partial_m \partial_4 L(x-y,t) \cdot g(y)\, dy$ for $\delta \downarrow 0$ by Lemma 7.1 and Lebesgue's theorem. Moreover, by (4.4) with $l = 1,\ \alpha = 0,\ \epsilon = 1/4$ we get

$$|\mathfrak{A}_2(\delta)| \le \mathfrak{C} \cdot \|g\|_\infty \cdot t^{-5/4} \cdot \delta^{-3/2} \cdot \int_{\partial B_\delta(x)} do_x \le \mathfrak{C} \cdot \|g\|_\infty \cdot t^{-5/4} \cdot \delta^{1/2}$$

for $\delta \in (0,\infty)$. Therefore $\mathfrak{A}_2(\delta) \to 0$ for $\delta \downarrow 0$, and the lemma follows. □

Corollary 7.1. *Let $g \in C_0^\infty(\mathbb{R}^3),\ a \in (0,\infty),\ x \in \mathbb{R}^3$. Then*

$$\int_0^a \left| \int_{\mathbb{R}^3} \partial_m \partial_4 L(x-y,t) \cdot g(y)\, dy \right| dt \le \mathfrak{C} \cdot \|\partial_m g\|_\infty \cdot a^{1/2}.$$

Note that the integral $\int_0^a \int_{\mathbb{R}^3} |\partial_m \partial_4 L(x-y,t) \cdot g(y)|\, dy\, dt$ need not be finite.

Proof of Corollary 7.1. By Lemma 7.3, the left-hand side of the estimate in Corollary 7.1 is bounded by $\int_0^a \int_{\mathbb{R}^3} |\partial_4 L(x-y,t) \cdot \partial_m g(y)|\, dy\, dt$, and thus by $\mathfrak{C} \cdot \|\partial_m g\|_\infty \cdot \int_0^a t^{-1/2}\, dt$ (Lemma 7.1). □ □

Lemma 7.4. *Let $f \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3,\ x \in \mathbb{R}^3,\ t \in (0,\infty)$. Then the integral $\int_0^t | \int_{\mathbb{R}^3} \partial_m \partial_4 L(x-y,t-\sigma) \cdot f(y,\sigma)\, dy|\, d\sigma$ is finite, and*

$$\partial_m \partial_4 \mathfrak{M}(f)(x,t) = \int_0^t \int_{\mathbb{R}^3} \partial_m \partial_4 L(x-y,t-\sigma) \cdot f(y,\sigma)\, dy\, d\sigma. \tag{7.1}$$

Note that the integral $\int_0^t \int_{\mathbb{R}^3} |\partial_m \partial_4 L(x-y,t-\sigma) \cdot f(y,\sigma)|\, dy\, d\sigma$ need not be finite.

Proof of Lemma 7.4. Referring to Lemmas 5.7 and 5.8, we see that $\int_0^t | \int_{\mathbb{R}^3} \partial_4 L(x-y,t-\sigma) \cdot \partial_m f(y,\sigma)\, dy|\, d\sigma < \infty$ and

$$\partial_m \partial_4 \mathfrak{M}(f)(x,t) = \int_0^t \int_{\mathbb{R}^3} \partial_4 L(x-y,t-\sigma) \cdot \partial_m f(y,\sigma)\, dy\, d\sigma.$$

But for $\sigma \in (0,t)$, Lemma 7.3 with $g = f(\,\cdot\,,\sigma)$ yields that the integral $\int_{\mathbb{R}^3} \partial_4 L(x-y,t-\sigma)\cdot\partial_m f(y,\sigma)\,dy$ equals $\int_{\mathbb{R}^3} \partial_m\partial_4 L(x-y,t-\sigma)\cdot f(y,\sigma)\,dy$. Lemma 7.4 follows from the preceding relations. □ □

Lemma 7.5. *Let $v \in L^2(\mathbb{R}^3)^3,\ t \in (0,\infty)$. Then*

$$\Big(\int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3} |\partial_m\partial_4 L(x-y,t)\cdot v(y)|\,dy\Big)^2\,dx\Big)^{1/2} \le \mathfrak{C}\cdot t^{-1}\cdot\|v\|_2.$$

In particular, the integral $\int_{\mathbb{R}^3} |\partial_m\partial_4 L(x-y,t)\cdot v(y)|\,dy$ is finite for a.e. $x\in\mathbb{R}^3$.

Proof. Young's inequality (Theorem 3.2) and Lemma 7.1. □

In view of Lemma 7.5, for $t \in (0,\infty)$ we may define $E_m(t) : L^2(\mathbb{R}^3)^3 \mapsto L^2(\mathbb{R}^3)^3$ by setting $E_m(t)(v)(x) := \int_{\mathbb{R}^3} \partial_m\partial_4 L(x-y,t)\cdot v(y)\,dy$ for $v\in L^2(\mathbb{R}^3)^3$ and a.e. $x\in\mathbb{R}^3$. Let $\mathfrak{L}$ denote the space of linear bounded operators from $L^2(\mathbb{R}^3)^3$ into $L^2(\mathbb{R}^3)^3$. We equip this space with the usual operator norm, which we denote by $|||\ \ |||$.

We are going to check whether $k = E_m$ satisfies the assumptions of Theorem 3.6. Lemma 7.5 yields

Corollary 7.2. *Let $t\in(0,\infty)$. For $v \in L^2(\mathbb{R}^3)^3$, the inequality $\|E_m(t)(v)\|_2 \le \mathfrak{C}\cdot t^{-1}\cdot\|v\|_2$ holds. In particular, $E_m(t)$ is an operator from $L^2(\mathbb{R}^3)^3$ into $L^2(\mathbb{R}^3)^3$. This operator is linear. These relations mean that $E_m(t)\in\mathfrak{L}$ and $|||E_m(t)||| \le \mathfrak{C}\cdot t^{-1}$.*

Lemma 7.6. *The mapping $E_m : (0,\infty)\mapsto\mathfrak{L}$ is (strongly $\mathfrak{L}$-) measurable ([51, p. 130]), as well as integrable on compact subsets of $(0,\infty)$. Moreover, $\int_{4\cdot|t|}^{\infty} |||E_m(s-t)-E_m(s)|||\,ds \le \mathfrak{C}$ for $t\in\mathbb{R}\backslash\{0\}$.*

Proof. Let $t\in\mathbb{R}\backslash\{0\},\ v\in L^2(\mathbb{R}^3)^3,\ s\in(4\cdot|t|,\infty)$. Then $s-\vartheta\cdot t \ge s-|t| \ge 3\cdot s/4 > 0$ for $\vartheta\in[0,1]$. Thus we find for $z\in\mathbb{R}^3\backslash\{0\}$ that $\partial_m\partial_4(L(z,s-t)-L(z,s)) = -t\cdot\int_0^1 \partial_m\partial_4^2 L(z,\,s-\vartheta\cdot t)\,d\vartheta$. Therefore Minkowski's inequality yields that $\|(E_m(s-t)-E_m(s))(v)\|_2$ may be estimated by

$$|t|\cdot\int_0^1\Big(\int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}|\partial_m\partial_4^2 L(x-y,\,s-\vartheta\cdot t)\cdot v(y)|\,dy\Big)^2\,dx\Big)^{1/2}\,d\vartheta.$$

Thus, by Young's inequality (Theorem 3.1), $\|(E_m(s-t)-E_m(s))(v)\|_2$ admits the upper bound

$$|t|\cdot\int_0^1\Big(\int_{\mathbb{R}^3}|\partial_m\partial_4^2 L(z,\,s-\vartheta\cdot t)|\,dz\Big)\,d\vartheta\cdot\|v\|_2$$

But the preceding integral with respect to z is bounded by $\mathfrak{C}\cdot(s-\vartheta\cdot t)^{-2}$ for $\vartheta\in[0,1]$, as follows from Lemma 7.1. Using the estimate $s-\vartheta\cdot t\ge 3\cdot s/4$ ($\vartheta\in[0,1]$) mentioned above, we may conclude

$$|||E_m(s-t)-E_m(s)||| \le \mathfrak{C}\cdot|t|\cdot s^{-2} \quad \text{for}\ \ s\in(4\cdot|t|,\infty). \tag{7.2}$$

