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An Integral Operator Related to the Stokes System in Exterior Domains

Paul Deuring

Technische Hochschule Darmstadt, Fachbereich 4-Mathematik, Schlossgartenstrasse 7, D-6100 Darmstadt, Federal Republic of Germany

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Consider a bounded domain Ω in $\mathbb{R}^3$ with C^2-boundary $\partial\Omega$. In [1] the Stokes problem in the exterior domain $\mathbb{R}^3\backslash\bar{\Omega}$, with resolvent parameter $\lambda \in \mathbb{C}\backslash]-\infty, 0]$, is solved by using the method of integral equations. However, for estimating the corresponding solutions in L^p norms, it turns out that a certain operator defined on the spaces $L^r(\partial\Omega)^3$, for $r \in]1, \infty[$, has to be evaluated in the norm of $L^r(\partial\Omega)^3$. This estimate is proved in the present paper.

1. Introduction, outline of Proofs

Let Ω be a bounded domain in $\mathbb{R}^3$, with C^2-boundary $\partial\Omega$. Then consider the Stokes system in the exterior domain $\mathbb{R}^3\backslash\bar{\Omega}$, with resolvent parameter $\lambda \in \mathbb{C}\backslash]-\infty, 0]$:

$$-\Delta \mathbf{u} + \lambda \cdot \mathbf{u} + \nabla \pi = \mathbf{f}, \qquad \operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad \mathbb{R}^3\backslash\bar{\Omega}, \tag{1.1}$$

and with Dirichlet boundary conditions

$$\mathbf{u} \,|\, \partial\Omega = 0. \tag{1.2}$$

The function $\mathbf{f}$ is assumed to be given in $L^p(\mathbb{R}^3\backslash\bar{\Omega})^3$, for some $p \in]1, \infty[$. Solutions to (1.1, 1.2), and their estimate in L^p norms, are of interest for numerical purposes, and for treating the full Navier–Stokes system by means of functional analysis; see [1] for a more detailed background. In [1], we adapt the method of integral equations in order to construct a function

$$\mathbf{u} \in W^{2,p}(\mathbb{R}^3\backslash\bar{\Omega})^3 \cap W_0^{1,p}(\mathbb{R}^3\backslash\bar{\Omega})^3,$$

and a function $\pi \in L^p_{\mathrm{loc}}(\mathbb{R}^3\backslash\bar{\Omega})$, with $\nabla\pi$ belonging to $L^p(\mathbb{R}^3\backslash\bar{\Omega})^3$, and with $(\mathbf{u}, \pi)$ solving (1.1). $\mathbf{u}$ is uniquely determined, and π is unique up to an additive constant. Of course, the condition $\mathbf{u} \in W_0^{1,p}(\mathbb{R}^3\backslash\bar{\Omega})^3$ implies that $\mathbf{u}$ solves (1.2) in a weak sense. Concerning L^p estimates of solutions, one would like to find a constant $\mathscr{D}_1(\theta, p, E, \Omega) > 0$, for $\theta \in [0, \pi[$, $p \in]1, \infty[$, $E > 0$, which satisfies the relation

$$\|\mathbf{u}\|_p \leqslant \mathscr{D}_1(\theta, p, E, \Omega) \,|\lambda|^{-1} \,\|\mathbf{f}\|_p, \tag{1.3}$$

where $\mathbf{f} \in L^p(\mathbb{R}^3\backslash\bar{\Omega})^3$, $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant E$ and $|\arg(\lambda)| \leqslant \theta$, and where $(\mathbf{u}, \pi)$ is the

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solution of (1.1, 1.2) corresponding to λ and $\mathbf{f}$. Up to now, estimate (1.3) could not be proved directly; see [1] for more details and for references. Since the approach in [1] yields an integral representation of $\mathbf{u}$, it is natural to try and estimate $\|\mathbf{u}\|_p$ by using this representation. As shown in [1], this method reduces (1.3) to estimating a certain integral operator $\tilde{\mathscr{T}}^\lambda$, which maps $L^r(\partial\Omega)^3$ into itself ($r \in]1, \infty[$). It is the purpose of the present paper to prove this estimate. We still have to introduce some notations before we can define $\tilde{\mathscr{T}}^\lambda$. However, to give an idea of what we are aiming at, let us first state the result on $\tilde{\mathscr{T}}^\lambda$ that is needed in [1] (see [1, Theorem 1.5]): for $\theta \in [0, \pi[$, $p \in]1, \infty[$, there are constants $\mathscr{D}_2(\theta, p, \Omega)$, $\mathscr{D}_3(\theta, p, \Omega) > 0$ such that

$$\|\mathbf{\Phi}\|_p \leqslant \mathscr{D}_3(\theta, p, \Omega)\, \|\mathbf{\Phi} + \tilde{\mathscr{T}}^\lambda(\mathbf{\Phi})\|_p$$
$$\text{for } \mathbf{\Phi} \in L^p(\partial\Omega)^3,\ \lambda \in \mathbb{C} \text{ with } |\arg(\lambda)| \leqslant \theta, \qquad |\lambda| \geqslant \mathscr{D}_2(\theta, p, \Omega). \tag{1.4}$$

Now we turn to introducing our notations. We write $\mathbf{n}$ for the outward unit normal to Ω. Among the equivalent norms of $L^p(\partial\Omega)$, we choose the norm $\|\ \|_p$ given by

$$\|\psi\|_p := \left(\int_{\partial\Omega} |\psi(y)|^p \, d\Omega(y)\right)^{1/p}, \quad (\psi \in L^p(\partial\Omega), \quad p \in]1, \infty[).$$

The symbols D_k, $D^\mathbf{a}$, div, ∇, Δ, with $k \in \mathbb{N}$, $\mathbf{a} \in \mathbb{N}_0^n$, $n \in \mathbb{N}$, are to denote differential operators, with obvious meanings. For any matrix $\mathbf{B}$, the term $\mathbf{B}^\mathrm{T}$ stands for the transpose of $\mathbf{B}$. The symbol $\langle\ ,\ \rangle$ means the inner product in $\mathbb{R}^n (n \in \mathbb{N})$. For $n \in \mathbb{N}$, $\mathbf{x} \in \mathbb{R}^n$, $\sigma > 0$, we write $B_\sigma^n(\mathbf{x})$ for a ball in $\mathbb{R}^n$, with centre $\mathbf{x}$ and radius σ, whereas $Q^n(\mathbf{x}, \sigma)$ means a cube in $\mathbb{R}^n$ with centre $\mathbf{x}$ and edge-length 2σ. For any set A, the symbol $\mathrm{id}(A)$ denotes the identity function on A, and χ_A the characteristic function with respect to A. If n, $k \in \mathbb{N}$, A an open subset of $\mathbb{R}^n$, and $u \in C^k(A)$, then $|u|_k$ abbreviates the sum of the terms $|D^\mathbf{a} u|_0$, taken over $\mathbf{a} \in \mathbb{N}_0^n$ with $|\mathbf{a}|_* \leqslant k$, where $|\ |_0$ denotes the supremum norm, and $|\mathbf{a}|_*$ means the length $a_1 + \cdots + a_n$ of $\mathbf{a} \in \mathbb{N}_0^n$. For $g \in L^2(\mathbb{R}^n)$, $\hat{g}$ denotes the Fourier transform of the function g, and $\check{g}$ its inverse Fourier transform. Take $n \in \mathbb{N}$, B a subset of $\mathbb{R}^n$ and $p \in]1, \infty[$. If $(f_\sigma)_{\sigma > 0}$ is a family of functions in $L^p(B)$, converging in $L^p(B)$ for $\sigma \downarrow 0$, then we write $L^p(B) - \lim_{\sigma \downarrow 0} f_\sigma$ for the corresponding limit function.

