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Pointwise Spatial Decay of Weak Solutions to the Navier–Stokes System in 3D Exterior Domains

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Abstract. We consider L^2-weak solutions to the Navier–Stokes system in a 3D exterior domain. Under the assumptions that the initial data decrease as $O(|x|^{-\mu})$ when $|x| \to \infty$, for some $\mu \geq 7/6$, and that the volume force decays sufficiently fast, we show that the velocity decreases pointwise with the rate $O(\,|x|^{-\min\{\mu,\,13/5\}})$ for $|x| \to \infty$, uniformly with respect to time.

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1. Introduction

Consider the time-dependent Navier–Stokes system

$$\partial_t v - \Delta_x v + \tau \cdot (v \cdot \nabla_x) v + \nabla_x \varrho \;=\; g, \quad \operatorname{div}_x v = 0 \quad \text{in } \; Z_\infty := (\mathbb{R}^3 \backslash \overline{\Omega}) \times (0, \infty), \tag{1.1}$$

under a homogeneous Dirichlet boundary condition on $S_\infty := \partial\Omega \times (0, \infty)$ and with vanishing velocity at infinity,

$$v|S_\infty = 0, \quad v(x,t) \to 0 \;\; (|x| \to \infty) \;\text{ for }\; t \in (0, \infty), \tag{1.2}$$

and with the initial condition

$$v(x, 0) = v_0(x) \;\text{ for }\; x \in \overline{\Omega}^c := \mathbb{R}^3 \backslash \overline{\Omega}, \tag{1.3}$$

where Ω is a bounded open set in $\mathbb{R}^3$ with C^2-boundary $\partial\Omega$. This initial-boundary value problem is a mathematical model for a viscous incompressible fluid moving around a rigid body described by the set Ω. The velocity at infinity of the fluid is supposed to be zero. The initial velocity $v_0 : \overline{\Omega}^c \mapsto \mathbb{R}^3$ of the flow, the Reynold's number $\tau \in (0, \infty)$ and the volume force $g : Z_\infty \mapsto \mathbb{R}^3$ are given, whereas the velocity field $v : Z_\infty \mapsto \mathbb{R}^3$ and the pressure field $\varrho : Z_\infty \mapsto \mathbb{R}$ of the flow are unknown.

There is an extensive literature dealing with temporal decay of spatial L^p-norms of solutions to (1.1). As examples, we mention [6–10, 16–18, 28, 33–35, 40, 42, 45, 46, 49, 52–55, 62, 65]. Also pointwise decay of strong solutions to the Cauchy problem

$$\partial_t w - \Delta_x w + \tau \cdot (w \cdot \nabla_x) w + \nabla_x \pi \;=\; h, \quad \operatorname{div}_x w = 0 \quad \text{in } \; \mathbb{R}^3 \times (0, \infty), \tag{1.4}$$

$$w(x, 0) = w_0(x) \quad (x \in \mathbb{R}^3) \tag{1.5}$$

has been studied rather frequently. As far as we know, Knightly [37, 38] was the first to take up this subject. He constructed analytical solutions to (1.4), (1.5) exhibiting spatial and temporal decay. However, his estimates of pointwise spatial decay do not hold uniformly with respect to time and require smallness assumptions on the data. Takahashi [59] established uniform in time estimates, under the assumption that the velocity is small in some $L^p(L^q)$-norm. In subsequent papers [3, 5, 13, 36, 41, 48], Takahashi's smallness

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condition is replaced by various other smallness assumptions, some of which take the form of asymptotic decay of certain quantities when t tends to infinity. Other authors [43, Chapter 25], [61] established existence results implying spatial decay of the velocity without any smallness condition, but this decay does not seem to be uniform in the lifespan of the solution in question. If the initial data exhibit certain symmetries, decay rates increase [13,14]. Results on spatial asymptotics of the velocity are presented in [5,15,18], under the constraint that the velocity is bounded by a term of the form $C\cdot(1+|x|)^{-\alpha}\cdot(1+t)^{-\beta}$, with suitable constants C, α, β.

He and Miyakawa [35] constructed strong solutions to problem (1.1)–(1.3) (exterior domain problem) which decay pointwise with respect to space and to time, provided that $g=0$ and the initial data are small in $L^3(\overline{\Omega}^c)^3$. The spatial decay is uniform with respect to $t\in[1,\infty)$.

In the work at hand, we consider L^2-weak solutions of (1.1)–(1.3). We will show that if the initial data v_0 decrease as $O(|x|^{-\mu})$ when $|x|$ tends to ∞, for some $\mu\geq 7/6$, and if the function g decays sufficiently fast for large values of $|x|$, then the velocity part of an L^2-weak solution to (1.1)–(1.3) exhibits the decay rate $O(|x|^{-\min\{\mu,13/5\}})$ for $|x|\to\infty$, uniformly in $t\in(0,\infty)$. Important features of this result are that we consider weak solutions in an exterior domain instead of strong ones in the whole space $\mathbb{R}^3$, we do not need any smallness condition, and our estimates are uniform in time for $t\in(0,\infty)$. It remains an open question how to obtain combined estimates of spatial and temporal decay of weak solutions, and whether a spatial decay rate $O(|x|^{-3+\epsilon})$ may be established for $\epsilon>0$ arbitrarily small, as was done in [35] for the case of strong solutions.

Let us state our theory in more detail. First we give a list of our assumptions, using notation as specified above and in Sect. 2.

(A1) $v_0\in L^2_\sigma(\overline{\Omega}^c)\cap L^{5/4}_\sigma(\overline{\Omega}^c)$; there is $S_0,C_0>0,\ \mu\in[7/6,\infty)$ with $|v_0(y)|\leq C_0\cdot|y|^{-\mu}$ for $y\in B^c_{S_0}$.

(A2) The function $g:Z_\infty\mapsto\mathbb{R}^3$ is measurable with $g|Z_{S_0,\infty}\in L^{5/4}(Z_{S_0,\infty})^3$; there is $\gamma\in L^{5/4}((0,\infty))$, $\widetilde{\mu}\in(3,\infty)$ such that $|g(y,s)|\leq\gamma(s)\cdot|y|^{-\widetilde{\mu}}$ for $y\in B^c_{S_0},\ s\in(0,\infty)$.

(A3) The function v is the velocity part of a weak solution to (1.1)–(1.3) in the sense that $v\in L^\infty(0,\infty,\,L^2_\sigma(\overline{\Omega}^c))\cap L^2_{loc}([0,\infty),\,V),\ \nabla_x v\in L^2(Z_\infty)^9$,

$$\int_0^\infty\int_{\overline{\Omega}^c}\left[-\varphi'(t)\cdot v(t)\cdot\vartheta+\varphi(t)\cdot(\nabla_x v(t)\cdot\nabla\vartheta)+\varphi(t)\cdot\tau\cdot(v(t)\cdot\nabla_x)\,v(t)\cdot\vartheta-\varphi(t)\cdot g(t)\cdot\vartheta\right]\,dx\,dt$$
$$=\int_{\overline{\Omega}^c}v_0\cdot\vartheta\,dx\cdot\varphi(0)$$

for $\varphi\in C_0^\infty([0,\infty)),\ \vartheta\in C_0^\infty(\overline{\Omega}^c)^3$ with $\operatorname{div}\vartheta=0$.

An important role in our theory is played by Theorem 6.2, which states that $|v(x,t)|$ is bounded uniformly in $t\in(0,\infty)$ if x is located outside a sufficiently large ball—a deep-lying but well known regularity result for weak solutions to (1.1)–(1.3). For this theorem to hold, some additional conditions on the data and on v are needed. According to [63, Satz III.5.1], Theorem 6.2 is valid if (A3) and the following assumptions (A4)–(A8) are satisfied. We introduce the conditions in (A4)–(A8) only in view of that theorem; they will not be exploited in any other context. Functions v, ϱ with properties as in (A3) and (A5)–(A8) exist by [63, Satz II.4.1; p. 306/307; Satz II.5.1].

(A4) There are constants $\overline{q}\in(5/2,\infty),\ \ \delta\in(0,1),\ \ \overline{\gamma}\in(0,1/4]$ such that $v_0\in D(A_2^{3/4+\overline{\gamma}})\cap L^{9/8}_\sigma(\overline{\Omega}^c)$, and such that g belongs to $C^\delta([0,\infty),\,L^2(\overline{\Omega}^c)^3)\cap L^1_{loc}([0,\infty),\,L^{9/8}(\overline{\Omega}^c)^3)$, as well as to $L^r_{loc}(0,\infty,\,L^s(\overline{\Omega}^c)^3)$ for $r=s=\overline{q}$ and for $r=2,\ s=9/8$. Moreover $g\in L^1(0,\infty,\,L^2(\overline{\Omega}^c)^3)$ and $\|g(t)\|_2\to 0\ (t\to\infty)$.

(A5) $v'\in L^q_{loc}(0,\infty,\,L^r(\overline{\Omega}^c)^3)$ with $q=3/2,\ r=9/8$ and $q=r=5/4$,
$v\in L^q_{loc}(0,\infty,\mathfrak{B})$ with $q=3/2,\ \mathfrak{B}=D(A_{9/8})$ and $q=5/4,\ \mathfrak{B}=D(A_{5/4})$.

(A6) There is a measurable mapping $\varrho:(0,\infty)\mapsto L^{15/7}(\overline{\Omega}^c)\cap L^{9/5}(\overline{\Omega}^c)$ such that $\varrho\in L^q_{loc}(0,\infty,\ L^r(\overline{\Omega}^c))$ with $q=5/4,\ r=15/7$ and $q=3/2,\ r=9/5$, and $\nabla_x\varrho$ belongs to $L^q_{loc}(0,\infty,\ L^r(\overline{\Omega}^c)^3)$ with $q=r=5/4$ and $q=3/2,\ r=9/8$, and such that the pair of functions (v,ϱ) satisfies (1.1).

(A7) The function v is a "suitable weak solution", in the sense that

$$\int_{\overline{\Omega}^c} \varphi(t) \cdot |v(t)|^2 \, dx + \int_s^t \int_{\overline{\Omega}^c} \varphi \cdot |\nabla_x v|^2 \, dx \, dr$$
$$\leq \int_{\overline{\Omega}^c} \varphi(0) \cdot |v_0|^2 \, dx + \int_s^t \int_{\overline{\Omega}^c} \left[|v|^2 \cdot (\varphi_r + \Delta_x \varphi) + (|v|^2 + 2 \cdot \varrho) \cdot v \cdot \nabla_x \varphi + 2 \cdot (P_2 g \cdot v) \cdot \varphi \right] \, dx \, dr$$

for a.e. $s \in (0, \infty)$, for any $t \in [s, \infty)$ and for $\varphi \in C^2(\Omega^c \times [0, \infty))$ with $\varphi \geq 0$.

(A8) $\|v(t)\|_2^2 + 2 \cdot \int_s^t \|\nabla_x v(r)\|_2^2 \, dr \leq \|v(s)\|_2^2 + 2 \cdot \int_s^t \int_{\overline{\Omega}^c} g \cdot v \, dx \, dr$ for a.e. $s \in (0, \infty)$, for $s = 0$ and for any $t \in [s, \infty)$.

Our main result may now be stated as follows.

Theorem 1.1. *Under the assumptions* (A1)–(A3)*, the relation* $v \in C^0([0, \infty), L^2(\overline{\Omega}^c)^3) + C^0([0, \infty), L^{5/4}(\overline{\Omega}^c)^3)$ *holds.*

Moreover, if the conditions in (A1)–(A8) *are valid, then there are constants* $R_0, c_0 \in (0, \infty)$ *such that* $\overline{\Omega} \subset B_{R_0}$ *and*

$$|v(x,t)| \leq c_0 \cdot |x|^{-\min\{\mu, 13/5\}} \quad \textit{for any } t \in (0, \infty) \textit{ and a.e. } x \in B_{R_0}^c. \tag{1.6}$$

We remark that the assumption $\mu \geq 7/6$ in (A1) may probably be replaced by the requirement $\mu \geq 1 + \epsilon$, for some $\epsilon > 0$. But such a generalization would give rise to additional technical problems we wanted to avoid. Of course, if in (A1) we drop the condition $v_0 \in L_\sigma^{5/4}(\overline{\Omega}^c)$ and require instead that $\mu > 12/5$, we still have $v_0 \in L^{5/4}(\overline{\Omega}^c)^3$. Hence, by [29, p. 119] and the assumption $v_0 \in L_\sigma^2(\overline{\Omega}^c)$, we recover the relation $v_0 \in L_\sigma^{5/4}(\overline{\Omega}^c)$. Thus a slightly less general but somewhat more explicit variant of (A1) reads like this:

(A1′) $v_0 \in L_\sigma^2(\overline{\Omega}^c)$, and there is $S_0, C_0 > 0$, $\mu \in (12/5, \infty)$ with $|v_0(y)| \leq C_0 \cdot |y|^{-\mu}$ for $y \in B_{S_0}^c$.

Inequality (1.6) is optimal for $\mu \in [7/6, 13/5]$, in the sense that for such μ, the velocity part of the solution in question reproduces the decay rate of the initial data. The upper bound 13/5 is ultimately due to the fact that for weak solutions as in (A3), the nonlinearity $(v \cdot \nabla)v$ belongs to $L^{5/4}(\overline{\Omega}^c)^3$ ([32]; Theorem 6.1). We did not address the question as to which is the largest range of functions g such that inequality (1.6) remains valid. Instead we restricted ourselves to giving an example, that is, (A2), of a list of suitable assumptions.

Our results are similar to those in [25], where the Navier–Stokes system with Oseen term,

$$\partial_t v - \Delta_x v + \tau \cdot \partial_{x_1} v + \tau \cdot (v \cdot \nabla_x) v + \nabla_x \varrho \;=\; g, \quad \mathrm{div}_x v = 0 \quad \text{in } Z_T := \overline{\Omega}^c \times (0, T) \tag{1.7}$$

is considered for a given number $T \in (0, \infty]$. But there is an important difference between the theory in [25] and the one presented here: whereas in [25], we consider only strong solutions of (1.7), here we admit weak solutions to (1.1). There are essentially two reasons for this disparity. The first concerns boundedness of solutions to (1.1) and (1.7), respectively, outside a large ball. As mentioned further above, such a boundedness property is well known for weak solutions of (1.1)–(1.3) (Theorem 6.2). However, no such result seems to be available for solutions of (1.7), neither for weak nor for strong ones, so an appropriate estimate had to be proved in [25, Lemma 4.2]. But the proof of that estimate carries through only for strong solutions.

The second major reason why weak solutions are admitted here but not in [25] is related to a key feature of our theory both here and in [25], that is, integral representations of the velocity part of solutions to a linearized problem—the Stokes system (1.8) here and the Oseen system in [25]. These integral representations give access to decay estimates of solutions to the respective linearized problem; in a second step, these estimates are then extended to the nonlinear case. The integral representations established in the work at hand are Green's formulas, whereas in [25, Theorem 3.4] we solved an integral equation on the boundary S_T in order to obtain a suitable representation. In the context of (1.1) or (1.7), Green's formulas have the disadvantage of featuring a critical term involving the trace on $\partial\Omega$ of the pressure and of the first-order space derivatives of the velocity. In the work at hand, this critical term is handled in Corollary 5.1 below, which is perhaps the key observation of the present theory, giving rise to the number 13/5 in the decay rate $O(|x|^{-\min\{\mu,\,13/5\}})$ in (1.6). But it is not self-evident that Corollary 5.1 can be exploited in the nonlinear case. In fact, here this will be possible—via Theorems 5.2 and 6.3—because of the relation $(v \cdot \nabla_x)v \in L^{5/4}(Z_\infty)^3$ (stated in Theorem 6.1) and due to maximal $L^{5/4}$-regularity of solutions to the linearized problem (1.8)–(1.10) (stated in Theorem 3.5). In the context of the Oseen system, though—used as linearized problem in [25], as mentioned above—such regularity does not seem to be available. By choosing an approach via an integral equation in order to arrive at a suitable representation formula, we were able in [25] to circumvent this difficulty. In fact, the formula obtained in this way does not feature the critical term just mentioned. But the requirements we had to impose on the data of the integral equation in question are such that when we applied our linear theory to the nonlinear equation (1.7) [25, Section 4], we could only admit strong solutions.

Some remarks are perhaps in order as to how we structure our argument. As already mentioned above, we use the Stokes system

$$\partial_t u - \Delta_x u + \nabla_x p = f, \quad \mathrm{div}_x u = 0 \quad \text{in } Z_\infty \tag{1.8}$$

as linearization of (1.1). Equation (1.8) is supplemented by boundary and initial conditions as in (1.2) and (1.3), respectively:

$$u|S_\infty = 0, \quad u(x,t) \to 0 \ \ (|x| \to \infty) \ \text{ for } \ t \in (0,\infty), \tag{1.9}$$

$$u(x,0) = u_0(x) \ \text{ for } \ x \in \overline{\Omega}^c. \tag{1.10}$$

In Sect. 3, we will present various auxiliary results related to problem (1.8)–(1.10). These results are essentially well known, but some technical subtleties seem to be new, or at least we do not know any reference for them. This is true in particular for a uniqueness theorem for mild solutions of (1.8)–(1.10) (Theorem 3.4), which slightly differs from the usual one, as stated for example in [50, Corollary 4.1.5]. This theorem allows to identify L^2-weak solutions and certain mild solutions to (1.8)–(1.10) (proof of Corollary 3.2), and thus gives access to a result on spatial regularity of weak solutions to (1.8)–(1.10) (Corollary 3.2). A much more deep-lying regularity property—maximal $L^{5/4}$-regularity—is also valid for problem (1.8)–(1.10) and is needed for our argument, as indicated above. We will present this property in Theorem 3.5, referring to [31] and [47] for a proof.

Section 4 provides integral representations of the velocity part u of solutions to (1.8)–(1.10). In the first part of that section, we will require two sets of assumptions: either $u_0 \in D^{1/5,\,5/4}_{5/4}$, $f \in L^{5/4}(Z_\infty)^3$, or $u_0 \in V$, $f \in L^2(Z_\infty)^3$. Actually, in the second case, it would be sufficient for our purposes to assume $f = 0$, but we will admit more general functions f as long as they do not give rise to additional difficulties. For both sets of assumptions, we will establish a Green's formula, expressing u by a sum of certain integrals, one of them involving the pressure (Theorem 4.2). In the second part of Sect. 4, we will derive a representation theorem for u under the conditions that $f = 0$ and the initial data only belong to $L^2_\sigma(\overline{\Omega}^c)$. In this situation, the difficulty arises that the pressure has very low regularity near $t = 0$. In order to circumvent that obstacle, we will use the fact that $u(t_0) \in V$ for $t_0 > 0$. This means that the representation formula derived in the first part of Sect. 4 is valid with $u(\cdot\ + t_0)$ in the role of u. After transforming this formula in such a way that the pressure is eliminated, we will then let t_0 tend to zero, obtaining an integral representation without pressure term (Theorem 4.3).

In Sect. 5, we will exploit our representation formulas in order to study the asymptotic behaviour of the velocity part u of solutions to (1.8)–(1.10), under the assumptions $u_0 = v_0$ with v_0 from (A1), and $f = g + \widetilde{g}$ with g from (A2) and $\widetilde{g} \in L^{5/4}(Z_\infty)^3$. The decay result we will obtain is stated in Theorem 5.2. In the proof of this theorem, we will split u into a sum $u^{(1)} + u^{(2)}$, with $u^{(1)}$ being a solution of (1.8)–(1.10) for f as above and $u_0 = 0$, whereas $u^{(2)}$ solves (1.8)–(1.10) for u_0 as above and $f = 0$. We may apply to $u^{(1)}$ the representation formula valid under the conditions $u_0 \in D^{1/5,\,5/4}_{5/4}$ and $f \in L^{5/4}(Z_\infty)^3$ (see Theorem 4.2). The spatial asymptotics of $u^{(1)}$ will then be determined by evaluating the integrals appearing in that formula—with one exception: there is a volume potential, denoted by $\mathfrak{R}(\widetilde{g})$ and involving the function $\widetilde{g}$, that will not be analyzed until we consider the nonlinear problem (1.1)–(1.3) in Sect. 6, where we will take $\widetilde{g} = -\tau \cdot (v \cdot \nabla_x)v$. Concerning $u^{(2)}$, we will derive suitable decay estimates of $u^{(2)}|(0,1]$ by using the representation formula that does not involve the pressure (see Theorem 4.3). Unfortunately we could not see how this formula might yield such estimates uniformly in $t \in (0,\infty)$. In order to get around this difficulty, we will use the fact, established in Sect. 3 (Corollary 3.2), that $u^{(2)}(1) \in D(A_{5/4})$, where $A_{5/4}$ is the Stokes operator in $L^{5/4}(\overline{\Omega}^c)^3$. Since $D(A_{5/4}) \subset D^{1/5,\,5/4}_{5/4}$, we will be able to obtain an integral representation of $u^{(2)}(\cdot\ +1)$ by again using the formula valid in the case $u_0 \in D^{1/5,\,5/4}_{5/4},\ f \in L^{5/4}(Z_\infty)^3$ (see Corollary 3.4, Theorem 4.2). This representation will then lead to decay estimates of $u^{(2)}$ on $(1,\infty)$. We remark that Corollary 5.1—the corollary we discussed further above—enters into the estimate of $u^{(1)}$ and $u^{(2)}|(1,\infty)$.

The last section—Sect. 6—deals with the asymptotic behaviour of the function v introduced in (A3) as velocity part of a weak solution to the nonlinear problem (1.1)–(1.3). As indicated above, we will consider v as a solution to the linear problem (1.8)–(1.10) with the right-hand side f replaced by $g - \tau \cdot (v \cdot \nabla_x)v$, a point of view allowing us to apply the decay results from Theorem 5.2 to v. In this way the problem reduces to determining the spatial asymptotics of the volume potential $\mathfrak{R}(\widetilde{g})$ mentioned above, with $-\tau \cdot (v \cdot \nabla_x)v$ in the role of $\widetilde{g}$. In a first step in this direction, this potential is transformed by an integration by parts into a term—denoted by $\mathfrak{A}$—with a space derivative acting on the original kernel and with the nonlinearity $-\tau \cdot ((v_l \cdot \partial_l)v_k)_{1\le k,l\le 3}$ replaced by $\tau \cdot (v_l \cdot v_k)_{1\le k,l\le 3}$. Thus we will have to study the asymptotic behaviour of the potential $\mathfrak{A}$. Our starting point in this regard is Theorem 6.2, which was already mentioned above and which states, as we may recall, that v is bounded pointwise outside a large ball, uniformly with respect to time. On referring to this theorem, using an argument by Babenko [4], and applying the decay estimates from Theorem 5.2, we may show that $|v(x,t)| = O(|x|^{-7/6})$ for $|x| \to \infty$, uniformly in $t \in (0,\infty)$ (Theorem 6.4). Once this result is available, it will be possible to prove that $\mathfrak{A}$ decays with a rate higher than $-7/6$. Referring to Theorem 5.2 once more, we will then conclude that the function v also decays with this higher rate. The technical details of this argument may be found in the proof of Lemma 6.6. Depending on the size of the parameter μ from (A1), this reasoning either yields the looked-for inequality (1.6) directly, or has to be repeated, leading to (1.6) after a finite number of iterations.

2. Notation: Preliminary Results

Recall that in Sect. 1, we introduced the open bounded set Ω with C^2-boundary $\partial\Omega$. This set will be kept fixed throughout. The notation $\overline{\Omega}^c$, also introduced in Sect. 1, is generalized to $A^c := \mathbb{R}^3\backslash A$ for any $A \subset \mathbb{R}^3$. We additionally recall that we defined $Z_T := \overline{\Omega}^c \times (0,T)$ and $S_T := \partial\Omega \times (0,T)$ for $T \in (0,\infty]$. Let $n^{(\Omega)}$ denote the outward unit normal to Ω. We put $B_r(x) := \{y \in \mathbb{R}^3\ :\ |x-y| < r\}$ for $x \in \mathbb{R}^3,\ r > 0,\ B_S := B_S(0),\ \Omega_S := B_S\backslash\overline{\Omega},\ Z_{S,T} := \Omega_S \times (0,T)$ for $S \in (0,\infty),\ T \in (0,\infty]$. Set $e_l = (\delta_{jl})_{1\le j\le 3}$. For a multiindex $\alpha \in \mathbb{N}_0^3$, we use the abbreviation $|\alpha|$ for the length $\alpha_1+\alpha_2+\alpha_3$ of α.

We write C for numerical constants, and $C(\gamma_1,\ldots,\gamma_n)$ for constants depending on quantities $\gamma_1,\ldots,\gamma_n$, for some $n \in \mathbb{N}$. If $\mathfrak{V}$ is a vector space consisting of functions from an arbitrary nonempty set A into $\mathbb{K}$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and if $m \in \mathbb{N}$, we put $\mathfrak{V}^m := \{F : A \mapsto \mathbb{K}^m\ :\ F_1,\ldots,,F_m \in \mathfrak{V}\}$. If $\|\ \|$ is a norm on $\mathfrak{V}$, we will use the same notation $\|\ \|$ for the norm $(\sum_{j=1}^m \|F_j\|^2)^{1/2}$ $(F \in \mathfrak{V}^m)$ on $\mathfrak{V}^m$.

For $n \in \mathbb{N},\ A \subset \mathbb{R}^3$ measurable, $q \in [1,\infty]$, the usual norm of the Lebesgue space $L^q(A)$ is denoted by $\|\ \|_q$. The same notation is used for L^q-norms on $\partial\Omega$ and on S_T. For $q \in (1,\infty)$ and $A \subset \mathbb{R}^3$ open, we define $L^q_\sigma(A)$ as the closure of the set $\{v \in C_0^\infty(A)^3\ :\ \operatorname{div} v = 0\}$ with respect to the norm of $L^q(A)^3$.