Let $a\in(0,\infty)$. For $\epsilon\in(0,\,a/4),\ s\in[a,\infty)$, we may deduce from (7.2) that $|||E_m(s+\epsilon)-E_m(s)||| \le \mathfrak{C}\cdot\epsilon\cdot a^{-2}$, so E_m is continuous on $[a,\infty)$. This proves that E_m is continuous on $(0,\infty)$, and hence measurable and locally integrable. The inequality at the end of Lemma 7.6 follows from (7.2). □

Corollary 7.3. *Let $v\in L^2(\mathbb{R}^3)^3$. Then the mapping $t\mapsto E_m(t)(v)$ from $(0,\infty)$ into $L^2(\mathbb{R}^3)^3$ is measurable, as well as integrable on compact subsets of $(0,\infty)$. For such a subset K, we have $(\mathfrak{L}-\int_K E_m(t)\,dt)(v) = L^2(\mathbb{R}^3)^3-\int_K E_m(t)(v)\,dt$. Moreover, for a.e. $x\in\mathbb{R}^3$,*

$$\Big(\mathfrak{L}-\int_K E_m(t)\,dt\Big)(v)(x) = \int_K\int_{\mathbb{R}^3}\partial_m\partial_4 L(x-y,t)\cdot v(y)\,dy\,dt. \tag{7.3}$$

Proof. Define $T : \mathfrak{L} \mapsto L^2(\mathbb{R}^3)^3$ by $T(B) := B(v)$ for $B \in \mathfrak{L}$. Then T is a linear bounded operator, so all the statements of the corollary except (7.3) follow from [51, p. 134]. Equation (7.3) is a consequence of the first equation in the corollary, Lemma 3.5 and the definition of $E_m(t)(v)$. □

Lemma 7.7. *Let $\epsilon,\, \delta \in (0,\infty)$ with $\epsilon < \delta$. Then $|||\mathfrak{L} - \int_\epsilon^\delta E_m(t)\, dt||| \le \mathfrak{C}$.*

Proof. Let $v \in L^2(\mathbb{R}^3)^3$ with $v \neq 0$. Since $C_0^\infty(\mathbb{R}^3)^3$ is dense in $L^2(\mathbb{R}^3)^3$, we may choose $w \in C_0^\infty(\mathbb{R}^3)^3$ with $\|v - w\|_2 \le \min\{1,\ 1/\ln(\delta/\epsilon)\} \cdot \|v\|_2$. Let $x \in \mathbb{R}^3$, and denote the term $(\mathfrak{L} - \int_\epsilon^\delta E_m(t)\ dt)(w)(x)$ by $\mathfrak{A}_1(x)$. Then $\mathfrak{A}_1(x)$ is given by the right-hand side of (7.3) with $K = [\epsilon,\, \delta]$ and v replaced by w, so by Lemma 7.2, $|\mathfrak{A}_1(x)| \le \sum_{\mu\in\{\epsilon,\,\delta\}} |\int_{\mathbb{R}^3} \partial_m L(x-y,\mu) \cdot w(y)\, dy|$. It follows with Young's inequality and Lemma 7.1 that

$$\|\mathfrak{A}_1\|_2 \le \mathfrak{C} \cdot \sum_{\mu\in\{\epsilon,\,\delta\}} \left(\int_{\mathbb{R}^3} |\partial_m L(z,\mu)|\ dz \right) \cdot \|w\|_2 \le \mathfrak{C} \cdot \|w\|_2.$$

Thus, by the choice of w, we may conclude that $\|\mathfrak{A}_1\|_2 \le \mathfrak{C} \cdot \|v\|_2$. Using the abbreviation $\mathfrak{A}_2(x) := (\mathfrak{L} - \int_\epsilon^\delta E_m(t)\ dt)(v-w)(x)$ for $x \in \mathbb{R}$, we find with Corollary 7.3 that $\|\mathfrak{A}_2\|_2 = \|L^2(\mathbb{R}^3)^3 - \int_\epsilon^\delta E_m(t)(v-w)\ dt\|_2$. Therefore with Corollary 7.2,

$$\begin{aligned} \|\mathfrak{A}_2\|_2 &\le \int_\epsilon^\delta \|E_m(t)(v-w)\|_2\ dt \le \mathfrak{C} \cdot \left(\int_\epsilon^\delta t^{-1}\, dt \right) \cdot \|v - w\|_2 \\ &\le \mathfrak{C} \cdot \ln(\delta/\epsilon) \cdot \|v - w\|_2. \end{aligned}$$

Again referring to the choice of w, we may conclude that $\|\mathfrak{A}_2\|_2 \le \mathfrak{C} \cdot \|v\|_2$. Lemma 7.7 follows from the preceding estimates of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. □

Lemma 7.8. *Let $v \in L^2(\mathbb{R}^3)^3$. Then there is $\gamma \in L^2(\mathbb{R}^3)^3$ such that $(\mathfrak{L} - \int_\delta^1 E_m(t)\ dt)(v) \to \gamma$ for $\delta \downarrow 0$.*

Proof. By Lemma 7.7, there is $C_0 > 0$ such that $\|(\mathfrak{L} - \int_\epsilon^\delta E_m(t)\ \ dt)(\varphi)\|_2 \le C_0 \cdot \|\varphi\|_2$ for $\varphi \in L^2(\mathbb{R}^3)^3$, $\epsilon,\, \delta \in (0,\infty)$ with $\epsilon < \delta$. Let $\kappa \in (0,\infty)$. Choose $w \in C_0^\infty(\mathbb{R}^3)^3$ such that $\|v - w\|_2 \le \kappa/(2 \cdot C_0)$. The two preceding inequalities yield $\|(\mathfrak{L} - \int_\epsilon^\delta E_m(t)\ dt)(v-w)\|_2 < \kappa/2$ for any $\epsilon,\, \delta \in (0,\infty)$ with $\epsilon < \delta$. On the other hand, by Corollary 7.1,

$$\int_0^\delta \left| \int_{\mathbb{R}^3} \partial_4 \partial_m L(x-y,t) \cdot w(y)\ dy \right|\ dt \downarrow 0 \quad \text{for} \quad \delta \downarrow 0. \tag{7.4}$$

Moreover, by Minkowski's inequality and Lemma 7.3,

$$\begin{aligned} &\left(\int_{\mathbb{R}^3} \left(\int_0^1 \left| \int_{\mathbb{R}^3} \partial_4 \partial_m L(x-y,t) \cdot w(y)\, dy \right|\ dt \right)^2\ dx \right)^{1/2} \\ &\quad \le \int_0^1 \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} G(x,y,t)\ dy \right)^2\ dx \right)^{1/2}\ dt, \end{aligned}$$

with $G(x,y,t) := |\partial_4 L(x-y,t) \cdot \partial_m w(y)|\ (x,y \in \mathbb{R}^3,\ x \neq y,\ t > 0)$. Now we apply Young's inequality, which implies that the right-hand side of the preceding inequality is bounded by $\int_0^1 (\int_{\mathbb{R}^3} |\partial_4 L(z,t)|\ dz)\, dt \cdot \|\partial_m w\|_2$, hence by $\mathfrak{C} \cdot \int_0^1 t^{-1/2}\, dt \cdot \|\partial_m w\|_2$, where we used Lemma 7.1. Thus we see that

$$\left(\int\limits_{\mathbb{R}^3}\left(\int\limits_0^1\left|\int\limits_{\mathbb{R}^3}\partial_4\partial_m L(x-y,t)\cdot w(y)\,dy\right|\,dt\right)^2\,dx\right)^{1/2}\leq \mathfrak{C}\cdot\|\partial_m w\|_2. \tag{7.5}$$