Next, turning to the definition of $\tilde{\mathscr{T}}^\lambda$, we set

$$g_1(r) := e^{-r} + r^{-2}(re^{-r} + e^{-r} - 1),$$
$$g_2(r) := e^{-r} + 3r^{-2}(re^{-r} + e^{-r} - 1), \quad (r \in \mathbb{C} \backslash \{0\});$$
$$\tilde{E}_{jk}^\lambda(\mathbf{x}) := (4\pi|\mathbf{x}|)^{-1} [\delta_{jk} g_1(\sqrt{\lambda}|\mathbf{x}|) - x_j x_k |\mathbf{x}|^{-2} g_2(\sqrt{\lambda}|\mathbf{x}|)],$$
$$E_{4k}(\mathbf{x}) := (4\pi|\mathbf{x}|^3)^{-1} x_k, \quad (1 \leqslant j, k \leqslant 3, \quad \lambda \in \mathbb{C} \backslash \{0\}, \quad \mathbf{x} \in \mathbb{R}^3 \backslash \{0\});$$
$$\tilde{S}_{jkm}^\lambda := \delta_{jk} E_{4m} - D_j \tilde{E}_{km}^\lambda - D_k \tilde{E}_{jm}^\lambda, \quad (1 \leqslant j, k, m \leqslant 3, \quad \lambda \in \mathbb{C} \backslash \{0\});$$
$$\tilde{\mathscr{T}}_m^\lambda(\mathbf{\Phi})(\mathbf{x}) := 2 \int_{\partial\Omega} \sum_{1 \leqslant j, k \leqslant 3} \tilde{S}_{jkm}^\lambda(\mathbf{x} - \mathbf{y}) \Phi_j(\mathbf{y}) n_k(\mathbf{y}) \, d\Omega(\mathbf{y})$$
$$(1 \leqslant m \leqslant 3,\ \mathbf{\Phi} \in L^r(\partial\Omega)^3 \text{ for some } r \in]1, \infty[,\ \mathbf{x} \in \partial\Omega,\ \lambda \in \mathbb{C} \backslash]-\infty, 0]);$$

$\tilde{\mathscr{T}}^\lambda$ is well defined, as explained in [1, section 3].

We know of only one reference pertaining to (1.4): In [3] McCracken uses the method of integral equations in order to investigate system (1.1) in the half-space in $\mathbb{R}^3$, under Dirichlet boundary conditions. The present paper was inspired by that article. The role that $\tilde{\mathscr{T}}^{\lambda}$ plays in [1] is taken in [3] by an operator $\mathbf{R}^{\lambda}$ defined as follows: set

$$f_1(r) := 3e^{-r} + re^{-r} + 6\cancel{e^{-1}}(e^{-r} + re^{-r} - 1),$$ H r^{-2}

$$f_2(r) := 1 + 2e^{-r} + 6\cancel{e^{-1}}(e^{-r} + re^{r} - 1), \quad (r \in \mathbb{C}\backslash\{0\});$$ H r^{-2}

$$X_j^{\lambda}(\rho) := (4\pi|\boldsymbol{\rho}|^3)^{-1}\rho_j f_2(\sqrt{\lambda}\,|\boldsymbol{\rho}|), \qquad Y_j^{\lambda}(\boldsymbol{\rho}) := (4\pi|\rho|^3)^{-1}\rho_j f_1(\sqrt{\lambda}\,|\boldsymbol{\rho}|),$$

$$(j = 1, 2, \quad \lambda \in \mathbb{C}\backslash]-\infty, 0], \quad \rho \in \mathbb{R}^2\backslash\{0\});$$

$$\mathbf{R}^{\lambda}(\boldsymbol{\Phi}) := \left(X_1^{\lambda} * \psi_3,\ X_2^{\lambda} * \psi_3,\ \sum_{1 \leqslant j \leqslant 2} Y_j^{\lambda} * \psi_j \right) \quad (\lambda \in \mathbb{C}\backslash]-\infty, 0],$$ L ϕ

$$\boldsymbol{\Phi} \in L^s(\mathbb{R}^2)^3 \text{ for some } s \in]1, \infty[),$$

where the symbol $*$ means convolution. Note that X_j^{λ}, $Y_j^{\lambda} \in L^s(\mathbb{R}^2)$ for $1 \leqslant j \leqslant 2$, $\lambda \in \mathbb{C}\backslash]-\infty, 0]$, $s \in]1, \infty[$, so that $\mathbf{R}^{\lambda}$ is well defined. $\mathbf{R}^{\lambda}$ will enter into the computations leading to (1.4).

At this point let us mention the tools that will be needed in the following. From [3] we shall use [3, equation (5.5)], and [3, Lemma 5.6]. Furthermore, we shall apply the multiplier theorem from [5, p. 96], the Calderón–Zygmund theorem for odd kernels, as given in [4, p. 89], the open-mapping theorem, and some standard results on Fourier transforms. In section 2, when studying the operator $\mathscr{S}_p$ introduced there, we shall implicitly use the maximum principle for harmonic functions. Otherwise we shall only use elementary analysis.

In order to give an idea of our reasoning, and to reduce our proof of (1.4) to its main difficulties, let us state the main steps leading to (1.4). The first one concerns some particular descriptions of $\partial\Omega$ by local parameters:

Theorem 1.1. *There are constants* $\mathscr{A}_1(\Omega)$, $\mathscr{A}_2(\Omega) > 0$, *and for any* $\varepsilon \in]0, 1]$ *a finite index-set* $I(\varepsilon)$, *and for* $t \in I(\varepsilon)$ *an orthonormal matrix* $\mathbf{D}_t^{\varepsilon} \in \mathbb{R}^{3\times 3}$, *a vector* $\mathbf{C}_t^{\varepsilon} \in \mathbb{R}^3$, *as well as a* C^2*-function* β_t^{ε}, *mapping* $Q^2(0, \varepsilon\mathscr{A}_1(\Omega))$ *into* $\mathbb{R}$, *such that the following assertions hold: take* $\varepsilon \in]0, 1]$, *and set*

$$\Delta_{\varepsilon}^{\delta} := Q^2(\mathbf{0}, \delta\varepsilon\mathscr{A}_1(\Omega)) \quad \textit{for} \quad \delta \in]0, 1], \quad \Delta_{\varepsilon} := \Delta_{\varepsilon}^{1};$$

$$\boldsymbol{\gamma}_t^{\varepsilon}(\boldsymbol{\rho}) := \mathbf{D}_t^{\varepsilon}(\boldsymbol{\rho}, \beta_t^{\varepsilon}(\boldsymbol{\rho})) + \mathbf{C}_t^{\varepsilon}, \qquad J_t^{\varepsilon}(\boldsymbol{\rho}) := (1 + |\nabla\beta^{\varepsilon}(\boldsymbol{\rho})|^2)^{1/2}$$

$$\textit{for} \quad \boldsymbol{\rho} \in \Delta_{\varepsilon},\ t \in I(\varepsilon).$$

then we have

$$|I(\varepsilon)| \leqslant \mathscr{A}_2(\Omega)\varepsilon^{-5}; \qquad |\boldsymbol{\gamma}_t^{\varepsilon}|_2 \leqslant \mathscr{A}_2(\Omega); \qquad |\beta_t^{\varepsilon}|_1 \leqslant \varepsilon; \tag{1.5}$$

$$\mathbf{n} \circ \boldsymbol{\gamma}_t^{\varepsilon}(\boldsymbol{\eta}) = \mathbf{D}_t^{\varepsilon}(-\nabla\beta_t^{\varepsilon}(\boldsymbol{\eta}), 1)(1/J_t^{\varepsilon}(\boldsymbol{\eta})) \qquad (\boldsymbol{\eta} \in \Delta_t^{\varepsilon},\ t \in I(\varepsilon)); \tag{1.6}$$

$$\mathscr{A}_2(\Omega)\,\|f\|_p \leqslant \left(\sum_{t \in I(\varepsilon)} \|f \circ \boldsymbol{\gamma}_t^{\varepsilon}|\Delta_{\varepsilon}^{\delta}\|_p^p \right)^{1/p} \leqslant \mathscr{A}_2(\Omega)^2\,\|f\|_p$$

$$\textit{for} \quad p \in]1, \infty[,\ f \in L^p(\partial\Omega), \quad \delta \in [\tfrac{1}{4}, 1]. \tag{1.7}$$

The local parameters $\boldsymbol{\gamma}_t^{\varepsilon}$ may be constructed by breaking up $\partial\Omega$ into small parts, and by projecting these parts on suitable tangent planes to $\partial\Omega$. This approach must be carried through in such a way that both inequalities (1.5) and (1.7) are satisfied. This is

L $P^{\lambda}_{jkm}(z) := (4\pi|z|^3)^{-1}\,[\delta_{km} z_j f_1(\sqrt{\lambda}|z|) + \delta_{jk} z_m f_2(\sqrt{\lambda}|z|)]$,

not completely trivial since the constant in (1.7) must not depend on ε. A detailed proof is lengthy, but straightforward; so we do not give it here.

If $\varepsilon \in]0, 1]$ and $\mathbf{F}$ is a function from Δ_ε into $\mathbb{C}^3$, then we shall denote the trivial extension of $\mathbf{F}$ to $\mathbb{R}^2$ by $EX^\varepsilon(\mathbf{F})$.