Let $A \subset \mathbb{R}^3$ be open. Then, for $m \in \mathbb{N},\ q \in [1,\infty)$, we write $W^{m,q}(A)$ for the usual Sobolev space on A of order m and exponent q. The standard norm of $W^{m,q}(A)$ is denoted by $\|\ \|_{m,q}$. In the context of the resolvent equation (2.1), Lebesgue and Sobolev spaces are to be understood as spaces of complex valued functions; otherwise they are to consist of real valued functions. For brevity, we write V for the closure of the set $\{v \in C_0^\infty(A)^3\ :\ \operatorname{div} v = 0\}$ in $W^{1,2}(A)^3$. The symbol V' stands for the usual dual space of V. The subspace $W_0^{m,q}(A)$ of $W^{m,q}(A)$ is defined in the standard way. By $W_{loc}^{m,q}(A)$, we designate the set of all functions $w : A \mapsto \mathbb{R}$ with $w|K \in W^{m,q}(K)$ for any bounded open set $K \subset \mathbb{R}^3$ with $\overline{K} \subset A$.

Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a < b$. If $\mathfrak{B}$ is a Banach space and $q \in [1, \infty]$, then the usual norm of the space $L^q(a, b, \mathfrak{B})$ is denoted by $\| \ \|_{L^q(a,b,\mathfrak{B})}$. However, if $q, r \in [1, \infty]$, $\sigma \in \{1, 3\}$, $T \in (0, \infty]$ and A a measurable subset of $\mathbb{R}^3$, we write $\| \ \|_{r,q;T}$ for the norm of the space $L^q(0, T, L^r(A)^\sigma)$.

Let $T \in (0, \infty]$, $q \in [1, \infty]$, and let $\mathfrak{B}$ again denote a Banach space. Then we write $L^q_{loc}(0, T, \mathfrak{B})$ and $W^{1,q}_{loc}(0, T, \mathfrak{B})$, respectively, for the set of all functions $v : (0, T) \mapsto \mathfrak{B}$ such that $v|(a, b) \in L^q(a, b, \mathfrak{B})$ and $v|(a, b) \in W^{1,q}(a, b, \mathfrak{B})$, respectively, for any $a, b \in (0, T)$ with $a < b$. On the other hand, we use the notation $L^q_{loc}([0, T), \mathfrak{B})$ for the space of all functions $v : (0, T) \mapsto \mathfrak{B}$ with $v|(0, T') \in L^q(0, T', \mathfrak{B})$ for any $T' \in (0, T)$. The space $W^{1,q}_{loc}([0, T), \mathfrak{B})$ is to be understood in an analogous way.

If $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a < b$, $A \subset \mathbb{R}^3$ open, $\sigma \in \{1, 3\}$ and $w : A \times (a, b) \mapsto \mathbb{R}^\sigma$ a function such that for any $x \in A$ and $t \in (0, \infty)$, the term $\nabla_x w(x, t)$ is well defined, we write $\nabla_x w$ for the function $(x, t) \mapsto \nabla_x w(x, t)$. The notations $\Delta_x w$ and $\operatorname{div}_x w$ are used with an analogous meaning. For a Banach space $\mathfrak{B}$, a function $w : (0, \infty) \mapsto \mathfrak{B}$ and for $t_0 \in (0, \infty)$, we define $w(\cdot\ + t_0) : (0, \infty) \mapsto \mathfrak{B}$ in an obvious way. The same notation will be used for functions $w : [0, \infty) \mapsto \mathfrak{B}$.

For $q \in (1, \infty)$, the set $D(A_q)$ and the operator A_q (Stokes operator) are defined in the passage following Theorem 2.4. The space $D^{1/5,5/4}_{5/4}$ is introduced in the remark preceding Theorem 3.5. For the definition of the fractional powers of A_2 appearing in (A4), we refer to [56, p. 133–134].

Let us now point out some properties of L^p- and Sobolev spaces.

Theorem 2.1 [29, Theorem III.1.2]. *Let $q \in (1, \infty)$. Then, for any $F \in L^q(\overline{\Omega}^c)^3$, there are functions $P_q(F) \in L^q_\sigma(\overline{\Omega}^c)$ and $G_q(F) \in W^{1,q}_{loc}(\overline{\Omega}^c)$ with $\nabla G_q(F) \in L^q(\overline{\Omega}^c)^3$, $F = P_q(F) + G_q(F)$, and $\|P_q(F)\|_q + \|\nabla G_q(F)\|_q \le C(\Omega, q) \cdot \|F\|_q$.*

Theorem 2.2 [29, Section III.4.2]. *Let $q \in (1, \infty)$, $v \in W_0^{1,q}(\overline{\Omega}^c)^3$ with $\operatorname{div} v = 0$. Then there is a sequence (φ_n) in $C_0^\infty(\overline{\Omega}^c)^3$ with $\operatorname{div} \varphi_n = 0$ for $n \in \mathbb{N}$ and $\|\varphi_n - v\|_{1,q} \to 0$.*

Inversely, if $v \in W^{1,q}(\overline{\Omega}^c)^3$, and if there is a sequence (φ_n) with the above properties, then $v \in W_0^{1,q}(\overline{\Omega}^c)^3$ with $\operatorname{div} v = 0$.

Lemma 2.1. *Let $q \in (1, 3)$, $\kappa \in (1, \infty)$, $v \in W^{1,1}_{loc}(\overline{\Omega}^c) \cap L^\kappa(\overline{\Omega}^c)$ with $\nabla v \in L^q(\overline{\Omega}^c)^3$. Then $v \in L^{3\cdot q/(3-q)}(\overline{\Omega}^c)$ and $\|v\|_{3\cdot q/(3-q)} \le C(\Omega, q) \cdot \|\nabla v\|_q$.*

Proof. Proceed as in the proof of [24, Lemma 2.4], where the case $q = 2$ is treated. □

The ensuing lemma gives a link between Bochner and Lebesgue integrals.

Lemma 2.2. *Let $J \subset \mathbb{R}$ be an interval, $n \in \mathbb{N}$, $U \subset \mathbb{R}^n$ open, $q \in [1, \infty)$, $f : J \mapsto L^q(U)$ integrable as a Bochner integral in $L^q(U)$. Then $\int_J |f(t)(x)|\, dt < \infty$ and $\int_J f(t)(x)\, dt = (\int_J f(t)\, dt)(x)$ for a.e. $x \in U$, where the first integral in the preceding equation is a usual Lebesgue integral and the second a Bochner integral.*

Proof. Compare the proof of [23, Lemma 3.5], where the case $q = 2$ is treated. □

The next lemma should be well known. But since we cannot give a reference, we indicate a proof for the convenience of the reader. This lemma will be applied frequently but mostly implicitly in Sect. 4.

Lemma 2.3. *Let $q \in (1, \infty)$, $u \in W^{1,q}_{loc}([0, \infty), L^q(\overline{\Omega}^c))$. Let u' denote the weak derivative of u as a function from $(0, \infty)$ into $L^q(\overline{\Omega}^c)$. Then the weak derivative $\partial_t u$ of u as a function from Z_∞ into $\mathbb{R}$ exists and $\partial_t u(\cdot\,, t) = u'(t)$ for a.e. $t \in (0, \infty)$.*

Proof. Let $\alpha \in C_0^\infty(\mathbb{R}^3)$ with $\alpha \ge 0$, $supp(\alpha) \subset B_1$, $\int_{B_1} \alpha(x)\, dx = 1$. For $\epsilon > 0$, $W \in L^q(\overline{\Omega}^c)$, $x \in \mathbb{R}^3$, put $J_\epsilon(W)(x) := \int_{\mathbb{R}^3} \epsilon^{-3} \cdot \alpha(\epsilon^{-1} \cdot (x - y)) \cdot W(y)\, dy$ ("Friedrich's mollifier"; see [1, Section 2.17]). The mapping J_ϵ is a linear and bounded operator from $L^q(\overline{\Omega}^c)$ into $L^q(\overline{\Omega}^c)$.

Let $\epsilon > 0$, $w \in L^1(0, \infty, L^q(\overline{\Omega}^c))$. Since the operator J_ϵ is bounded, we have $J_\epsilon(\int_0^\infty w(t)\,dt) = \int_0^\infty J_\epsilon(w(t))\,dt$, where the preceding integrals are to be understood as Bochner integrals in $L^q(\overline{\Omega}^c)$. Due to the standard properties of Friedrich's mollifier, the preceding equation means that the term $(\int_0^\infty J_\epsilon(w(t))\,dt)(y)$ is a continuous function of $y \in \overline{\Omega}^c$. On the other hand, the Lebesgue integral $\int_0^\infty J_\epsilon(w(t))(y)\,dt$ is a continuous function of $y \in \overline{\Omega}^c$, as follows by Lebesgue's theorem and again by standard properties of Friedrich's mollifier. Now we may conclude with Lemma 2.2 that $(\int_0^\infty J_\epsilon(w(t))\,dt)(y) = \int_0^\infty J_\epsilon(w(t))(y)\,dt$ for any (and not only for a.e.) $y \in \overline{\Omega}^c$.

Let $\epsilon > 0$ and $\zeta \in C_0^\infty(Z_\infty)$. For $x \in \overline{\Omega}^c$, we have $\zeta(x, \cdot\,) \in C_0^\infty((0,\infty))$, so by the preceding part of this proof and because $u \in W^{1,q}_{loc}([0,\infty), L^q(\overline{\Omega}^c))$, we get

$$\int_0^\infty \zeta(x,t)\cdot J_\epsilon\left(u'(t)\right)(y)dt = J_\epsilon\left(\int_0^\infty \zeta(x,t)\cdot u'(t)\,dt\right)(y) = -\int_0^\infty \partial_t\zeta(x,t)\cdot J_\epsilon\left(u(t)\right)(y)dt$$

for any $y \in \overline{\Omega}^c$. On choosing $y = x$, integrating with respect to $x \in \overline{\Omega}^c$, and using Fubini's theorem, we get $\int_{Z_\infty} \zeta(x,t)\cdot J_\epsilon(u'(t))(x)\,d(x,t) = -\int_{Z_\infty} \partial_t\zeta(x,t)\cdot J_\epsilon(u(t))(x)\,d(x,t)$. On the other hand, by a standard property of Friedrich's mollifier and by Lebesgue's theorem, $\int_0^\infty \|F_\epsilon(\cdot\,,t)\|_q\,dt \to 0$ $(\epsilon \downarrow 0)$ in the case $F_\epsilon(x,t) = \zeta(x,t)\cdot[J_\epsilon(u'(t)) - u'(t)](x)$ for $(x,t) \in Z_\infty$, $\epsilon > 0$, as well as if $F_\epsilon(x,t) = -\partial_t\zeta(x,t)\cdot[J_\epsilon(u(t)) - u(t)](x)$ for (x,t), ϵ as before, Thus we may conclude that $\int_{Z_\infty} \zeta\cdot u'\,d(x,t) = -\int_{Z_\infty} \partial_t\zeta\cdot u\,d(x,t)$. This proves the lemma. □

We will frequently make use of the fact that the fundamental theorem of calculus is valid for functions from $W^{1,1}(a,b,\mathfrak{B})$ if $\mathfrak{B}$ is a Banach space:

Lemma 2.4. *Let $\mathfrak{B}$ be a Banach space, $a, b \in \mathbb{R}$ with $a < b$, $v \in W^{1,1}(a,b,\mathfrak{B})$. Then, possibly after a modification of v on a subset of (a,b) of measure zero, we have $v \in C^0([a,b],\mathfrak{B})$ and $v(t) - v(t_0) = \int_{t_0}^t v'(s)\,ds$ for $t, t_0 \in [a,b]$, where the preceding integral is to be understood as a Bochner integral in $\mathfrak{B}$.*

Proof. We refer to [56, Lemma IV.1.3.1] and the remark in [56, p. 193 below]; also see [60, Lemma 3.1.1]. □

Next we turn to the divergence equation.

Theorem 2.3. *Let $R > 0$ with $\overline{\Omega} \subset B_R$. Then, for any $F \in C_0^\infty(\Omega_R)$ with $\int_{\Omega_R} F\,dx = 0$, there is a function $\mathfrak{D}(F) = \mathfrak{D}_{\Omega,R}(F) \in C_0^\infty(\Omega_R)^3$ with $div\,\mathfrak{D}(F) = F$. The operator $\mathfrak{D}$ may be chosen in such a way that it is linear, and such that for $q \in (1,\infty)$, there is a constant $C_0(\Omega,R,q) > 0$ with $\|\mathfrak{D}(F)\|_{1,q} \le C_0(\Omega,R,q)\cdot\|F\|_q$ for F as before.*

Proof. [11, Theorem 2.4]; also see [29, Section III.3], in particular [29, Theorem III.3.2]. □

We will need some results about the Stokes resolvent problem.

Theorem 2.4. *Let $q \in (1,\infty)$, $\lambda \in \mathbb{C}\backslash(-\infty,0]$, $F \in L^q(\overline{\Omega}^c)^3$. Then there is a unique function $U \in W^{2,q}(\overline{\Omega}^c)^3 \cap W_0^{1,q}(\overline{\Omega}^c)^3 \cap L^q_\sigma(\overline{\Omega}^c)$ and a function $\Pi \in W^{1,q}_{loc}(\overline{\Omega}^c)$, unique up to a constant, such that $\nabla\Pi \in L^q(\overline{\Omega}^c)^3$ and*

$$-\Delta U + \lambda\cdot U + \nabla\Pi = F, \quad div\,U = 0. \tag{2.1}$$

Let $\vartheta \in [0,\pi)$. Then, for $\lambda \in \mathbb{C}\backslash\{0\}$ with $|\arg\lambda| \le \vartheta$ and for F, U as above, we have $\|U\|_q \le C(\Omega,\vartheta,q)\cdot|\lambda|^{-1}\cdot\|F\|_q$.

Proof. See [12,30], or [19–21]. □

Theorem 2.4 may be reformulated in terms of the Stokes operator on $\overline{\Omega}^c$, which we denote by A_q and define as follows:

For $q \in (1,\infty)$, put $D(A_q) := W^{2,q}(\overline{\Omega}^c)^3 \cap W_0^{1,q}(\overline{\Omega}^c)^3 \cap L^q_\sigma(\overline{\Omega}^c)$, $\quad A_q v := -P_q(\Delta v)$ for $v \in D(A_q)$, with P_q introduced in Theorem 2.1.

An operator-theoretical variant of Theorem 2.4 may now be stated like this:

Corollary 2.1. *Let $q \in (1,\infty)$. For $\lambda \in \mathbb{C}\backslash(-\infty,0]$, the operator $v \mapsto A_q v + \lambda \cdot v$, $v \in D(A_q)$, is a bijective linear mapping onto $L^q_\sigma(\overline{\Omega}^c)$. We denote its inverse (Stokes resolvent) by $R(\lambda, A_q)$, so that $R(\lambda, A_q) : L^q_\sigma(\overline{\Omega}^c) \mapsto D(A_q)$ with $R(\lambda, A_q)\, A_q W = -\lambda \cdot R(\lambda, A_q) W + W$ for $W \in D(A_q)$.*

Let $\vartheta \in [0,\pi)$. Then $\|R(\lambda, A_q)\, F\,\|_q \le C(\Omega, \vartheta, q) \cdot |\lambda|^{-1} \cdot \|F\|_q$ for $\lambda \in \mathbb{C}\backslash\{0\}$ with $|\arg\lambda| \le \vartheta$, $F \in L^q_\sigma(\overline{\Omega}^c)$.

Proof. This corollary follows from Theorems 2.4 and 2.1. □

Corollary 2.1 allows to introduce "mild solutions" to (1.8)–(1.10). We recall some facts in this respect which will be relevant later on:

Theorem 2.5 [27, p. 101–105]. *Put $\Gamma := \{r \cdot e^{-i\cdot\vartheta} : r \in [1,\infty)\} \cup \{e^{i\cdot\varphi} : \varphi \in [-\vartheta, \vartheta]\} \cup \{r \cdot e^{i\cdot\vartheta} : r \in [1,\infty)\}$, for some fixed $\vartheta \in (\pi/2, \pi)$. Let $q \in (1,\infty)$, $u_0 \in L^q_\sigma(\overline{\Omega}^c)$, and put $e^{-t\cdot A_q} u_0 := (2\cdot\pi\cdot i)^{-1} \cdot \int_\Gamma e^{\lambda\cdot t} \cdot R(\lambda, A_q)\, u_0\, dt$ for $t \in [0,\infty)$, where the integral is to be understood as a Bochner integral in $L^q_\sigma(\overline{\Omega}^c)$, and where Γ is to be oriented from $-\infty \cdot e^{-i\cdot\vartheta}$ to $\infty \cdot e^{i\cdot\vartheta}$.*

Then the function $t \mapsto e^{-t\cdot A_q} u_0$, $t \in [0,\infty)$, belongs to $C^0([0,\infty), L^q_\sigma(\overline{\Omega}^c)) \cap C^1((0,\infty), L^q_\sigma(\overline{\Omega}^c))$, and $e^{-t\cdot A_q} u_0 \in D(A_q)$ with $d/dt\,(e^{-t\cdot A_q} u_0) = -A_q e^{-t\cdot A_q} u_0$ for $t \in (0,\infty)$, $e^{-0\cdot A_q} u_0 = u_0$.

In the next lemma, we settle a technical point involving Eq. (2.1).

Lemma 2.5. *Let $\lambda \in \mathbb{C}\backslash(-\infty,0]$, $q_1, q_2 \in (1,\infty)$, $F \in L^{q_1}(\overline{\Omega}^c)^3 \cap L^{q_2}(\overline{\Omega}^c)^3$. For $i \in \{1, 2\}$, let $U^{(q_i)} \in D(A_{q_i})$, $\Pi^{(q_i)} \in W^{1,q_i}_{loc}(\overline{\Omega}^c)$ such that $\nabla\Pi^{(q_i)} \in L^{(q_i)}(\overline{\Omega})^3$ and the pair $(U^{(q_i)}, \Pi^{(q_i)})$ solves (2.1) with $U^{(q_i)}, \Pi^{(q_i)}$ in the place of U, Π, respectively. Then $U^{(q_1)} = U^{(q_2)}$.*

Proof. For $\varphi \in C^\infty_0(\overline{\Omega}^c)^3$, $r \in (1,\infty)$, let $W^{(r)}(\varphi) \in D(A_r)$, $\Lambda^{(r)}(\varphi) \in W^{1,r}_{loc}(\overline{\Omega}^c)$ such that $\nabla\Lambda^{(r)}(\varphi) \in L^r(\overline{\Omega}^c)^3$ and the pair $(W^{(r)}(\varphi), \Lambda^{(r)}(\varphi))$ solves (2.1) with U, Π, F replaced by $W^{(r)}(\varphi)$, $\Lambda^{(r)}(\varphi)$, φ, respectively.

Take $r, q \in (1,\infty)$ with $W^{1,q}(\overline{\Omega}^c) \subset L^{r'}(\overline{\Omega}^c)$ and $W^{1,r}(\overline{\Omega}^c) \subset L^{q'}(\overline{\Omega}^c)$. Further take $\varphi, \psi \in C^\infty_0(\overline{\Omega}^c)^3$, and abbreviate $U := W^{(r)}(\varphi)$, $\Pi := \Lambda^{(r)}(\varphi)$, $V := W^{(q)}(\psi)$, $R := \Lambda^{(q)}(\psi)$. By Theorem 2.2, there is a sequence (γ_n) in $C^\infty_0(\overline{\Omega}^c)^3$ with $\operatorname{div}\gamma_n = 0$ for $n \in \mathbb{N}$ and $\|\gamma_n - U\|_{1,r} \to 0$. Then

$$\begin{aligned}\int_{\overline{\Omega}^c} U \cdot \psi\, dx &= \lim_{n\to\infty} \int_{\overline{\Omega}^c} \gamma_n \cdot \psi\, dx = \lim_{n\to\infty} \int_{\overline{\Omega}^c} \gamma_n \cdot (-\Delta V + \lambda \cdot V + \nabla R)\, dx \\ &= \lim_{n\to\infty} \int_{\overline{\Omega}^c} (\nabla\gamma_n \cdot \nabla V + \lambda \cdot \gamma_n \cdot V)\, dx = \int_{\overline{\Omega}^c} (\nabla U \cdot \nabla V + \lambda \cdot U \cdot V)\, dx, \end{aligned} \tag{2.2}$$

where the second equation is valid because of (2.1), and the last one holds since $W^{1,q}(\overline{\Omega}^c) \subset L^{r'}(\overline{\Omega}^c)$. Again referring to Theorem 2.2, we choose a sequence (ϱ_n) in $C^\infty_0(\overline{\Omega}^c)^3$ with $\|\varrho_n - V\|_{1,q} \to 0$ and $\operatorname{div}\varrho_n = 0$ for $n \in \mathbb{N}$. Then we deduce from (2.2) that

$$\int_{\overline{\Omega}^c} U \cdot \psi\, dx = \lim_{n\to\infty} \int_{\overline{\Omega}^c} (\nabla U \cdot \nabla\varrho_n + \lambda \cdot U \cdot \varrho_n)\, dx = \int_{\overline{\Omega}^c} \varphi \cdot V\, dx, \tag{2.3}$$

where the first equation holds due to the relation $W^{1,r}(\overline{\Omega}^c) \subset L^{q'}(\overline{\Omega}^c)$, and the last one due to (2.1).

Now let $r \in (1, 3/2]$ so that $W^{1,r}(\overline{\Omega}^c) \subset L^{3/2}(\overline{\Omega}^c)$, $W^{1,3}(\overline{\Omega}^c) \subset L^{r'}(\overline{\Omega}^c)$. Thus by (2.3) with $q = 3$, we get $\int_{\overline{\Omega}^c} W^{(r)}(\varphi) \cdot \psi\, dx = \int_{\overline{\Omega}^c} \varphi \cdot W^{(3)}(\psi)\, dx$ for $\varphi, \psi \in C^\infty_0(\overline{\Omega}^c)^3$. This implies that $W^{(r)}(\varphi) = W^{(3/2)}(\varphi)$ for $\varphi \in C^\infty_0(\overline{\Omega}^c)^3$, $r \in (1, 3/2]$.

Next let $r \in [3/2, 3]$ so that $W^{1,r}(\overline{\Omega}^c) \subset L^3(\overline{\Omega}^c)$ and $W^{1,3/2}(\overline{\Omega}^c) \subset L^{r'}(\overline{\Omega}^c)$. Thus we may again use (2.3), this time with $q = 3/2$, to obtain $\int_{\overline{\Omega}^c} W^{(r)}(\varphi) \cdot \psi\, dx = \int_{\overline{\Omega}^c} \varphi \cdot W^{(3/2)}(\psi)\, dx$ for $\varphi, \psi \in C^\infty_0(\overline{\Omega}^c)^3$. Therefore $W^{(r)}(\varphi) = W^{(3/2)}(\varphi)$ for $\varphi \in C^\infty_0(\overline{\Omega}^c)^3$, $r \in [3/2, 3]$.

Finally take $r \in [3,\infty)$. Obviously $W^{1,r}(\overline{\Omega}^c) \subset L^r(\overline{\Omega}^c)$, $W^{1,r'}(\overline{\Omega}^c) \subset L^{r'}(\overline{\Omega}^c)$ and $r' \le 3$, so by (2.3) with $q = r'$ we have $\int_{\overline{\Omega}^c} W^{(r)}(\varphi) \cdot \psi\, dx = \int_{\overline{\Omega}^c} \varphi \cdot W^{(r')}(\psi)\, dx$ for $\varphi, \psi \in C^\infty_0(\overline{\Omega}^c)^3$. But since $r' \le 3/2$, we already know that $W^{(r')}(\psi) = W^{(3/2)}(\psi)$. Thus we may conclude that $W^{(r)}(\varphi) = W^{(3)}(\varphi)$

for $r \in [3, \infty)$, $\varphi \in C_0^\infty(\overline{\Omega}^c)^3$, hence $W^{(r)}(\varphi) = W^{(3/2)}(\varphi)$ for such φ by what we showed before. Now Lemma 2.5 follows by a density argument based on the previous equation and the inequality at the end of Theorem 2.4. □

In the rest of this section, we define and discuss various potential functions. We begin by introducing the usual heat kernel in $\mathbb{R}^3$, which we denote by $\mathfrak{H}$:

$$\mathfrak{H}(z,t) := (4 \cdot \pi \cdot t)^{-3/2} \cdot e^{-|z|^2/(4 \cdot t)} \quad \text{for } (z,t) \in \mathbb{R}^3 \times (0,\infty),$$
$$\mathfrak{H}(z,0) := 0 \quad \text{for } z \in \mathbb{R}^3 \backslash \{0\}.$$

Furthermore, we introduce a fundamental solution of the time-dependent Stokes system by setting as in [51]

$$\Gamma_{jk}(z,t) := \delta_{jk} \cdot \mathfrak{H}(z,t) + \int_t^\infty \partial_j \partial_k \mathfrak{H}(z,s)\, ds, \quad E_k(x) := (4 \cdot \pi)^{-1} \cdot x_k \cdot |x|^{-3}$$

for $(z,t) \in \mathfrak{B} := (\mathbb{R}^3 \times (0,\infty)) \cup (\mathbb{R}^3 \backslash \{0\} \times \{0\})$, $x \in \mathbb{R}^3 \backslash \{0\}$, $1 \le j,k \le 3$.

Theorem 2.6. *$\mathfrak{H}, \Gamma_{jk} \in C^\infty(\mathfrak{B})$; for $z \in \mathbb{R}^3$, $t \in (0,\infty)$, $\alpha \in \mathbb{N}_0^3$, $l \in \mathbb{N}_0$, $j,k \in \{1, 2, 3\}$, the estimate $|\partial_t^l \partial_z^\alpha \mathfrak{H}(z,t)| + |\partial_t^l \partial_z^\alpha \Gamma_{jk}(z,t)| \le C(l, |\alpha|) \cdot (|z|^2 + t)^{-3/2-|\alpha|/2-l}$ holds.*

Proof. See [57] for the estimate of $\mathfrak{H}$, and [51] for that of Γ. □

Lemma 2.6. $\int_{\mathbb{R}^3} \mathfrak{H}(z,t)\, dt = 1$ *for* $z \in \mathbb{R}^3$.

