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The Resolvent Problem for the Stokes System in Exterior Domains: An Elementary Approach

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We consider the Stokes system with resolvent parameter in an exterior domain:

$$-\nu\Delta\mathbf{u} + \lambda\mathbf{u} + \nabla\pi = \mathbf{f},$$
$$\operatorname{div}\mathbf{u} = 0 \text{ in } \mathbb{R}^3\backslash\bar{\Omega},$$

under Dirichlet boundary conditions. Here Ω is a bounded domain with C^2 boundary, and $\lambda \in \mathbb{C}\backslash\,]-\infty, 0]$, $\nu > 0$. Using the method of integral equations, we are able to construct solutions $(\mathbf{u}, \pi)$ in L^p spaces. Our approach yields an integral representation of these solutions. By evaluating the corresponding integrals, we obtain L^p estimates that imply in particular that the Stokes operator on exterior domains generates an analytic semigroup in L^p.

1. Introduction, main results, and outline of proofs

Let Ω be a bounded domain in $\mathbb{R}^3$, with C^2 boundary $\partial\Omega$. This domain will be kept fixed throughout. Take $\lambda \in \mathbb{C}\backslash\,]-\infty, 0]$, $\nu \in\,]0, \infty[$, $p \in\,]1, \infty[$, $\mathbf{f} \in L^p(\mathbb{R}^3\backslash\bar{\Omega})^3$. Then we consider the Stokes system in the exterior domain $\mathbb{R}^3\backslash\bar{\Omega}$, with resolvent parameter λ, right-hand side $\mathbf{f}$, and viscosity ν:

$$-\nu\Delta\mathbf{u} + \lambda\mathbf{u} + \nabla\pi = \mathbf{f}, \qquad \operatorname{div}\mathbf{u} = 0 \text{ in } \mathbb{R}^3\backslash\bar{\Omega}. \tag{1.1}$$

We prescribe Dirichlet boundary conditions:

$$\mathbf{u}|\partial\Omega = 0. \tag{1.2}$$

The unknown functions $\mathbf{u}$, π are to take values in $\mathbb{C}^3$, and $\mathbb{C}$, respectively. We shall study the boundary-value problem (1.1, 1.2) in L^p spaces, giving new and elementary proofs for our results, which concern the existence and uniqueness of solutions, L^p estimates of these solutions, and applications to the Stokes operator in exterior domains. In particular, our results imply that an analytic semigroup is generated by the latter operator. Before giving more details, let us remark on some notations.

The spaces $L^p(A)$, $W^{k,p}(B)$ and $W_o^{k,p}(B)$, defined in the usual way, are to relate to complex-valued functions ($p \in\,]1, \infty[$, $A, B \subset \mathbb{R}^3$, B open, $k \in \mathbb{N}$). For $\sigma \in \mathbb{N}_o^3$, we denote the length $\sigma_1 + \sigma_2 + \sigma_3$ of σ by $|\sigma|_*$. For $R > 0$, B_R denotes the set $\{x \in \mathbb{R}^3$:

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$|x| < R\}$. For $k \in \mathbb{N}$, and for $A \subset \mathbb{R}^k$, $\bar{A}$ means the closure of A. We shall use the differential operators $D^{\mathbf{a}}$, D_l, Δ, div and ∇ ($\mathbf{a} \in \mathbb{N}_0^3$, $1 \leq l \leq 3$). The meaning of these operators should be clear. It will be convenient to use a special symbol—namely $\langle \mathbf{x}, \mathbf{y} \rangle$—for the inner product of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$. $\mathbf{n}$ denotes the *outward* unit normal to Ω. We shall write $H_p(\mathbb{R}^3 \backslash \bar{\Omega})$ for the closure in $L^p(\mathbb{R}^3 \backslash \bar{\Omega})^3$ of the set

$$\{\boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^3 \backslash \bar{\Omega})^3 : \operatorname{div} \boldsymbol{\varphi} = 0\}, \quad (p \in]1, \infty[).$$