Next we are going to introduce a certain operator which does not depend on λ. For this purpose we define L

L $\lambda \in \mathbb{C}\setminus(-\infty, 0]$ $$P^\infty_{jkm}(\mathbf{z}) := (4\pi)^{-1}\delta_{jk} z_m |\mathbf{z}|^{-3} \quad (\mathbf{z} \in \mathbb{R}^3\setminus\{0\},\ 1 \leqslant j, k, m \leqslant 3).$$

Now we are able to state

Theorem 1.2. *If $\Phi \in C^\alpha(\partial\Omega)^3$ for some $\alpha \in]0, 1[$, then the limit*

$$Z_m(\Phi)(\mathbf{x}) := \lim_{\sigma\downarrow 0} \int_{\partial\Omega\setminus B^3_\sigma(\mathbf{x})} \sum_{1\leqslant j,k\leqslant 3} P^\infty_{jkm}(\mathbf{x}-\mathbf{y})\Phi_j(\mathbf{y}) n_k(\mathbf{y})\, d\Omega(\mathbf{y}) \tag{1.8}$$

exists for $\mathbf{x} \in \partial\Omega$, $1 \leqslant m \leqslant 3$. For $p \in]1, \infty[$, there is a constant $\mathscr{D}_4(\Omega, p) > 0$ such that

$$\|\mathbf{Z}(\Phi)\|_p \leqslant \mathscr{D}_4(\Omega, p)\, \|\Phi\|_p \quad \text{for} \quad \Phi \in C^\alpha(\partial\Omega)^3,\ \alpha \in]0, 1[. \tag{1.9}$$

Thus, for any $p \in]1, \infty[$, the definition in (1.8) induces a continuous operator S_p from $L^p(\partial\Omega)^3$ into itself. Finally, for $p \in]1, \infty[$, there is a constant $\mathscr{D}_5(\Omega, p) > 0$ such that

$$\|\psi\|_p \leqslant \mathscr{D}_5(\Omega, p)\, \|\tfrac{1}{2}\psi + \mathbf{S}_p(\psi)\|_p \quad \text{for} \quad \psi \in L^p(\partial\Omega)^3. \tag{1.10}$$

Before we are able to explain why this theorem is of interest, we still need another set of definitions. Put

$$X^\infty_j(\rho) := (4\pi|\rho|^3)^{-1}\rho_j \quad (\rho \in \mathbb{R}^2\setminus\{0\},\ 1 \leqslant j \leqslant 2);$$

L∞ $$\mathbf{R}^\infty(\psi) := (X_1 * \psi_3,\ X_2 * \psi_3,\ 0), \qquad (\psi \in L^s(\mathbb{R}^2)^3 \quad \text{for some} \quad s \in]1, \infty[),$$

where the convolution $*$ is to be understood as a principal-value integral. For $\lambda \in \mathbb{C}\setminus]-\infty, 0]$ and for $\lambda = \infty$, let the matrix-valued function $\mathbf{E}^\lambda$ from $\mathbb{R}^2$ into $\mathbb{C}^{3\times 3}$ be defined by the condition that the Fourier transform of $\frac{1}{2}\Phi + \mathbf{R}^\lambda(\Phi)$ is equal to $\mathbf{E}^\lambda\Phi$, for $\Phi \in L^2(\mathbb{R}^2)^3$. This means in the case $\lambda = \infty$ (see [5, pp. 57/8]) that $E^\infty_{jk} = 0$ for $1 \leqslant j, k \leqslant 3$ with $j \neq k$ ~~or~~ $k \neq 3$, and

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$$E^\infty_{kk} = \tfrac{1}{2},\ E^\infty_{j3}(\rho) = \tfrac{i}{2}|\rho|^{-1}\rho_j \quad \text{for} \quad \rho \in \mathbb{R}^2\setminus\{0\}, \quad 1 \leqslant k \leqslant 3,\ 1 \leqslant j \leqslant 2.$$

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In the case $\lambda \in \mathbb{C}\setminus]-\infty, 0]$, an explicit form of $\mathbf{E}^\lambda$ is given in [3, (5.5)]. The function $\mathbf{E}^\lambda$ is invertible, as proved in [3, Lemma 5.6] for the case $\lambda \in \mathbb{C}\setminus]-\infty, 0]$. For $\lambda = \infty$ this fact is obvious. Now define for $\lambda \in \mathbb{C}\setminus]-\infty, 0]$, $\Phi \in C^0(\partial\Omega)^3$, $\varepsilon \in]0, 1]$, $t \in I(\varepsilon)$, $1 \leqslant m \leqslant 3$:

L m $$V_m(\varepsilon, t, \Phi, \lambda)(\rho) := \int_{\Delta^{1/2}_\varepsilon} \sum_{1\leqslant j,k\leqslant 3} P^\lambda_{jkl}(\mathbf{D}^\varepsilon_t(\rho - \eta, 0))\,(D^\varepsilon_t)_{k3}\Phi_j \circ \gamma^\varepsilon_t(\eta)\, d\eta \quad (\rho \in \mathbb{R}^2);$$

$$V_m(\varepsilon, t, \Phi, \infty) := L^p(\mathbb{R}^2) - \lim_{\sigma\downarrow 0} \int_{\Delta^{1/2}_\varepsilon} \chi_{]\sigma,\infty[}(|\mathrm{id}(\mathbb{R}^2) - \eta|)$$
$$\times \sum_{1\leqslant j,k\leqslant 3} P^\infty_{jkm}[\mathbf{D}^\varepsilon_t(\mathrm{id}(\mathbb{R}^2) - \eta, 0)]\,(D^\varepsilon_t)_{k3}\Phi_j \circ \gamma^\varepsilon_t(\eta)\, d\eta;$$

L- $$\mathbf{W}(\varepsilon, t, \Phi, \lambda) := (\mathbf{D}^\varepsilon_t)^{\mathrm{T}}[\tfrac{1}{2}EX^\varepsilon(\mathscr{T}^\lambda(\Phi) \circ \gamma^\varepsilon_t|\Delta^{1/2}_\varepsilon) + \mathbf{V}(\varepsilon, t, \Phi, \lambda)];$$

$$\mathbf{W}(\varepsilon, t, \Phi, \infty) := (\mathbf{D}^\varepsilon_t)^{\mathrm{T}}[-EX^\varepsilon(\mathbf{S}_p(\Phi) \circ \gamma^\varepsilon_t|\Delta^{1/2}_\varepsilon) + \mathbf{V}(\varepsilon, t, \Phi, \infty)];$$

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$$\mathbf{b}(\varepsilon, t, \mathbf{\Phi}, \lambda) := (\mathbf{E}^\lambda)^{-1}\{ + \tfrac{1}{2}(\mathbf{D}_t^\varepsilon)^{\mathrm{T}}[(\mathbf{\Phi} + \tilde{\mathscr{T}}^\lambda(\mathbf{\Phi}))\circ \boldsymbol{\gamma}_t^\varepsilon | \Delta_\varepsilon^{1/2}]\}^{\wedge}$$
$$+ (\mathbf{E}^\lambda)^{-1}\hat{\mathbf{W}}(\varepsilon, t, \mathbf{\Phi}, \lambda) - (\mathbf{E}^\infty)^{-1}\hat{\mathbf{W}}(\varepsilon, t, \mathbf{\Phi}, \infty). \tag{1.11}$$

After a short computation we obtain for λ, ε, t and $\mathbf{\Phi}$ as above:

$$(\mathbf{D}_t^\varepsilon)^{\mathrm{T}}(\tfrac{1}{2}\mathbf{\Phi} + \mathbf{S}_p(\mathbf{\Phi}))\circ \boldsymbol{\gamma}_t^\varepsilon|\Delta_\varepsilon^{1/4} = [\mathbf{E}^\infty \mathbf{b}(\varepsilon, t, \mathbf{\Phi}, \lambda)]^{\vee}|\Delta_\varepsilon^{1/4}. \tag{1.12}$$

This equation is the key to the proof of (1.4), for the following reason: on the one hand, $\|\mathbf{\Phi}\|_p$ is less than a constant times the sum over $t\in I(\varepsilon)$ of the L^p norm of the left-hand side of (1.12); see (1.7) and (1.10). On the other hand, the L^p norm of the right-hand side of (1.12) is bounded by a constant times $\|\mathbf{\Phi} + \tilde{\mathscr{T}}^\lambda(\mathbf{\Phi})\|_p$, plus a perturbation that becomes small if $|\lambda|$ is large, and ε small. The latter fact will be made precise in