Lemma 2.7. $|\partial^\alpha E_k(z)| \le C(\alpha) \cdot |z|^{-2-|\alpha|}$ *for* $z \in \mathbb{R}^3 \backslash \{0\}$, $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$.

Theorem 2.7. *Let $\nu, q \in [1,\infty]$ with $q < \nu$, $s \in [1,\infty]$. Let $h \in L^s(0,\infty, L^q(\mathbb{R}^3))$, $M \in (0,\infty)$, $j,k \in \{1, 2, 3\}$. Let $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. Suppose that $1 - |\alpha|/2 + 3 \cdot (1/\nu - 1/q)/2 > 1/s$. Then, if $\nu < \infty$,*

$$\left[\int_{\mathbb{R}^3} \left(\int_0^\infty \int_{\mathbb{R}^3} \chi_{(0,M)}(t-\sigma) \cdot |\partial_x^\alpha \Gamma_{jk}(x-y, t-\sigma)| \cdot |h(y,\sigma)|\, dy\, d\sigma \right)^\nu dx \right]^{1/\nu}$$
$$\le C(\nu, q, s) \cdot M^{3 \cdot (1/\nu - 1/q)/2 + 1 - |\alpha|/2 - 1/s} \cdot \|h\|_{q,s;\infty} \quad \text{for any } t \in (0,\infty).$$

If $\nu = \infty$, the preceding inequality has to be modified in an obvious way (pointwise estimate for any $x \in \mathbb{R}^3$). If $1 - |\alpha|/2 + 3 \cdot (1/\nu - 1/q)/2 < 1/s$, the preceding inequality holds with $\chi_{(M,\infty)}$ in the place of $\chi_{(0,M)}$ if $\nu < \infty$, and with obvious modifications in the case $\nu = \infty$ (pointwise estimate for any $x \in \mathbb{R}^3$).

Proof. Theorem 2.7 follows by the proof of [25, Theorem 2.8] and [22, Lemma 2.7]. In these references an Oseen fundamental solution takes the place of the Stokes fundamental solution Γ appearing above. But the argument in the proof of [25, Theorem 2.8] and [22, Lemma 2.7] consists in first applying Minkowski's and Young's inequality for integrals, then performing a change of variables which turns the Oseen into the Stokes fundamental solution, and finally calculating the integrals arising in this way. So this reasoning also works here; it even simplifies because the change of variables just mentioned is no longer necessary. □

Lemma 2.8. *Let $q \in [1,\infty)$, $s \in (1,\infty]$, $A \subset \mathbb{R}^3$ measurable, $T \in (0,\infty]$. Suppose that h is a function belonging to the space $L^s(0,T, L^q(A)^3)$, and write $\widetilde{h}$ for the zero extension of h to Z_∞. Then the integral $\int_0^t \int_{\mathbb{R}^3} |\Gamma(x-y, t-\sigma) \cdot \widetilde{h}(y,\sigma)|\, dy\, d\sigma$ is finite for any $t \in (0,\infty)$ and $x \in \mathbb{R}^3 \backslash N_t$, with $N_t \subset \mathbb{R}^3$ a set of measure zero that may depend on t.*

If $s > 2$, an analogous statement is true for $\int_0^t \int_{\mathbb{R}^3} |\partial_l \Gamma(x-y, t-\sigma) \cdot \widetilde{h}(y,\sigma)|\, dy\, d\sigma$, with $l \in \{1, 2, 3\}$.

Proof. Choose $\nu \in (q,\infty)$ so close to q that $1 + 3 \cdot (1/\nu - 1/q)/2 > 1/s$. (Note that $s > 1$.) Then the first part of Lemma 2.8 follows from Theorem 2.7 with $M = t$. The second part may be shown by the same argument, with $\nu \in (q,\infty)$ chosen in such a way that $1/2 + 3 \cdot (1/\nu - 1/q)/2 > 1/s$. □

We remark that the condition $s > 2$ at the end of Lemma 2.8 may be dropped if the last statement of the lemma is to hold only for a.e. $(x,t) \in \mathbb{R}^3 \times (0,\infty)$, instead of for any $t \in (0,\infty)$ and a.e. $x \in \mathbb{R}^3$; see [22, p. 898 below].

Lemma 2.9. *Let $q \in [1, \infty]$, $c \in L^q(\mathbb{R}^3)$. Then $\int_{\mathbb{R}^3} \mathfrak{H}(x-y,t) \cdot |c(y)|\, dy < \infty$ for $y \in \mathbb{R}^3$, $t > 0$.*

Proof. Use Lemma 2.6 in the case $q = \infty$ and $q = 1$, and Theorem 2.6 combined with Hölder's inequality else. □

Now we are in a position to define two volume potentials we will need in the following. Put

$$\mathfrak{R}(h)(x,t) := \int_0^t \int_{\mathbb{R}^3} \Gamma(x-y, t-\sigma) \cdot \widetilde{h}(y,\sigma)\, dy\, d\sigma$$

for h, $\widetilde{h}$ as in Lemma 2.8, for $t \in (0, \infty)$ and for a.e. $x \in \mathbb{R}^3$,

$$\mathfrak{I}(c)(x,t) := \int_{\mathbb{R}^3} \mathfrak{H}(x-y,t) \cdot c(y)\, dy$$

for $c \in L^q(\mathbb{R}^3)^3$ with some $q \in [1, \infty]$, and for $x \in \mathbb{R}^3$, $t > 0$.

Theorem 2.8. *Let $\psi \in C_0^0(\mathbb{R}^3)^3$. Then $\mathfrak{I}(\psi)(x,\epsilon) \to \psi(x)$ $(\epsilon \downarrow 0)$, uniformly in $x \in \mathbb{R}^3$.*

Proof. Well known; compare [26, Theorem 1.2.1]. □

Lemma 2.10 [23, Lemma 5.10]. *Let $q \in (1, \infty)$, $F \in L^q_\sigma(\overline{\Omega}^c)$, $x \in \mathbb{R}^3$, $t > 0$. Then $\int_{\overline{\Omega}^c} \Gamma(x-y,t) \cdot F(y)\, dy = \mathfrak{I}(F)(x,t)$.*

Next we introduce a single layer potential related to Γ. In fact, by Theorem 2.6 and Lebesgue's theorem we get

Lemma 2.11. *Let $s, q \in [1, \infty]$, $T \in (0, \infty]$, $\phi \in L^s(0, T, L^q(\partial\Omega)^3)$. Then, for $x \in \mathbb{R}^3 \backslash \partial\Omega$, $t \in (0, \infty)$, the integral $\int_0^t \int_{\partial\Omega} |\Gamma(x-y, t-\sigma) \cdot \widetilde{\phi}(y,\sigma)|\, do_y\, d\sigma$ is finite, where $\widetilde{\phi}$ denotes the zero extension of ϕ to S_∞. Thus we may define $\mathfrak{V}(\phi) : (\mathbb{R}^3 \backslash \partial\Omega) \times (0, \infty) \mapsto \mathbb{R}^3$ by setting*

$$\mathfrak{V}(\phi)(x,t) := \int_0^t \int_{\partial\Omega} \Gamma(x-y, t-\sigma) \cdot \widetilde{\phi}(y,\sigma)\, do_y\, d\sigma$$

for x, t as before. We have $\mathfrak{V}(\phi)(\cdot\,, t) \in C^0(\mathbb{R}^3 \backslash \partial\Omega)^3$ for $t \in (0, \infty)$.

In the last part of this section, we deal with a truncated version of Γ appearing in the potential function $\mathfrak{K}_S(u)$ defined below. This potential function will be part of a representation formula without pressure term (Theorem 4.3). For the definition of this modification of Γ, we introduce a family of cut-off functions. Put $R(\Omega) := \inf\{r \in (0, \infty) : \overline{\Omega} \subset B_r\}$. For any $S \in (0, \infty)$ with $\overline{\Omega} \subset B_S$, we have $S > R(\Omega)$, so we may fix a function $\varphi_S \in C_0^\infty(\mathbb{R}^3)$ with $0 \le \varphi_S \le 1$, $\varphi_S(x) = 1$ for $x \in B_{(R(\Omega)+S)/2}$, $\varphi_S(x) = 0$ for $x \in \mathbb{R}^3$ with $|x| \ge (R(\Omega) + 3 \cdot S)/4$.

Lemma 2.12. *Let $S \in (0, \infty)$ with $\overline{\Omega} \subset B_S$. Then $\varphi_S|\partial B_S = 0$, $\varphi_S|\overline{\Omega} = 1$ and $\nabla\varphi_S \in C_0^\infty(\Omega_S)^3$.*

For $S \in (0, \infty)$ with $\overline{\Omega} \subset B_S$, $x \in \overline{B_S}^c$, $y \in \Omega_S$, $r \in [0, \infty)$, we put

$$M_S(x,y,r) := \left(\sum_{k=1}^3 \partial_k \varphi_S(y) \cdot \Gamma_{jk}(x-y, r) \right)_{1 \le j \le 3}.$$

Lemma 2.13. *Let $S \in (0, \infty)$ with $\overline{\Omega} \subset B_S$, $x \in \overline{B_S}^c$, $r \in [0, \infty)$, $l \in \{0, 1\}$. Then the function $\partial_r^l M_S(x, \cdot\,, r)$ belongs to $C_0^\infty(\Omega_S)^3$ and $\int_{\Omega_S} \partial_r^l M_S(x,y,r)\, dy = 0$.*

Proof. Since $x \in \overline{B_S}^c$, the function $y \mapsto \Gamma(x-y, r)$ $(y \in \overline{B_S})$ is in $C^1(\overline{B_S})^{3\times 3}$ (Theorem 2.6). Let $j \in \{1, 2, 3\}$. We note that $\sum_{k=1}^3 \partial_{y_k} \Gamma_{jk}(x-y, r) = 0$ for $y \in \Omega_S$, so the mean value of the function $y \mapsto \partial_r^l M_{S,j}(x,y,r)$ on Ω_S equals $-\int_{\partial\Omega} \sum_{k=1}^3 \partial_r^l \Gamma_{jk}(x-y, r) \cdot n_k^{(\Omega)}(y)\, do_y$. The preceding integral coincides with $\int_\Omega \sum_{k=1}^3 \partial_{y_k} \partial_r^l \Gamma_{jk}(x-y, r)\, dy$, and thus vanishes. □

Due to Lemma 2.13 and Theorem 2.3, we may define a function $G_S : \overline{B_S}^c \times \overline{\Omega_S} \times [0,\infty) \mapsto \mathbb{R}^{3\times 3}$, for $S \in (0,\infty)$ with $\overline{\Omega} \subset B_S$ by setting

$$G_S(x,y,r) := \left[\varphi_S(y) \cdot \Gamma_{jl}(x-y,r) - \mathfrak{D}_{\Omega,S}\left(M_{S,j}(x,\,\cdot\,,r)\right)_l(y)\right]_{1\le j,l\le 3}$$

for $x \in \overline{B_S}^c,\ y \in \overline{\Omega_S},\ r \in [0,\infty)$, with the operator $\mathfrak{D}_{\Omega,S}$ introduced in Theorem 2.3.

Lemma 2.14. *Let $S \in (0,\infty)$ with $\overline{\Omega} \subset B_S,\ x \in \overline{B_S}^c,\ r \in [0,\infty)$. Then $G_S(x,\ \cdot\ ,r) \in C^\infty(\overline{\Omega_S})^{3\times 3},\ \sum_{k=1}^3 \partial_{y_k} G_{S,jk}(x,y,r) = 0$ for $y \in \Omega_S,\ 1 \le j \le 3,\ G_S(x,y,r) = \Gamma(x-y,r)$ for $y \in \partial\Omega,\ G_S(x,y,r) = 0$ for $y \in \partial B_S$.*

Proof. Lemma 2.14 follows from Lemma 2.12, 2.13 and Theorem 2.3. □

Lemma 2.15. *Let $S \in (0,\infty)$ with $\overline{\Omega} \subset B_S,\ x \in \overline{B_S}^c$, and define $L(y,r) := \Gamma(x-y,r)$ for $y \in \Omega_S,\ r \in [0,\infty)$. Let $q \in (1,\infty)$. Then the mapping $r \mapsto L(\cdot\,,r)$ $(r \in [0,\infty))$ belongs to $C^1([0,\infty),\,W^{1,q}(\Omega_S)^{3\times 3})$, with $d/dr(L(\cdot\,,r))(y) = \partial_r\Gamma(x-y,r)$ for $r \in [0,\infty),\ y \in \Omega_S$.*

Proof. Since $\operatorname{dist}(x,\overline{\Omega_S}) > 0$, the function L and its derivatives are uniformly continuous and bounded on $\Omega_S \times [0,T]$, for any $T \in (0,\infty)$ (Theorem 2.6). □

Lemma 2.16. *Let $S \in (0,\infty)$ with $\overline{\Omega} \subset B_S,\ x \in \overline{B_S}^c,\ q \in (1,\infty)$. Then the function $r \mapsto G_S(x,\ \cdot\ ,r)$ $(r \in [0,\infty))$ belongs to $C^1([0,\infty),\,W^{1,q}(\Omega_S)^{3\times 3})$, and*

$$d/dr\left(G_{S,jk}(x,\ \cdot\ ,r)\right)(y) = \varphi_S(y)\cdot\partial_r\Gamma_{jk}(x-y,r) - \mathfrak{D}_{\Omega,S}\left(\partial_r M_{S,j}(x,\ \cdot\ ,r)\right)_k(y)$$

for $y \in \Omega_S,\ r \in [0,\infty),\ 1 \le j,k \le 3$.

Proof. Lemma 2.16 follows from Theorem 2.3 and Lemma 2.15. □

Lemma 2.17. *Let $S \in (0,\infty)$ with $\overline{\Omega} \subset B_S,\ S_1 \in (S,\infty),\ q \in (1,\infty),\ x \in B^c_{S_1},\ r \in [0,\infty),\ l \in \{0,1\},\ \nu \in \{1,\,2,\,3\}$. Then $\|d^l/dr^l(G_S(x,\ \cdot\ ,r))\|_q \le C(\Omega,q,S,S_1)\cdot|x|^{-3-2\cdot l}$ and $\|\partial_\nu(G_S(x,\ \cdot\ ,r))\|_q \le C(\Omega,q,S,S_1)\cdot|x|^{-3}$.*

Proof. We find with Lemma 2.16, Theorem 2.3 and 2.6 that

$$\begin{aligned}\|d^l/dr^l\left(G_S(x,\ \cdot\ ,r)\right)\|_q &\le \left(\int_{\Omega_S}|\partial_r^l\Gamma(x-y,r)|^q\,dy\right)^{1/q} + \sum_{j,k=1}^3\|\mathfrak{D}_{\Omega,S}\left(\partial_r^l M_{S,j}(x,\ \cdot\ ,r)\right)_k\|_q \\ &\le C(\Omega,q,S)\cdot\left(\int_{\Omega_S}|\partial_r^l\Gamma(x-y,r)|^q\,dy\right)^{1/q} \le C(\Omega,q,S)\cdot\left(\int_{\Omega_S}|x-y|^{(-3-2\cdot l)\cdot q}\,dy\right)^{1/q}.\end{aligned}\tag{2.4}$$

But for $y \in \Omega_S$, we have

$$|x-y| \ge |x| - |y| = (1-S/S_1)\cdot|x| + (S/S_1)\cdot|x| - S \ge (1-S/S_1)\cdot|x|. \tag{2.5}$$

Thus the first estimate in Lemma 2.17 follows from (2.4). As for the second one, we observe that by the same references as above,

$$\begin{aligned}\|\partial_\nu\left(G_S(x,\ \cdot\ ,r)\right)\|_q &\le \left(\int_{\Omega_S}|\partial_{y_\nu}\Gamma(x-y,r)|^q\,dy\right)^{1/q} + C\cdot\sum_{j,k=1}^3\|\mathfrak{D}_{\Omega,S}\left(M_{S,j}(x,\ \cdot\ ,r)\right)_k\|_{1,q} \\ &\le C\cdot\left(\int_{\Omega_S}|x-y|^{-4\cdot q}\,dy\right)^{1/q} + C(\Omega,q,S)\cdot\left(\int_{\Omega_S}|\Gamma(x-y,r)|^q\,dy\right)^{1/q} \\ &\le C(S,S_1)\cdot|x|^{-4} + C(\Omega,q,S)\cdot\left(\int_{\Omega_S}|x-y|^{-3\cdot q}\,dy\right)^{1/q} \le C(\Omega,q,S,S_1)\cdot|x|^{-3}.\end{aligned}$$

□

Lemma 2.18. *Let $S \in (0, \infty)$ with $\overline{\Omega} \subset B_S$, $S_1 \in (S, \infty)$, $u \in C^0([0, \infty),\, L^2(\Omega_S)^3)$, $w \in L^2(Z_{S,\infty})^3$, $x \in B^c_{S_1}$, $t \in (0, \infty)$, $\alpha \in \mathbb{N}^3_0$ with $|\alpha| \le 1$. Then*

$$\int_0^t \int_{\Omega_S} |d/dr\,(G_S(x,\ \cdot\ , t-r))\,(y) \cdot u(y,r)|\; dy\, dr \le C(\Omega, S, S_1) \cdot |x|^{-5} \cdot \|u|Z_{S,t}\|_{2,1;t},$$

$$\int_0^t \int_{\Omega_S} |\partial^\alpha_y G_S(x,y,t-r) \cdot w(y,r)|\; dy\, dr \le C(\Omega, S, S_1) \cdot |x|^{-3} \cdot t^{1/2} \cdot \|w|Z_{S,t}\|_2.$$

Proof. Using Hölder's inequality, we see that the left-hand side of the first inequality in the lemma is bounded by $\int_0^t (\int_{\Omega_S} |d/dr(G_S(x,\ \cdot\ , t-r))(y)|^2 dy)^{1/2} \cdot \|u(r)|\Omega_S\|_2\, dr$, hence by $C(\Omega, S, S_1) \cdot |x|^{-5} \cdot \int_0^t \|u(r)\|_2\, dr$ (Lemma 2.17), and thus by $C(\Omega, S, S_1) \cdot |x|^{-5} \cdot \|u|Z_{S,t}\|_{2,1;t}$. This proves the first estimate. The second follows by an analogous argument. □

Lemma 2.19. *Let $S \in (0, \infty)$ with $\overline{\Omega} \subset B_S$, $S_1 \in (S, \infty)$, $u \in C^0([0, \infty),\, L^2(\Omega_S)^3)$, $x \in B^c_{S_1}$, $s, t \in [0, \infty)$. Then $\int_{\Omega_S} |G_S(x,y,t) \cdot u(y,s)|\; dy \le C(\Omega, S, S_1) \cdot |x|^{-3} \cdot \|u(\cdot\,, s)\|_2$.*

Proof. Hölder's inequality and Lemma 2.17. □

In view of Lemmas 2.18 and 2.19, we may introduce a volume potential $\mathfrak{K}_S(u)$ for $S \in (0, \infty)$ with $\overline{\Omega} \subset B_S$ and $u \in C^0([0, \infty),\, L^2(\Omega_S)^3)$ with $\nabla_x u \in L^2(Z_{S,\infty})^9$ by defining $\mathfrak{K}_S(u)(x,t)$ as

$$\int_0^t \int_{\Omega_S} \left(\sum_{l=1}^3 \partial_{y_l} G_S(x,y,t-r) \cdot \partial_l u(y,r) - d/dr\,(G_S(x,\ \cdot\ , t-r))\,(y) \cdot u(y,r) \right) dy\, dr$$
$$+ \int_{\Omega_S} G_S(x,y,0) \cdot u(y,t)\; dy - \int_{\Omega_S} G_S(x,y,t) \cdot u(y,0)\; dy \quad \text{for } x \in \overline{B_S}^c,\ t \in [0, \infty).$$

3. Some Results on the Instationary Stokes System

In this section, we present some aspects of the L^p-theory of the instationary Stokes system (1.8). Most of the results in question are well known, but we will give a proof of some details for which we could not find a clear reference. We begin with a uniqueness theorem for weak solutions to (1.8)–(1.10).

Theorem 3.1. *Let $u \in L^1_{loc}([0, \infty),\, V)$ with $\int_0^\infty \int_{\overline{\Omega}^c} (-\varphi'(t) \cdot u(t) \cdot \vartheta + \varphi(t) \cdot \nabla_x u(t) \cdot \nabla\vartheta)\, dx\, dt = 0$ for $\varphi \in C^\infty_0([0, \infty))$, $\vartheta \in C^\infty_0(\overline{\Omega}^c)^3$ with $\operatorname{div}\vartheta = 0$. Then $u = 0$.*

Proof. See [56, Lemma IV.2.4.2, IV.2.2.1 a)]. □

Next we state a basic existence result for L^2-weak solutions of problem (1.8)–(1.10).

Theorem 3.2. *Let $u_0 \in L^2_\sigma(\overline{\Omega}^c)$, $f \in L^2(0, \infty, V')$. Then there is a unique function $u \in L^2_{loc}([0, \infty),\, V)$ such that*

$$\int_0^\infty \left(\int_{\overline{\Omega}^c} (-\varphi'(t) \cdot u(t) \cdot \vartheta + \varphi(t) \cdot \nabla_x u(t) \cdot \nabla\vartheta)\; dx - \varphi(t) \cdot f(t)(\vartheta) \right) dt = \int_{\overline{\Omega}^c} u_0 \cdot \vartheta\; dx \cdot \varphi(0) \tag{3.1}$$

for $\varphi \in C^\infty_0([0, \infty))$, $\vartheta \in C^\infty_0(\overline{\Omega}^c)^3$ with $\operatorname{div}\vartheta = 0$. This function u belongs to the spaces $W^{1,2}_{loc}([0, \infty),\, V')$, $L^\infty(0, \infty,\, L^2_\sigma(\overline{\Omega}^c))$, $C^0([0, \infty),\, L^2_\sigma(\overline{\Omega}^c))$ and $L^2(0, \infty,\, L^6(\overline{\Omega}^c)^3)$. Moreover, $\nabla_x u \in L^2(Z_\infty)^9$, $u(0) = u_0$, $\operatorname{div} u = 0$, $u|S_\infty = 0$.

Define $\mathfrak{B}(\vartheta)(\kappa) := \int_{\overline{\Omega}^c} \nabla\vartheta \cdot \nabla\kappa\; dx$ for $\vartheta, \kappa \in V$. Then $\mathfrak{B}(\vartheta) \in V'$ for $\vartheta \in V$ and the equation $u'(t) + \mathfrak{B}(u(t)) - f(t) = 0$ holds for $t \in (0, \infty)$.

Proof. The claim on uniqueness follows from Theorem 3.1. The rest of the theorem, except the relations $\operatorname{div} u = 0$ and $u \in L^2(0,\infty,\, L^6(\overline{\Omega}^c)^3)$, hold according to [60, Section 3.1.3, 3.1.4 and p. 180 above]. The equation $\operatorname{div} u = 0$ is valid by Theorem 2.2 and because $u(t) \in V$ for $t \in (0,\infty)$, whereas the relation $u \in L^2(0,\infty,\, L^6(\overline{\Omega}^c)^3)$ is a consequence of Lemma 2.1 with $\kappa = q = 2$ and the fact that $u \in L^2_{loc}([0,\infty),\, V)$ and $\nabla_x u \in L^2(Z_\infty)^9$. □

We state in precise form that the last claim in Theorem 3.2 implies (3.1):

Lemma 3.1. *Let $u_0 \in L^2_\sigma(\overline{\Omega}^c)$, $f \in L^2(0,\infty, V')$, and let $u \in L^2_{loc}([0,\infty),\, V)$ satisfy the relations $u \in W^{1,2}_{loc}([0,\infty),\, V')$, $u'(t) + \mathfrak{B}(u(t)) - f(t) = 0$ for a.e. $t \in (0,\infty)$, and $u(0) = u_0$. Then the function u verifies (3.1).*

Lemma 3.2. *Let $q \in (1,3)$, $h : (0,\infty) \mapsto W^{1,q}_{loc}(\overline{\Omega}^c)$ with $\nabla_x h \in L^q_{loc}([0,\infty),\, L^q(\overline{\Omega}^c)^3)$. Then there is $\widetilde{h} : (0,\infty) \mapsto W^{1,q}_{loc}(\overline{\Omega}^c)$ with $\widetilde{h} \in L^q_{loc}([0,\infty),\, L^{3\cdot q/(3-q)}(\overline{\Omega}^c))$ and $\nabla_x \widetilde{h} = \nabla_x h$. If $\nabla_x h \in L^q(Z_\infty)^3$, then $\widetilde{h} \in L^q(0,\infty,\, L^{3\cdot q/(3-q)}(\overline{\Omega}^c))$.*

Proof. Since $\nabla_x h(t) \in L^q(\overline{\Omega}^c)^3$ for $t > 0$, we know by [29, Lemma II.5.2, Theorem II.5.1] that for $t \in (0,\infty)$, there is $h_0(t) \in \mathbb{R}$ with $\|h(t) - h_0(t)\|_{3\cdot q/(3-q)} \le C(\Omega)\cdot \|\nabla_x h(t)\|_q$. Define $\widetilde{h}(x,t) := h(x,t) - h_0(t)$ for $(x,t) \in Z_\infty$. Then $\widetilde{h} : (0,\infty) \mapsto W^{1,q}_{loc}(\overline{\Omega}^c) \cap L^{3\cdot q/(3-q)}(\overline{\Omega}^c)$ and $\nabla_x \widetilde{h} = \nabla_x h$.