By Lebesgue's theorem, (7.4) and (7.5),

$$\left(\int\limits_{\mathbb{R}^3}\left(\int\limits_0^\delta\left|\int\limits_{\mathbb{R}^3}\partial_4\partial_m L(x-y,t)\cdot w(y)\,dy\right|\,dt\right)^2\,dx\right)^{1/2}\to 0\ \ (\delta\downarrow 0).$$

On the other hand, by (7.3) we know that for $\epsilon,\ \delta\in(0,\infty)$ with $\epsilon<\delta$, the term $\|(\mathfrak{L}-\int_\epsilon^\delta E_m(t)\,dt)(w)\|_2$ is bounded by the preceding triple integral. Thus there is $\delta_0>0$ with $\|(\mathfrak{L}-\int_\epsilon^\delta E_m(t)\,dt)(w)\|_2\leq\kappa/2$ for $\epsilon,\ \delta\in(0,\delta_0]$ with $\epsilon<\delta$. Lemma 7.8 now follows with the estimate of $\|(\mathfrak{L}-\int_\epsilon^\delta E_m(t)\,dt)(v-w)\|_2$ given at the beginning of this proof. □

Corollary 7.4. *Let $v\in L^2(\mathbb{R}^3)^2,\ \varrho\in(0,\infty)$. Then $\int_0^\varrho t\cdot\|E_m(t)(v)\|_2\,dt\leq\mathfrak{C}\cdot\varrho\cdot\|v\|_2$.*

Proof. Corollary 7.4 is an immediate consequence of Corollary 7.2. □

Now we are in a position to apply Theorem 3.6. In fact, Theorem 3.6, Lemma 7.6–7.8, Corollary 7.4 yield

Theorem 7.1. *Let $f\in C_0^\infty(\mathbb{R}^3\times(0,\infty))^3$. For $\epsilon,\ t\in(0,\infty)$, define*

$$A_{m,\epsilon}(f)(t):=L^2(\mathbb{R}^3)^3-\int\limits_0^\infty\chi_{(\epsilon,\,\infty)}(t-s)\cdot E_m(t-s)(f(s))\,ds.$$

Then $A_{m,\epsilon}(f)\in L^2(0,\infty,\,L^2(\mathbb{R}^3)^3)$ for $\epsilon>0$, and there is a function $A_m(f)$ belonging to $L^2(0,\infty,\,L^2(\mathbb{R}^3)^3)$ such that $\|A_{m,\epsilon}(f)-A_m(f)\|_{2,2;\infty}\to 0$ for $\epsilon\downarrow 0$. Let $p\in(1,\infty)$. Then $A_m(f)\in L^p(0,\infty,L^2(\mathbb{R}^3)^3)$ and $\|A(f)\|_{2,p;\infty}\leq\mathfrak{C}(p)\cdot\|f\|_{2,p;\infty}$.

Lemma 7.9. *Let $f\in C_0^\infty(\mathbb{R}^3\times(0,\infty))^3$. Then $A_m(f)=\partial_4\partial_m\mathfrak{M}(f)$ a.e.*

Proof. Due to Lemma 7.1, we may define a function $\gamma_\epsilon:\mathbb{R}^3\times(0,\infty)\mapsto\mathbb{R}^3$ by setting $F(x,t,s):=\int_{\mathbb{R}^3}\partial_4\partial_m L(x-y,t-s)\cdot f(y,s)\,dy$ for $x\in\mathbb{R}^3,\ \ t\in(0,\infty),\ \ s\in(0,t),\ F(x,t,s):=0$ for x,t as before and $s\in[t,\infty),\ \gamma_\epsilon(x,t):=\int_0^\infty\chi_{(\epsilon,\,\infty)}(t-s)\cdot F(x,t,s)\,ds$ again for x,t as before.

Let $x\in\mathbb{R}^3,\ \ t>0$. Since $|\chi_{(\epsilon,\,\infty)}(t-s)\cdot F(x,t,s)|\leq|F(x,t,s)|$ for $\epsilon,\ s\in(0,\infty)$, we may conclude from Lemma 7.4 and Lebesgue's theorem that $\gamma_\epsilon(x,t)\to\partial_4\partial_m\mathfrak{M}(f)(x,t)$ if $\epsilon\downarrow 0$. On the other hand, by the definition of E_m and by Lemma 3.5, we find for $\epsilon,\ t\in(0,\infty)$ that

$$\gamma_\epsilon(x,t)=\int\limits_0^\infty\chi_{(\epsilon,\infty)}(t-s)\cdot E_m(t-s)(f(s))(x)\,ds=A_{m,\epsilon}(f)(t)(x)$$

for a.e. $x\in\mathbb{R}^3$. Since $\|A_{m,\epsilon}(f)-A_m(f)\|_{2,2;\infty}\to 0$ if $\epsilon\downarrow 0$ by Theorem 7.1, we thus have $\|\gamma_\epsilon-A_m(f)\|_2\to 0\ (\epsilon\downarrow 0)$. Therefore we may choose a sequence (ϵ_n) in $(0,\infty)$ with $\epsilon_n\downarrow 0$ and $\gamma_{\epsilon_n}(x,t)\to A_m(f)(x,t)\ (n\to\infty)$ for a.e. $(x,t)\in\mathbb{R}^3\times(0,\infty)$. Now Lemma 7.9 follows from the relation $\gamma_\epsilon(x,t)\to\partial_4\partial_m\mathfrak{M}(f)(x,t)\ (\epsilon\downarrow 0)$ proved above. □

Corollary 7.5. *The inequality $\|\partial_m\partial_4\mathfrak{M}(f)\|_{2,p;\infty}\leq\mathfrak{C}(p)\cdot\|f\|_{2,p;\infty}$ holds for $p\in(1,\infty),\ \ f\in C_0^\infty(\mathbb{R}^3\times(0,\infty))^3$.*

Proof. Lemma 7.9 and Theorem 7.1. □

The ensuing relation is proved in the Appendix.

Lemma 7.10. *Let* $f \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3,\ x \in \mathbb{R}^3,\ t \in (0,\infty)$. *Then*

$$\partial_m \partial_4 \mathfrak{M}(f)(x,t) = \int_{\mathbb{R}} \int_{\mathbb{R}^3} e^{i\cdot(x\cdot\xi + t\cdot\varrho)} \cdot (i - \varrho/|\varrho|) \cdot |\varrho|^{1/2} \cdot \xi_m \cdot (2^{5/2} \cdot \pi^{3/2} \cdot (|\xi|^2 - i \cdot \varrho))^{-1} \cdot \widehat{f}(\xi,\, \varrho)\, d\xi\, d\varrho.$$

The proof of the next lemma is obvious.

Lemma 7.11. *There is* $M > 0$ *such that for* $\xi \in \mathbb{R}^3\backslash\{0\},\ \varrho \in \mathbb{R}\backslash\{0\},\ \kappa \in \{0,1\}^3,\ \mu \in \{0,1\}$,

$$|\partial_\varrho^\mu \partial_\xi^\kappa (|\varrho|^{1/2} \cdot \xi_m \cdot (|\xi|^2 - i \cdot \varrho)^{-1})| \cdot |\varrho|^\mu \cdot \prod_{i=1}^{3} |\xi_i|^{\kappa_i} \le M.$$

Lemmas 7.10, 7.11 and [36, Theorem 8] yield an estimate of $\partial_4 \partial_m \mathfrak{M}(f)$ in L^p-norms:

Theorem 7.2. *Let* $p \in (1,\infty),\ f \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3$. *Then* $\|\partial_4 \partial_m \mathfrak{M}(f)\|_p \le \mathfrak{C}(p) \cdot \|f\|_p$.

In the next step, we use interpolation in order to estimate $\partial_4 \partial_m \mathfrak{M}(f)$ in the norm of $\|\ \cdot\ \|_{3/2,2;\infty}$.

Corollary 7.6. *Let* $f \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3$. *Then* $\|\partial_4 \partial_m \mathfrak{M}(f)\|_{3/2,2;\infty} \le \mathfrak{C} \cdot \|f\|_{3/2,2;\infty}$.