I_p denotes the identity mapping of $H_p(\mathbb{R}^3 \backslash \bar{\Omega})$ onto itself. For $p \in]1, \infty[$, we define $\tilde{H}^{1,p}(\mathbb{R}^3 \backslash \bar{\Omega})$ as the set of all measurable functions g from $\mathbb{R}^3 \backslash \bar{\Omega}$ into $\mathbb{C}$, with the weak derivatives $D_j g$ ($1 \leq j \leq 3$) existing and belonging to $L^p(\mathbb{R}^3 \backslash \bar{\Omega})$, and with $g|B_R \backslash \bar{\Omega}$ contained in $L^p(\mathbb{R}^3 \backslash \bar{\Omega})$, for $R \in]0, \infty[$. For $r \in]1, \infty[$, the function space A^r is to contain any pair of functions $(\mathbf{u}, \pi)$ with

$$\mathbf{u} \in W^{2,p}(\mathbb{R}^3 \backslash \bar{\Omega})^3 \cap W_0^{1,p}(\mathbb{R}^3 \backslash \bar{\Omega})^3, \quad \pi \in \tilde{H}^{1,p}(\mathbb{R}^3 \backslash \bar{\Omega}).$$

If $(\mathbf{u}, \pi) \in A^r$, then of course $\mathbf{u}$ solves (1.2) in a weak sense.

Now we may state our existence and uniqueness result:

Theorem 1.1. *Let* $\nu \in]0, \infty[$, $\lambda \in \mathbb{C} \backslash]-\infty, 0]$, $p \in]1, \infty[$, $\mathbf{f} \in L^p(\mathbb{R}^3 \backslash \bar{\Omega})^3$. *Then there is a pair* $(\mathbf{u}, \pi) \in A^p$ *solving* (1.1). $\mathbf{u}$ *is determined uniquely, and* π *is unique up to an additive constant.*

The next theorem gives L^p estimates of our solutions to (1.1, 1.2):

Theorem 1.2. *Let* $\nu \in]0, \infty[$, $\theta \in [0, \pi[$, $p \in]1, \infty[$. *Then there are constants* $\mathscr{D}_1(\nu, \theta, p, \Omega)$, $\mathscr{D}_2(\nu, \theta, p, \Omega) > 0$ *with the following properties:*

Let $\lambda \in \mathbb{C}$ *with* $|\arg(\lambda)| \leq \theta$, $|\lambda| \geq \mathscr{D}_1(\nu, \theta, p, \Omega)$. *Take* $\mathbf{f} \in L^p(\mathbb{R}^3 \backslash \bar{\Omega})^3$. *Let* $(\mathbf{u}, \pi) \in A^p$ *be the solution to* (1.1, 1.2) *corresponding to* $\mathbf{f}$ *and* λ. *Let* $\mathbf{c} \in \mathbb{N}_0^3$ *with* $|\mathbf{c}|_* \leq 2$. *Then*

$$|\lambda|^{1-|\mathbf{c}|_*/2} \|D^{\mathbf{c}}\mathbf{u}\|_p + \|\nabla \pi\|_p \leq \mathscr{D}_2(\nu, \theta, p, \Omega) \|\mathbf{f}\|_p. \tag{1.3}$$

In addition, let $E > 0$. *Then there is a constant* $\mathscr{D}_3(\nu, \theta, p, \Omega, E) > 0$ *such that for* $\lambda \in \mathbb{C}$ *with* $|\arg(\lambda)| \leq \theta$, $|\lambda| \geq E$, *and for* $\mathbf{f}$, $(\mathbf{u}, \pi)$ *as above, the following inequality holds:*

$$\|\mathbf{u}\|_p \leq \mathscr{D}_3(\nu, \theta, p, \Omega, E) |\lambda|^{-1} \|\mathbf{f}\|_p. \tag{1.4}$$

Next we introduce the Helmholtz decomposition of $L^p(\mathbb{R}^3 \backslash \bar{\Omega})^3$—a fundamental tool when dealing with problems related to the Stokes system in $\mathbb{R}^3 \backslash \bar{\Omega}$:

Theorem 1.3. (*Helmholtz decomposition*): *Let* $p \in]1, \infty[$, $\mathbf{f} \in \mathbf{L}^p(\mathbb{R}^3 \backslash \bar{\Omega})^3$. *Then there exists a uniquely determined function* $\mathbf{g} \in H_p(\mathbb{R}^3 \backslash \bar{\Omega})$, *as well as some* $h \in \tilde{H}^{1,p}(\mathbb{R}^3 \backslash \bar{\Omega})$, *uniquely determined up to an additive constant, such that* $\mathbf{f} = \mathbf{g} + \nabla h$. *There is a constant* $\mathscr{A}_1(p, \Omega) > 0$ *with*

$$\|\mathbf{g}\|_p + \|\nabla h\|_p \leq \mathscr{A}_1(p, \Omega) \|\mathbf{g} + \nabla h\|_p$$

for

$$\mathbf{g} \in H_p(\mathbb{R}^3 \backslash \bar{\Omega}), \quad h \in \tilde{H}^{1,p}(\mathbb{R}^3 \backslash \bar{\Omega}). \tag{1.5}$$