Put $r := (3\cdot q/(3-q))'$ and let $\psi \in L^r(\overline{\Omega}^c)$. Note that $r \in (1,3)$ and $3\cdot r/(3-r) = q'$. By [29, Remark II.5.1, Theorem II.5.1, II.6.1 and III.3.4], there is $F \in W^{1,r}_{loc}(\overline{\Omega}^c)^3 \cap L^{q'}(\overline{\Omega}^c)^3$ with $\nabla F \in L^r(\overline{\Omega}^c)^9$, $\operatorname{div} F = \psi$, and such that there is a sequence (φ_n) in $C_0^\infty(\overline{\Omega}^c)^3$ with $\|\nabla(\varphi_n - F)\|_r \to 0$ and $\|\varphi_n - F\|_{q'} \to 0$. Recalling that $\widetilde{h}(t) \in L^{3\cdot q/(3-q)}(\overline{\Omega}^c)$, $\nabla_x \widetilde{h}(t) = \nabla_x h(t) \in L^q(\overline{\Omega}^c)^3$, we get $\int_{\overline{\Omega}^c} \widetilde{h}(t)\cdot\psi\, dx = -\int_{\overline{\Omega}^c} \sum_{l=1}^3 \partial_l \widetilde{h}(t)\cdot F_l\, dx$ for $t \in (0,\infty)$. Since $\nabla_x \widetilde{h} \in L^q_{loc}([0,\infty),\, L^q(\overline{\Omega}^c)^3)$ and $F \in L^{q'}(\overline{\Omega}^c)^3$, the function $t \mapsto \int_{\overline{\Omega}^c} \sum_{l=1}^3 \partial_l \widetilde{h}(t)\cdot F_l\, dx$, $t \in (0,\infty)$, is measurable [64, p. 131–132]. Thus we may conclude that the integral $\int_{\overline{\Omega}^c} \widetilde{h}(t)\cdot\psi\, dx$ as a function of $t \in (0,\infty)$ is measurable. As a consequence [64, p. 131–132], the function $\widetilde{h} : (0,\infty) \mapsto L^{3\cdot q/(3-q)}(\overline{\Omega}^c)$ is measurable. Due to the inequality $\|\widetilde{h}(t)\|_{3\cdot q/(3-q)} \le C(\Omega)\cdot\|\nabla_x h(t)\|_q$ for $t \in (0,\infty)$ (see above), we thus get that $\widetilde{h} \in L^q_{loc}([0,\infty),\, L^{3\cdot q/(3-q)}(\overline{\Omega}^c))$, with the preceding relation valid without the index "loc" if $\nabla_x h \in L^q(Z_\infty)^3$. □

The next theorem gives an example on how additional regularity of the data of problem (1.8)–(1.10) yields additional regularity of the solution.

Theorem 3.3. *Let $u_0 \in V$, $f \in L^2(0,\infty,\, L^2(\overline{\Omega}^c)^3)$, and let u be the solution of (3.1) associated to u_0 and f according to Theorem 3.2. Then, in addition to the regularity results in Theorem 3.2, the function u belongs to $L^2_{loc}([0,\infty),\, W^{2,2}(\overline{\Omega}^c)^3)$ and to $W^{1,2}_{loc}([0,\infty),\, L^2_\sigma(\overline{\Omega}^c))$. Moreover $u' \in L^2(0,\infty,\, L^2_\sigma(\overline{\Omega}^c))$ and*

$$u'(t) + A_2\,(u(t)) = P_2 f(t) \quad \text{for } t \in (0,\infty). \tag{3.2}$$

Moreover there is a unique function $p : (0,\infty) \mapsto W^{1,2}_{loc}(\overline{\Omega}^c)$ such that $p \in L^2_{loc}([0,\infty),\, L^6(\overline{\Omega}^c))$ and $\nabla_x p \in L^2_{loc}([0,\infty),\, L^2(\overline{\Omega}^c)^3)$, and such that (1.8) holds.

Proof. The first part of this theorem, up to and including Eq. (3.2), holds according to [56, Theorem IV.2.5.1, IV.2.5.2, IV.2.5.4]. Moreover, according to [56, Theorem IV.2.6.3], there is a function $\widetilde{p} : (0,\infty) \mapsto W^{1,2}_{loc}(\overline{\Omega}^c)$ such that $\nabla_x \widetilde{p} \in L^2_{loc}([0,\infty),\, L^2(\overline{\Omega}^c)^3)$ and such that (1.8) holds with p replaced by $\widetilde{p}$. By Lemma 3.2 with $q = 2$ we know there is a function $p : (0,\infty) \mapsto W^{1,2}_{loc}(\overline{\Omega}^c)$ with $p \in L^2_{loc}([0,\infty),\, L^6(\overline{\Omega}^c))$ and $\nabla_x p = \nabla_x \widetilde{p}$. The latter equation implies that $\nabla_x p \in L^2_{loc}([0,\infty),\, L^2(\overline{\Omega}^c)^3)$ and that the pair (u,p) satisfies (1.8).

Let $\overline{p} : (0,\infty) \mapsto W^{1,2}_{loc}(\overline{\Omega}^c)$ be another function satisfying the properties stated for p in Theorem 3.3. Then we have $\nabla_x p = \nabla_x \overline{p}$ due to Eq. (1.8), so Lemma 2.1 with $\kappa = 6$, $q = 2$ and $v = (p - \overline{p})(t)$ for $t \in (0,\infty)$ implies $p = \overline{p}$. □

In the case $f=0$, the preceding theorem yields a result on local regularity in time and global regularity in space for the L^2-weak solution of (1.8) from Theorem 3.2. We express this result in terms of time shifts of this solution:

Corollary 3.1. *Consider the situation of Theorem 3.2 with $f=0$. Then, for $t_0 \in (0,\infty)$, all the statements of Theorems 3.2 and 3.3 are valid if u_0 and u are replaced by $u(t_0)$ and $u(\cdot+t_0)$, respectively. In particular, Eqs. (3.1) and (3.2) hold with $f=0$ and with $u(t_0)$ and $u(\cdot+t_0)$ in the place of u_0 and u, respectively, for $t_0 \in (0,\infty)$.*

There is a unique function $p:(0,\infty) \mapsto W^{1,2}_{loc}(\overline{\Omega}^c)$ with $p(\cdot+t_0) \in L^2_{loc}([0,\infty),\, L^6(\overline{\Omega}^c))$ and $\nabla_x p(\cdot+t_0) \in L^2_{loc}([0,\infty),\, L^2(\overline{\Omega}^c)^3)$, and such that (1.8) holds with $u(\cdot+t_0)$, $p(\cdot+t_0)$ in the place of u and p, respectively, for $t_0 \in (0,\infty)$.

Proof. Let $t_0 \in (0,\infty)$. By Theorem 3.2, we know that $u(\cdot+t_0) \in L^2_{loc}([0,\infty),\, V) \cap W^{1,2}_{loc}([0,\infty),\, V')$, and that the equation $u(\cdot+t_0)'(t)+\mathfrak{B}(u(t+t_0))=0$ holds for $t \in (0,\infty)$. Thus, in view of Lemma 3.1, we may conclude that Theorem 3.2 remains valid if $f=0$ and u_0, u are replaced by $u(t_0)$ and $u(\cdot+t_0)$, respectively. Since $u(t_0) \in V$ and $f=0$, it is obvious that also the statements of Theorem 3.3 hold with $u(t_0)$ and $u(\cdot+t_0)$ in the place of u_0 and u, respectively. This leaves us to show that there is a function p with properties as described in the corollary. To see this, we observe that for $t_0 \in (0,\infty)$, Theorem 3.3 yields existence of a unique function $p_{t_0}:(0,\infty) \mapsto W^{1,2}_{loc}(\overline{\Omega}^c)$ such that $p_{t_0} \in L^2_{loc}([0,\infty),\, L^6(\overline{\Omega}^c))$, $\nabla_x p_{t_0} \in L^2_{loc}([0,\infty),\, L^2(\overline{\Omega}^c)^3)$, and such that (1.8) holds with $u=u(\cdot+t_0)$, $p=p_{t_0}$. But uniqueness of p_{t_0} for $t_0 \in (0,\infty)$ implies $p_{t_1}(t)=p_{t_0}(t+t_1-t_0)$ for $t,\, t_0, t_1 \in (0,\infty)$ with $t_0<t_1$. Thus, setting $p(t)=p_{t_0}(t-t_0)$ for $t \in (0,\infty)$, $t_0 \in (0,t)$, we obtain a function p with properties a stated in Corollary 3.1. □

Next we prove a uniqueness theorem for mild solutions of (1.8)–(1.10).

Theorem 3.4. *Let $u \in C^0([0,\infty),\, L^2_\sigma(\overline{\Omega}^c)) \cap W^{1,2}_{loc}(0,\infty,\, L^2_\sigma(\overline{\Omega}^c))$ with $u(t) \in D(A_2)$ and $u'(t)+A_2u(t)=0$ for a.e. $t \in (0,\infty)$, and with $u(0)=0$. Then $u=0$.*

Proof. We use a reasoning as in [50, p. 100–102]. Let $\lambda \in \mathbb{C}\backslash(-\infty,0]$, and put $K(t):=R(\lambda,A_2)u(t)$, $k(t):=\lambda\cdot R(\lambda,A_2)u(t)-u(t)$ for $t \in [0,\infty)$. By our assumptions on u, and because $R(\lambda,A_2): L^2_\sigma(\overline{\Omega}^c) \mapsto L^2_\sigma(\overline{\Omega}^c)$ is linear and bounded, the functions K and k belong to $C^0([0,\infty),\, L^2_\sigma(\overline{\Omega}^c))$, $K|(T_1,T_2) \in W^{1,2}(T_1,T_2,\, L^2_\sigma(\overline{\Omega}^c))$ for $T_1,T_2 \in (0,\infty)$ with $T_1<T_2$, and $(K|(T_1,T_2))'(t)=R(\lambda,A_2)u'(t)$ for $t \in (T_1,T_2)$.

The last equation, the relation $u'(t)=-A_2u(t)$ for a.e. $t \in (0,\infty)$ and Corollary 2.1 imply that $(K|(T_1,T_2))'(t)=k(t)$ for a.e. $t \in (T_1,T_2)$, with T_1,T_2 as above. Now Lemma 2.4 yields that $K(t)=\int_1^t k(s)\,ds+K(1)$ for $t \in [T_1,T_2]$, $T_1 \in (0,1)$, $T_2 \in (1,\infty)$, where the integral is to be understood as a Bochner integral in $L^2_\sigma(\overline{\Omega}^c)$. Since K and k are continuous on $[0,\infty)$, we may conclude that the preceding equation even holds for $t \in [0,\infty)$. Thus we obtain that $K \in C^1([0,\infty),\, L^2_\sigma(\overline{\Omega}^c))$ and $K'(t)=k(t)$ for any $t \in [0,\infty)$. On the other hand, since $u \in C^0([0,\infty),\, L^2_\sigma(\overline{\Omega}^c))$, the function $\widetilde{u}(t):=-\int_0^t e^{\lambda\cdot(t-s)}u(s)\,ds$, $t \in [0,\infty)$, also belongs to $C^1([0,\infty),\, L^2_\sigma(\overline{\Omega}^c))$, with $\widetilde{u}'(t)=\lambda\cdot\widetilde{u}(t)-u(t)$. Thus the difference $w(t):=K(t)-\widetilde{u}(t)$, $t \in [0,\infty)$, is in $C^1([0,\infty),\, L^2_\sigma(\overline{\Omega}^c))$ as well, and $w'(t)=\lambda\cdot w(t)$. Obviously $w(0)=0$. These relations imply by a standard argument that $w(t)=0$ for any $t \geq 0$. In fact, for $t>0$, we get with Lemma 2.4 that

$$\|w(t)\|_2=\left\|\int_0^t \lambda\cdot w(s)\,ds\right\|_2 \leq |\lambda|\cdot t\cdot \sup\{\|w(s)\|_2 : s \in [0,t]\}.$$

This means that $\|w(t)\|_2 \leq \sup\{\|w(s)\|_2 : s \in [0,\, (2\cdot|\lambda|)^{-1}]\}/2$ for $t \in [0,\, (2\cdot|\lambda|)^{-1}]$, which is only possible if $w(t)=0$ for such t. Thus, by induction, we obtain that w vanishes everywhere on $[0,\infty)$. We may conclude that $R(\lambda,A_2)u(t)=\widetilde{u}(t)$ for $t \in [0,\infty)$. In this situation, referring to the last part of the proof of [50, Theorem 4.1.2] and to Corollary 2.1, we see that $u(t)=0$ for $t \in [0,\infty)$. □

We exploit Theorem 3.4 in order to get the following result on global spatial regularity in $W^{2,5/4}$ of L^2-weak solutions to (1.8)–(1.10) with $f=0$.

Corollary 3.2. *Let* $u_0 \in L^2_\sigma(\overline{\Omega}^c) \cap L^{5/4}_\sigma(\overline{\Omega}^c)$, *and let* u *be given as in Theorem 3.2 with* $f = 0$. *Then* $u(t) \in D(A_{5/4})$ *for any* $t \in (0, \infty)$.

Proof. Define $\widetilde{u}(t) := e^{-t \cdot A_2} u_0$ for $t \in [0, \infty)$; see Theorem 2.5. By that reference we know that $\widetilde{u}$ belongs to the spaces $C^0([0,\infty),\ L^2_\sigma(\overline{\Omega}^c))$ and $C^1((0,\infty),\ L^2_\sigma(\overline{\Omega}^c))$, and that $\widetilde{u}(0) = u_0$, $\widetilde{u}(t) \in D(A_2)$ as well as $\widetilde{u}'(t) = -A_2 \widetilde{u}(t)$ for $t \in (0,\infty)$. In particular $\widetilde{u} \in W^{1,2}_{loc}(0,\infty,\ L^2_\sigma(\overline{\Omega}^c))$. By Theorem 3.2 we have $u \in C^0([0,\infty),\ L^2_\sigma(\overline{\Omega}^c))$ and $u(0) = u_0$. Corollary 3.1 implies that $u \in W^{1,2}_{loc}(0,\infty,\ L^2_\sigma(\overline{\Omega}^c))$, $u(t) \in D(A_2)$ and $u'(t) = -A_2 u(t)$ for $t \in (0,\infty)$. Now Theorem 3.4 yields that $u = \widetilde{u}$. Put $w(t) := e^{-t \cdot A_{5/4}} u_0$ for $t \in [0,\infty)$. By Lemma 2.5, our assumptions on u_0 and the definition of the operator $e^{-t \cdot A_p}$ in Theorem 2.5, we get $\widetilde{u} = w$. Therefore $u = w$, so Corollary 3.2 follows from Theorem 2.5. □

Following [31, Remark 2.10], [58, p. 487], we define $D^{1/5,\,5/4}_{5/4}$ as the completion of $D(A_{5/4})$ in the norm of the Sobolev space $W^{2/5,\,5/4}(\overline{\Omega}^c)^3$. (For the definition of fractional order Sobolev spaces, see [1, Section 7.48] for example.) Obviously $D(A_{5/4}) \subset D^{1/5,\,5/4}_{5/4}$. The space $D^{1/5,\,5/4}_{5/4}$ appears in the ensuing result on maximal $L^{5/4}$-regularity of solutions to (1.8)–(1.10).

Theorem 3.5. *Let* $u_0 \in D^{1/5,\,5/4}_{5/4}$, $f \in L^{5/4}(Z_\infty)^3$. *Then there are functions* u, p *with the following properties: The function* u *belongs to* $L^{5/4}_{loc}([0,\infty),\ W^{2,5/4}(\overline{\Omega}^c)^3)$, $W^{1,5/4}_{loc}([0,\infty),\ L^{5/4}_\sigma(\overline{\Omega}^c))$ *and* $C^0([0,\infty),\ L^{5/4}_\sigma(\overline{\Omega}^c))$ *and satisfies the equations* $u|S_\infty = 0$ *and* $u(0) = u_0$. *The function* p *maps from* $(0,\infty)$ *into* $W^{1,5/4}_{loc}(\overline{\Omega}^c)$. *Moreover* $p \in L^{5/4}(0,\infty,\ L^{15/7}(\overline{\Omega}^c))$, u', $\nabla_x p$, $\partial_l \partial_m u \in L^{5/4}(Z_\infty)^3$ *for* $1 \le l, m \le 3$, *and the pair* (u,p) *satisfies* (1.8).

There is only one such pair (u,p), *and* u *belongs to* $L^1_{loc}([0,\infty), V)$ *and verifies* (3.1).

Proof. By [31, Theorem 2.8], [47, Theorem 1.1], there exist functions $u, \widetilde{p}$ satisfying the claims in Theorem 3.5 with $\widetilde{p}$ in the place of p, except the relations $u \in C^0([0,\infty),\ L^{5/4}_\sigma(\overline{\Omega}^c))$ and $\widetilde{p} \in L^{5/4}(0,\infty,\ L^{15/7}(\overline{\Omega}^c))$, and the statements in the last sentence of the theorem. The function $\widetilde{p}$ is defined by $\widetilde{p}(t) := G_{5/4}(-u'(t) + \Delta u(t) + f(t))$ for $t \in (0,\infty)$, with $G_{5/4}$ from Theorem 2.1, which means in particular that $\widetilde{p}$ maps into $W^{1,5/4}_{loc}(\overline{\Omega}^c)$, a fact not explicitly mentioned in [31].

By Lemma 2.4 and because $u \in W^{1,5/4}_{loc}([0,\infty),\ L^{5/4}_\sigma(\overline{\Omega}^c))$, the function u may be modified on a subset of $(0,\infty)$ of measure zero so that $u \in C^0([0,\infty),\ L^{5/4}_\sigma(\overline{\Omega}^c))$. By Lemma 3.2 with $q = 5/4$, there is a function $p : (0,\infty) \mapsto W^{1,q}_{loc}(\overline{\Omega}^c)$ with $p \in L^{5/4}([0,\infty),\ L^{15/7}(\overline{\Omega}^c))$ and $\nabla_x p = \nabla_x \widetilde{p}$. Thus obviously $\nabla_x p \in L^{5/4}(Z_\infty)$ and the pair (u,p) satisfies (1.8).

In view of justifying the last sentence in the theorem, let $(\overline{u}, \overline{p})$ be any pair of functions with properties as shown up to this point for u and p, respectively. By a Sobolev inequality, we have $W^{2,5/4}(\overline{\Omega}^c) \subset W^{1,2}(\overline{\Omega}^c)$. Thus, since $\overline{u} \in L^{5/4}_{loc}([0,\infty),\ W^{2,5/4}(\overline{\Omega}^c)^3)$, we obtain that $\overline{u} \in L^1_{loc}([0,\infty),\ W^{1,2}(\overline{\Omega}^c)^3)$. But $\overline{u}|S_\infty = 0$ and $\operatorname{div}_x \overline{u} = 0$, so we conclude by Theorem 2.2 that $\overline{u} \in L^1_{loc}([0,\infty),\ V)$. Let $\zeta \in C^\infty_0([0,\infty))$. By the properties of $\overline{u}$, the function $\zeta \cdot \overline{u}$ belongs to $C^0([0,\infty),\ L^{5/4}_\sigma(\overline{\Omega}^c))$ and to $W^{1,5/4}([0,\infty),\ L^{5/4}_\sigma(\overline{\Omega}^c))$. Therefore by Lemma 2.4 and (1.8)

$$
\begin{aligned}
0 &= \int_0^\infty (\zeta \cdot \overline{u})'(t)\, dt - (\zeta \cdot \overline{u})(0) \\
&= \int_0^\infty \big(\zeta'(t) \cdot \overline{u}(t) + \zeta(t) \cdot (\Delta_x \overline{u}(t) - \nabla_x \overline{p}(t) + f(t))\big)\, dt - \zeta(0) \cdot u_0,
\end{aligned}
$$

where the preceding integrals are to be understood as Bochner integrals in $L^2_\sigma(\overline{\Omega})$. Now Lemma 2.2 yields that (3.1) holds with $\overline{u}$ in the place of u. In particular the function u considered in the first part of this proof belongs to $L^1_{loc}([0,\infty),\ V)$ and satisfies (3.1). It follows from Theorem 3.1 that $u = \overline{u}$. Therefore $\nabla_x p = \nabla_x \overline{p}$ by (1.8), so Lemma 2.1 with $\kappa = 15/7$, $q = 5/4$, $v = (p - \overline{p})(\cdot\ , t)$ for $t \in (0,\infty)$ yields $p = \overline{p}$. □

Corollary 3.3. *Let $u_0 \in L^2_\sigma(\overline{\Omega}^c) \cap L^{5/4}_\sigma(\overline{\Omega}^c)$, $f \in L^{5/4}(Z_\infty)^3$, $u \in L^2_{loc}([0,\infty), V)$ such that (3.1) holds. Let $(u^{(1)}, p^{(1)})$ be the solution of (1.8) associated by Theorem 3.5 to the preceding function f and to $u_0 = 0$. Let $u^{(2)}$ be the solution of (3.1) associated by Theorem 3.2 to the preceding function u_0 and to $f = 0$. Then $u = u^{(1)} + u^{(2)}$.*

Proof. According to Theorem 3.5, $u^{(1)}$ belongs to $L^1_{loc}([0,\infty), V)$ and satisfies (3.1) with $u = u^{(1)}$, $u_0 = 0$. Thus $u^{(1)} + u^{(2)} \in L^1_{loc}([0,\infty), V)$ fulfills (3.1) for u_0 and f as given in the corollary, so Theorem 3.1 yields $u = u^{(1)} + u^{(2)}$. $\square$

Corollary 3.4. *Consider the situation of Corollary 3.3. For $t_0 \in (0,\infty)$, we have $u^{(2)}(t_0) \in D^{1/5,\,5/4}_{5/4}$, with the space $D^{1/5,\,5/4}_{5/4}$ defined in the remark preceding Theorem 3.5. In addition, the function $u^{(2)}(\cdot + t_0)$ belongs to $L^{5/4}_{loc}([0,\infty), W^{2,5/4}(\overline{\Omega}^c)^3)$, $W^{1,5/4}_{loc}([0,\infty), L^{5/4}_\sigma(\overline{\Omega}^c))$ and $C^0([0,\infty), L^{5/4}_\sigma(\overline{\Omega}^c))$, and we have $u^{(2)}(\cdot + t_0)'$, $\partial_l \partial_m u^{(2)}(\cdot + t_0) \in L^{5/4}(Z_\infty)^3$ for $1 \le l, m \le 3$, $t_0 \in (0,\infty)$.*

Moreover there is a function $p^{(2)} : (0,\infty) \mapsto W^{1,5/4}_{loc}(\overline{\Omega}^c)$ such that the relations $p^{(2)}(\cdot + t_0) \in L^{5/4}(0,\infty, L^{15/7}(\overline{\Omega}^c))$, $\nabla_x p^{(2)}(\cdot + t_0) \in L^{5/4}(Z_\infty)^3$ and $(u^{(2)})'(t + t_0) - \Delta_x u^{(2)}(t + t_0) + \nabla_x p^{(2)}(t + t_0) = 0$ hold for $t_0, t \in (0,\infty)$.

Proof. By Corollary 3.1, the function $u^{(2)}(\cdot + t_0)$ is in $L^2_{loc}([0,\infty), W^{2,2}(\overline{\Omega}^c)^3)$ and in $W^{1,2}_{loc}([0,\infty), L^2_\sigma(\overline{\Omega}^c))$ for $t_0 \in (0,\infty)$, and there is a function $p^{(2)} : (0,\infty) \mapsto W^{1,2}_{loc}(\overline{\Omega}^c)$ with $p^{(2)}(\cdot + t_0) \in L^2_{loc}([0,\infty), L^6(\overline{\Omega}^c))$ and $\nabla_x p^{(2)}(\cdot + t_0) \in L^2_{loc}([0,\infty), L^2(\overline{\Omega}^c)^3)$, and such that equation (1.8) holds with $f = 0$ and with $(u^{(2)}(\cdot + t_0), p^{(2)}(\cdot + t_0))$ in the place of (u, p), for $t_0 \in (0,\infty)$.

Now fix $t_0 \in (0,\infty)$. We have $u^{(2)}(t_0) \in D(A_{5/4})$ by Corollary 3.2, so $u^{(2)}(t_0) \in D^{1/5,5/4}_{5/4}$. Theorem 3.5, applied with $f = 0$ and $u_0 = u^{(2)}(t_0)$, yields existence of a pair of functions $(\widetilde{u}, \widetilde{p}) = (\widetilde{u_{t_0}}, \widetilde{p_{t_0}})$ such that $\widetilde{u}$ and $\widetilde{p}$ belong to the function spaces listed in the corollary for $u^{(2)}(\cdot + t_0)$ and $p^{(2)}(\cdot + t_0)$, respectively, and such that $\mathrm{div}_x \widetilde{u} = 0$, $\widetilde{u}|S_\infty = 0$, $\widetilde{u}(0) = u^{(2)}(t_0)$. Moreover this reference implies that (1.8) holds with $f = 0$, $(u, p) = (\widetilde{u}, \widetilde{p})$. Theorem 3.5 further states that $\widetilde{u}$ belongs to $L^1_{loc}([0,\infty), V)$ and verifies (3.1) with $f = 0$ and u_0, u replaced by $u^{(2)}(t_0)$, $\widetilde{u}$, respectively. The preceding sentence is true in an analogous way for $u^{(2)}(\cdot + t_0)$; see Corollary 3.1. Therefore Theorem 3.1 yields $\widetilde{u} = u^{(2)}(\cdot + t_0)$. As a consequence the function $u^{(2)}(\cdot + t_0)$ possesses all the qualities listed in the first part of Corollary 3.4. Since Eq. (1.8) is valid with $f = 0$ for $(u, p) = (u^{(2)}(\cdot + t_0), p^{(2)}(\cdot + t_0))$, as mentioned above, as well as for $(u, p) = (\widetilde{u}, \widetilde{p})$, it follows that $\nabla_x \widetilde{p} = \nabla_x p^{(2)}(\cdot + t_0)$. Therefore $\nabla_x p^{(2)}(\cdot + t_0) \in L^{5/4}(Z_\infty)^3$, so from Lemma 2.1 with $\kappa = 6$, $q = 5/4$ we get $p^{(2)}(t + t_0) \in L^{15/7}(\overline{\Omega}^c)$ for $t \in (0,\infty)$. Once more using Lemma 2.1, this time with $\kappa = 15/7$, $q = 5/4$, we may conclude that $p^{(2)}(t + t_0) - \widetilde{p}(t) = 0$ for $t \in (0,\infty)$. It follows that $p^{(2)}(\cdot + t_0) \in L^{5/4}(0,\infty, L^{15/7}(\overline{\Omega}^c))$. $\square$

In the situation of Theorem 3.5 and Corollary 3.3, the $W^{1,5/4}(\Omega_S)$-norm of the pressure and of the spatial gradient of the velocity is globally $L^{5/4}$-integrable with respect to time, where $S > 0$ is arbitrary but fixed. This observation is stated in the following two lemmas. Although obvious, it is a key ingredient of our theory because later on (Corollary 5.1), it will allow us to control the trace on $\partial\Omega$ of the pressure and of the space derivatives of the velocity.