Proof. Recall that $C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3$ is dense in $L^p(0,\infty,\, L^q(\mathbb{R}^3)^3)$ for $p, q \in [1,\infty)$ (Lemma 2.3). Thus Corollary 7.6 follows from Corollary 7.5 with $p = 6$, Theorem 7.2 with $p = 6/5$, and Theorem 3.5. $\square$

Theorem 7.3. $\|\partial_4 \partial_m \mathfrak{M}(\mathfrak{F}_{3/2}(f))\|_{3/2,2;\infty} \le \mathfrak{C} \cdot \|f\|_{3/2,2;\infty}$ *for functions* f *from* $C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3$.

Proof. Since $\mathfrak{F}_{3/2}(f) \in L^2(0,\infty, L^{3/2}(\mathbb{R}^3)^3)$ (Lemma 3.4), we may choose a function $g_n \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3$ for $n \in \mathbb{N}$ such that $\|\mathfrak{F}_{3/2}(f) - g_n\|_{3/2,2;\infty} \to 0$ (Lemma 2.3). Thus $\|[\mathfrak{M}(\mathfrak{F}_{3/2}(f)) - \mathfrak{M}(g_n)]\,|\,\mathbb{R}^3 \times (0,T)\|_{3/2,2;T} \to 0$ $(n \to \infty)$ for $T \in (0,\infty)$, by Corollary 5.1. Moreover Corollary 7.6 yields that $(\partial_4 \partial_m \mathfrak{M}(g_n))$ is a Cauchy sequence in $L^2(0,\infty,\, L^{3/2}(\mathbb{R}^3)^3)$. Thus by Corollary 7.6 with $f = g_n$, and by a passage to the limit with respect to n, we obtain that $\|\partial_4 \partial_m \mathfrak{M}(\mathfrak{F}_{3/2}(f))\|_{3/2,2;\infty} \le \mathfrak{C} \cdot \|\mathfrak{F}_{3/2}(f)\|_{3/2,2;\infty}$. Theorem 7.3 now follows with Lemma 3.4. $\square$

An analogous result holds for $\partial_m \partial_n \mathfrak{K}(\mathfrak{F}_{3/2}(f))$ and $\partial_4 \mathfrak{K}(\mathfrak{F}_{3/2}(f))$:

Theorem 7.4. $\|\partial_4^l \partial^\alpha \mathfrak{K}(\mathfrak{F}_{3/2}(f))\|_{3/2,2;\infty} \le \mathfrak{C} \cdot \|f\|_{3/2,2;\infty}$ *for* $l \in \{0,1\},\ \alpha \in \mathbb{N}_0^3$ *with* $2 \cdot l + |\alpha| = 2,\ f \in C_0^\infty(\mathbb{R}^3 \times [0,\infty))^3$.

Theorem 7.4 may be proved by imitating the reasoning that led to Theorem 7.3. This approach starts from Lemmas 5.5 and 5.12, instead of Lemma 5.4, Corollary 5.1 and Lemma 5.11, and then repeats the sequence of arguments we presented since the beginning of this section. There are even some points that simplify. For example, the somewhat unwieldy inequalities (4.4) and (4.7) are replaced by the estimate of H stated in Lemma 4.2. Instead of Lemma 7.10, we may use the well-known fact that the Fourier transform of $\mathfrak{H}$ in $\mathbb{R}^4$ equals $((2 \cdot \pi)^2 \cdot (i \cdot \sigma + |\xi|^2))^{-1}$, with $\sigma \in \mathbb{R},\ \xi \in \mathbb{R}^3,\ (\xi,\, \sigma) \ne 0$. But there is one point that needs special care: the equivalent of Eq. (7.1) features an additional term. In fact, we have

$$\partial_t^l \partial_x^\alpha \mathfrak{K}(f)(x,t) = \int_0^t \int_{\mathbb{R}^3} \partial_l^t \partial_x^\alpha \mathfrak{H}(x-y, t-s) \cdot f(y,s)\, dy\, ds + \delta_{l1} \cdot f(x,t)$$

for f, l, α as in Theorem 7.4, $x \in \mathbb{R}^3,\ t > 0$; compare [22, Theorem 1.3.4, 1.3.5].

But actually Theorem 7.4 states a known result on maximal regularity of solutions to the heat equation. Indeed it would not even be necessary to replace $\mathfrak{R}(f)$ by $\mathfrak{K}(\mathfrak{F}_{3/2}(f))$ at the passage from Theorems 5.5 to 6.1. Instead one may use the fact that $\mathfrak{R}(f)$ is the velocity part of a solution of the time-dependent Stokes system in the whole space $\mathbb{R}^3$ with right-hand side f and vanishing initial data. Thus Theorem 7.4 with $\mathfrak{R}(f)$ in the place of $\mathfrak{K}(\mathfrak{F}_{3/2}(f))$ follows from [27, Theorem 2.8] or from [46, Theorem 2.2]. But the theory in [27,46] covers even boundary value problems associated with the evolutionary Stokes system, in contrast to the Cauchy problem which is of interest here. So the full depth of the results of these references is not exploited when they are applied to $\mathfrak{R}(f)$.

8. Proof of Theorem 1.1

Collecting our results, we are now able to establish Theorem 1.1. To begin with, we observe that Theorem 6.1 and the results of Sect. 7 yield an estimate of $\|\mathfrak{R}^{(\tau)}(f)|S_\infty\|_{H_\infty}$ if $f \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3$:

Corollary 8.1. *Let $q_0 \in [1, 3/2)$, $f \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3$. Then*

$$\|\mathfrak{R}^{(\tau)}(f)|S_\infty\|_{H_\infty} \le \mathfrak{C}(q_0) \cdot (\|f\|_{q_0,2;\infty} + \|f\|_{3/2,2;\infty}). \tag{8.1}$$

Proof. Theorems 6.1, 7.3 and 7.4. □

Next we extend inequality (8.1) to the case that $f \in L^2(0,\infty,\, L^p(\mathbb{R}^3)^3)$ for any $p \in [q_0, 3/2]$, where $q_0 \in [1, 3/2)$ is arbitrary but fixed.

Theorem 8.1. *Let $f \in L^2(0,\infty,\, L^{q_0}(\mathbb{R}^3)^3) \cap L^2(0,\infty,\, L^{3/2}(\mathbb{R}^3)^3)$ for some $q_0 \in [1, 3/2)$. Then $\mathfrak{R}^{(\tau)}(f)|S_\infty \in H_\infty$ and inequality (8.1) holds.*

Proof. By Lemma 2.3, there is a sequence (f_n) in $C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3$ converging to f with respect to the norms $\|\cdot\|_{3/2,2;\infty}$ and $\|\cdot\|_{q_0,2;\infty}$. It follows from Corollary 8.1 that the sequence $(\mathfrak{R}^{(\tau)}(f_n)|S_\infty)$ converges in the norm of H_∞. On the other hand, for $T \in (0,\infty)$, the function $\mathfrak{R}^{(\tau)}(f)\,|\,\mathbb{R}^3 \times (0,T)\big)$ belongs to $L^2(0,T,\, H^1(\mathbb{R}^3)^3)$, and the sequence $(\mathfrak{R}^{(\tau)}(f_n)\,|\,\mathbb{R}^3 \times (0,T))$ converges in the norm of $L^2(0,T,H^1(\mathbb{R}^3)^3)$ to the former function. This follows from Lemma 5.1 with $M = T$, $p = 2$, $s = r = 2$, $q = 3/2$. Thus, by a standard trace theorem and Lemma 2.2, the limit function in $L^2(S_T)^3$ of the sequence $(\mathfrak{R}^{(\tau)}(f_n)|S_T)$ coincides with the boundary value of $\mathfrak{R}^{(\tau)}(f)\,|\,\mathbb{R}^3 \times (0,T)$ in $L^2(S_T)^3$. As a consequence, the boundary value $\mathfrak{R}^{(\tau)}(f)|S_\infty$ belongs to H_∞ and is the limit in H_∞ of the sequence $(\mathfrak{R}^{(\tau)}(f_n)|S_\infty)$. The estimate stated in Theorem 8.1 now follows from Corollary 8.1. □