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Theorem 1.3. *Take $p\in]1, \infty[$, $\theta\in[0, \pi[$. Then there exists a constant $\mathscr{D}_6(\theta, p, \Omega) > 0$ such that the ensuing inequality holds for $\lambda\in\mathbb{C}$ with $|\arg(\lambda)| \leqslant \theta$, $|\lambda| \geqslant 1$, $\varepsilon\in]0, 1/8[$, $\mathbf{\Phi}\in C^0(\partial\Omega)^3$:*

$$\left(\sum_{t\in I(\varepsilon)} \|[\mathbf{E}^\infty \mathbf{b}(\varepsilon, t, \mathbf{\Phi}, \lambda)]^{\vee}|\Delta_\varepsilon^{1/4}\|_p^p\right)^{1/p}$$
$$\leqslant \mathscr{D}_6(\theta, p, \Omega)\left[\|\mathbf{\Phi} + \tilde{\mathscr{T}}^\lambda(\mathbf{\Phi})\|_p + \varepsilon^{-10}|\lambda|^{-1/4}\|\mathbf{\Phi}\|_p\right.$$
$$\left. + \varepsilon\left(\sum_{t\in I(\varepsilon)} \|\mathbf{\Phi}\circ \boldsymbol{\gamma}_t^\varepsilon|\Delta_\varepsilon^{1/2}\|_p^p\right)^{1/p}\right]. \tag{1.13}$$

(1.4) may now easily be derived from (1.13), (1.12), (1.10) and (1.7). Note that the constant $\mathscr{A}_2(\Omega)$ from (1.7) does not depend on ε. Note further that we need only consider $\mathbf{\Phi}\in C^0(\partial\Omega)^3$, since $\tilde{\mathscr{T}}^\lambda$ is continuous, even compact, as an operator on $L^p(\partial\Omega)^3$ (see [1, section 3]).

2. Proof of Theorem 1.2

Take $\varepsilon\in]0, \frac{1}{8}[$, $t\in I(\varepsilon)$. Set for $\sigma\in]0, \infty[$, $\boldsymbol{\rho}\in\Delta_\varepsilon^{1/2}$:

$$V(\boldsymbol{\rho}, \sigma) := \{\mathbf{z}\in\mathbb{R}^2 : |\mathbf{z}|^2 + \langle\nabla\beta_t^\varepsilon(\boldsymbol{\rho}), \mathbf{z}\rangle^2 < \sigma^2\}.$$

For any bounded, measurable function K from Δ_ε into $\mathbb{C}$, and for $\psi\in L^1(\Delta_\varepsilon)$, we set

$$H(K, \psi) := \int_{\Delta_\varepsilon} K(\boldsymbol{\eta})\psi(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}.$$

Since $|\beta_t^\varepsilon|_1 \leqslant \varepsilon \leqslant 1/8$, we have for $\boldsymbol{\rho}$, $\boldsymbol{\eta}\in\Delta_\varepsilon$ with $\boldsymbol{\rho}\neq\boldsymbol{\eta}$:

$$\left|\left(\boldsymbol{\rho} - \boldsymbol{\eta}, \sum_{j=1}^{2} D_j\beta_t^\varepsilon(\boldsymbol{\rho})(\rho_j - \eta_j)\right)\right|^{-3} - |\boldsymbol{\rho} - \boldsymbol{\eta}|^{-3}$$
$$= \sum_{\nu=1}^{\infty} \cancel{\{(-4)}\binom{1/2}{\nu+2}\cancel{(\nu+2)(\nu+1)\}} \qquad H\binom{-3/2}{\nu}$$
$$\times \sum_{0\leqslant s\leqslant 2\nu} \binom{2\nu}{s} D_1\beta_t^\varepsilon(\boldsymbol{\rho})^s D_2\beta_t^\varepsilon(\boldsymbol{\rho})^{2\nu - s}|\boldsymbol{\rho} - \boldsymbol{\eta}|^{-2\nu-\cancel{3}}, \tag{2.1}$$

$$L\,(\rho-\eta)_1^s\,(\rho-\eta)_2^{2\nu-s}$$

with the infinite sum on the right-hand side converging absolutely. Now take $\psi \in C^\alpha(\Delta_\varepsilon)$ for some $\alpha \in]0, 1[$, $\boldsymbol{\rho} \in \Delta_\varepsilon^{1/2}$, $j, k, m \in \{1, 2, 3\}$. We have

$$\int_V z_1^q z_2^n |\mathbf{z}|^{-n-q-2}\, d\mathbf{z} = 0, \tag{2.2}$$

for $n, q \in \mathbb{N}$ with $q + n$ odd, and for V coinciding with one of the sets $V(\boldsymbol{\rho}, \sigma) \setminus V(\boldsymbol{\rho}, \tilde{\sigma})$, $B_\sigma^2(0) \setminus B_{\tilde{\sigma}}^2(0)$, $B_\sigma^2(0) \setminus V(\boldsymbol{\rho}, \sigma)$, where $\sigma, \tilde{\sigma} \in]0, \infty[$ with $\sigma > \tilde{\sigma}$. If $\sigma > 0$ is small enough, then for V as before, the sets $\{\boldsymbol{\eta} \in \Delta_\varepsilon : \boldsymbol{\rho} - \boldsymbol{\eta} \in V\}$ and $\{\boldsymbol{\eta} \in \mathbb{R}^2 : \boldsymbol{\rho} - \boldsymbol{\eta} \in V\}$ are equal. Thus, setting

$$L(\boldsymbol{\rho}, \boldsymbol{\eta}) := L_{jkm}^{\varepsilon, t}(\boldsymbol{\rho}, \boldsymbol{\eta}) := P_{jkm}^{\infty}\Big(D_t^\varepsilon\big[\boldsymbol{\rho} - \boldsymbol{\eta}, \sum_{1 \leqslant r \leqslant 2} D_r \beta_t^\varepsilon(\boldsymbol{\rho})(\boldsymbol{\rho} - \boldsymbol{\eta})_r\big]\Big), \tag{2.3}$$

$L\gamma_t^\varepsilon$

$$M(\boldsymbol{\rho}, \boldsymbol{\eta}) := M_{jkm}^{\varepsilon, t}(\boldsymbol{\rho}, \boldsymbol{\eta}) := P_{jkm}^{\infty}(\cancel{\Gamma_t^\varepsilon}(\boldsymbol{\rho}) - \cancel{\Gamma_t^\varepsilon}(\boldsymbol{\eta})), \tag{2.4}$$

$H\chi_{\mathbb{R}^2\setminus B_\sigma^2(0)}$

$$K_1(\boldsymbol{\rho}, \boldsymbol{\eta}, \sigma) := \chi_{\mathbb{R}^2 \setminus V(\boldsymbol{\rho}, \sigma)}(\boldsymbol{\rho} - \boldsymbol{\eta}) L(\boldsymbol{\rho}, \boldsymbol{\eta}), \quad K_2(\boldsymbol{\rho}, \boldsymbol{\eta}, \sigma) := \cancel{\chi^{\mathbb{R}^2 \setminus B_\sigma^2(0)}}(\boldsymbol{\rho} - \boldsymbol{\eta}) L(\boldsymbol{\rho}, \boldsymbol{\eta}),$$

$H\chi_{\mathbb{R}^2\setminus B_\sigma^2(0)}$

$$K_3(\boldsymbol{\rho}, \boldsymbol{\eta}, \sigma) := \chi_{\mathbb{R}^2 \setminus V(\boldsymbol{\rho}, \sigma)}(\boldsymbol{\rho} - \boldsymbol{\eta}) M(\boldsymbol{\rho}, \boldsymbol{\eta}), \quad K_4(\boldsymbol{\rho}, \boldsymbol{\eta}, \sigma) := \cancel{\chi^{\mathbb{R}^2 \setminus B_\sigma^2(0)}}(\boldsymbol{\rho} - \boldsymbol{\eta}) M(\boldsymbol{\rho}, \boldsymbol{\eta}),$$

$L\gamma_t^\varepsilon$

$$K_5(\boldsymbol{\rho}, \boldsymbol{\eta}, \sigma) := \chi_{\{\boldsymbol{\eta} \in \Delta_\varepsilon : |\cancel{\Gamma_t^\varepsilon}(\boldsymbol{\rho}) - \cancel{\Gamma_t^\varepsilon}(\boldsymbol{\eta})| \geqslant \sigma\}}(\boldsymbol{\rho} - \boldsymbol{\eta}) M(\boldsymbol{\rho}, \boldsymbol{\eta}), \quad (\boldsymbol{\eta} \in \Delta_\varepsilon, \sigma > 0),$$

we may in a first step conclude from (2.1, 2.2):

$$H(K_j(\boldsymbol{\rho}, \cdot, \sigma) - K_i(\rho, \cdot, \tilde{\sigma}), 1) = 0 \text{ for } \sigma > \tilde{\sigma} > 0, \text{ with } \sigma \text{ small, and for } 1 \leqslant i \leqslant j \leqslant 2. \tag{2.5}$$