Lemma 3.3. *Let $u \in L^{5/4}_{loc}([0,\infty), W^{2,5/4}(\overline{\Omega}^c)^3)$ with $\partial_l \partial_m u \in L^{5/4}(Z_\infty)^3$ for $1 \le l, m \le 3$. Then $\nabla_x u \in L^{5/4}(0,\infty, L^{15/7}(\overline{\Omega}^c)^9)$. Let $S \in (0,\infty)$ with $\overline{\Omega} \subset B_S$. Then $\nabla_x u|Z_{S,\infty} \in L^{5/4}(0,\infty, W^{1,5/4}(\Omega_S)^9)$.*

Proof. Let $j, k \in \{1, 2, 3\}$. By Lemma 3.2 with $q = 5/4$, there is a function $\widetilde{h} : (0,\infty) \mapsto W^{1,5/4}_{loc}(\overline{\Omega}^c)$ with $\widetilde{h} \in L^{5/4}(0,\infty, L^{15/7}(\overline{\Omega}^c))$ and $\nabla_x \widetilde{h} = \nabla_x D_j u_k$. On the other hand, $D_j u_k(t) \in L^{5/4}(\overline{\Omega}^c)$, so by Lemma 2.1 with $\kappa = q = 5/4$, we may conclude that $D_j u_k(t) \in L^{15/7}(\overline{\Omega}^c)$, for $t \in (0,\infty)$. Another application of Lemma 2.1, this time with $\kappa = 15/7$, $q = 5/4$, then yields $D_j u_k(t) = \widetilde{h}(t)$ for $t \in (0,\infty)$, so $D_j u_k \in L^{5/4}(0,\infty, L^{15/7}(\overline{\Omega}^c))$. This relation and our assumptions on u imply the statement at the end of Lemma 3.4 $\square$

Lemma 3.4. *Let* $q \in \{5/4, 2\}$*. Consider a function* $p : (0,\infty) \mapsto W^{1,q}_{loc}(\overline{\Omega}^c)$ *with* p *belonging to* $L^q(0,\infty, L^{3\cdot q/(3-q)}(\overline{\Omega}^c))$ *and with* $\nabla_x p \in L^q(Z_\infty)^3$*. Let* $S \in (0,\infty)$ *with* $\overline{\Omega} \subset B_S$*. Then* $p|Z_{S,\infty} \in L^q(0,\infty, W^{1,q}(\Omega_S))$*.*

4. Representation Formulas for Solutions of the Time-Dependent Stokes System

We will derive two integral representations for the velocity part of solutions to (1.8)–(1.10). These representations are obtained by partial integration in integrals involving the Stokes fundamental solution Γ and solutions to (1.8)–(1.10). The difficulty with this approach consists in some subtle passages to the limit. We deal with them in a series of lemmas which will enter into the proof of Theorem 4.2, where we will state our first formula. The second (Theorem 4.3), which does not involve the pressure, will be deduced from the first.

Lemma 4.1. *Let* $x \in \mathbb{R}^3$, $t \in (0,\infty)$, $\epsilon \in (0,t)$, (R_n) *a sequence in* $(0,\infty)$ *with* $R_n \to \infty$ *and* $\overline{\Omega} \subset B_{R_n}$ *for* $n \in \mathbb{N}$*. Then, in the situation of Theorem 3.3 or 3.5,*

$$\lim_{n\to\infty} \int_0^{t-\epsilon} \int_{\Omega_{R_n}} \Gamma(x-y,t-s) \cdot \partial_s u(y,s)\, dy\, ds = \int_0^{t-\epsilon} \int_{\overline{\Omega}^c} -\partial_s \Gamma(x-y,t-s) \cdot u(y,s)\, dy\, ds$$
$$+ \int_{\overline{\Omega}^c} \Gamma(x-y,\epsilon) \cdot u(y,t-\epsilon)\, dy - \int_{\overline{\Omega}^c} \Gamma(x-y,t) \cdot u_0(y)\, dy.$$

Proof. Put $q = 5/4$ if we are in the situation of Theorem 3.5, and $q = 2$ else. Let $n \in \mathbb{N}$. By Theorem 3.3 or Theorem 3.5, we know that $u \in W^{1,q}_{loc}([0,\infty), L^q(\overline{\Omega}^c)^3)$. Thus $u|Z_{R_n,t-\epsilon} \in W^{1,q}(0,t-\epsilon, L^q(\Omega_{R_n})^3)$. The function $(y,s) \mapsto \Gamma(x-y,t-s)$, with $y \in \mathbb{R}^3$, $s \in [0,t-\epsilon]$, belongs to $C^1(\mathbb{R}^3 \times [0,t-\epsilon])^{3\times 3}$, and all its derivatives are bounded, as follows from Theorem 2.6 and the inequality $t-s \geq \epsilon$ for $s \in (0,t-\epsilon)$. Therefore the function $F(y,s) := \Gamma(x-y,t-s) \cdot u(y,s)$ ($(y,s) \in \Omega_{R_n} \times [0,t-\epsilon]$) is in $W^{1,q}(0,t-\epsilon, L^q(\Omega_{R_n})^3)$. By Theorem 3.2 or 3.5, we have $u \in C^0([0,t-\epsilon], L^q(\overline{\Omega}^c)^3)$, so the mapping $s \mapsto F(\cdot\,,s)$ is continuous from $[0,t-\epsilon]$ into $L^q(\Omega_{R_n})^3$. Thus with Lemma 2.4 we get $\int_0^{t-\epsilon} F'(s)\, ds = F(t-\epsilon) - F(0)$. Integrating this equation over Ω_{R_n} and applying Lemma 2.2 and 2.3, we obtain

$$\int_0^{t-\epsilon} \int_{\Omega_{R_n}} \Gamma(x-y,t-s) \cdot \partial_s u(y,s)\, dy\, ds = -\int_0^{t-\epsilon} \int_{\Omega_{R_n}} \partial_s \Gamma(x-y,t-s) \cdot u(y,s)\, dy\, ds$$
$$+ \int_{\Omega_{R_n}} \Gamma(x-y,\epsilon) \cdot u(y,t-\epsilon)\, dy - \int_{\Omega_{R_n}} \Gamma(x-y,t) \cdot u_0(y)\, dy.$$

But for $s = 0$ and $s = t-\epsilon$, by Theorem 2.6 and Hölder's inequality, and because $t-s \geq \epsilon$,

$$\int_{\overline{\Omega}^c} |\Gamma(x-y,t-s) \cdot u(y,s)|\, dy \leq C \cdot \left(\int_{\overline{\Omega}^c} \left(|x-y| + (t-s)^{1/2} \right)^{-3\cdot q'} dy \right)^{1/q'} \cdot \|u(s)\|_q$$
$$\leq C(\epsilon,q) \cdot \|u(s)\|_q.$$

Similarly

$$\int_0^{t-\epsilon} \int_{\overline{\Omega}^c} |\partial_s \Gamma(x-y,t-s) \cdot u(y,s)|\, dy\, ds \leq C(\epsilon,t,q) \cdot \|u|Z_{t-\epsilon}\|_q,$$
$$\int_0^{t-\epsilon} \int_{\overline{\Omega}^c} |\Gamma(x-y,t-s) \cdot \partial_s u(y,s)|\, dy\, ds \leq C(\epsilon,t,q) \cdot \|u'|Z_{t-\epsilon}\|_q.$$

But $\|u|Z_{t-\epsilon}\|_q < \infty$ and $\|u'|Z_{t-\epsilon}\|_q < \infty$ since $u \in W^{1,q}_{loc}([0,\infty), L^q(\overline{\Omega}^c)^3)$, as mentioned above. Lemma 4.1 follows from these observations and Lebesgue's theorem. $\square$

Lemma 4.2. *Let* $K \in L^r(\overline{\Omega}^c)^{3\times 3}$ *for any* $r \in (1,\infty)$. *Let* $s_1, s_2 \in [0,\infty)$. *Then, for* u *as in Theorem 3.3 or 3.5,*

$$\int_{\overline{\Omega}^c} K(y)\cdot (u(y,s_1)-u(y,s_2))\, dy = \int_{s_2}^{s_1}\int_{\overline{\Omega}^c} K(y)\cdot \partial_s u(y,s)\, dy\, ds.$$

Proof. Put $q=2$ in the situation of Theorem 3.3, and $q=5/4$ else. Recall that u belongs to $C^0([0,\infty),\, L^q(\overline{\Omega}^c)^3)$ and to $W^{1,q}_{loc}([0,\infty),\, L^q(\overline{\Omega}^c)^3)$ (Theorems 3.2, 3.3, 3.5). Thus Lemmas 2.3 and 2.4 yield $\int_{s_2}^{s_1} \partial_s u(s)\, ds = u(s_1)-u(s_2)$, where the integral is to be understood as a Bochner integral in $L^q(\overline{\Omega}^c)^3$. Using Lemma 2.2, multiplying by $K(y)$ and integrating with respect to $y \in \overline{\Omega}^c$ yields the lemma. □

Theorem 4.1. *Let* $t \in (0,\infty)$ *and let* (ϵ_n) *be a sequence in* $(0,t)$ *with* $\epsilon \downarrow 0$. *Then, in the situation of Theorem 3.3 or 3.5, there is a subsequence* $(\widetilde{\epsilon_n})$ *of* (ϵ_n) *with* $\int_{\overline{\Omega}^c} \mathfrak{H}(x-y,\widetilde{\epsilon_n})\cdot u(y,t-\widetilde{\epsilon_n})\, dy \to u(x,t)$ $(n\to\infty)$ *for a.e.* $x \in \overline{\Omega}^c$.

Proof. Put $q=2$ in the situation of Theorem 3.3 and $q=5/4$ else. Let $\epsilon \in (0,t)$, $x\in\mathbb{R}^3$. The function $y \mapsto \mathfrak{H}(x-y,\epsilon)$ $(y\in\mathbb{R}^3)$ belongs to $L^\kappa(\mathbb{R}^3)$ for any $\kappa\in(1,\infty)$, as follows from Theorem 2.6. Thus we get by Lemma 4.2 that $\int_{\overline{\Omega}^c} \mathfrak{H}(x-y,\epsilon)\cdot(u(y,t-\epsilon)-u(y,t))\, dy = \int_t^{t-\epsilon}\int_{\overline{\Omega}^c} \mathfrak{H}(x-y,\epsilon)\cdot\partial_s u(y,s)\, dy\, ds$. We may conclude with Minkowski's and Young's inequality for integrals ([2, Theorem 2.9, Corollary 2.25]) and with Lemma 2.6 that

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^3}\Big|\int_{\overline{\Omega}^c} \mathfrak{H}(x-y,\epsilon)\cdot(u(y,t-\epsilon)-u(y,t))\; dy\Big|^q dx\Big)^{1/q}\\
&\quad\le \int_{t-\epsilon}^t\Big(\int_{\mathbb{R}^3}\Big(\int_{\overline{\Omega}^c}\mathfrak{H}(x-y,\epsilon)\cdot|\partial_s u(y,s)|\, dy\Big)^q dx\Big)^{1/q} ds\\
&\quad\le \int_{t-\epsilon}^t\Big(\int_{\mathbb{R}^3}\mathfrak{H}(z,\epsilon)\, dz\Big)\cdot\|\partial_s u(\cdot\,,s)\|_q\, ds = \int_{t-\epsilon}^t \|\partial_s u(\cdot\,,s)\|_q\, ds \le \epsilon^{1/q'}\cdot\|\partial_s u|Z_t\|_q.
\end{aligned}\tag{4.1}$$

Let $\kappa>0$. Since $\partial_s u|Z_t \in L^q(Z_t)^3$ (Theorem 3.3 or 3.5), there is $\epsilon_1\in(0,t)$ such that the left-hand side of (4.1) is bounded by $\kappa/5$ for any $\epsilon\in(0,\epsilon_1]$. Recalling that $u(t)\in L^q(\overline{\Omega}^c)^3$, we may choose $\psi\in C_0^\infty(\overline{\Omega}^c)^3$ with $\|u(\cdot\,,t)-\psi\|_q \le \kappa/5$. It follows with Young's inequality for integrals and Lemma 2.6 that for $\epsilon\in(0,\infty)$,

$$\Big(\int_{\mathbb{R}^3}\Big|\int_{\overline{\Omega}^c}\mathfrak{H}(x-y,\epsilon)\cdot(u(y,t)-\psi(y))\, dy\Big|^q dx\Big)^{1/q} \le \int_{\mathbb{R}^3}\mathfrak{H}(z,\epsilon)\, dz\cdot\|u(\cdot\,,t)-\psi\|_q \le \kappa/5. \tag{4.2}$$

Abbreviate $M(x,\epsilon) := \int_{\overline{\Omega}^c}\mathfrak{H}(x-y,\epsilon)\cdot\psi(y)\, dy$ for $x\in\mathbb{R}^3$, $\epsilon>0$. Let $R\in(S,\infty)$ be so large that $supp(\psi)\subset B_{R/2}$. By Theorem 2.8 we know that $M(x,\epsilon)\to\psi(x)$ $(\epsilon\downarrow 0)$, uniformly in $x\in\mathbb{R}^3$, so we may choose $\epsilon_2>0$ such that

$$\|\,(M(\cdot\,,\epsilon)-\psi)\,|B_R\|_q \le \kappa/5 \quad \text{for}\;\; \epsilon\in(0,\epsilon_2]. \tag{4.3}$$

By the choice of R, we have $|x-y|\ge|x|/2$ for $x\in B_R^c$, $y\in supp(\psi)$. Therefore the relation $\mathfrak{H}(x-y,\delta)\to 0$ $(\delta\downarrow 0)$ is valid for such x and y, so for $\epsilon>0$,

$$\begin{aligned}
\mathfrak{H}(x-y,\epsilon) &= \lim_{\delta\downarrow 0}\big(\mathfrak{H}(x-y,\epsilon)-\mathfrak{H}(x-y,\delta)\big)\\
&= \lim_{\delta\downarrow 0}\int_0^1 \partial_4\mathfrak{H}\,(x-y,\,\delta+\vartheta\cdot(\epsilon-\delta))\, d\vartheta\cdot(\epsilon-\delta).
\end{aligned}$$

But $|\partial_4\mathfrak{H}(x-y,\,\delta+\vartheta\cdot(\epsilon-\delta))| \le C\cdot(|x-y|+[\delta+\vartheta\cdot(\epsilon-\delta)]^{1/2})^{-5}$ for $x,\,y,\,\epsilon$ as before, and for $\delta\in(0,\epsilon)$ (Theorem 2.6). Therefore, in view of our previous estimate of $|x-y|$, we may conclude that $0\le\mathfrak{H}(x-y,\epsilon)\le C\cdot\epsilon\cdot|x|^{-5}$ for $x,\,y,\,\epsilon$ as before. It follows that

$$\| (M(\cdot\,,\epsilon) - \psi) \mid B_R^c\|_q = \|M(\cdot\,,\epsilon)|B_R^c\|_q \le C\cdot\epsilon\cdot\|\psi\|_1\cdot\Big(\int_{B_R^c}|x|^{-5\cdot q}\,dx\Big)^{1/q} \le C(R)\cdot\|\psi\|_1\cdot\epsilon,$$

for $\epsilon > 0$. As a consequence, there is $\epsilon_3 > 0$ with $\|(M(\cdot\,,\epsilon) - \psi) \,|\, B_R^c\|_q \le \kappa/5$ for $\epsilon \in (0, \epsilon_3]$. Define $L(x,\epsilon) := \int_{\overline{\Omega}^c} \mathfrak{H}(x-y,\epsilon)\cdot u(y,t-\epsilon)\,dy$ for $x\in\mathbb{R}^3$, $\epsilon\in(0,t)$. Then, by the preceding inequality and (4.1)–(4.3), we have $\|L(\cdot\,,\epsilon)-\psi\|_q \le 4\cdot\kappa/5$ for $\epsilon\in(0,\min\{\epsilon_1,\epsilon_2,\epsilon_3\}]$. By the choice of ψ, this means that $\|L(\cdot\,,\epsilon)-u(\cdot\,,t)\|_q\le\kappa$ for ϵ as before. Therefore $\|L(\cdot\,,\epsilon)-u(\cdot\,,t)\|_q\to 0$ $(\epsilon\downarrow 0)$, so the theorem follows. □

Lemma 4.3. *Let $t\in(0,\infty)$, $x\in\overline{\Omega}^c$. Then, in the situation of Theorem 3.3 or 3.5, there is a sequence (R_n) in $(0,\infty)$ with $R_n\to\infty$, $\overline{\Omega}^c\subset B_{R_n}$ for $n\in\mathbb{N}$, and*

$$\int_0^t\int_{\partial B_{R_n}}\big(|\Gamma(x-y,t-s)|\cdot(|\nabla_y u(y,s)|+|p(y,s)|)+|\partial_{y_l}\Gamma(x-y,t-s)|\cdot|u(y,s)|\big)\,do_y\,ds \to 0 \quad (n\to\infty)\quad \textit{for } 1\le l\le 3.$$

Proof. Put $q = 5/4$ in the situation of Theorem 3.5, and $q = 2$ else. Abbreviate $\widetilde{q} = 3\cdot q/(3-q)$. Fix some $S\in(0,\infty)$ with $\overline{\Omega}\subset B_S$ and $S\ge 2\cdot|x|$. Note that $u\in L^q_{loc}([0,\infty),\,W^{2,q}(\overline{\Omega}^c)^3)$ and $p\in L^q_{loc}([0,\infty),\,L^{\widetilde{q}}(\overline{\Omega}^c))$. By Minkowski's inequality for integrals [2, Theorem 2.9],

$$\Big(\int_{B_S^c}\Big(\int_0^t|u(y,s)|\,ds\Big)^q dy\Big)^{1/q} \le \int_0^t\Big(\int_{B_S^c}|u(y,s)|^q\,dy\Big)^{1/q} ds \le t^{1/q'}\cdot\|u|Z_t\|_q<\infty,$$

and similarly $(\int_{B_S^c}(\int_0^t|p(y,s)|\,ds)^{\widetilde{q}}\,dy)^{1/\widetilde{q}}<\infty$ and $(\int_{B_S^c}(\int_0^t|\nabla_y u(y,s)|\,ds)^q dy)^{1/q}<\infty$. Put

$$\mathfrak{A}(y) := \Big(\int_0^t|u(y,s)|\,ds\Big)^q + \Big(\int_0^t|p(y,s)|\,ds\Big)^{\widetilde{q}} + \Big(\int_0^t|\nabla_y u(y,s)|\,ds\Big)^q \quad \text{for } y\in B_S^c.$$

Using this notation, we have shown that $\int_{B_S^c}\mathfrak{A}(y)\,dy<\infty$, so $\int_S^\infty\int_{\partial B_r}\mathfrak{A}(y)\,do_y\,dr<\infty$. Suppose for a contradiction there is some $R_0\in[S,\infty)$ with $\int_{\partial B_r}\mathfrak{A}(y)\,do_y\ge r^{-1}$ for any $r\in[R_0,\infty)$. Then $\int_{R_0}^\infty\int_{\partial B_r}\mathfrak{A}(y)\,do_y\,dr=\infty$, which contradicts the relation $\int_S^\infty\int_{\partial B_r}\mathfrak{A}(y)\,do_y<\infty$. So we may choose a sequence (R_n) in (S,∞) such that $R_n\to\infty$ and $\int_{\partial B_{R_n}}\mathfrak{A}(y)\,do_y\le R_n^{-1}$ for $n\in\mathbb{N}$. Let $n\in\mathbb{N}$, $l\in\{1,2,3\}$. Then, for $y\in\partial B_{R_n}$, we have $|y|=R_n\ge S\ge 2\cdot|x|$, so $|y-x|\ge|y|/2=R_n/2$. It follows with Theorem 2.6,

$$\begin{aligned}\int_0^t\int_{\partial B_{R_n}}|\partial_{y_l}\Gamma(x-y,t-s)|\cdot|u(y,s)|\,do_y\,ds &\le C\cdot\int_{\partial B_{R_n}}\int_0^t|x-y|^{-4}\cdot|u(y,s)|\,ds\,do_y\\ &\le C\cdot R_n^{-4}\cdot\int_{\partial B_{R_n}}\int_0^t|u(y,s)|\,ds\,do_y\\ &\le C\cdot R_n^{-4}\cdot\Big(\int_{\partial B_{R_n}}do_y\Big)^{1/q'}\cdot\Big(\int_{\partial B_{R_n}}\mathfrak{A}(y)\,do_y\Big)^{1/q}\le C\cdot R_n^{-4+2/q'-1/q},\end{aligned}\tag{4.4}$$

where the last inequality holds by the choice of the sequence (R_n). In a similar way it may be shown on the one hand that $\int_0^t\int_{\partial B_{R_n}}|\Gamma(x-y,t-s)|\cdot|p(y,s)|\,do_y\,ds\le C\cdot R_n^{-3+2/\widetilde{q}'-1/\widetilde{q}}$, and on the other hand that the estimate $\int_0^t\int_{\partial B_{R_n}}|\Gamma(x-y,t-s)|\cdot|\partial_l u(y,s)|\,do_y\,ds\le C\cdot R_n^{-3+2/q'-1/q}$ is valid. The lemma follows from (4.4) and the two preceding inequalities. □

Now we are in a position to prove the first of the two representation formulas we derive in this section.

Theorem 4.2. *Let u_0, f, u, p be given as in Theorem 3.3 or 3.5. Let $t \in (0,\infty)$. Then there is a subset N_t of $\overline{\Omega}^c$ with measure zero such that for $x \in \overline{\Omega}^c \backslash N_t$,*

$$u(x,t) = \mathfrak{R}(f)(x,t) + \mathfrak{I}(u_0)(x,t) + \mathfrak{V}\left(\sum_{l=1}^{3} n_l^{(\Omega)} \cdot (-\partial_l u + p \cdot e_l)\right)(x,t). \tag{4.5}$$

A similar integral representation, but under different assumptions, was proved by Kozono [39].