Collecting our results, we arrive at a

Proof of Theorem 1.1. By Theorem 5.4, we have $\mathfrak{I}^{(\tau)}(a)|S_T \in H_T$. Moreover, Theorem 8.1 yields $\mathfrak{R}^{(\tau)}(f)|S_T \in H_T$. Thus, by Theorem 5.3, there is a unique function $\Phi \in L^2_n(S_T)$ with (1.16). We further note that $f \in L^2(0,T,V')$ by Lemma 2.4, and that the inequality $7/4 > 1/s + 3/(2 \cdot \widetilde{q})$ holds for $s = 2$ and $\widetilde{q} = 3/2$. Now existence and uniqueness of a function $v : (0,T) \mapsto H^1(\overline{\Omega}^c)^3$ with (1.13)–(1.15) and the representation formula (1.17) follows from Theorem 5.2, with $s = 2$, $\widetilde{q} = 3/2$, $\epsilon = \epsilon_0 + 1/2$. □

Appendix

In this appendix, we indicate a proof of some results that, although probably not new, are perhaps not so well known, and for which we cannot provide a reference.

Proof of Lemma 2.3. Suppose that $T = \infty$. A similar argument holds if $T < \infty$. Put $f_k := \chi_{B_k \times (0,k)} \cdot f$ for $k \in \mathbb{N}$. Then $\|f - f_k\|_{q_i,p_i;T} \to 0$ for $i \in \{1, 2\}$. Let $m \in \mathbb{N}$, and put $h_l := \max\{-l,\, \min\{l,\, f_m|B_m \times (0,m)\}\}$ for $l \in \mathbb{N}$. Obviously the sequence (h_l) converges to $f_m|B_m \times (0,m)$ with respect to the norm of $L^{p_i}(0,m,L^{q_i}(B_m))$, again for $i \in \{1, 2\}$. Let $l \in \mathbb{N}$. The function h_l is measurable and bounded. Thus, if we set $p_0 := \max\{p_1, p_2, q_1, q_2\}$, we have $h_l \in L^{p_0}(B_m \times (0,m))$, so by a well-known density result, there is a sequence (ψ_j) in $C_0^\infty(B_m \times (0,m))$ with $\|h_l - \psi_j\|_{p_0} \to 0$ $(j \to \infty)$. By the choice of p_0, we may conclude that (ψ_j) converges to h_l in the norm of $L^{p_i}(0,m,L^{q_i}(B_m))$ for $i \in \{1, 2\}$. The lemma follows from these considerations. □

Proof of Theorem 3.7. We proceed as in [49, p. 172; p. 175-176]. Let $u : (0,T) \mapsto H^1(\overline{\Omega}^c)^3$ verify (1.13)–(1.15). Define $A : (0,T) \mapsto V'$ by

$$A(t)(\vartheta) := \int_{\overline{\Omega}^c} ((\nabla_x u(x,t) \cdot \nabla\vartheta(x)) + \tau \cdot \partial_1 u(x,t) \cdot \vartheta(x))\, dx$$

for $t \in (0,T)$, $\vartheta \in V$. Due to (1.13), we have $(f-A)|(0,T') \in L^2(0,T',V')$ for $T' \in (0,T)$. Consider u as a function from $(0,T)$ into V'. Then

$$V'-\int_0^T (-\varphi'(t)\cdot u(t)+\varphi(t)\cdot(A(t)-f(t)))\,dt = 0$$

for $\varphi \in C_0^\infty((0,T))$ by (1.15) and a density argument with respect to ϑ. This means that the weak derivative u' of $u:(0,T)\mapsto V'$ exists and equals $f-A$. In particular, $u|(0,T') \in H^1(0,T',V')$ for $T' \in (0,T)$, hence u is a continuous function from $[0,T)$ into V' ([49, Lemma 3.1.1]). Now it follows

$$-\int_0^T \varphi'(t)\cdot u(t)(\vartheta)\,dt-\int_0^T \varphi(t)\cdot u'(t)(\vartheta)\,dt=\varphi(0)\cdot u(0)(\vartheta) \tag{9.2}$$

for $\vartheta \in V$, $\varphi \in C_0^\infty([0,T))$. Since we want to prove uniqueness of u, we may now suppose $b=0$, $a=0$ and $f=0$. But then the equation $u'+A=f$ reduces to $u'+A=0$, so (9.2) and (1.15) imply $u(0)=0$. Moreover, by (1.14), [25, p. 149–150] and the assumption $b=0$, we have $u(t)\in V$ for $t\in(0,T)$. We further note that $\int_{\overline{\Omega}^c} \partial_1 u(t)\cdot u(t)\,dx = 0$ for $t\in(0,T)$. Thus an argument as in [49, p. 176] yields $u=0$. □

Proof of Lemma 3.5. By the definition of a Bochner integral in $L^2(\mathbb{R}^3)$, there is a sequence (v_n) of simple functions from J into $L := L^2(\mathbb{R}^3)$ such that

$$\int_J \|v_n(t)-v(t)\|_2\,dt \to 0 \quad\text{and}\quad \left\| L-\int_J (v_n(t)-v(t))\,dt \right\|_2 \to 0 \tag{9.3}$$

for $n\to\infty$; see [51, p. 132-134] for example. The second of the preceding relations implies there is a subsequence (w_n) of (v_n) such that

$$\left(L-\int_J w_n(t)\,dt\right)(x) \longrightarrow \left(L-\int_J v(t)\,dt\right)(x) \quad \text{for a.e. } x\in\mathbb{R}^3. \tag{9.4}$$

On the other hand, by Minkowski's inequality (Theorem 3.1),

$$\left(\int_{\mathbb{R}^3}\left(\int_J |w_n(t)(x)-v(t)(x)|\,dt\right)^2 dx\right)^{1/2} \le \int_J \|w_n(t)-v(t)\|_2\,dt, \tag{9.5}$$

so that we may conclude by (9.3) that $\int_J |w_n(t)(x)-v(t)(x)|\,dt<\infty$ for $n\in\mathbb{N}$ and for a.e. $x\in\mathbb{R}^3$, in particular $\int_J |v(t)(x)|\,dt<\infty$ for a.e. $x\in\mathbb{R}^3$. It further follows that the left-hand side of (9.5) as a sequence of n tends to zero. So there is a subsequence (z_n) of (w_n) such that $\int_J z_n(t)(x)\,dt \to \int_J v(t)(x)\,dt$ for a.e. $x\in\mathbb{R}^3$. Since z_n is a simple function, it is obvious that $\int_J z_n(t)(x)\,dt = (L-\int_J z_n(t)\,dt)(x)$ for $x\in\mathbb{R}^3$, $n\in\mathbb{N}$. Therefore the last statement of the lemma follows from (9.4). □

Proof of Lemma 4.5. Take $Y\in C^1((\mathbb{R}^3\backslash\{0\})\times[0,\infty))$. Let $U\subset\mathbb{R}^3\backslash\{0\}$ be open and bounded, with $\overline{U}\subset\mathbb{R}^3\backslash\{0\}$, and let $T>0$. Since $\overline{U}\subset\mathbb{R}^3\backslash\{0\}$, there is $M>0$ with

$$|\partial_s^l\partial_z^\alpha Y(z,s-r)|\le M \quad \text{for } l\in\{0,1\},\ \alpha\in\mathbb{N}_0^3 \text{ with } l+|\alpha|\le 1, \tag{9.6}$$

$z\in U$, $s,r\in[0,T]$ with $s\ge r$. Therefore $\int_0^T r^{-1/2}\cdot|\partial_s^l\partial_z^\alpha Y(z,s-r)|\,dr<\infty$ for l,α,z,s as in (9.6).