Next we note the inequality

$$\left| \sum_{1 \leqslant r \leqslant 2} D_r \beta_t^\varepsilon(\boldsymbol{\rho})(\boldsymbol{\rho} - \boldsymbol{\eta})_r - \beta_t^\varepsilon(\boldsymbol{\rho}) + \beta_t^\varepsilon(\boldsymbol{\eta}) \right| \leqslant \mathscr{A}_2(\Omega)|\boldsymbol{\rho} - \boldsymbol{\eta}|^2, \quad (\boldsymbol{\eta} \in \Delta_\varepsilon). \tag{2.6}$$

The relations in (2.5, 2.6), and the Hölder continuity of ψ yield the following convergence result, for $J = \{3, 4\}$:

$$\text{The limit } \lim_{\sigma \downarrow 0} H(K_j(\boldsymbol{\rho}, \cdot, \sigma), \psi) \text{ exists for } j \in J, \text{ and takes the same value for any } j \in J. \tag{2.7}$$

On the other hand, for $\sigma > 0$, σ sufficiently small, we get after some computations:

$$\left| \int_{\Delta_\varepsilon} (K_3 - K_5)(\boldsymbol{\rho}, \boldsymbol{\eta}, \sigma)\, d\boldsymbol{\eta} \right| \leqslant \mathscr{E}_1 \ln[(1 + 4\mathscr{A}_2(\Omega)\sigma)/(1 - 4\mathscr{A}_2(\Omega)\sigma)], \tag{2.8}$$

with $\mathscr{E}_1$ independent of σ. By letting σ tend to zero in (2.8), it follows that (2.7) also holds for $J = \{3, 4, 5\}$. However, (2.7) with $J = \{5\}$ implies the first assertion of Theorem 1.2.

Turning to the proof of (1.9), we first note that the integrand of the integral in (2.2) is bounded by one. This implies by (2.1) and [4, p. 89] that for any $p \in]1, \infty[$, there exists a constant $\mathscr{D}_7(p) > 0$ with

$$\left(\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} (x - y)_1^q \cdot (x - y)_2^n |\mathbf{x} - \mathbf{y}|^{-q-n-2} \varphi(\mathbf{y})\, d\mathbf{y} \right|^p d\mathbf{x} \right)^{1/p} \leqslant \mathscr{D}_7(p) \|\varphi\|_p \tag{2.9}$$

for $\varphi \in L^p(\mathbb{R}^2)$, and $q, n \in \mathbb{N}_0$ with $q + n$ odd.

, where the inner integral is to be understood as a principal value integral

Let $p \in]1, \infty[$ be fixed for the rest of this section. Constants that only depend on p or Ω will be denoted by $\mathscr{E}$. As a consequence of (1.5), (2.1) and (2.9), the following assertion holds for $j = 2$:

$$H(K_j(\mathbf{p}, \cdot, \sigma), \psi), \text{ as a function of } \mathbf{p} \in \Delta_\varepsilon^{1/2}, \text{ converges in } L^p(\Delta_\varepsilon^{1/2}) \text{ for } \sigma \downarrow 0. \text{ The corresponding limit function is bounded by } \mathscr{E}\|\psi\|_p. \tag{2.10}$$

From (2.6) we may conclude that (2.10) also holds for $j = 4$. Now (1.9) is an easy consequence of (2.10), with $j = 4$, and of (2.7), with $J = \{4, 5\}$.

In order to prove inequality (1.10), we propose to show that $\frac{1}{2}\mathrm{id}(L^p(\partial\Omega)^3) + \mathbf{S}_p$ is bijective. (1.10) then follows by the open-mapping theorem. Consider the operator $\mathscr{S}_p$ from $L^p(\partial\Omega)$ into $L^p(\partial\Omega)$ defined by

$$\mathscr{S}_p(\varphi)(\mathbf{x}) := (4\pi)^{-1} \int_{\partial\Omega} \langle \mathbf{n}(x), \mathbf{x} - \mathbf{y} \rangle |\mathbf{x} - \mathbf{y}|^{-3} \varphi(\mathbf{y}) d\Omega(\mathbf{y}), \quad (\mathbf{x} \in \partial\Omega,\ \varphi \in L^p(\partial\Omega)). \tag{2.11}$$

Note that $|\langle \mathbf{n}(x), \mathbf{x} - \mathbf{y} \rangle|$ is less than $\mathscr{E}|\mathbf{x} - \mathbf{y}|^2$ for $\mathbf{x}, \mathbf{y} \in \partial\Omega$ (see [2, (2.22)]). Thus, the integral on the right-hand side of (2.9) exists, and defines a function belonging to $L^p(\partial\Omega)$. The operator $\frac{1}{2}\mathrm{id}(L^p(\partial\Omega)) + \mathscr{S}_p$ is bijective. In fact, [2, Satz 5.1] implies compactness of $\mathscr{S}_p$. Thus we only have to prove that $\frac{1}{2}\mathrm{id}(L^p(\partial\Omega)) + \mathscr{S}_p$ is one to one. Taking Φ from the kernel of this operator, we may conclude from [2, Lemma 5.4]: $\Phi \in C^0(\partial\Omega)$. Now [2, Satz 4.1] yields that the functions $-\frac{1}{2}\Phi + \mathscr{S}_p(\Phi)$, $\frac{1}{2}\Phi + \mathscr{S}_p(\Phi)$ represent boundary values of certain harmonic functions in Ω, and $\mathbb{R}^3 \backslash \bar{\Omega}$, respectively. This situation implies $\Phi = 0$. Returning to the operator $\mathbf{S}_p$, we note the equation

$$\langle \mathbf{n}, \mathbf{S}_p(f\mathbf{n}) \rangle = \mathscr{S}_p(f) \quad \text{for } f \in L^p(\partial\Omega), \tag{2.12}$$

which may be derived from (1.8, 2.10) by a density argument. Now let $\mathbf{\Gamma} \in L^p(\partial\Omega)^3$ with $\frac{1}{2}\mathbf{\Gamma} + \mathbf{S}_p(\mathbf{\Gamma}) = 0$. From (2.12) we conclude that $\frac{1}{2}\langle \mathbf{n}, \mathbf{\Gamma} \rangle + \mathscr{S}_p(\langle \mathbf{n}, \mathbf{\Gamma} \rangle)$ is vanishing. Thus we have $\langle \mathbf{n}, \mathbf{\Gamma} \rangle = 0$. By (1.8, 2.10), and a density argument, it follows that $\mathbf{S}_p(\mathbf{\Gamma}) = 0$, so that $\mathbf{\Gamma} = 0$. Hence $\mathbf{S}_p$ is one to one. Next take $\mathbf{a} \in L^p(\partial\Omega)^3$. There exists some $f \in L^p(\partial\Omega)$ with $\frac{1}{2}f + \mathscr{S}_p(f) = \langle \mathbf{n}, \mathbf{a} \rangle$. Set

$$\boldsymbol{\varphi} := f\mathbf{n}, \qquad \boldsymbol{\psi} := -\mathbf{S}_p(\boldsymbol{\varphi}) + \langle \mathbf{S}_p(\boldsymbol{\varphi}), \mathbf{n} \rangle \mathbf{n}, \qquad \mathbf{\Gamma} := \boldsymbol{\varphi} + 2\boldsymbol{\psi} + 2\mathbf{a} - 2\langle \mathbf{n}, \mathbf{a} \rangle \mathbf{n}.$$

Then we have $\mathbf{\Gamma} \in L^p(\partial\Omega)^3$, $\mathbf{S}_p(\boldsymbol{\varphi}) = \mathbf{S}_p(\mathbf{\Gamma})$. Equation (2.12) now yields: $\frac{1}{2}\mathbf{\Gamma} + \mathbf{S}_p(\mathbf{\Gamma}) = \mathbf{a}$. Thus we have shown the function $\frac{1}{2}\mathrm{id}(L^p(\partial\Omega)^3) + \mathbf{S}_p$ to be bijective, so that the proof of Theorem 1.2 is completed.

3. Proof of Theorem 1.3

We begin by setting

$$f_3(r) := 6e^{-r} + re^{-r} + 15r^{-2}(e^{-r} + re^{-r} - 1), \ (r \in \mathbb{C} \backslash \{0\});$$

$$Q^\lambda_{jkm}(\mathbf{x}) := (4\pi|\mathbf{x}|^3)^{-1}(\delta_{jm} x_k f_1(\sqrt{\lambda}|\mathbf{x}|) - 2x_j x_k x_m |\mathbf{x}|^{-2} f_3(\sqrt{\lambda}|\mathbf{x}|)),$$

$$(\mathbf{x} \in \mathbb{R}^3 \backslash \{0\},\ 1 \leqslant j, k, m \leqslant 3).$$

Note that $\tilde{S}^\lambda_{jkm} = P^\lambda_{jkm} + Q^\lambda_{jkm}$ $(1 \leqslant j, k, m \leqslant 3)$.