Proof. Put $q = 5/4$ in the situation of Theorem 3.5, and $q = 2$ else. Let (ϵ_m) be a sequence in $(0,t)$ with $\epsilon_m \downarrow 0$. Let $m \in \mathbb{N}$, $x \in \overline{\Omega}^c$. Choose a sequence (R_n) in $(0,\infty)$ associated to t and x as in Lemma 4.3. By Eq. (1.8) we have

$$0 = \lim_{n\to\infty} \int_0^{t-\epsilon_m} \int_{\Omega_{R_n}} \Gamma(x-y,t-s) \cdot (\partial_s u - \Delta_y u + \nabla_y p - f)(y,s)\, dy\, ds. \tag{4.6}$$

Note that for $s \in (0,t)$, the function $y \mapsto \Gamma(x-y,t-s)$ $(y \in \mathbb{R}^3)$ belongs to $C^\infty(\mathbb{R}^3)^{3\times 3}$ (Theorem 2.6), and that $\sum_{k=1}^{3} \partial_{y_k}\Gamma_{jk}(x-y,t-s) = 0$ $(y \in \mathbb{R}^3,\ 1 \le j \le 3)$. In view of the relations $u \in L^q_{loc}([0,\infty), W^{2,q}(\overline{\Omega}^c)^3)$, $p|Z_{R_n,\infty} \in L^q_{loc}([0,\infty), W^{1,q}(\Omega_{R_n})^3)$ (Theorem 3.3 or 3.5, Lemma 3.4), we may thus integrate by parts in (4.6) with respect to the space variables, to obtain

$$0 = \lim_{n\to\infty} \int_0^{t-\epsilon_m} \left(\mathfrak{A}_1(x,s) + \mathfrak{A}_2(x,s) + \sum_{\nu=3}^{7} \mathfrak{A}_\nu(n,x,s)\right) ds, \tag{4.7}$$

with

$$\mathfrak{A}_1(x,s) := \int_{\partial\Omega} \Gamma(x-y,t-s) \cdot \sum_{l=1}^{3} n_l^{(\Omega)}(y) \cdot (\partial_l u - p \cdot e_l)(y,s)\, do_y,$$

$$\mathfrak{A}_2(x,s) := -\int_{\partial\Omega} \sum_{l=1}^{3} \partial_{y_l}\Gamma(x-y,t-s) \cdot \left(n_l^{(\Omega)}(y) \cdot u(y,s)\right) do_y,$$

$$\mathfrak{A}_5(n,x,s) := -\int_{\Omega_{R_n}} \Delta_y\Gamma(x-y,t-s) \cdot u(y,s)\, do_y,$$

$$\mathfrak{A}_6(n,x,s) := -\int_{\Omega_{R_n}} \Gamma(x-y,t-s) \cdot f(y,s)\, do_y$$

for $s \in (0,\, t-\epsilon_m)$. The terms $\mathfrak{A}_3(n,x,s)$ and $\mathfrak{A}_4(n,x,s)$ are defined as $\mathfrak{A}_1(x,s)$ and $\mathfrak{A}_2(x,s)$, respectively, but with the domain of integration $\partial\Omega$ replaced by ∂B_{R_n}, and the factor $n_l^{(\Omega)}(y)$ by $-y_l/R_n$. As concerns the term $\mathfrak{A}_7(n,x,s)$, it is defined as $\mathfrak{A}_6(n,x,s)$, but with $-\partial_s u(y,s)$ in the place of $f(y,s)$. Since $u|S_\infty = 0$, we have $\mathfrak{A}_2(x,s) = 0$ for $s \in (0,\, t-\epsilon_m)$. By the choice of the sequence (R_n) and by Lemma 4.3, we further get $\lim_{n\to\infty} \int_0^{t-\epsilon_m} (\mathfrak{A}_3(n,x,s) + \mathfrak{A}_4(n,x,s))\, ds = 0$. Moreover, according to Lemma 4.1, $\lim_{n\to\infty} \int_0^{t-\epsilon_m} \mathfrak{A}_7(n,x,s)\, ds = \sum_{r=1}^{3} \mathfrak{B}_r(m,x)$, where $\mathfrak{B}_1(m,x)$, $\mathfrak{B}_2(m,x)$, $\mathfrak{B}_3(m,x)$ are defined as the first, second and third term, respectively, in the sum on the right-hand side of the equation in Lemma 4.1, with ϵ replaced by ϵ_m. (Actually the term $\mathfrak{B}_3(m,x)$ is independent of m.) By Theorem 2.6, we know that for $y \in \mathbb{R}^3$, $s \in (0,\, t-\epsilon_m)$, the inequality $|\Delta_y\Gamma(x-y,t-s)| \le C \cdot (|x-y| + \epsilon_m^{1/2})^{-5}$ holds. Therefore the function $(y,s) \mapsto \Delta_y\Gamma(x-y,t-s)$, with $(y,s) \in Z_{t-\epsilon_m}$, belongs to $L^{q'}(Z_{t-\epsilon_m})^{3\times 3}$. Since $u|Z_{t-\epsilon_m} \in L^q(Z_{t-\epsilon_m})^3$, we may thus conclude by Lebesgue's theorem that the function $(y,s) \mapsto \Delta_y\Gamma(x-y,t-s) \cdot u(y,s)$ $((y,s) \in Z_{t-\epsilon_m})$ is integrable, and $\int_0^{t-\epsilon_m} \mathfrak{A}_5(n,x,s)\, ds \to \mathfrak{B}_4(m,x)$ $(n \to \infty)$, with

$$\mathfrak{B}_4(m,x) := -\int_0^{t-\epsilon_m} \int_{\overline{\Omega}^c} \Delta_y\Gamma(x-y,t-s) \cdot u(y,s)\, dy\, ds.$$

Since $f \in L^q(Z_\infty)^3$, and because $|\Gamma(x-y,t-s)| \le C \cdot (|x-y| + \epsilon_m^{1/2})^{-3}$ for $y \in \mathbb{R}^3,\ s \in (0, t-\epsilon_m)$ by Theorem 2.6, the same argument yields $\int_0^{t-\epsilon_m} \mathfrak{A}_6(n,x,s)\,ds \to \mathfrak{B}_5(m,x)\ (n \to \infty)$, with $\mathfrak{B}_5(m,x) := -\int_0^{t-\epsilon_m} \int_{\overline{\Omega}^c} \Gamma(x-y,t-s) \cdot f(y,s)\,dy\,ds$. Thus we get from (4.7) that

$$0 = \int_0^{t-\epsilon_m} \mathfrak{A}_1(x,s)\,ds + \sum_{i=1}^{5} \mathfrak{B}_i(m,x). \tag{4.8}$$

Since $-\partial_s \Gamma(x-y,t-s) - \Delta_y \Gamma(x-y,t-s) = 0$, the terms $\mathfrak{B}_1(m,x)$ and $\mathfrak{B}_4(m,x)$ cancel. By Lemma 2.10 we have

$$\mathfrak{B}_2(m,x) = \int_{\overline{\Omega}^c} \mathfrak{H}(x-y,\epsilon_m) \cdot u(y,t-\epsilon_m)\,dy, \tag{4.9}$$

$$\mathfrak{B}_3(m,x) = \int_{\overline{\Omega}^c} -\mathfrak{H}(x-y,t) \cdot u_0(y)\,dy = -\mathfrak{I}(u_0)(x,t). \tag{4.10}$$

The results from (4.6) to this point hold for any $m \in \mathbb{N},\ x \in \overline{\Omega}^c$. Now we are going to let m tend to infinity. To this end, we observe that for $x \in \overline{\Omega}^c$, there is $\delta(x) > 0$ with $|x-y| \ge \delta(x)$ for $y \in \partial\Omega$, hence $|\Gamma(x-y,t-s)| \le C \cdot \delta(x)^{-3}$ for $s \in (0,t),\ y \in \partial\Omega$ by Theorem 2.6. Thus by Lebesgue's theorem we get $\int_0^{t-\epsilon_m} \mathfrak{A}_1(x,s)\,ds \to \int_0^t \mathfrak{A}_1(x,s)\,ds\ (m \to \infty)$ for any $x \in \overline{\Omega}^c$. Note that $\int_0^t \mathfrak{A}_1(x,s)\,ds$ coincides with $\mathfrak{V}(\sum_{l=1}^3 n_l^{(\Omega)} \cdot (\partial_l u - p \cdot e_l))(x,t)$. As indicated in (4.10), $\mathfrak{B}_3(m,x) = -\mathfrak{I}(u_0)(x,t)$ for $m \in \mathbb{N}$. Since $f \in L^{5/4}(Z_\infty)^3$, we see by Lemma 2.8 and Lebesgue's theorem that $\mathfrak{B}_5(m,x)$ tends to $-\mathfrak{R}(f)(x,t)$ for $m \to \infty$, for a.e. $x \in \overline{\Omega}^c$. Finally Theorem 4.1 and (4.9) imply that $\mathfrak{B}_2(m_j,x) \to u(x,t)\ (j \to \infty)$ for a.e. $x \in \overline{\Omega}^c$, where $(m_j)_{j \ge 1}$ is some strictly increasing sequence in $\mathbb{N}$. Equation (4.5) now follows from (4.8). □

Next we represent the velocity part of a solution to (1.8)–(1.10) by integrals not involving the pressure. In a first step (Lemma 4.4), we suppose that $u_0 \in V,\ f = 0$. Afterwards (Theorem 4.3), we reduce the assumptions on u_0 to $u_0 \in L^2_\sigma(\overline{\Omega}^c)$. The integral representations in question only hold at points outside a ball B_S with $\overline{\Omega} \subset B_S$.

Lemma 4.4. *Let u_0, u, p be given as in Theorem 3.3 with $f = 0$. Let $S \in (0,\infty)$ with $\overline{\Omega} \subset B_S,\ t \in (0,\infty)$. Then there is a set $N_t \subset \overline{B_S}^c$ of measure zero such that $u(x,t) = \mathfrak{I}(u_0)(x,t) + \mathfrak{K}_S(u)(x,t)$ for $x \in \overline{B_S}^c \backslash N_t$.*

Proof. Let $x \in \overline{B_S}^c$. In view of the properties of the function G_S (Lemma 2.14), in particular due to the relation $\sum_{k=1}^3 \partial_{y_k} G_{S,jk}(x,y,r) = 0\ (y \in \Omega_S,\ r \in [0,\infty),\ 1 \le j \le 3)$, we get

$$\mathfrak{V}\left(\sum_{l=1}^{3} n_l^{(\Omega)} \cdot (-\partial_l u + p \cdot e_l)\right)(x,t) = \mathfrak{B}^{(1)}(x,t) + \mathfrak{B}^{(2)}(x,t), \tag{4.11}$$

where $\mathfrak{B}^{(1)}(x,t) := \int_0^t \int_{\Omega_S} \sum_{l=1}^3 \partial_{y_l} G_S(x,y,t-\sigma) \cdot \partial_l u(y,\sigma)\,dy\,d\sigma$, and where $\mathfrak{B}^{(2)}(x,t)$ stands for the integral $\int_0^t \int_{\Omega_S} G_S(x,y,t-\sigma) \cdot (\Delta_y u - \nabla_y p)(y,\sigma)\,dy\,d\sigma$. Now we recall that u belongs to $C^0([0,\infty),\ L^2(\overline{\Omega}^c)^3)$ (Theorem 3.2) and to $W^{1,2}_{loc}([0,\infty),\ L^2(\overline{\Omega}^c)^3)$ with $\Delta_y u - \nabla_y p = u'$ (Theorem 3.3), and that the function $r \mapsto G_S(x,\ \cdot\ ,r)$ is in $C^1([0,\infty),\ L^2(\Omega_S)^{3\times3})$ (Lemma 2.16). Thus, referring to [56, Lemma IV.1.3.2], we may integrate by parts, to obtain

$$\begin{aligned} \mathfrak{B}^{(2)}(x,t) = &\int_0^t \int_{\Omega_S} -d/dr\,(G_S(x,\ \cdot\ ,t-r))\,(y) \cdot u(y,r)\,dy\,dr \\ &+ \int_{\Omega_S} G_S(x,y,0) \cdot u(y,t)\,dy + \int_{\Omega_S} -G_S(x,y,t) \cdot u(y,0)\,dy. \end{aligned} \tag{4.12}$$

The lemma now follows from (4.11) and Theorem 4.2. □

Theorem 4.3. *Consider the situation of Theorem 3.2 with $f = 0$. Let $S \in (0,\infty)$ with $\overline{\Omega} \subset B_S,\ t \in (0,\infty)$. Then there is a set $N_t \subset \overline{B_S}^c$ of measure zero such that $u(x,t) = \mathfrak{I}(u_0)(x,t) + \mathfrak{K}_S(u)(x,t)$ for $x \in \overline{B_S}^c \backslash N_t$.*

Proof. Choose a sequence (t_n) in $(0,1)$ with $t_n \downarrow 0$. Let $n \in \mathbb{N}$. By Corollary 3.1, all the statements of Theorems 3.2 and 3.3 are valid if $f=0$ and if u_0, u are replaced by $u(t_n)$ and $u(\cdot, t_n)$, respectively. Therefore Lemma 4.4 with $u(t_n)$, $u(\cdot\,, t_n)$ in the place of u_0, u, respectively, implies for a.e. $x \in \overline{B_S}^c$ that

$$u(x,\, t+t_n) = \mathfrak{I}\,(u(t_n)) + \sum_{i=1}^{4} \mathfrak{A}_n^{(i)}(x), \tag{4.13}$$

with $\mathfrak{A}_n^{(1)}(x) := \int_0^t \int_{\Omega_S} \sum_{l=1}^3 \partial_{y_l} G_S(x,y,t-r)\cdot \partial_l u(y,r+t_n)\;dy\;dr$, and with $\mathfrak{A}_n^{(2)}(x)$, $\mathfrak{A}_n^{(3)}(x)$ and $\mathfrak{A}_n^{(4)}(x)$ defined as the first, second and third integral on the right-hand side of (4.12), but with $u(y,r)$, $u(y,t)$, $u(y,0)$ replaced by $u(y,r+t_n)$, $u(y,t+t_n)$ and $u(y,t_n)$, respectively. Let $x \in \overline{B_S}^c$. We have $u \in C^0([0,\infty),\, L^2(\overline{\Omega}^c)^3)$ (Theorem 3.2). The function $r \mapsto d/dr(G_S(x,\,\cdot\,,r))$ with $r \in [0,\infty)$ maps continuously into $L^2(\Omega_S)^{3\times 3}$ (Lemma 2.16). Therefore the integral $\int_{\Omega_S} d/dr(G_S(x,\,\cdot\,,t-r))(y)\cdot u(y,r+\kappa)\,dy$ as a function of $(\kappa,r) \in [0,1]\times[0,t]$ is continuous with respect to $\kappa \in [0,1]$, for any fixed $r \in [0,t]$, and is bounded uniformly with respect to $\kappa \in [0,1]$, $r \in [0,t]$. Hence by Lebesgue's theorem

$$\mathfrak{A}_n^{(2)}(x) \to \int_0^t \int_{\Omega_S} -d/dr\,(G_S(x,\,\cdot\,,t-r))\,(y)\cdot u(y,r)\,dy\,dr \quad \text{for } \ n\to\infty.$$

The function $G_S(x,\,\cdot\,,0)$ is L^2-integrable on Ω_S (Lemma 2.16). Thus, by Hölder's inequality and the relation $u \in C^0([0,\infty),\, L^2(\overline{\Omega}^c)^3)$, we get that $\mathfrak{A}_n^{(3)}(x) \to \int_{\Omega_S} G_S(x,y,0)\cdot u(y,t)\,dy$ $(n\to\infty)$. The same argument yields $\mathfrak{A}_n^{(4)}(x) \to \int_{\Omega_S} -G_S(x,y,t)\cdot u(y,0)\,dy$ $(n\to\infty)$. Since the function $r\mapsto G_S(x,\,\cdot\,,r)$, $r\in[0,\infty)$, maps continuously into $W^{1,2}(\Omega_S)^{3\times 3}$ (Lemma 2.16), and because $\partial_l u(\kappa) \in L^2(\overline{\Omega}^c)^3$ for $1\le l\le 3$, $\kappa \in (0,\infty)$, Hölder's inequality yields that the function

$$H(r,\kappa) := \int_{\Omega_S} \sum_{l=1}^3 \partial_{y_l} G_S(x,y,r)\cdot \partial_l u(y,\kappa)\,dy \quad (r\in[0,\,t+1],\ \kappa\in(0,\,t+1))$$

is continuous with respect to $r \in [0,\,t+1]$, for any fixed $\kappa \in (0,\,t+1)$. Therefore, for $\kappa \in (0,\,t+1)$,

$$\chi_{(0,\infty)}(t_n+t-\kappa)\cdot\chi_{(0,\infty)}(\kappa-t_n)\cdot H(t_n+t-\kappa,\,\kappa) \to \chi_{[0,\infty)}(t-\kappa)\cdot H(t-\kappa,\kappa)\ \ (n\to\infty).$$

Moreover, again recalling that the function $r\mapsto G_S(x,\,\cdot\,,r)$ is continuous from $[0,\infty)$ into $W^{1,2}(\Omega_S)^{3\times 3}$, we see there is $c>0$ with $|H(r,\kappa)| \le c\cdot \|\nabla_x u(\kappa)\|_2$ for $r\in[0,\,t+1]$, $\kappa\in(0,\,t+1)$. Since $\nabla_x u \in L^2(Z_\infty)^9 \subset L^1_{loc}([0,\infty),\, L^2(\overline{\Omega}^c)^9)$ and $\mathfrak{A}_n^{(1)}(x,t) = \int_{t_n}^{t_n+t} H(t_n+t-\kappa,\,\kappa)\,d\kappa$, hence

$$\mathfrak{A}_n^{(1)}(x,t) = \int_0^{t+1} \chi_{(0,\infty)}(t_n+t-\kappa)\cdot\chi_{(0,\infty)}(\kappa-t_n)\cdot H(t_n+t-\kappa,\,\kappa)\;d\kappa,$$

it follows with Lebesgue's theorem that $\mathfrak{A}_n^{(1)}(x) \to \int_0^t H(t-\kappa,\,\kappa)\,d\kappa$ for $n\to\infty$. Moreover, because $|\mathfrak{H}(x-y,t)| \le C\cdot(|x-y|+t^{1/2})^{-3}$ for $y\in\mathbb{R}^3$ by Theorem 2.6, the function $y\mapsto \mathfrak{H}(x-y,t)$, $y\in\overline{\Omega}^c$, belongs to $L^2(\overline{\Omega}^c)$. Therefore the relations $u\in C^0([0,\infty),\, L^2(\overline{\Omega}^c)^3)$ and $u(0)=u_0$ together with Hölder's inequality imply that $\mathfrak{I}(u(t_n))(x,t) \to \mathfrak{I}(u_0)(x,t)$ for $n\to\infty$.

Thus we have shown that for any $x\in\overline{B_S}^c$, the right-hand side of (4.13) converges to $\mathfrak{I}(u_0)(x,t) + \mathfrak{K}_S(u)(x,t)$ when n tends to infinity. Again using the relation $u\in C^0([0,\infty),\, L^2(\overline{\Omega}^c)^3)$, we further obtain $\|u(t_n+t)-u(t)\|_2 \to 0$ $(n\to\infty)$, so there is a subsequence of (t_n), which we also denote by (t_n), such that $u(x,t+t_n) \to u(x,t)$ $(n\to\infty)$ for a.e. $x\in\overline{\Omega}^c$. Therefore representation formula in Theorem 4.3 holds for a.e. $x\in\overline{B_S}^c$. □

5. Pointwise Spatial Decay of Solutions of the Time-Dependent Stokes System

In this section, we study the spatial asymptotics of the velocity part of L^2-weak solutions to (1.8)–(1.10). To this end, we use the representation formulas derived in Sect. 4, estimating each term appearing in them, under the assumption $f = g + \widetilde{g}$, with g from (A2) and $\widetilde{g} \in L^{5/4}(\overline{\Omega}^c)^3$. We will further use the quantities v_0, S_0, C_0, $\gamma, \widetilde{\mu}$ introduced in (A1) and (A2), respectively. As an obvious consequence of (A2), we note

Lemma 5.1. $g \in L^{5/4}(Z_\infty)^3$.

Due to this observation and (A2), we may give a pointwise decay estimate of $\mathfrak{R}(g)$:

Theorem 5.1. *Let* $S_1 \in (S_0, \infty),\ x \in B^c_{S_1},\ t \in (0, \infty)$. *Then*

$$|\mathfrak{R}(g)(x,t)| \le C(S_0, S_1, \widetilde{\mu}) \cdot (\|\gamma\|_{5/4} + \|g|Z_{S_0,\infty}\|_{5/4}) \cdot |x|^{-13/5}.$$

Proof. We have $|\mathfrak{R}(g)(x,t)| \le \sum_{i=1}^4 |\mathfrak{A}_i|$, with $\mathfrak{A}_i := \int_0^t \int_{A_i} \Gamma(x-y, t-s) \cdot g(y,s)\, dy$ for $i \in \{1, \dots, 4\}$, with $A_1 := \Omega_{S_0},\ A_2 := B_{|x|/2} \backslash B_{S_0},\ A_3 := B_{2\cdot|x|} \backslash (B_{|x|/2} \cup B_{S_0}),\ A_4 := B^c_{2\cdot|x|}$. For $y \in A_1$, we find as in (2.5) that the estimate $|x - y| \ge (1 - S_0/S_1) \cdot |x|$ holds, so we may refer to Theorem 2.6, arriving at the inequality $|\mathfrak{A}_1| \le C(S_0, S_1) \cdot \int_0^t \int_{\Omega_{S_0}} (|x|^2 + t - s)^{-3/2} \cdot |g(y,s)|\, dy\, ds$. Hence

$$|\mathfrak{A}_1| \le C(S_0, S_1) \cdot \int_{\Omega_{S_0}} \left(\int_0^t (|x|^2 + t - s)^{-15/2}\, ds \right)^{1/5} \cdot \|g(y, \cdot\,)\|_{5/4}\, dy.$$

Thus $|\mathfrak{A}_1| \le C(S_0, S_1) \cdot |x|^{-13/5} \cdot \|g|Z_{S_0,\infty}\|_{5/4}$. We further observe that $|x - y| \ge |x|/2$ for $y \in B_{|x|/2}$, so with Theorem 2.6, Hölder's inequality and (A2),

$$\begin{aligned} |\mathfrak{A}_2| &\le C \cdot \int_0^t \int_{B_{|x|/2} \backslash B_{S_0}} (|x|^2 + t - s)^{-3/2} \cdot \gamma(s) \cdot |y|^{-\widetilde{\mu}}\, dy\, ds \\ &\le C \cdot \int_{B_{|x|/2} \backslash B_{S_0}} |x|^{-13/5} \cdot \|\gamma\|_{5/4} \cdot |y|^{-\widetilde{\mu}}\, dy \le C(S_0, \widetilde{\mu}) \cdot |x|^{-13/5} \cdot \|\gamma\|_{5/4}, \end{aligned}$$

where in the last inequality we used that $\widetilde{\mu} > 3$. Moreover, again applying Theorem 2.6, Hölder's inequality and (A2), we get

$$\begin{aligned} |\mathfrak{A}_3| &\le C \cdot \int_0^t \int_{B_{2\cdot|x|} \backslash (B_{|x|/2} \cup B_{S_0})} (|x-y|^2 + t - s)^{-3/2} \cdot \gamma(s) \cdot |y|^{-\widetilde{\mu}}\, dy\, ds \\ &\le C \cdot \|\gamma\|_{5/4} \cdot \int_{B_{2\cdot|x|} \backslash (B_{|x|/2} \cup B_{S_0})} |x-y|^{-13/5} \cdot |y|^{-\widetilde{\mu}}\, dy \\ &\le C(\widetilde{\mu}) \cdot \|\gamma\|_{5/4} \cdot |x|^{-\widetilde{\mu}} \cdot \int_{B_{2\cdot|x|} \backslash B_{|x|/2}} |x-y|^{-13/5}\, dy. \end{aligned}$$

But $B_{2\cdot|x|} \subset B_{3\cdot|x|}(x)$, so $|\mathfrak{A}_3| \le C(\widetilde{\mu}) \cdot \|\gamma\|_{5/4} \cdot |x|^{-\widetilde{\mu}+2/5} \le C(S_1, \widetilde{\mu}) \cdot \|\gamma\|_{5/4} \cdot |x|^{-13/5}$. Finally, for $y \in \mathfrak{A}_4$, the relation $|x - y| \ge |y|/2$ holds, so by the same techniques as above,

$$\begin{aligned} |\mathfrak{A}_4| &\le C \cdot \int_0^t \int_{B^c_{2\cdot|x|}} (|y|^2 + t - s)^{-3/2} \cdot \gamma(s) \cdot |y|^{-\widetilde{\mu}}\, dy\, ds \\ &\le C \cdot \|\gamma\|_{5/4} \cdot \int_{B^c_{2\cdot|x|}} |y|^{-\widetilde{\mu}-13/5}\, dy \le C(\widetilde{\mu}) \cdot \|\gamma\|_{5/4} \cdot |x|^{-\widetilde{\mu}+2/5} \le C(S_1, \widetilde{\mu}) \cdot \|\gamma\|_{5/4} \cdot |x|^{-13/5}. \end{aligned}$$

Combining the preceding estimates of $\mathfrak{A}_1$ to $\mathfrak{A}_4$ yields the theorem. □

Lemma 5.2. *Let* $S \in (0, \infty)$ *with* $\overline{\Omega} \subset B_S,\ S_1 \in (S, \infty),\ u_0 \in L^2_{loc}(\overline{\Omega}^c)^3,\ \widetilde{C} \in (0, \infty),\ \nu \in (0, 3)$ *with* $|u_0(y)| \le \widetilde{C} \cdot |y|^{-\nu}$ *for* $y \in B^c_S$. *(This means in particular that* $\chi_{\Omega_S} \cdot u_0 \in L^2(\overline{\Omega}^c)^3$ *and* $\chi_{B^c_S} \cdot u_0 \in L^\infty(\overline{\Omega}^c)^3$ *so that* $\mathfrak{I}(u_0)$ *is well defined by Lemma 2.9.) Let* $x \in B^c_{S_1},\ t \in (0, \infty)$. *Then* $|\mathfrak{I}(u_0)(x,t)| \le C(S, S_1, \nu) \cdot (\widetilde{C} + \|u_0|\Omega_S\|_1) \cdot |x|^{-\nu}$.

Proof. We start by observing that $|\mathfrak{I}(u_0)(x,t)| \le \mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_3$, with $\mathfrak{A}_i := \int_{A_i} \mathfrak{H}(x-y,t) \cdot |u_0(y)|\, dy$, for $i \in \{1, 2, 3\}$, where $A_1 := \Omega_S,\ A_2 := B_{|x|/2} \backslash B_S,\ A_3 := B^c_{|x|/2} \backslash B_S$. For $y \in \Omega_S$, we obtain by an estimate as in (2.5) that $|x-y| \ge (1 - S/S_1) \cdot |x|$. Therefore we may conclude with Theorem 2.6 that $\mathfrak{A}_1 \le C \cdot \int_{\Omega_S} |x-y|^{-3} \cdot |u_0(y)|\, dy \le C \cdot |x|^{-3} \cdot \|u_0|\Omega_S\|_1$. Moreover, noting that $|x-y| \ge |x|/2$ for $y \in B_{|x|/2}$, we obtain

$$\mathfrak{A}_2 \le C \cdot \widetilde{C} \cdot \int_{B_{|x|/2} \backslash B_S} |x-y|^{-3} \cdot |y|^{-\nu}\, dy \le C \cdot \widetilde{C} \cdot |x|^{-3} \cdot \int_{B_{|x|/2}} |y|^{-\nu}\, dy \le C(\nu) \cdot \widetilde{C} \cdot |x|^{-\nu},$$

where we used Theorem 2.6 again. Note that $\nu < 3$. Finally, for $y \in B^c_{|x|/2}$, we have

$$\mathfrak{A}_3 \le \widetilde{C} \cdot \int_{B^c_{|x|/2} \backslash B_S} \mathfrak{H}(x-y,t) \cdot |y|^{-\nu}\, dy \le C(\nu) \cdot \widetilde{C} \cdot |x|^{-\nu} \cdot \int_{B^c_{|x|/2} \backslash B_S} \mathfrak{H}(x-y,t)\, dy.$$

Now Lemma 2.6 yields $\mathfrak{A}_3 \le C(\nu) \cdot \widetilde{C} \cdot |x|^{-\nu}$. The preceding estimates imply Lemma 5.2. □

Lemma 5.3. *Let $S \in (0,\infty)$ with $\overline{\Omega} \subset B_S,\ S_1 \in (S,\infty),\ \phi \in L^{5/4}(S_\infty)^3,\ x \in B^c_{S_1},\ t \in (0,\infty)$. Then $|\mathfrak{V}(\phi)(x,t)| \le C(\Omega, S, S_1) \cdot \|\phi\|_{5/4} \cdot |x|^{-13/5}$.*

Proof. Theorem 2.6 yields that $|\mathfrak{V}(\phi)(x,t)| \le C \cdot \int_0^t \int_{\partial\Omega} (|x-y|^2 + t - s)^{-3/2} \cdot |\phi(y,s)|\, do_y\, ds$. Therefore by Hölder's inequality

$$\begin{aligned} |\mathfrak{V}(\phi)(x,t)| &\le C \cdot \int_{\partial\Omega} \left(\int_0^t (|x-y|^2 + t - s)^{-15/2}\, ds \right)^{1/5} \cdot \|\phi(y,\ \cdot\ )\|_{5/4}\, do_y \\ &\le C \cdot \int_{\partial\Omega} |x-y|^{-13/5} \cdot \|\phi(y,\ \cdot\ )\|_{5/4}\, do_y. \end{aligned} \tag{5.1}$$

On the other hand, since $\overline{\Omega} \subset B_S,\ S_1 > S$ and $x \in B^c_{S_1}$, we have $|x-y| \ge (1 - S/S_1) \cdot |x|$ for $y \in \partial\Omega$ [compare (2.5)]. Hence inequality (5.1) implies $|\mathfrak{V}(\phi)(x,t)| \le C(\Omega, S, S_1) \cdot \|\phi\|_{5/4} \cdot |x|^{-13/5}$. □

Corollary 5.1. *Let $S \in (0,\infty)$ with $\overline{\Omega} \subset B_S$ and $S_1 \in (S,\infty)$. Let $u : (0,\infty) \mapsto W^{2,1}_{loc}(\Omega_S)^3$ with $\nabla_x u \in L^{5/4}(0,\infty, W^{1,5/4}(\Omega_S)^9)$. Let $p \in L^{5/4}(0,\infty, W^{1,5/4}(\Omega_S))$, $x \in B^c_{S_1}$, $t \in (0,\infty)$. Then*

$$\begin{aligned} &\left| \mathfrak{V}\left(\sum_{l=1}^{3} n_l^{(\Omega)} \cdot (-\partial u + p \cdot e_l) \right)(x,t) \right| \\ &\quad \le C(\Omega, S, S_1) \cdot (\|\nabla_x u\|_{L^{5/4}\left(0,\infty,\, W^{1,5/4}(\Omega_S)^9\right)} + \|p\|_{L^{5/4}\left(0,\infty,\, W^{1,5/4}(\Omega_S)\right)}) \cdot |x|^{-13/5}. \end{aligned}$$

Proof. Let $l \in \{1, 2, 3\}$. A standard trace theorem on Ω_S yields that $\|\partial_l u|S_\infty\|_{5/4} \le C(\Omega, S) \cdot \|\partial_l u|Z_{S,\infty}\|_{L^{5/4}(0,\infty,\, W^{1,5/4}(\Omega_S)^3)}$. An analogous estimate is valid for $p|S_\infty$. Thus Corollary 5.1 follows from Lemma 5.3. □

Now we are in a position to give decay estimates for solutions of (1.8)–(1.10) under assumptions which correspond to the case that the nonlinear flow from Theorem 1.1 is considered as a solution of (1.8)–(1.10) with $f = g - \tau \cdot (v \cdot \nabla_x) v$.