Let $\delta\in(0,T)$. Then, if $l\in\{0,1\}$, $\alpha\in\mathbb{N}_0^3$ with $l+|\alpha|\le 1$, the term $r^{-1/2}\cdot\partial_s^l\partial_z^\alpha Y(z,s-r)$ is uniformly continuous as a function of $r,s\in[\delta,T]$ with $s\ge r$, $z\in\overline{U}$. It follows that the function $F^{(1)}(z,s) := \int_\delta^s r^{-1/2}\cdot Y(z,s-r)\,dr$ $(z\in U,\ s\in[\delta,T])$ belongs to $C^1(U\times[\delta,T])$, with

$$\partial_s^l \partial_z^\alpha F^{(1)}(z,s) = \int_\delta^s r^{-1/2} \cdot \partial_s^l \partial_z^\alpha Y(z, s-r)\, dr + \delta_{l1} \cdot s^{-1/2} \cdot Y(z,0) \tag{9.7}$$

for $l \in \{0,1\},\ \alpha \in \mathbb{N}_0^3$ with $l + |\alpha| \le 1,\ z \in U,\ s \in [\delta, T]$. By (9.6) and Lebesgue's theorem, the function $F^{(2)}(z,s) := \int_0^\delta r^{-1/2} \cdot Y(z, s-r)\, dr$ $(z \in U,\ s \in [\delta, T])$ is of class C^1 on $U \times [\delta, T]$, with $\partial_s^l \partial_z^\alpha F^{(2)}(z,s) = \int_0^\delta r^{-1/2} \cdot \partial_s^l \partial_z^\alpha Y(z, s-r)\, dr$ for l, α, z, s as in (9.7). Now we may conclude that the function $F(z,s) := \int_0^s r^{-1/2} \cdot Y(z, s-r)\, dr$ $(z \in U,\ s \in [\delta, T])$ also is of class C^1 on $U \times [\delta, T]$, with

$$\partial_s^l \partial_z^\alpha F(z,s) = \int_0^s r^{-1/2} \cdot \partial_s^l \partial_z^\alpha Y(z, s-r)\, dr + \delta_{l1} \cdot s^{-1/2} \cdot Y(z,0) \tag{9.8}$$

$(l, \alpha, z, s$ as in (9.7)). This is true for any U, δ, T with the above properties, so an analogous result holds if F is defined on $(\mathbb{R}^3 \backslash \{0\}) \times (0, \infty)$ instead of $U \times [\delta, T]$. Since for any $l \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^3$, the function $\partial_4^l \partial^\alpha X$ verifies the assumptions required above for Y, it follows by induction that all statements of the lemma up to and including (4.2) are valid. Equation (4.3) follows from (9.8), an integration by parts on $[s/2, s]$ and a change of variables. $\square$

Proof of Lemma 5.6. Let $g \in C^1(\mathbb{R}^3 \times \mathbb{R})$ with $supp(g) \subset \mathbb{R}^3 \times (0, \infty)$. Then there are numbers $S, M \in (0, \infty)$ with

$$|\partial_r^l \partial_z^\alpha g(z,r)| \le M \cdot \chi_{B_S}(z) \quad \text{for } r \in \mathbb{R},\ z \in \mathbb{R}^3,\ l \in \{0,1\}, \tag{9.9}$$

$\alpha \in \{0,1\}^3$ with $l + |\alpha| \le 1$. Let $\delta, T, R \in (0, \infty)$ with $\delta < T$, and let $\kappa \in (0, \delta]$. Then, for l, α as in (9.9), the term $U(y,s) \cdot \partial_t^l \partial_x^\alpha g(x-y, t-s)$ as a function of $y \in B_{R+S} \backslash B_\kappa,\ s \in [\kappa, T],\ x \in B_R,\ t \in [\delta, \gamma]$ is uniformly continuous. Thus the function

$$G_\kappa(x,t) := \int_\kappa^t \int_{B_{R+S} \backslash B_\kappa} U(y,s) \cdot g(x-y, t-s)\, dy\, ds$$

$(x \in B_R,\ t \in [\delta, \gamma])$ is well defined, belongs to $C^1(B_R \times [\delta, \gamma])$, and verifies the equation

$$\partial_t^l \partial_x^\alpha G_\kappa(x,t) = \int_\kappa^t \int_{B_{R+S} \backslash B_\kappa} U(y,s) \cdot \partial_t^l \partial_x^\alpha g(x-y, t-s)\, dy\, ds \tag{9.10}$$

for $\kappa \in (0, \delta],\ x \in B_R,\ t \in [\delta, T],\ \alpha \in \{0,1\}^3,\ l \in \{0,1\}$ with $|\alpha| + l \le 1$. Note that $g(x-y, 0) = 0$ for $x, y \in \mathbb{R}^3$ so that a derivative with respect to t does not generate additional terms. Now put $H_{\kappa,\alpha,l}(x,t,y,s) := \chi_{B_\kappa^c}(y) \cdot \chi_{[\kappa, T]}(s) \cdot \chi_{(0,\infty)}(t-s) \cdot U(y,s) \cdot \partial_t^l \partial_x^\alpha g(x-y, t-s)$ for $y \in B_{R+S},\ s \in (0, T]$, and for κ, x, t, α, l as in (9.10). By that latter equation, we have $\partial_t^l \partial_x^\alpha G_\kappa(x,t) = \int_0^T \int_{B_{R+S}} H_{\kappa,\alpha,l}(x,t,y,s)\, dy\, ds$ [κ, x, t, α, l as in (9.10)]. Define $H_{\alpha,l}$ in the same way as $H_{\kappa,\alpha,l}$, except that the factors $\chi_{B_\kappa^c}(y)$ and $\chi_{[\kappa,T]}(s)$ are to be dropped. Then we deduce from (9.9) that $|H_{\alpha,l}(x,t,y,s)|$ is bounded by $M \cdot |U(y,s)|$, and the difference $|H_{\alpha,l}(x,t,y,s) - H_{\kappa,\alpha,l}(x,t,y,s)|$ by $M \cdot (\chi_{B_\kappa}(y) + \chi_{(0,\kappa)}(s)) \cdot |U(y,s)|$, for $y \in B_{R+S},\ s \in (0, T]$ and κ, x, t, α, l as in (9.10). Since $\int_0^T \int_{B_{R+S}} |U(y,s)|\, dy\, ds < \infty$, we may define $V_{\alpha,l}(x,t) := \int_0^T \int_{B_{R+S}} H_{\alpha,l}(x,t,y,s)\, dy\, ds$ for x, t, α, l as in (9.10), and we obtain by Lebesgue's theorem and (9.10) that $\partial_t^l \partial_x^\alpha G_\kappa(x,t) \to V_{\alpha,l}(x,t)$ $(\kappa \downarrow 0)$ uniformly in $(x,t) \in B_R \times [\delta, T]$. Now we may conclude that $V_{0,0} \in C^1(B_R \times [\delta, T])$ and $\partial_t^l \partial_x^\alpha V_{0,0}(x,t) = V_{\alpha,l}(x,t)$.

But by (9.9) we have $\partial_t^l \partial_x^\alpha g(x-y, t-s) = 0$ for $y \in B_{R+S}^c,\ s \in (0, T]$ and x, t, α, l as in (9.10), so that

$$\partial_t^l \partial_x^\alpha V_{0,0}(x,t) = \int_0^t \int_{\mathbb{R}^3} U(y,s) \cdot \partial_t^l \partial_x^\alpha g(x-y, t-s)\, dy\, ds.$$

Since R, T were arbitrarily chosen in $(0, \infty)$, the preceding results on $V_{0,0}$ remain valid if $V_{0,0}$ is defined on $\mathbb{R}^3 \times (0, \infty)$ instead of $B_R \times [\delta, T]$. Thus all statements in Lemma 5.6 under the restriction $t > 0$ follow by induction because for $l \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^3$, the function $\partial_4^l \partial^\alpha f$ satisfies the conditions imposed above on g. We further note there is $\delta > 0$ with $f|\mathbb{R}^3 \times (-\infty, \delta] = 0$, hence $F|\mathbb{R}^3 \times (-\infty, \delta] = 0$. This observation removes the restriction $t > 0$. □

The following proof fills a gap in [13], where we justified the relation $\mathfrak{R}^{(\tau)}(f)|S_\infty \in H_\infty$ by claiming that $\mathfrak{R}^{(\tau)}(f)$ belongs to $C_0^\infty(\mathbb{R}^3 \times (0, \infty))^3$ [13, Theorem 2.16 and p. 909], which is, of course, not true in general.