Take $\theta \in [0, \pi[$, $p \in]1, \infty[$, $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$, $|\arg(\lambda)| \leqslant \theta$. These parameters will be kept fixed for the rest of this paper. Constants that only depend on θ, p or Ω will be denoted by $\mathscr{D}$.

The proof of inequality (1.13) is based on the following estimates:

$$|D^{\mathbf{a}} P^{\lambda}_{jkm}(\mathbf{x})| \leqslant \mathscr{D} |\lambda|^{\gamma} |\mathbf{x}|^{-2-|\mathbf{a}|_* + 2\gamma}, \tag{3.1}$$

$$|D^{\mathbf{a}}(P^{\lambda}_{jkm} - P^{\infty}_{jkm})(\mathbf{x})| \leqslant \mathscr{D} |\lambda|^{-1} |\mathbf{x}|^{-4-|\mathbf{a}|_*}, \tag{3.2}$$

$$\left| \sum_{1 \leqslant r \leqslant 3} n_r(\mathbf{z}) Q^{\lambda}_{jrm}(\mathbf{z} - \tilde{\mathbf{z}}) \right| \leqslant \mathscr{D} |\lambda|^{-\gamma} |\mathbf{z} - \tilde{\mathbf{z}}|^{-1-2\gamma} \tag{3.3}$$

L, $\gamma \in [0,1]$, $a \in \mathbb{N}_0^3$ with $|a|_* \leq 3$

$$(\mathbf{x} \in \mathbb{R}^3 \backslash \{0\},\ \mathbf{z}, \tilde{\mathbf{z}} \in \partial\Omega,\ \mathbf{z} \neq \tilde{\mathbf{z}},\ 1 \leqslant j, k, m \leqslant 3);$$

$$|D^{\mathbf{a}} E^{\infty}(\boldsymbol{\rho})_{jk}|,\ |D^{\mathbf{a}}(E^{\infty})^{-1}(\boldsymbol{\rho})_{jk}|,\ |D^{\mathbf{a}} E^{\lambda}(\boldsymbol{\rho})_{jk}|,\ |D^{\mathbf{a}}(E^{\lambda})^{-1}(\boldsymbol{\rho})_{jk}| \leqslant \mathscr{D} |\boldsymbol{\rho}|^{-|\mathbf{a}|_*}, \tag{3.4}$$

$$|D^{\mathbf{a}}((E^{\lambda})^{-1} - (E^{\infty})^{-1})(\boldsymbol{\rho})_{jk}| \leqslant \mathscr{D} |\boldsymbol{\rho}|^{-|\mathbf{a}|_* + \gamma} |\lambda|^{-\gamma/2} \tag{3.5}$$

L, $a \in \mathbb{N}_0^3$ with $|a|_* \leq 4$

$$(\boldsymbol{\rho} \in \mathbb{R}^2 \backslash \{0\},\ 1 \leqslant j, k \leqslant 3,\ \gamma \in [0, 1]).$$

Concerning the proof of these inequalities, we remark that $\det \mathbf{E}^{\lambda}(\boldsymbol{\rho}) \neq 0$ for $\rho \in \mathbb{R}^2$ (see [2, Lemma 5.6]). Using this fact, (3.4) and (3.5) may be shown by easy but tedious calculations. Inequality (3.3) is a consequence of (2.6).

Now fix $\boldsymbol{\Phi} \in C^0(\partial\Omega)^3$, $\varepsilon \in]0, \frac{1}{8}[$. Take $t \in N(\varepsilon)$. By (1.12, 3.4), and by the multiplier theorem from [5, p. 96] we may conclude, with notations as introduced in section 1:

$$\begin{aligned} &\| [\mathbf{E}^{\infty} \cdot \mathbf{b}(\varepsilon, t, \boldsymbol{\Phi}, \lambda)]^{\check{}} |\Delta_{\varepsilon}^{1/4} \|_p^p \leqslant \mathscr{D} \| (\boldsymbol{\Phi} + \tilde{\mathscr{T}}^{\lambda}(\boldsymbol{\Phi})) \circ \boldsymbol{\gamma}_t^{\varepsilon} |\Delta_{\varepsilon}^{1/2} \|_p^p \\ &+ \| [\mathbf{E}^{\infty} [(\mathbf{E}^{\lambda})^{-1} - (\mathbf{E}^{\infty})^{-1}] \hat{\mathbf{W}}(\varepsilon, t, \boldsymbol{\Phi}, \lambda)]^{\check{}} |\Delta_{\varepsilon}^{1/4} \|_p^p \\ &+ \| (\mathbf{W}(\varepsilon, t, \boldsymbol{\Phi}, \lambda) - \mathbf{W}(\varepsilon, t, \boldsymbol{\Phi}, \infty)) |\Delta_{\varepsilon}^{1/4} \|_p^p. \end{aligned} \tag{3.6}$$

This leaves us to estimate the second and third summands of the sum appearing on the right-hand side of (3.6). To this end, we choose $\varphi \in C_0^{\infty}(\mathbb{R}^2)$ with $\varphi | B_1^2(\mathbf{0}) = 1$, $\varphi | \mathbb{R}^2 \backslash B_2^2(\mathbf{0}) = 0$, $0 \leqslant \varphi \leqslant 1$. For $h \in]0, \infty[$, $\boldsymbol{\rho} \in \mathbb{R}^2$, we set $\varphi_h(\boldsymbol{\rho}) := \varphi((1/h)\boldsymbol{\rho})$. Furthermore, we choose $\tilde{\zeta} \in C^{\infty}(\mathbb{R})$ with $0 \leqslant \tilde{\zeta} \leqslant 1$,

$$\tilde{\zeta} | [-\tfrac{5}{16}\mathscr{A}_1(\Omega), \tfrac{5}{16}\mathscr{A}_1(\Omega)] = 1, \qquad \tilde{\zeta} | \mathbb{R} \backslash [-\tfrac{3}{8}\mathscr{A}_1(\Omega), \tfrac{3}{8}\mathscr{A}_1(\Omega)] = 0,$$

with $\mathscr{A}_1$ from Theorem 1.1. Set

$$\zeta_{\varepsilon}(\boldsymbol{\rho}) := \tilde{\zeta}(\rho_1/\varepsilon)\tilde{\zeta}(\rho_2/\varepsilon) \text{ for } \boldsymbol{\rho} \in \mathbb{R}^2,\ \varepsilon \in]0, 1[.$$

The second summand on the right-hand side of (3.6) will now be split into terms that can be estimated in a suitable way. With $L^{\varepsilon, t, \lambda}_{jkm}$ and $M^{\varepsilon, t, \lambda}_{jkm}$ defined as in (2.3), but with the superscript ∞ replaced by λ, we set for $1 \leqslant m \leqslant 3$, $\boldsymbol{\rho} \in \Delta_{\varepsilon}^{1/2}$, $\boldsymbol{\sigma} \in \mathbb{R}^2$:

$$\begin{aligned} H_{1,m}(\boldsymbol{\rho}) &:= \sum_{1 \leqslant j, k \leqslant 3} \int_{\Delta_{\varepsilon}^{1/2}} (L^{\varepsilon, t, \lambda}_{jkm} - M^{\varepsilon, t, \lambda}_{jkm})(\boldsymbol{\rho}, \boldsymbol{\eta})(\Phi_j n_k) \circ \boldsymbol{\gamma}_t^{\varepsilon}(\boldsymbol{\eta}) J_t^{\varepsilon}(\boldsymbol{\eta}) d\boldsymbol{\eta}, \\ H_{2,m}(\boldsymbol{\rho}) &:= \sum_{1 \leqslant j, k \leqslant 3} \int_{\Delta_{\varepsilon}^{1/2}} [P^{\lambda}_{jkm}(\mathbf{D}_t^{\varepsilon}(\boldsymbol{\rho} - \boldsymbol{\eta}, 0)) - L^{\varepsilon, t, \lambda}_{jkm}(\boldsymbol{\rho}, \boldsymbol{\eta})](\Phi_j n_k) \circ \boldsymbol{\gamma}_t^{\varepsilon}(\boldsymbol{\rho}) J_t^{\varepsilon}(\boldsymbol{\rho}) d\boldsymbol{\rho}, \end{aligned} \tag{3.7}$$