Theorem 5.2. *Let $\widetilde{g} \in L^{5/4}(Z_\infty)^3$. Let $u \in L^2_{loc}([0,\infty), V)$ satisfy (3.1) with $f = g + \widetilde{g}$, and with u_0 replaced by the function v_0 from* (A1). (*The function g was introduced in* (A2)*; it belongs to $L^{5/4}(Z_\infty)^3$ according to Lemma 5.1.*) *Then $\nabla_x u|\overline{\Omega}^c \times (1,\infty) \in L^{5/4}(1,\infty, L^{15/7}(\overline{\Omega}^c)^9)$ and $u \in C^0([0,\infty), L^{5/4}(\overline{\Omega}^c)^3) + C^0([0,\infty), L^2(\overline{\Omega}^c)^3)$.*

Let $S_1 \in (S_0,\infty)$. Then there is $\mathcal{C}(S_1)$ with

$$|u(x,t)| \le \mathcal{C}(S_1) \cdot |x|^{-\min\{\mu,\, 13/5\}} + |\mathfrak{R}(\widetilde{g})(x,t)| \quad \textit{for } t \in (0,\infty) \textit{ and for a.e. } x \in B^c_{S_1}. \tag{5.2}$$

The constant $\mathcal{C}(S_1)$ depends on Ω, S_0, C_0, μ, γ, $\widetilde{\mu}$ (see (A1), (A2)), S_1, and on certain norms of $v_0|\Omega_{S_0}$, $g|Z_{S_0,\infty}$, u and p, with p a pressure function associated to u.

Proof. Choose $u^{(1)}$, $u^{(2)}$, $p^{(1)}$ as in Corollary 3.3 and $p^{(2)}$ as in Corollary 3.4. Then $u = u^{(1)} + u^{(2)}$ (Corollary 3.3). By Theorem 3.5, Corollary 3.4 and Lemma 3.3, we have $\nabla_x u^{(1)}$, $\nabla_x u^{(2)}(\cdot\ +1) \in L^{5/4}(0,\infty,\, L^{15/7}(\overline{\Omega}^c)^9)$, so $\nabla_x u|\overline{\Omega}^c \times (1,\infty) \in L^{5/4}(1,\infty,\, L^{15/7}(\overline{\Omega}^c)^9)$. Corollary 3.3 and Theorem 3.5 further yield that $u^{(1)} \in C^0([0,\infty),\, L^{5/4}_\sigma(\overline{\Omega}^c))$. Moreover by Theorem 4.2,

$$u^{(1)}(x,t) = \mathfrak{R}(g)(x,t) + \mathfrak{R}(\widetilde{g})(x,t) + \mathfrak{V}\left(\sum_{l=1}^{3} n_l^{(\Omega)} \cdot (-\partial_l u^{(1)} + p^{(1)} \cdot e_l)\right)(x,t) \tag{5.3}$$

for $t \in (0,\infty)$ and for a.e. $x \in \overline{\Omega}^c$. Corollary 3.3 and Theorem 3.2 yield that $u^{(2)} \in C^0([0,\infty),\, L^2_\sigma(\overline{\Omega}^c))$, and Theorem 4.3 implies that

$$u^{(2)}(x,t) = \mathfrak{I}(v_0)(x,t) + \mathfrak{K}_{S_0}(u^{(2)})(x,t) \quad \text{for any } \ t \in (0,\infty) \ \text{ and for a.e. } \ x \in \overline{B_{S_0}}^c. \tag{5.4}$$

Thus we have found in particular that u is continuous in the sense stated in the theorem. By Corollary 3.4 with $t_0 = 1$ and by Theorem 4.2

$$u^{(2)}(x,\, t+1) = \mathfrak{I}\left(u^{(2)}(1)\right)(x,t) + \mathfrak{V}\left(\sum_{l=1}^{3} n_l^{(\Omega)} \cdot [-\partial_l u^{(2)}(\cdot\ +1) + p^{(2)}(\cdot\ +1) \cdot e_l]\right)(x,t) \tag{5.5}$$

for $t \in (0,\infty)$ and for a.e. $x \in \overline{\Omega}^c$.

The constants $\mathcal{C}_1, \dots, \mathcal{C}_7$ appearing below may depend on the same quantities as the constant $\mathcal{C}(S_1)$ in (5.2); in this respect see the remark following Theorem 5.2. Referring to Corollary 3.3, Theorem 3.5, Lemma 3.3 and 3.4, we get $\partial_l u_m^{(1)}|Z_{S_0,\infty},\ p^{(1)}|Z_{S_0,\infty} \in L^{5/4}(0,\infty,\, W^{1,5/4}(\Omega_{S_0}))$ for $1 \le l,m \le 3$. Thus Corollary 5.1 yields $|\mathfrak{V}(\sum_{l=1}^3 n_l^{(\Omega)} \cdot (-\partial u^{(1)} + p^{(1)} \cdot e_l))(x,t)| \le \mathcal{C}_1 \cdot |x|^{-13/5}$ for $x \in B^c_{S_1},\ t \in (0,\infty)$. Therefore by (5.3) and Theorem 5.1

$$|u^{(1)}(x,t)| \le \mathcal{C}_2 \cdot |x|^{-13/5} + \mathfrak{R}(\widetilde{g})(x,t) \quad \text{for } \ t \in (0,\infty) \ \text{ and a.e. } \ x \in B^c_{S_1}. \tag{5.6}$$

Lemma 5.2 with u_0, S, $\widetilde{C}$, ν, S_1 replaced by v_0, S_0, C_0, μ, $(S_0+S_1)/2$, respectively, implies

$$\mathfrak{I}(v_0)(x,t) \le \mathcal{C}_3 \cdot |x|^{-\mu} \quad \text{for } \ x \in B^c_{(S_0+S_1)/2},\ t \in (0,\infty). \tag{5.7}$$

By Corollary 3.3, we have $u^{(2)} \in C^0([0,\infty),\, L^2(\overline{\Omega}^c)^3)$, $\nabla_x u^{(2)} \in L^2(Z_\infty)^9$. Thus, from Lemma 2.18 and 2.19 with S, S_1 replaced by S_0, $(S_0+S_1)/2$, respectively, we may conclude that

$$\begin{aligned} |\mathfrak{K}_{S_0}\left(u^{(2)}\right)(x,t)| &\le C(\Omega, S_0, S_1) \cdot |x|^{-3} \cdot (1 + t^{1/2} + t) \\ &\quad \cdot (\max\{\|u^{(2)}(r)|\Omega_{S_0}\|_2 \,:\, r \in [0,t]\} + \|\nabla_x u^{(2)}|Z_{S_0,t}\|_2), \end{aligned}$$

so that $|\mathfrak{K}_{S_0}(u^{(2)})(x,t)| \le \mathcal{C}_4 \cdot |x|^{-3}$, for $x \in B^c_{(S_0+S_1)/2},\ t \in (0,1]$. This estimate, (5.7) and (5.4) imply

$$|u^{(2)}(x,t)| \le \mathcal{C}_4 \cdot |x|^{-\mu} \quad \text{for } \ t \in (0,1] \ \text{ and for a.e. } \ x \in B^c_{(S_0+S_1)/2}. \tag{5.8}$$

In particular $|u^{(2)}(x,1)| \le \mathcal{C}_4 \cdot |x|^{-\mu}$ for a.e. $x \in B^c_{(S_0+S_1)/2}$. Recalling that $u^{(2)}(1) \in L^2(\overline{\Omega}^c)^3$ (Corollary 3.3), we may apply Lemma 5.2 with u_0, S, $\widetilde{C}$, ν replaced by $u^{(2)}(1)$, $(S_0+S_1)/2$, $\mathcal{C}_4$, μ, respectively, to obtain

$$|\mathfrak{I}\left(u^{(2)}(1)\right)(x,t)| \le \mathcal{C}_5 \cdot |x|^{-\mu} \quad \text{for } \ x \in B^c_{S_1},\ t \in (0,\infty). \tag{5.9}$$

We know from Corollary 3.4, Lemmas 3.3 and 3.4 that $\partial_l u_m^{(2)}(\cdot\ +1)\,|\,Z_{S_0,\infty}$ and $p^{(2)}(\cdot\ +1)\,|\,Z_{S_0,\infty}$ belong to $L^{5/4}(0,\infty,\, W^{1,5/4}(\Omega_{S_0}))$. Therefore we may deduce from Corollary 5.1 that

$$\left|\mathfrak{V}\left(\sum_{l=1}^{3} n_l^{(\Omega)} \cdot [-\partial_l u^{(2)}(\cdot\ +1) + p^{(2)}(\cdot\ +1) \cdot e_l]\right)(x,t)\right| \le \mathcal{C}_6 \cdot |x|^{-13/5}$$

for $t \in (0,\infty)$, $x \in B^c_{S_1}$. As a consequence of this inequality, (5.5) and (5.9), we arrive at the estimate $|u^{(2)}(x, t+1)| \le \mathcal{C}_7 \cdot |x|^{-\min\{\mu, 13/5\}}$ for $t \in (0,\infty)$ and for a.e. $x \in B^c_{S_1}$. Inequality (5.2) follows from the preceding observation, (5.6), (5.8) and the equation $u = u^{(1)} + u^{(2)}$. □

6. The Nonlinear Case

Our aim in this section is to deduce Theorem 1.1 from Theorem 5.2, considering a solution of (1.1)–(1.3) as a solution of (1.8)–(1.10) with right-hand side $f = g - \tau \cdot (v \cdot \nabla_x)v$, with g from (A2). We recall that the parameters τ and S_0 and the velocity part v of an L^2-weak solution of (1.1)–(1.3) were fixed in Sect. 1. In the following, they will be used without further indications. The starting point of our estimates are the ensuing three theorems.

Theorem 6.1 [32]. $(v \cdot \nabla_x)v \in L^{5/4}(Z_\infty)^3$.

Theorem 6.2. *There is $r_0 \in (0,\infty)$ with $\overline{\Omega} \subset B_{r_0}$ and $A_0 \in (0,\infty)$ such that $|v(x,t)| \le A_0$ for a.e. $(x,t) \in B^c_{r_0} \times (0,\infty)$.*

Proof. [44], [63, Satz III.5.1], (A3), (A4)–(A8). □

Theorem 6.3. *The relation $\nabla_x v|\overline{\Omega}^c \times (1,\infty) \in L^{5/4}(1,\infty, L^{15/7}(\overline{\Omega}^c)^9)$ holds. Moreover, the function v belongs to $C^0([0,\infty), L^{5/4}(\overline{\Omega}^c)^3) + C^0([0,\infty), L^2(\overline{B_\Omega}^c)^3)$.*

For $S_1 \in (S_0,\infty)$, there exists a constant $A(S_1) > 0$ with

$$|v(x,t)| \le A(S_1) \cdot |x|^{-\min\{\mu, 13/5\}} + |\mathfrak{R}\left(-\tau \cdot (v \cdot \nabla_x)v\right)(x,t)|$$

for any $t \in (0,\infty)$ and for a.e. $x \in B^c_{S_1}$.

Proof. According to (A3), the function v satisfies (3.1) with $f = g - \tau \cdot (v \cdot \nabla_x)v$, where $\tau \cdot (v \cdot \nabla_x)v \in L^{5/4}(Z_\infty)^3$ by Theorem 6.1. Thus Theorem 6.3 follows from Theorem 5.2. □

Lemma 6.1. $v \in L^2(0,\infty, L^6(\overline{\Omega}^c)^3)$.

Proof. We have $v \in L^\infty(0,\infty, L^2(\overline{\Omega}^c)^3)$ and $\nabla_x v \in L^2(Z_\infty)^9$ (see (A3)). Thus we may apply Lemma 2.1 with $\kappa = q = 2$. □

Next we transform $\mathfrak{R}((v \cdot \nabla_x)v)$ by a partial integration.

Lemma 6.2. *Let $x \in \overline{\Omega}^c$, $t \in (0,\infty)$, $s \in (0,t)$. Then*

$$\int_{\overline{\Omega}^c} \Gamma(x-y,t-s) \cdot \left((v \cdot \nabla_y)v\right)(y,s)\, dy = -\int_{\overline{\Omega}^c} \sum_{l=1}^3 \partial_{y_l}\Gamma(x-y,t-s) \cdot (v_l \cdot v)(y,s)\, dy.$$

Proof. According to Lemma 6.1, we have $v(\cdot\,,s) \in L^6(\overline{\Omega}^c)^3$. Since $v \in L^\infty(0,\infty, L^2(\overline{\Omega}^c)^3)$, we thus get $v(\cdot\,,s) \in L^3(\overline{\Omega}^c)^3$. In view of Theorem 6.1, we have $((v \cdot \nabla)v)(\cdot\,,s) \in L^{5/4}(\overline{\Omega}^c)^3$. Moreover Theorem 2.6 implies that the terms $\Gamma(x-y,t-s)$, $\partial y_l \Gamma(x-y,t-s)$, considered as functions of $y \in \overline{\Omega}^c$, belong to $L^5(\overline{\Omega}^c)^{3\times 3}$ and to $L^{3/2}(\overline{\Omega}^c)^{3\times 3}$, Thus the lemma follows by the proof of [25, Lemma 3.8] and the assumption $v(\cdot\,,s) \in V$, which implies $v(\cdot\,,s)|\partial\Omega = 0$. □

Corollary 6.1. *$\int_0^t \int_{\overline{\Omega}^c} |\partial_l \Gamma_{jk}(x-y,t-s) \cdot (v_k \cdot v_l)(y,s)|\, dy\, ds < \infty$ for $1 \le j,k,l \le 3$, $t \in (0,\infty)$ and for a.e. $x \in \overline{\Omega}^c$. Therefore we may define*

$$\mathfrak{A}(x,t) := \int_0^t \int_{\overline{\Omega}^c} \sum_{l=1}^3 \partial_l \Gamma(x-y,t-s) \cdot \left((v_k \cdot v_l)(y,s)\right)_{1\le k\le 3}\, dy\, ds$$

for x, t as above. The equation $\mathfrak{R}((v \cdot \nabla_x)v)(x,t) = \mathfrak{A}(x,t)$ holds for such x and t.

Proof. We have $v_k \cdot v_l \in L^\infty(0,\infty,\, L^1(\overline{\Omega}^c))$ for $1 \le k,l \le 3$, so Corollary 6.1 follows from Lemma 6.2 and 2.8. □

Now we turn to pointwise decay estimates of v, adapting an approach by Babenko [4]. To this end we fix a number $R_0 \in (\max\{S_0, r_0\}, \infty)$, with r_0 from Theorem 6.2 and S_0 from (A1). We define

$$\psi(R) := \sup\{|v(x,t)| \,:\, x \in B_R^c,\ t>0\},\ \ \varphi(R) := \|v|B_R^c \times (0,\infty)\|_{6,2;\infty} \text{ for } R \in [R_0,\infty). \tag{6.1}$$

Note that by Lemma 6.1, we have $\varphi(R) < \infty$ for $R \in [R_0,\infty)$. In the rest of this section, we will write $\mathfrak{C}$ for constants that may depend on Ω, τ, S_0, on μ (see (A1)), A_0 (see Theorem 6.2), R_0, $A_1(R_0)$ (see Theorem 6.3), on the function φ from (6.1), and on certain norms of v. Constants that may additionally depend on $\gamma_1, \ldots, \gamma_n \in (0,\infty)$ for some $n \in \mathbb{N}$ will be denoted by $\mathfrak{C}(\gamma_1, \ldots, \gamma_n)$.

Lemma 6.3. *Let $t \in (0,\infty)$, $x \in B_{R_0}^c$, $1 \le j,k \le 3$. Then*

$$\int_0^t \int_{\overline{\Omega}^c} |\partial_l \Gamma_{jk}(x-y,t-s)\cdot(v_k\cdot v_l)(y,s)|\,dy\,ds \le \mathfrak{C}.$$

Proof. The integral in Lemma 6.3 is bounded by $\mathfrak{A}^{(1)} + \mathfrak{A}^{(2)} + \mathfrak{A}^{(3)}$, with

$$\mathfrak{A}^{(1)} := \int_0^t \int_{\Omega_{r_0}} |\partial_l \Gamma_{jk}(x-y,t-s)| \cdot |v(y,s)|^2\,dy\,ds,$$
$$\mathfrak{A}^{(2)} := \int_0^t \int_{B_{r_0}^c} \chi_{(1,\infty)}(t-s)\cdot|\partial_l \Gamma_{jk}(x-y,t-s)| \cdot |v(y,s)|^2\,dy\,ds.$$

The term $\mathfrak{A}^{(3)}$ is to be defined in the same way as $\mathfrak{A}^{(2)}$, but with the function $\chi_{(1,\infty)}$ replaced by $\chi_{(0,1]}$. Since $x \in B_{R_0}^c$ and $R_0 > r_0$, we find for $y \in \Omega_{r_0}$ that $|x-y| \ge (1 - r_0/R_0)\cdot|x|$ (see (2.5) and $1 - r_0/R_0 > 0$. Therefore with Theorem 2.6,

$$\begin{aligned}\mathfrak{A}^{(1)} &\le C\cdot \int_0^t \int_{\Omega_{r_0}} (|x-y|^2 + t - s)^{-2}\cdot|v(y,s)|^2\,dy\,ds\\ &\le C(r_0,R_0)\cdot|x|^{-4}\cdot\int_0^t\int_{\Omega_{r_0}} |v(y,s)|^2\,dy\,ds\\ &\le C(r_0,R_0)\cdot|x|^{-4}\cdot\Big(\int_{\Omega_{r_0}} dy\Big)^{2/3}\cdot\int_0^t \|v(\,\cdot\,,s)|\Omega_{r_0}\|_6^2\,ds \le C(r_0,R_0)\cdot\|v\|_{6,2;\infty}^2.\end{aligned}$$

Moreover, by Theorem 2.7 with $\nu = s = \infty$, $q = 12/5$,

$$\begin{aligned}\mathfrak{A}^{(2)} &\le A_0^{7/6}\cdot\int_0^t\int_{B_{r_0}^c} \chi_{(1,\infty)}(t-s)\cdot|\partial_l\Gamma_{jk}(x-y,t-s)|\cdot|v(y,s)|^{5/6}\,dy\,ds\\ &\le C\cdot A_0^{7/6}\cdot\|v\|_{2,\infty;\infty}^{5/6}.\end{aligned}$$

Finally Theorem 6.2 and Theorem 2.7 with $\nu = s = \infty$, $q = 6$ yield

$$\begin{aligned}\mathfrak{A}^{(3)} &\le C\cdot A_0^{1/2}\cdot\int_0^t\int_{B_{r_0}^c} \chi_{(0,1)}(t-s)\cdot|\partial_l\Gamma_{jk}(x-y,t-s)|\cdot|v(y,s)|^{3/2}\,dy\,ds\\ &\le C\cdot A_0^{1/2}\cdot\|v\,|\,B_{r_0}^c\times(0,\infty)\|_{9,\infty;\infty}^{3/2} \le C\cdot A_0^{5/3}\cdot\|v\|_{2,\infty;\infty}^{1/3}.\end{aligned}$$

Lemma 6.3 follows from (A3), Lemma 6.1, the preceding estimates of $\mathfrak{A}^{(1)}$, $\mathfrak{A}^{(2)}$ and $\mathfrak{A}^{(3)}$. □

Corollary 6.2. *Without loss of generality, we may suppose that*

$$|v(x,t)| \le A(R_0)\cdot|x|^{-\min\{\mu,\,13/5\}} + \tau\cdot|\mathfrak{A}(x,t)| \ \textit{ for any } t\in(0,\infty) \textit{ and for any } x \in B_{R_0}^c \tag{6.2}$$

and $\psi(R) \le \mathfrak{C}$ for $R \in [R_0,\infty)$.

Proof. In view of Theorem 6.3, Corollary 6.1 and Lemma 6.3, and because $R_0 > S_0$, we may modify $v(t)$ on a set N_t of measure zero, for any $t \in (0,\infty)$, in such a way that (6.2) holds for any $x \in B^c_{R_0}$. This means by Lemma 6.3 that $|v(x,t)| \le \mathfrak{C}$ for $t \in (0,\infty)$ and $x \in B^c_{R_0}$, so the last claim in the corollary is proved. □

Lemma 6.4. *$\psi(R) \le \mathfrak{C} \cdot (R^{-7/6} + \psi(R/2)^{7/6})$ for $R \in [2 \cdot R_0, \infty)$.*

Proof. We use a similar approach as in the proof of Lemma 6.3. Let $R \in [2 \cdot R_0, \infty)$, $(x,t) \in B^c_R \times (0,\infty)$ and $j \in \{1, 2, 3\}$. Inequality (6.2) yields

$$|v_j(x,t)| \le A_1(R_0) \cdot |x|^{-\min\{\mu,\, 13/5\}} + \tau \cdot \sum_{k,l=1}^{3} (\mathfrak{A}^{(1)}_{kl} + \mathfrak{A}^{(2)}_{kl} + \mathfrak{A}^{(3)}_{kl}), \tag{6.3}$$

where $\mathfrak{A}^{(i)}_{kl}$ is defined in the same way as the term $\mathfrak{A}^{(i)}$ in the proof of Lemma 6.3, except that the domain of integration Ω_{r_0} is replaced by $\Omega_{R/2}$ in the case $i = 1$, and $B^c_{r_0}$ by $B^c_{R/2}$ if $i = 2$ or $i = 3$. For $y \in B_{R/2}$, we have $|x - y| \ge |x|/2$, so by proceeding as in the estimate of the term $\mathfrak{A}^{(1)}$ in the proof of Lemma 6.3, we get

$$\mathfrak{A}^{(1)}_{kl} \le C \cdot |x|^{-4} \cdot \left(\int_{\Omega_{R/2}} dy \right)^{2/3} \cdot \int_0^t \|v(\,\cdot\,,s)|\Omega_{R/2}\|_6^2 \, ds.$$

Hence $\mathfrak{A}^{(1)}_{kl} \le C \cdot |x|^{-4} \cdot R^2 \cdot \|v\|^2_{6,2;\infty} \le C \cdot R^{-2} \cdot \|v\|^2_{6,2;\infty}$, where the last inequality holds because $x \in B^c_R$. Theorem 2.7 with $\nu = s = \infty$, $q = 12/5$ yields

$$\begin{aligned}\mathfrak{A}^{(2)}_{kl} &\le C \cdot \psi(R/2)^{7/6} \cdot \int_0^t \int_{B^c_{R/2}} \chi_{(1,\infty)}(t-s) \cdot |\partial_l \Gamma_{jk}(x-y, t-s)| \cdot |v(y,s)|^{5/6} \, dy \, ds \\ &\le C \cdot \psi(R/2)^{7/6} \cdot \|v\|^{5/6}_{2,\infty;\infty}.\end{aligned}$$

Theorem 6.2 and again Theorem 2.7, this time with $\nu = s = \infty$, $q = 6$, imply

$$\begin{aligned}\mathfrak{A}^{(3)}_{kl} &\le C \cdot A_0^{1/2} \cdot \int_0^t \int_{B^c_{R/2}} \chi_{(0,1]}(t-s) \cdot |\partial_l \Gamma_{jk}(x-y, t-s)| \cdot |v(y,s)|^{3/2} \, dy \, ds \\ &\le C \cdot A_0^{1/2} \cdot \|v \,|\, B^c_{R/2} \times (0,\infty)\|^{3/2}_{9,\infty;\infty} \le C \cdot A_0^{1/2} \cdot \psi(R/2)^{7/6} \cdot \|v\|^{1/3}_{2,\infty;\infty}.\end{aligned}$$

Lemma 6.4 follows from (6.3), (A3), Lemma 6.1, the preceding estimates of $\mathfrak{A}^{(1)}_{kl}$, $\mathfrak{A}^{(2)}_{kl}$ and $\mathfrak{A}^{(3)}_{kl}$ and from the assumption $\mu \ge 7/6$. □

Lemma 6.5. *Let $\kappa \in (0,\infty)$. Then there is $R_\kappa \in [2 \cdot R_0, \infty)$, depending on the quantities κ, A_0, $A_1(R_0)$, $\|v\|_{6,2;\infty}$ and on the function φ introduced in (6.1), such that $|v(x,t)| \le \kappa$ for $x \in B^c_{R_\kappa}$, $t \in (0,\infty)$.*

Proof. Let $R \in [2 \cdot R_0, \infty)$, $M \in (0,\infty)$, $(x,t) \in B^c_R \times (0,\infty)$, $j \in \{1, 2, 3\}$. Then by (6.2) we have $|v_j(x,t)| \le A_1(R_0) \cdot |x|^{-\min\{\mu,\, 13/5\}} + \tau \cdot \sum_{k,l=1}^3 (\mathfrak{B}^{(1)}_{kl} + \mathfrak{B}^{(2)}_{kl} + \mathfrak{B}^{(3)}_{kl})$, with $\mathfrak{B}^{(1)}_{kl}$ defined in the same way as $\mathfrak{A}^{(1)}$ in the proof of Lemma 6.3, except that the domain of integration Ω_{r_0} is replaced by $\Omega_{R/2}$ (so that $\mathfrak{B}^{(1)}_{kl}$ coincides with the term $\mathfrak{A}^{(1)}_{kl}$ from the proof of Lemma 6.4),

$$\mathfrak{B}^{(2)}_{kl} := \int_0^t \int_{B^c_{R/2}} \chi_{(M,\infty)}(t-s) \cdot |\partial_l \Gamma_{jk}(x-y, t-s)| \cdot |v(y,s)|^2 \, dy \, ds,$$

and with $\mathfrak{B}^{(3)}_{kl}$ defined in the same way as $\mathfrak{B}^{(2)}_{kl}$, but with the term $\chi_{(M,\infty)}(t-s)$ replaced by $\chi_{(0,M]}(t-s)$. Since $x \in B^c_R$, we have $|x-y| \ge R/2$ for $y \in B_{R/2}$, so $|\partial_l \Gamma(x-y, t-s)| \le C \cdot R^{-4}$ for $y \in B_{R/2}$, $s \in (0,t)$ by Theorem 2.6. As a consequence, proceeding as in the estimate of $\mathfrak{A}^{(1)}$ in the proof of Lemma 6.3 and of $\mathfrak{A}^{(1)}_{kl}$ in the proof of Lemma 6.4, we find $\mathfrak{B}^{(1)}_{kl} \le C \cdot R^{-2} \cdot \|v\|^2_{6,2;\infty}$.