Proof of Lemma 5.13. There are numbers ϵ, δ, $Q, S \in (0, \infty)$ with $\epsilon < \delta$ and

$$|\partial_s^l \partial_y^\alpha f(y, \sigma)| \leq Q \cdot \chi_{(\epsilon, \delta)}(\sigma) \cdot \chi_{B_S}(y) \quad \text{for } y \in \mathbb{R}^3, \ \sigma \in (0, \infty), \tag{9.11}$$

$l \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^3$ with $l + |\alpha| \leq 1$. In particular $\mathfrak{R}^{(\tau)}(f)|\mathbb{R}^3 \times [0, \epsilon] = 0$. By Lemma 5.7, we know that $\mathfrak{R}^{(\tau)}(f) \in C^\infty(\mathbb{R}^3 \times [0, \infty))^3$. Let $\varphi \in C^\infty(\mathbb{R})$ with $\varphi|[-1, 1] = 1$, $\varphi|\mathbb{R}\backslash[-2, 2] = 0$. Put $\varphi_n(t) := \varphi(n^{-1} \cdot t)$ for $n \in \mathbb{N}$, $t \in \mathbb{R}$. Then we obtain that $\varphi_n(t) = 1$ for $t \in [0, n]$, $\varphi_n(t) = 0$ for $t \in [2 \cdot n, \infty)$, $|\varphi_n'(t)| \leq \mathfrak{C} \cdot \chi_{[n,\infty)}(t)$ for $t \in (0, \infty)$, $n \in \mathbb{N}$. Let $\psi \in C_0^\infty(\mathbb{R}^3)$ with $\overline{\Omega} \subset supp(\psi)$, and put $F_n(x, t) := \psi(x) \cdot \varphi_n(t) \cdot \mathfrak{R}^{(\tau)}(f)(x, t)$ $(n \in \mathbb{N}, \ x \in \mathbb{R}^3, \ t \in [0, \infty))$. Then we have $F_n|S_\infty \in \widetilde{H}_\infty$. For $n \in \mathbb{N}$, put $K_n := \mathfrak{R}^{(\tau)}(f) - F_n$,

$$R_n(t) := \|K_n(\,\cdot\,, t)|\partial\Omega\|_{1,2}^2 + \|n^{(\Omega)} \cdot \partial_t K_n(\,\cdot\,, t)|\partial\Omega\|_{H^1(\partial\Omega)'}^2 + \|\partial_4^{1/2} K_n(\,\cdot\,, t)|\partial\Omega\|_2^2 \quad (t \in (0, \infty)).$$

Obviously

$$\begin{aligned} R_n(t) \leq \mathfrak{C} \cdot (1 - \varphi_n(t))^2 \cdot \left(\|\mathfrak{R}^{(\tau)}(f)(\,\cdot\,, t)|\partial\Omega\|_2^2 + \sum_{l=1}^4 \|\partial_l \mathfrak{R}^{(\tau)}(f)(\,\cdot\,, t)|\partial\Omega\|_2^2 \right) \\ + \mathfrak{C} \cdot |\varphi_n'(t)|^2 \cdot \|\mathfrak{R}^{(\tau)}(f)(\,\cdot\,, t)|\partial\Omega\|_2^2 + A_n(t), \end{aligned}$$

with $A_n(t) := \int_{\partial\Omega} | \int_0^t (t - s)^{-1/2} \cdot \partial_s((1 - \varphi_n)(s) \cdot \mathfrak{R}^{(\tau)}(f)(x, s)) \ ds|^2 \ do_x$. We obtain the term $A_n(t)$ by performing the change of variables $r = t - s$, then taking the derivative with respect to t, and then performing the same change of variables again.

Now let $n \in \mathbb{N}$ with $n \geq 2 \cdot \delta$, with δ from (9.11). For $s \in (0, \infty)$ with $(1 - \varphi_n)(s) \neq 0$ or $\varphi_n'(s) \neq 0$, we have $s \geq n \geq 2 \cdot \delta$, so $s - \sigma \geq s/2$ for $\sigma \in (0, \delta)$. Thus by (9.11) and Lemma 4.3 with an arbitrary K (for example $K = 1$), for s as before, $x \in \partial\Omega$, $y \in \mathbb{R}^3$, $\sigma \in (0, s)$, $l \in \{1, \ldots, 4\}$, $1 \leq j, k \leq 3$, $g = f$ and $g = \partial_l f$,

$$|\Lambda_{jk}(x - y, s - \sigma) \cdot g_k(y, \sigma)| \leq \mathfrak{C} \cdot Q \cdot s^{-3/2} \cdot \chi_{(0,\delta]}(\sigma) \cdot \chi_{B_S}(y).$$

Recalling that $\partial_l \mathfrak{R}^{(\tau)}(f) = \mathfrak{R}^{(\tau)}(f)(\partial_l f)$ for $1 \leq l \leq 4$ (Lemma 5.7), we may conclude for $s \in (0, \infty)$ and x, l, g as before that

$$((1 - \varphi_n) + |\varphi_n'|)(s) \cdot |\mathfrak{R}^{(\tau)}(g)(x, s)| \leq \mathfrak{C}(S, \delta) \cdot Q \cdot s^{-3/2} \cdot \chi_{[n,\infty)}(s). \tag{9.12}$$

Therefore $R_n(t) \leq \mathfrak{C}(S, \delta) \cdot Q^2 \cdot \chi_{[n,\infty)}(t) \cdot t^{-3} + |A_n(t)|$ for $t \in (0, \infty)$, and the integral with respect to s in the definition of $A_n(t)$ extends from n to t if $t > n$, else $A_n(t) = 0$. Moreover, for $t \in [n, 2 \cdot n]$, we conclude with (9.12) that $|A_n(t)| \leq \mathfrak{C}(S, \delta) \cdot Q^2 \cdot (\int_n^t (t - s)^{-1/2} \cdot s^{-3/2} \, ds)^2$, so that $|A_n(t)| \leq \mathfrak{C}(S, \delta) \cdot Q^2 \cdot \chi_{[n,\infty)}(t) \cdot t^{-2}$ for such t. If $t \in [2 \cdot n, \infty)$, we perform an integration by parts on the interval $[n, t/2]$ of the integral with respect to s in the definition of $A_n(t)$, whereas the integral on $[t/2, t]$ is estimated directly by means of (9.12). We obtain that $|A_n(t)|$ is bounded by

$$\mathfrak{C}(S,\delta)\cdot\Bigg(Q^2\cdot t^{-2}\cdot\chi_{[n,\infty)}(t)+t^{-1}\cdot(1-\varphi_n)^2(t/2)\cdot\|\mathfrak{R}^{(\tau)}(f)(\,\cdot\,,t/2)|\partial\Omega\|_2^2$$

$$+\int_{\partial\Omega}\Bigg|\int_n^{t/2}(t-s)^{-3/2}\cdot(1-\varphi_n)(s)\cdot\mathfrak{R}^{(\tau)}(f)(x,s)\,ds\Bigg|^2\,do_x\Bigg)$$

for $t\in[2\cdot n,\infty)$. Using (9.12) again, and noting that $(t-s)^{-3/2}\le 2^{3/2}\cdot t^{-3/2}$ for $s\in[n,t/2]$, we thus get $|A_n(t)|\le\mathfrak{C}(S,\delta)\cdot Q^2\cdot\chi_{[n,\infty)}(t)\cdot t^{-2}$ for $t\in[2\cdot n,\infty)$. Therefore $R_n(t)\le\mathfrak{C}(S,\delta)\cdot Q^2\cdot\chi_{[n,\infty)}(t)\cdot t^{-2}$ for any $t\in(0,\infty)$. This estimate implies the lemma. □