$$H_{3,m}(\boldsymbol{\rho}) := \sum_{1 \leqslant j, k \leqslant 3} \int_{\Delta_{\varepsilon}^{1/2}} P^{\lambda}_{jkm}(\mathbf{D}_t^{\varepsilon}(\boldsymbol{\rho} - \boldsymbol{\eta}, 0))[(\mathbf{D}_t^{\varepsilon})_{k3} - n_k \circ \boldsymbol{\gamma}_t^{\varepsilon}(\boldsymbol{\eta}) J_t^{\varepsilon}(\boldsymbol{\eta})] \Phi_j \circ \boldsymbol{\gamma}_t^{\varepsilon}(\boldsymbol{\eta}) d\boldsymbol{\eta},$$

$$H_{4,m}(\boldsymbol{\rho}) := \sum_{1 \leqslant j,k \leqslant 3} \int_{\partial\Omega \setminus \Gamma_t^\varepsilon(\Delta_\varepsilon^{1/2})} P^\lambda_{jkm}(\gamma_t^\varepsilon(\boldsymbol{\rho}) - \mathbf{y}) n_k(\mathbf{y}) \Phi_j(\mathbf{y}) \, d\Omega(\mathbf{y}),$$

$$H_{5,m}(\boldsymbol{\sigma}) := \sum_{1 \leqslant j,k \leqslant 3} \int_{\Delta_\varepsilon^{1/2}} P^\lambda_{jkm}(\mathbf{D}_t^\varepsilon(\boldsymbol{\sigma} - \boldsymbol{\eta}, 0))(\mathbf{D}_t^\varepsilon)_{k3} \Phi_j \circ \gamma_t^\varepsilon(\boldsymbol{\eta}) \, d\boldsymbol{\eta},$$

$$H_{6,m}(\boldsymbol{\rho}) := \sum_{1 \leqslant j,k \leqslant 3} \int_{\partial\Omega} Q^\lambda_{jkm}(\gamma_t^\varepsilon(\boldsymbol{\rho}) - \mathbf{y}) n_k(\mathbf{y}) \Phi_j(\mathbf{y}) \, d\Omega(\mathbf{y}).$$

We further set

$H\chi_{\mathbb{R}^2\setminus\Delta_\varepsilon^{1/2}}$ $\tilde{\mathbf{H}}_5 := (\mathbf{D}_t^\varepsilon)^{\mathrm{T}} \cancel{\chi_{\mathbb{R}^2\setminus\Delta_\varepsilon^{1/2}}} \mathbf{H}_5, \quad \tilde{\mathbf{H}}_j := (\mathbf{D}_t^\varepsilon)^{\mathrm{T}} EX^\varepsilon(\mathbf{H}_j) \quad \text{for } 1 \leqslant j \leqslant 6, j \neq 5.$

Then $\mathbf{W}(\varepsilon, t, \boldsymbol{\Phi}, \lambda)$ is equal to the sum of the terms $\tilde{\mathbf{H}}_j$, for $1 \leqslant j \leqslant 6$. This implies that the second summand on the right-hand side of (3.6) is bounded by the sum $\|\mathbf{G}_1\|_p + \ldots + \|\mathbf{G}_7\|_p$, where the terms $\mathbf{G}_j$ $(1 \leqslant j \leqslant 7)$ are defined as follows:

$$\mathbf{B} := \mathbf{E}^\infty[(\mathbf{E}^\lambda)^{-1} - (\mathbf{E}^\infty)^{-1}],$$

$$\mathbf{G}_j := [\mathbf{B}(1 - \varphi_1)\{\zeta_\varepsilon \tilde{\mathbf{H}}_j\}^\wedge]^\vee | \Delta_\varepsilon^{1/4} \quad \text{for } 1 \leqslant j \leqslant 3,$$

$$\mathbf{G}_4 := [\mathbf{B}(1 - \varphi_1)\{\zeta_\varepsilon(\tilde{\mathbf{H}}_4 + \tilde{\mathbf{H}}_5)\}^\wedge]^\vee | \Delta_\varepsilon^{1/4}$$

$$\mathbf{G}_5 := [\mathbf{B}(1 - \varphi_1)\{(1 - \zeta_\varepsilon) \sum_{1 \leqslant j \leqslant 5} \tilde{\mathbf{H}}_j\}^\wedge]^\vee | \Delta_\varepsilon^{1/4},$$

$$\mathbf{G}_6 := \left[\mathbf{B}\varphi_1 \left(\sum_{1 \leqslant j \leqslant 5} \tilde{\mathbf{H}}_j \right)^\wedge \right]^\vee \Bigg| \Delta_\varepsilon^{1/4}, \qquad \mathbf{G}_7 := [\mathbf{B} \cdot \{\tilde{\mathbf{H}}_6\}^\wedge]^\vee | \Delta_\varepsilon^{1/4}.$$

Let us now estimate $\|\mathbf{G}_j\|_p$, for $1 \leqslant j \leqslant 7$. We begin by noting that

$$\mathbf{H}_1 \in C^1(\Delta_\varepsilon^{1/2})^3,$$

with

$$\|\mathbf{D}^\mathbf{a} \mathbf{H}_1\|_p \leqslant \mathscr{D} |\lambda|^{1/4} \|\boldsymbol{\Phi}\|_p \quad \text{for} \quad \mathbf{a} \in \mathbb{N}_0^2, \ |\mathbf{a}|_* \leqslant 1.$$

The preceding inequality is established by some careful calculations involving (2.6, 3.1), and the mean-value theorem. It follows that $\zeta_\varepsilon \tilde{\mathbf{H}}_1 \in C_0^1(\mathbb{R}^2)^3$, with

$$\|D_s(\zeta_\varepsilon \tilde{\mathbf{H}}_1)\|_p \leqslant \mathscr{D} |\lambda|^{1/4} \varepsilon^{-1} \|\boldsymbol{\Phi}\|_p, \quad (s = 1, 2).$$

Trivially the function $\zeta_\varepsilon \mathbf{H}_4$ also belongs to $C_0^1(\mathbb{R}^2)^3$, and the L^p-norm of its first derivatives is bounded by $\mathscr{D}\varepsilon^{-3}\|\boldsymbol{\Phi}\|_p$. On the other hand, take $1 \leqslant j, k \leqslant 3$, and consider the functions $F_1, F_2 \in C^1(\mathbb{R}^2)$ defined as follows:

$$F_r(\boldsymbol{\alpha}) := 0 \quad \text{for } \boldsymbol{\alpha} \in B^2_{1/2}(0),$$

$$F_r(\boldsymbol{\alpha}) := |\lambda|^{1/2} B_{jk}(\boldsymbol{\alpha})[1 - \varphi_1(\boldsymbol{\alpha})] |\boldsymbol{\alpha}|^{-2} \alpha_r \quad \text{for } \boldsymbol{\alpha} \in \mathbb{R}^2 \setminus B^2_{1/2}(\mathbf{0})$$

$(r = 1, 2)$. We have for $w \in C_0^1(\mathbb{R}^2)$:

$$[B_{jk}(1 - \varphi_1)\hat{w}]^\vee = |\lambda|^{-1/2} \sum_{1 \leqslant r \leqslant 2} [F_r \{D_r w\}^\wedge]^\vee.$$

It now follows by [5, p. 96] and (3.5), for $w \in C_0^1(\mathbb{R}^2)$:

$$\|[B_{jk}(1 - \varphi_1)\hat{w}]^\vee\|_p \leqslant \mathscr{D} |\lambda|^{-1/2} \sum_{1 \leqslant r \leqslant 2} \|D_r w\|_p.$$

Combining the previous estimates yields the relation

$$\|\mathbf{G}_j\|_p \leqslant \mathscr{D}|\lambda|^{-1/4}\varepsilon^{-3}\|\mathbf{\Phi}\|_p, \quad \text{for } j = 1, 4.$$