Concerning $\mathfrak{B}_{kl}^{(2)}$, we use Theorem 2.7 with $s=1,\ q=3,\ \nu=\infty$, to obtain

$$\mathfrak{B}_{kl}^{(2)} \le C \cdot M^{-1} \cdot \|v_k \cdot v_l \,|\, B_{R/2}^c \times (0,\infty)\|_{3,1;\infty} \le C \cdot M^{-1} \cdot \|v|B_{R/2}^c \times (0,\infty)\|_{6,2;\infty}^2,$$

so that $\mathfrak{B}_{kl}^{(2)} \le C \cdot M^{-1} \cdot \varphi(R/2)^2$. Turning to $\mathfrak{B}_{kl}^{(3)}$, we obtain with Theorem 2.6 and 6.2 that

$$\begin{aligned}\mathfrak{B}_{kl}^{(3)} &\le C \cdot A_0^2 \cdot \int_0^t \int_{\mathbb{R}^3} \chi_{(0,M]}(t-s) \cdot \Big(|x-y| + (t-s)^{1/2}\Big)^{-4}\, dy\, ds \\ &\le C \cdot A_0^2 \cdot \int_0^t \chi_{(0,M]}(t-s) \cdot (t-s)^{-1/2}\, ds \le C \cdot A_0^2 \cdot M^{1/2}.\end{aligned}$$

Altogether we have found that $|v(x,t)| \le C(A(R_0), A_0, \|u\|_{6,2;\infty}) \cdot (R^{-1} + M^{1/2} + M^{-1} \cdot \varphi(R/2))$ for $x \in B_R^c,\ t \in (0,\infty)$. (Recall that $\mu \ge 1$.) By Lemma 6.1 and Lebesgue's theorem, we have $\varphi(R/2) \to 0$ for $R \to \infty$. Therefore, by first choosing M sufficiently small and then R sufficiently large, we see that Lemma 6.5 follows from the preceding estimate of $|v(x,t)|$. □

Theorem 6.4. *$|v(x,t)| \le \mathfrak{C} \cdot |x|^{-7/6}$ for $x \in B_{R_0}^c,\ t \in (0,\infty)$.*

We adapt the proof of [25, Theorem 4.5], which goes back to Babenko [4]. By Lemma 6.4, we know there is a constant $C_1 > 0$ with

$$\psi(R) \le C_1 \cdot \Big(R^{-7/6} + \psi(R/2)^{7/6}\Big) \quad \text{for } R \in [2 \cdot R_0, \infty). \tag{6.4}$$

Without loss of generality, we may assume that $C_1 \ge 1$. Note that ψ is decreasing. By Lemma 6.5, we may choose $\widetilde{R} \in [2 \cdot R_0, \infty)$ with $\psi(\widetilde{R}) \le (4 \cdot C_1)^{-6} \cdot 2^{-1/7}$. Put $R_1 := \max\{\widetilde{R},\, (4 \cdot C_1)^7/2\}$. This means in particular that $R_1 \ge 1$ and $R_1 \ge \widetilde{R} \ge 2 \cdot R_0$. The fact that $R_1 \ge \widetilde{R}$ implies $\psi(R_1) \le \psi(\widetilde{R}) \le (4 \cdot C_1)^{-6} \cdot 2^{-1/7}$. It follows with (6.4) that

$$\psi(2 \cdot R_1) \le C_1 \cdot \Big((2 \cdot R_1)^{-7/6} + \psi(R_1)^{7/6}\Big) \le C_1 \cdot \Big((2 \cdot R_1)^{-7/6} + (4 \cdot C_1)^{-7} \cdot 2^{-1/6}\Big),$$

hence $\psi(2 \cdot R_1) \le C_1 \cdot 2^{-1/6} \cdot ((2 \cdot R_1^{7/6})^{-1} + (4 \cdot C_1)^{-7})$. Since $R_1 \ge 1$, we have $R_1^{7/6} \ge R_1$, so by the choice of R_1 we may conclude that $\psi(2 \cdot R_1) \le 2^{-7/6} \cdot (4 \cdot C_1)^{-6}$.

For a proof by induction let $n \in \mathbb{N}$ with $\psi(2^n \cdot R_1) \le 2^{-n \cdot 7/6} \cdot (4 \cdot C_1)^{-6}$. Then we find with (6.4) that $\psi(2^{n+1} \cdot R_1) \le C_1 \cdot ((2^{n+1} \cdot R_1)^{-7/6} + \psi(2^n \cdot R_1)^{7/6})$. By our assumption on n it follows that $\psi(2^{n+1} \cdot R_1) \le C_1 \cdot ((2^{n+1} \cdot R_1)^{-7/6} + [2^{n \cdot 49/36} \cdot (4 \cdot C_1)^7]^{-1})$. Again we use that $R_1^{7/6} \ge R_1 \ge (4 \cdot C_1)^7/2$, to obtain

$$\begin{aligned}\psi(2^{n+1} \cdot R_1) &\le C_1 \cdot \Big([2^{(n+1) \cdot 7/6} \cdot (4 \cdot C_1)^7/2]^{-1} + [2^{n \cdot 49/36} \cdot (4 \cdot C_1)^7]^{-1}\Big) \\ &= (4 \cdot C_1)^{-6} \cdot 2^{-1} \cdot \Big(2^{-(n+1) \cdot 7/6} + 2^{-n \cdot 49/36 - 1}\Big).\end{aligned}$$

But since $n \cdot 49/36 + 1 \ge (n+1) \cdot 7/6$, it follows that $\psi(2^{n+1} \cdot R_1) \le (4 \cdot C_1)^{-6} \cdot 2^{-(n+1) \cdot 7/6}$. Thus we have shown by induction that the inequality $\psi(2^n \cdot R_1) \le 2^{-n \cdot 7/6} \cdot (4 \cdot C_1)^{-6}$ holds for any $n \in \mathbb{N}$.

Let $R \in [2 \cdot R_1, \infty)$. Then there is $n \in \mathbb{N}$ with $2^n \cdot R_1 \le R < 2^{n+1} \cdot R_1$, so that $2^{n \cdot 7/6} \cdot R_1^{7/6} \le R^{7/6} < 2^{(n+1) \cdot 7/6} \cdot R_1^{7/6}$. It follows with the inequality just proved that

$$\begin{aligned}\psi(R) &\le \psi(2^n \cdot R_1) \le 2^{-n \cdot 7/6} \cdot (4 \cdot C_1)^{-6} \\ &\le 2^{(n+1) \cdot 7/6} \cdot R_1^{7/6} \cdot R^{-7/6} \cdot 2^{-n \cdot 7/6} \cdot (4 \cdot C_1)^{-6} = (2 \cdot R_1/R)^{7/6} \cdot (4 \cdot C_1)^{-6}.\end{aligned} \tag{6.5}$$

Now let $t \in (0,\infty),\ x \in B_{2 \cdot R_1}^c$. Then $x \in B_{|x|}^c$, so $|v(x,t)| \le \psi(|x|)$. Moreover, recalling the choice of x and referring to (6.5), we obtain the estimate $\psi(|x|) \le (2 \cdot R_1/|x|)^{7/6} \cdot (4 \cdot C_1)^{-6}$. Therefore we may conclude that $|v(x,t)| \le (2 \cdot R_1)^{7/6} \cdot (4 \cdot C_1)^{-6} \cdot |x|^{-7/6}$. For $x \in B_{2 \cdot R_1} \backslash B_{R_0}$, the second inequality in Lemma 6.3 yields $|v(x,t)| \le \mathfrak{C}$, so $|v(x,t)| \le \mathfrak{C} \cdot (2 \cdot R_1/|x|)^{7/6}$. Thus the theorem is proved. □

Lemma 6.6. *Suppose there are constants $c_0,\ \epsilon > 0$ with*

$$|v(y,s)| \le c_0 \cdot |y|^{-1-\epsilon} \quad \textit{for } y \in B^c_{R_0},\ s \in (0,\infty). \tag{6.6}$$

Let $x \in B^c_{2\cdot R_0},\ t \in (0,\infty)$. Then $|\mathfrak{A}(x,t)| \le \mathfrak{C} \cdot (1+c_0^2) \cdot (|x|^{-1-2\cdot\epsilon} + |x|^{-2})$, where $\mathfrak{A}(x,t)$ was defined in Corollary 6.1.

If $\epsilon \ge 1$, we further get $|\mathfrak{A}(x,t)| \le \mathfrak{C}(\epsilon) \cdot (1+c_0^2) \cdot |x|^{-13/5}$.

Proof. For $y \in B^c_{|x|/2}$, we have $|y| \ge |x|/2 \ge R_0$, so $|v(y,s)| \le c_0 \cdot |y|^{-1-\epsilon}$ for $y \in B^c_{|x|/2}$ due to (6.6). Thus, with Theorem 2.6 and because $\overline{\Omega} \subset B_{R_0}$, we find

$$|\mathfrak{A}(x,t)| \le C \cdot \int_0^t \int_{\overline{\Omega}^c} (|x-y|^2 + t - s)^{-2} \cdot |v(y,s)|^2\, dy\, ds \le C \cdot (c_0^2 \cdot \mathfrak{L}_1 + \mathfrak{L}_2),$$

with $\mathfrak{L}_1 := \int_0^t \int_{B^c_{|x|/2}} (|x-y|^2 + t - s)^{-2} \cdot |y|^{-2-2\cdot\epsilon}\, dy\, ds$, and with $\mathfrak{L}_2$ defined in the same way as $\mathfrak{L}_1$, except that the domain of integration $B^c_{|x|/2}$ is replaced by $B_{|x|/2}\backslash\overline{\Omega}$, and the factor $|y|^{-2-2\cdot\epsilon}$ by $|v(y,s)|^2$. But

$$\mathfrak{L}_1 \le C \cdot \int_{B^c_{|x|/2}} |x-y|^{-2} \cdot |y|^{-2-2\cdot\epsilon}\, dy \le C \cdot \sum_{i=1}^{2} \int_{A_i} |x-y|^{-2} \cdot |y|^{-2-2\cdot\epsilon}\, dy, \tag{6.7}$$

with $A_1 := B_{2\cdot|x|}\backslash B_{|x|/2},\ A_2 := B^c_{2\cdot|x|}$. We may conclude

$$\begin{aligned} \mathfrak{L}_1 &\le C \cdot \left(|x|^{-2-2\cdot\epsilon} \cdot \int_{A_1} |x-y|^{-2}\, dy + \int_{A_2} |y|^{-4-2\cdot\epsilon}\, dy \right) \\ &\le C \cdot \left(|x|^{-2-2\cdot\epsilon} \cdot \int_{B_{3\cdot|x|}(x)} |x-y|^{-2}\, dy + |x|^{-1-2\cdot\epsilon} \right) \le C \cdot |x|^{-1-2\cdot\epsilon}. \end{aligned} \tag{6.8}$$

Moreover, since $|x-y| \ge |x|/2$ for $y \in B_{|x|/2}$, we find

$$\begin{aligned} \mathfrak{L}_2 &\le C \cdot |x|^{-4} \cdot \int_0^t \int_{B_{|x|/2}\cap\overline{\Omega}^c} |v(y,s)|^2\, dy\, ds \\ &\le C \cdot |x|^{-4} \cdot \int_0^t \left(\int_{B_{|x|/2}} dy \right)^{2/3} \cdot \left(\int_{\overline{\Omega}^c} |v(y,s)|^6\, dy \right)^{1/3} ds \le C \cdot |x|^{-2} \cdot \|v\|^2_{6,2;\infty} \le \mathfrak{C} \cdot |x|^{-2}. \end{aligned}$$

Note that $\|v\|_{6,2;\infty} < \infty$ by Lemma 6.1. Thus we have proved the first estimate claimed in Lemma 6.6.

Now suppose that $\epsilon \ge 1$, so that by (6.6)

$$|v(y,s)| \le c_0 \cdot |y|^{-1-\epsilon} \le c_0 \cdot C(R_0,\epsilon) \cdot |y|^{-2} \quad \text{for } y \in B^c_{R_0},\ s > 0. \tag{6.9}$$

Since $|x-y| \ge |x|/2 > 0$ for $y \in B_{|x|/2}$, the function $y \mapsto \Gamma(x-y, t-s)$ $(y \in B_{|x|/2})$ is C^∞ for any $s \in (0,t)$ (Theorem 2.6), so by an integration by parts, we get $\mathfrak{A}_j(x,t) = \sum_{i=1}^4 \sum_{k,l=1}^3 \mathfrak{B}^{(i)}_{jkl}$, where for $1 \le j,k,l \le 3$, we put

$$\begin{aligned} \mathfrak{B}^{(1)}_{jkl} &:= \int_0^t \int_{B^c_{|x|/2}} \partial_l \Gamma_{jk}(x-y,t-s) \cdot (v_k \cdot v_l)(y,s)\, dy\, ds, \\ \mathfrak{B}^{(2)}_{jkl} &:= \int_0^t \int_{B_{|x|/2}\backslash\overline{\Omega}} -\Gamma_{jk}(x-y,t-s) \cdot ((v_l \cdot \partial_l) v_k)(y,s)\, dy\, ds, \\ \mathfrak{B}^{(3)}_{jkl} &:= \int_0^t \int_{\partial B_{|x|/2}} \Gamma_{jk}(x-y,t-s) \cdot (v_l \cdot v_k)(y,s) \cdot 2 \cdot y_l/|x|\, do_y\, ds. \end{aligned}$$

The term $\mathfrak{B}^{(4)}_{jkl}$ is to be defined in the same way as $\mathfrak{B}^{(3)}_{jkl}$, except that the domain of integration $\partial B_{|x|/2}$ is replaced by $\partial\Omega$, and the factor $2 \cdot y_l/|x|$ by $-n^{(\Omega)}_l(y)$. Since $v(\cdot\ ,s) \in V$ for $s \in (0,\infty)$, we have

$\mathfrak{B}^{(4)}_{jkl} = 0$ for $1 \le j,k,l \le 3$. After applying (6.6) and Theorem 2.6, an estimate as in (6.7) and (6.8) yields $|\mathfrak{B}^{(1)}_{jkl}| \le C \cdot c_0^2 \cdot |x|^{-1-2\cdot\epsilon}$. By Theorem 2.6

$$
\begin{aligned}
|\mathfrak{B}^{(2)}_{jkl}| &\le C \cdot \int_0^t \int_{B_{|x|/2}\setminus\overline{\Omega}} (|x-y|^2+t-s)^{-3/2} \cdot |(v\cdot\nabla_y)v|(y,s)\,dy\,ds \\
&\le C \cdot \int_0^t (|x|^2+t-s)^{-3/2} \cdot \int_{B_{|x|/2}\setminus\overline{\Omega}} |(v\cdot\nabla_y)v|(y,s)\,dy\,ds. \qquad (6.10)
\end{aligned}
$$

Since $v \in L^\infty(0,\infty,\, L^2(\overline{\Omega}^c)^3)$ and $\nabla_y v \in L^2(Z_\infty)^9$, we have $(v\cdot\nabla_y)v \in L^2(0,\infty,\, L^1(\overline{\Omega}^c)^3)$. Due to this fact, we get

$$
\begin{aligned}
\int_0^{\min\{1,t\}} (|x|^2+t-s)^{-3/2} \cdot \int_{B_{|x|/2}\setminus\overline{\Omega}} |(v\cdot\nabla_y)v|(y,s)\,dy\,ds &\le |x|^{-3} \cdot \|(v\cdot\nabla_x)v|Z_1\|_1 \\
&\le C\cdot|x|^{-3}\cdot\|(v\cdot\nabla_x)v|Z_1\|_{1,2;1} \le \mathfrak{C}\cdot|x|^{-3}. \qquad (6.11)
\end{aligned}
$$

The estimates (6.10) and (6.11) imply in the case $t \le 1$ that $|\mathfrak{B}^{(2)}_{jkl}| \le \mathfrak{C}\cdot|x|^{-3}$. Consider the case $t \ge 1$. Using (6.9), we get for $s \in (0,\infty)$ that

$$
\begin{aligned}
\|v(\cdot\,,s)\|_{15/8} &\le \|v(\cdot\,,s)\,|\,\Omega_{R_0}\|_{15/8} + \|v(\cdot\,,s)\,|\,B^c_{R_0}\|_{15/8} \\
&\le C(R_0)\cdot\|v(\cdot\,,s)\,|\,\Omega_{R_0}\|_2 + c_0\cdot C(R_0,\epsilon)\cdot\Big(\int_{B^c_{R_0}} |y|^{-15/4}\,dy\Big)^{8/15} \\
&\le C(R_0,\epsilon)\cdot(\|v(\cdot\,,t)\|_2 + c_0).
\end{aligned}
$$

Since $v \in L^\infty(0,\infty,\, L^2(\overline{\Omega}^c)^3)$, it follows that $\|v\|_{15/8,\infty;\infty} \le (1+c_0)\cdot\mathfrak{C}(\epsilon)$, so by Hölder's inequality

$$
\begin{aligned}
&\int_1^t (|x|^2+t-s)^{-3/2} \cdot \int_{B_{|x|/2}\setminus\overline{\Omega}} |(v\cdot\nabla_y)v|(y,s)\,dy\,ds \\
&\le \int_1^t (|x|^2+t-s)^{-3/2}\cdot\|v\|_{15/8,\infty;\infty}\cdot\|\nabla_x v(\cdot\,,s)\|_{15/7}\,ds \\
&\le \mathfrak{C}(\epsilon)\cdot(1+c_0)\cdot\Big(\int_0^t (|x|^2+t-s)^{-15/2}\,ds\Big)^{1/5}\cdot\|\nabla_x v|\overline{\Omega}^c\times(1,\infty)\|_{15/7,5/4;\infty}. \qquad (6.12)
\end{aligned}
$$

But $\|\nabla_x v|\overline{\Omega}^c\times(1,\infty)\|_{15/7,5/4;\infty} < \infty$ by Theorem 6.3, so in view of inequalities (6.10)–(6.12), we obtain $|\mathfrak{B}^{(2)}_{jkl}| \le \mathfrak{C}(\epsilon)\cdot(1+c_0)\cdot|x|^{-13/5}$. Moreover, by Theorem 2.6, (6.6) and the fact that $|x|/2 \ge R_0$,

$$
\begin{aligned}
|\mathfrak{B}^{(3)}_{jkl}| &\le C\cdot c_0^2\cdot\int_0^t\int_{\partial B_{|x|/2}} (|x-y|^2+t-s)^{-3/2}\cdot|y|^{-2-2\cdot\epsilon}\,do_y\,ds \\
&\le C\cdot c_0^2\cdot|x|^{-2-2\cdot\epsilon}\cdot\int_0^t(|x|^2+t-s)^{-3/2}\cdot\int_{\partial B_{|x|/2}} do_y\,ds \le C\cdot c_0^2\cdot|x|^{-1-2\cdot\epsilon}. \qquad (6.13)
\end{aligned}
$$

The second estimate claimed in Lemma 6.6 follows from the preceding estimates of $|\mathfrak{B}^{(1)}_{jkl}|$ to $|\mathfrak{B}^{(3)}_{jkl}|$ and from the equations $\mathfrak{A}_j(x,t) = \sum_{i=1}^4\sum_{k,l=1}^3 \mathfrak{B}^{(i)}_{jkl}$ and $\mathfrak{B}^{(4)}_{jkl} = 0$ stated above. □

Now we are able to carry out the

Proof of Theorem 1.1. The continuity property of v stated in Theorem 1.1 holds according to Theorem 6.3. Thus we have to show (1.6). By Corollary 6.2 we know that $|v(x,t)| \le \mathfrak{C}$ for $x \in B^c_{R_0},\ t>0$, so

$$
|v(x,t)| \le \mathfrak{C}\cdot|x|^{-\mu} \ \text{ for } \ x \in B_{2\cdot R_0}\setminus B_{R_0},\ t>0. \qquad (6.14)
$$

We know from Theorem 6.4 that $|v(x,t)| \le \mathfrak{C} \cdot |x|^{-7/6}$ for $x \in B^c_{R_0},\ t > 0$. Therefore from the first estimate in Lemma 6.6 with $\epsilon = 1/6$, we get that $|\mathfrak{A}(x,t)| \le \mathfrak{C} \cdot |x|^{-4/3}$ $(x \in B^c_{2\cdot R_0},\ t > 0)$. Thus by (6.2)

$$|v(x,t)| \le \mathfrak{C} \cdot (|x|^{-\min\{\mu,\,13/5\}} + |x|^{-4/3}) \quad \text{for } x \in B^c_{2\cdot R_0},\ t \in (0,\infty). \tag{6.15}$$

Due to (6.14) the preceding inequality even holds for $x \in B^c_{R_0},\ t > 0$. If $\mu \le 4/3$, we have proved (1.6). Now suppose that $\mu > 4/3$. Then inequality (6.15) and the remark following it yield that $|v(x,t)| \le \mathfrak{C} \cdot |x|^{-4/3}$ for $x \in B^c_{R_0},\ t \in (0,\infty)$. Therefore we may conclude from the first estimate in Lemma 6.6 with $\epsilon = 1/3$ that $|\mathfrak{A}(x,t)| \le \mathfrak{C} \cdot |x|^{-5/3}$ $(x \in B^c_{2\cdot R_0},\ t > 0)$, so by inequality (6.2) we obtain $|v(x,t)| \le \mathfrak{C} \cdot (|x|^{-\min\{\mu,\,13/5\}} + |x|^{-5/3})$ for $x \in B^c_{2\cdot R_0},\ t \in (0,\infty)$. By (6.14) the preceding inequality extends to $x \in B^c_{R_0},\ t > 0$. If $\mu \le 5/3$, inequality (1.6) is proved. Otherwise we may conclude that $|v(x,t)| \le \mathfrak{C} \cdot |x|^{-5/3}$ for $x \in B^c_{R_0},\ t \in (0,\infty)$. Again referring to the first estimate in Lemma 6.6, this time with $\epsilon = 2/3$, we get $|\mathfrak{A}(x,t)| \le \mathfrak{C} \cdot |x|^{-2}$, where $x \in B^c_{2\cdot R_0},\ t > 0$. Thus, by another application of (6.2) and (6.14), $|v(x,t)| \le \mathfrak{C} \cdot (|x|^{-\min\{\mu,\,13/5\}} + |x|^{-2})$ for $x \in B^c_{R_0},\ t > 0$. At this point estimate (1.6) is established if $\mu \le 2$. Otherwise $|v(x,t)| \le \mathfrak{C} \cdot |x|^{-2}$ $(x \in B^c_{R_0},\ t \in (0,\infty))$. In this situation we may deduce from the second inequality in Lemma 6.6 with $\epsilon = 1$ that $|\mathfrak{A}(x,t)| \le \mathfrak{C} \cdot |x|^{-13/5}$, for $x \in B^c_{2\cdot R_0},\ t > 0$. This estimate, (6.2) and (6.14) imply (1.6). □

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