Proof of Lemma 7.10. Let $g\in C_0^\infty(\mathbb{R}^3\times(0,\infty))^3$. Since g is bounded, and because of Lemma 4.1, the integral $\int_0^t\int_{\mathbb{R}^3}\int_0^{t-r}|(t-r-s)^{-1/2}\cdot\mathfrak{H}(x-y,s)\cdot g(y,r)|\,ds\,dy\,dr$ is finite. Therefore the integral appearing in the definition of $\mathfrak{M}(g)(x,t)$ [see (1.27)] may be transformed by Fubini's theorem, some simple changes of variables and Lebesgue's theorem. Setting $\mathfrak{H}(z,\kappa):=0$ for $z\in\mathbb{R}^3,\ \kappa<0$, we get

$$\mathfrak{M}(g)(x,t)=\lim_{n\to\infty}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^3}F_n(z,\kappa,\sigma)\,dz\,d\kappa\,d\sigma, \tag{9.13}$$

where $F_n(z,\kappa,\sigma):=\chi_{(0,n)}(\sigma)\cdot|\sigma|^{-1/2}\cdot\chi_{(0,\infty)}(\kappa+\sigma)\cdot\mathfrak{H}(z,\kappa)\cdot g(x-z,t-\sigma-\kappa)$ for $z\in\mathbb{R}^3\backslash\{0\},\ \kappa,\sigma\in\mathbb{R},\ n\in\mathbb{N}$. But $\chi_{(0,n)}(\sigma)\cdot\chi_{(0,\infty)}(\kappa+\sigma)\cdot\mathfrak{H}(z,\kappa)=\chi_{(0,n)}(\sigma)\cdot\mathfrak{H}(z,\kappa)$ for z,σ,κ,n as before. Thus, in the definition of the function F_n, the factor $\chi_{(0,\infty)}(\kappa+\sigma)$ may be dropped.

Let $\sigma\in\mathbb{R}$. Define $g_{x,t,\sigma}(z,\kappa):=g(x-z,t-\sigma-\kappa)$ for $z\in\mathbb{R}^3,\ \kappa\in\mathbb{R}$. Then $g_{x,t,\sigma}\in C_0^\infty(\mathbb{R}^4)^3$, hence $(g_{x,t,\sigma})^\vee$ is rapidly decreasing. Using Plancherel's equation and a well known formula for the Fourier transform of $\mathfrak{H}$, we get

$$\int_{\mathbb{R}}\int_{\mathbb{R}^3}\mathfrak{H}(z,\kappa)\cdot g_{x,t,\sigma}(z,\kappa)\,dz\,d\kappa=\int_{\mathbb{R}}\int_{\mathbb{R}^3}H^{(1)}(\xi,\varrho)\,d\xi\,d\varrho,$$

with $H^{(1)}(\xi,\,\varrho):=((2\cdot\pi)^2\cdot(|\xi|^2-i\cdot\varrho))^{-1}\cdot(g_{x,t,\sigma})^\vee(\xi,\,\varrho)$. But a simple computation shows that $(g_{x,t,\sigma})^\vee(\xi,\,\varrho)=e^{i\cdot(x\cdot\xi+(t-\sigma)\cdot\varrho)}\cdot\widehat{g}(\xi,\,\varrho)$ for $\xi\in\mathbb{R}^3,\ \ \varrho\in\mathbb{R}$. It follows from (9.13) that $\mathfrak{M}(g)(x,t)=\lim_{n\to\infty}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^3}H_n^{(2)}(\xi,\,\varrho,\,\sigma)\,d\xi\,d\varrho\,d\sigma$, with

$$H^{(2)}(\xi,\,\varrho,\,\sigma):=\chi_{(0,n)}(\sigma)\cdot|\sigma|^{-1/2}\cdot e^{-i\cdot\sigma\cdot\varrho}\cdot e^{i\cdot(x\cdot\xi+t\cdot\varrho)}$$
$$\cdot((2\cdot\pi)^2\cdot(|\xi|^2-i\cdot\varrho))^{-1}\cdot\widehat{g}(\xi,\varrho)\quad\text{for }\ \xi\in\mathbb{R}^3,\ \ \varrho,\sigma\in\mathbb{R},\ (\xi,\,\varrho)\ne 0.$$

Moreover $|H^{(2)}(\xi,\,\varrho,\,\sigma)|\le\mathfrak{C}\cdot\chi_{(0,n)}(\sigma)\cdot|\sigma|^{-1/2}\cdot|\xi|^{-2}\cdot|\widehat{g}(\xi,\varrho)|$. Since $\widehat{g}$ is rapidly decreasing, the right-hand side of the preceding inequality is integrable with respect to $\sigma,\ \xi$ and ϱ, so we may apply Fubini's theorem and perform the change of variables $\sigma=\gamma/|\varrho|$, to obtain

$$\mathfrak{M}(g)(x,t)=\lim_{n\to\infty}\int_{\mathbb{R}}\int_{\mathbb{R}^3}(\varphi_1(n\cdot|\varrho|)-(\varrho/|\varrho|)\cdot i\cdot\varphi_2(n\cdot|\varrho|))\cdot H^{(3)}(\xi,\,\varrho)\,d\xi\,d\varrho, \tag{9.14}$$

with $H^{(3)}(\xi,\,\varrho):=e^{i\cdot(x\cdot\xi+t\cdot\varrho)}\cdot(|\varrho|^{1/2}\cdot(2\cdot\pi)^2\cdot(|\xi|^2-i\cdot\varrho))^{-1}\cdot\widehat{g}(\xi,\,\varrho)$ for $\xi\in\mathbb{R}^3,\ \ \varrho\in\mathbb{R}\backslash\{0\},\ \ \varphi_1(s):=\int_0^s\gamma^{-1/2}\cdot\cos\gamma\ d\gamma,\ \ \varphi_2(s):=\int_0^s\gamma^{-1/2}\cdot\sin\gamma\ d\gamma$ for $s\in[0,\infty)$. The functions φ_1,φ_2 are continuous on $[0,\infty)$, with $\varphi_j(s)\to(\pi/2)^{1/2}$ for $s\to\infty$ $(j\in\{1,\,2\})$. In particular, these functions are bounded. This latter fact implies that the term $|(\varphi_1(n\cdot|\varrho|)-(\varrho/|\varrho|)\cdot i\cdot\varphi_2(n\cdot|\varrho|))\cdot H^{(3)}(\xi,\,\varrho)|$ is bounded by $\mathfrak{C}\cdot|\varrho|^{-1/2}\cdot|\xi|^{-2}\cdot|\widehat{g}(\xi,\varrho)|$ for $\varrho\in\mathbb{R}\backslash\{0\},\ \ \xi\in\mathbb{R}^3\backslash\{0\}$, with the preceding term being integrable with respect to ϱ and ξ. Thus we see by Lebesgue's theorem that we may perform the passage to the limit $n\to\infty$ inside the integral on the right-hand side of (9.14). Again using the relation $\varphi_j(s)\to(\pi/2)^{1/2}$ $(s\to\infty)$ for $j\in\{1,\,2\}$, we arrive at the equation

$$\mathfrak{M}(g)(x,t) = \int_{\mathbb{R}} \int_{\mathbb{R}^3} e^{i\cdot(x\cdot\xi+t\cdot\varrho)} \cdot (1-(\varrho/|\varrho|)\cdot i) \cdot (|\varrho|^{1/2} \cdot 2^{5/2} \cdot \pi^{3/2} \cdot (|\xi|^2 - i\cdot\varrho))^{-1} \cdot \widehat{g}(\xi,\,\varrho)\,d\xi\,d\varrho.$$

Replacing g by $\partial_4\partial_m f$ and recalling that $\partial_4\partial_m\mathfrak{M}(f) = \mathfrak{M}(\partial_4\partial_m f)$ (Lemma 5.7), we obtain Lemma 7.10. □

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