Next we note that $\|\mathbf{H}_2\|_p$ is bounded by $\|\mathbf{J}_1\|_p + \|\mathbf{J}_2\|_p$, where $J_{1,m}(\boldsymbol{\rho})$, $J_{2,m}(\boldsymbol{\rho})$ are given by the right-hand side in (3.7), with the domain of integration $\Delta_\varepsilon^{1/2}$ modified to

$$E(\boldsymbol{\rho}) := \{\boldsymbol{\eta} \in \Delta_\varepsilon^{1/2} : |\boldsymbol{\rho} - \boldsymbol{\eta}| \leqslant |\lambda|^{-1/2}\},$$

and $\Delta_\varepsilon^{1/2} \backslash E(\boldsymbol{\rho})$, respectively ($\boldsymbol{\rho} \in \Delta_\varepsilon^{1/2}$, $1 \leqslant m \leqslant 3$). The term $\|J_1\|_p$ is smaller than $\mathscr{D}\varepsilon\|\mathbf{\Phi} \circ \gamma_t^\varepsilon|\Delta_\varepsilon^{1/2}\|_p$, as follows from (2.1, 1.5). In order to evaluate $\mathbf{J}_2$, we write $\mathbf{J}_2 = \mathbf{J}_2^{\mathrm{I}} + \mathbf{J}_2^{\mathrm{II}}$, where the expressions $\mathbf{J}_{2,m}^{\mathrm{I}}(\boldsymbol{\rho})$, $J_{2,m}^{\mathrm{II}}(\boldsymbol{\rho})$ are defined by substituting $P_{jkm}^\lambda - P_{jkm}^\infty$ and P_{jkm}^∞, respectively, for the kernel P_{jkm}^λ in the definition of $J_{2,m}(\boldsymbol{\rho})$ ($\boldsymbol{\rho} \in \Delta_\varepsilon^{1/2}$, $1 \leqslant m \leqslant 3$). Applying (2.1, 1.5) and [5, p. 89], we obtain an upper bound $\mathscr{D}\varepsilon\|\mathbf{\Phi} \circ \gamma_t^\varepsilon|\Delta_\varepsilon^{1/2}\|_p$ for $\|\mathbf{J}_2^{\mathrm{II}}\|_p$. Inequality (3.2) yields an upper bound of the same form for $\|\mathbf{J}_2^{\mathrm{I}}\|_p$. By inequality (3.5) and [5, p. 96] it follows that $\|\mathbf{G}_2\|_p$ is less than $\mathscr{D}\varepsilon\|\mathbf{\Phi} \circ \gamma_t^\varepsilon|\Delta_\varepsilon^{1/2}\|_p$. Using (1.6) and [5, p. 96], we may construct an upper bound of the same form for $\|\mathbf{G}_3\|_p$. Turning to the estimate of $\|\mathbf{G}_5\|_p$ and $\|\mathbf{G}_6\|_p$, we abbreviate

$$\mathbf{h} := \sum_{1 \leqslant j \leqslant 5} \tilde{\mathbf{H}}_j, \qquad \mathbf{h}_\varepsilon := (1 - \zeta_\varepsilon)\mathbf{h}.$$

As we may readily deduce from (3.1), the expression $\|\mathbf{h}\|_p$, and hence $\|\mathbf{h}_\varepsilon\|_p$, is bounded by $\mathscr{D}|\lambda|^{1/4}\|\mathbf{\Phi}\|_p$. Let us transform $\mathbf{G}_5$ by a partial integration. To this end, take $1 \leqslant j, k \leqslant 3$, and observe that

$$[B_{jk}(1 - \varphi_1)\hat{h}_{\varepsilon,k}]^\vee|\Delta_\varepsilon^{1/4} = L^p(\Delta_\varepsilon^{1/4}) - \lim_{n \to \infty}[L^2(\Delta_\varepsilon^{1/4}) - \lim_{r \to \infty} M_{n,r}], \tag{3.8}$$

with

$$M_{n,r} := [B_{jk}(1 - \varphi_1)\varphi_r\{\varphi_n h_{\varepsilon,k}\}^\wedge]^\vee|\Delta_\varepsilon^{1/4} \quad (r, n \in \mathbb{N}).$$

Partial integration, along with some further transformations, yields for $n, r \in \mathbb{N}$, $\boldsymbol{\rho} \in \Delta_\varepsilon^{1/4}$, $1 \leqslant j, k \leqslant 3$:

$$M_{n,r}(\boldsymbol{\rho}) = (2\pi)^{-1}\int_{\mathbb{R}^2} \varphi_n(\boldsymbol{\eta})h_{\varepsilon,k}(\boldsymbol{\eta})\cancel{\chi_{[\varepsilon\mathscr{A}_1(\Omega)/16,\infty[}}(|\boldsymbol{\rho} - \boldsymbol{\eta}|)|\boldsymbol{\rho} - \boldsymbol{\eta}|^{-4}$$

$H\chi_{[\varepsilon A_1(\Omega)/16,\infty[}$

$$\times \sum_{1 \leqslant s,t \leqslant 2} \{D_t D_s(B_{jk}(1 - \varphi_1)\varphi_r)\}^\wedge(\boldsymbol{\rho} - \boldsymbol{\eta})\,d\boldsymbol{\eta}.$$

Now it follows by (3.5), for $n, r \in \mathbb{N}$, $\boldsymbol{\rho} \in \Delta_\varepsilon^{1/4}$:

$H\chi_{[\varepsilon A_1(\Omega)/16,\infty[}$

$$|M_{n,r}(\boldsymbol{\rho})| \leqslant \mathscr{D}|\lambda|^{-1/2}\int_{\mathbb{R}^2} \varphi_n(\boldsymbol{\eta})h_{\varepsilon,k}(\boldsymbol{\eta})\cancel{\chi_{[\varepsilon\mathscr{A}_1(\Omega)/16,\infty[}}(|\boldsymbol{\rho} - \boldsymbol{\eta}|)|\boldsymbol{\rho} - \boldsymbol{\eta}|^{-4}\,d\boldsymbol{\eta}. \tag{3.9}$$

By combining (3.8, 3.9) with our previous estimate of $\|\mathbf{h}_\varepsilon\|_p$, we arrive at the inequality

$$\|\mathbf{G}_5\|_p \leqslant \mathscr{D}\varepsilon^{-2}|\lambda|^{-1/4}\|\mathbf{\Phi}\|_p.$$

Using (3.5) with $\gamma = 1$, and referring to [5, p. 96], we may see that

$$\|[B_{jk}\varphi_1 h_k]\|_p \leqslant \mathscr{D}|\lambda|^{-1/2}\|h_k\|_p \quad (1 \leqslant j, k \leqslant 3).$$

This estimate implies $\|\mathbf{G}_6\|_p$ to be less than $\mathscr{D}|\lambda|^{-1/4}\|\mathbf{\Phi}\|_p$. For $\|\mathbf{G}_7\|_p$ we may find an upper bound $\mathscr{D}|\lambda|^{-1/2}\|\mathbf{\Phi}\|_p$ by applying (3.3, 3.5), and the multiplier theorem from [5, p. 96].

Combining the previous results, we see that the second summand on the right-hand side of (3.6) is less than

$$\mathscr{D}(\varepsilon^{-3}|\lambda|^{-1/4}+\varepsilon)\|\mathbf{\Phi}\circ\gamma_t^\varepsilon|\Delta_\varepsilon^{1/2}\|_p.$$

Using similar techniques as in the estimate of $\|\mathbf{H}_j\|_p$, for $j = 1, 2, 3$, we may find an upper bound

$$\mathscr{D}(\varepsilon^{-5}|\lambda|^{-1/2}+\varepsilon)\|\mathbf{\Phi}\circ\gamma_t^\varepsilon|\Delta_\varepsilon^{1/2}\|_p$$

for the third summand on the right-hand side of (3.6). Inequality (1.13) now follows with (1.5, 1.7). For the last step, as well as for the conclusion at the end of section 1, we need the fact that the constant $\mathscr{A}_2(\Omega)$ is independent of ε.

References

1. Deuring, P., 'The resolvent problem for the Stokes system in exterior domains: an elementary approach', *Math. Meth. in the Appl. Sci.*, **12** (1990).
2. Deuring, P., von Wahl, W. and Weidemaier, P., 'Das lineare Stokes Problem in $\mathbb{R}^3$. I. Vorlesung über das Innenraumproblem', *Bayreuth. Math. Schr.* **27**, 1–252 (1988).
3. McCracken, M., 'The resolvent problem for the Stokes equations on halfspace in L_p^*', *SIAM J. Math. Anal.*, **12**, 201–228 (1981).
4. Neri, U., 'Singular integrals', *Lecture Notes in Mathematics*, Vol. 200, Springer, Berlin, 1971.
5. Stein, E. M., 'Singular integrals and differentiability properties of functions', Princeton University Press, Princeton, NJ, 1970.

Addendum
An Integral Operator Related to the Stokes System in Exterior Domains

Paul Deuring

In [1], the auxiliary function P^{λ}_{jkm} plays a crucial role. However, as the result of an oversight, this function was left undefined in [1]. We want to give this definition here; it reads:

$$P^{\lambda}_{jkm}(z) := (4\pi|z|^3)^{-1}[\delta_{km}z_j f_1(\sqrt{\lambda}|z|) + \delta_{jk}z_m f_2(\sqrt{\lambda}|z|)],$$
$$(\lambda \in \mathbb{C}\backslash]-\infty, 0], \quad z \in \mathbb{R}^3\backslash\{0\}, \quad 1 \leqslant j, k, m \leqslant 3),$$

where f_1, f_2 were introduced in [1] p. 325.

1. Deuring, P., 'An integral operator related to the Stokes-system in exterior domains', *Math. Meth. in the Appl. Sci.*, **13**, 323–333 (1990).

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