The preceding theorem will be taken as given here. An elementary proof may be found in [11]. A linear operator P_p mapping $L^p(\mathbb{R}^3 \backslash \bar{\Omega})^3$ into $H_p(\mathbb{R}^3 \backslash \bar{\Omega})$ is defined by the first assertion of Theorem 1.3. Inequality (1.5) means, in particular, that P_p is bounded.

Take $\nu \in]0, \infty[$, $p \in]1, \infty[$. Then we set

$$A(p, \nu)\mathbf{u} := -\nu P_p(\Delta \mathbf{u})$$

for

$$\mathbf{u} \in H_p(\mathbb{R}^3 \setminus \bar{\Omega}) \cap W_o^{1,p}(\mathbb{R}^3 \setminus \bar{\Omega})^3 \cap W^{2,p}(\mathbb{R}^3 \setminus \bar{\Omega})^3.$$

$A(p, \nu)$ is called the Stokes operator (in $L^p(\mathbb{R}^3 \setminus \bar{\Omega})^3$). It is linear and densely defined in the Banach space $H_p(\mathbb{R}^3 \setminus \bar{\Omega})$, and in addition it satisfies the following assertions:

Theorem 1.4. *Let $\nu > 0$, $p \in]1, \infty[$. Then $A(p, \nu)$ is closed, and any $\lambda \in \mathbb{C} \setminus]-\infty, 0]$ belongs to the resolvent set of $-A(p, \nu)$. Let $\theta \in [0, \pi[$, $E > 0$. Then the ensuing inequality holds for $\lambda \in \mathbb{C}$ with $|\arg(\lambda)| \leqslant \theta$, $|\lambda| \geqslant E$.*

$$\|(A(p, \nu) + \lambda I_p)^{-1}\|_p \leqslant \mathscr{D}_3(\nu, \theta, p, \Omega, E)|\lambda|^{-1}. \tag{1.6}$$

Theorem 1.4 implies that $A(p, \nu)$ generates an analytic semigroup in the space $H_p(\mathbb{R}^3 \setminus \bar{\Omega})$—a fundamental property if one wants to apply functional analytic tools to the full Navier–Stokes system. This property of $A(p, \nu)$ was first proved by Solonnikov in [14]. There he estimates functions solving the time-dependent Stokes system in exterior domains, under Dirichlet boundary conditions ('Solonnikov's estimates'). Then, using a trick due to Sobolevskii (see [13, pp. 720/1]), he concludes that inequality (1.6) holds at least for some $\theta = \theta_o \in]\frac{1}{2}\pi, \pi[$, and for some $E = E_o > 0$. He does not consider the question of whether the resolvent set of $A(p, \nu)$ contains any element $\lambda \in \mathbb{C} \setminus]-\infty, 0]$ with $|\lambda|$ small. In [17, pp. 74/5], the resolvent problem for the Stokes system in bounded domains is dealt with in a similar way, that is, by reducing it to Solonnikov's estimates in [14]. However, these estimates represent a rather deep-lying theory, with an extensive proof. It should not be necessary to refer to them for the purpose of proving Theorem 1.4. In addition, the fact that $A(p, \nu)$ generates an analytic semigroup is interesting because it is useful for dealing with time-dependent situations. Thus, one would like to derive it without referring to the time-dependent Stokes problem.

Giga [6] uses the theory of pseudodifferential operators in order to show that the Stokes operator on *bounded* domains generates an analytical semigroup. He then goes on to remark (see [6, p. 327]) that his method may also be applied to the Stokes operator in exterior domains, yielding the result from Theorem 1.4, even for *any* space dimension. However, the theory of pseudodifferential operators that his proof is based on is very general, and not easily accessible.

This leaves the question of whether there is a direct and elementary proof for Theorem 1.4, as should be expected for a result related to a boundary-value problem as concrete as the system (1.1, 1.2). It is the purpose of this paper to give such a proof for Theorems 1.1, 1.2 and 1.4. We shall achieve this by using the method of integral equations, which allows us to represent solutions to (1.1, 1.2) by certain integrals. Such a representation may be useful for numerical purposes as well; see Varnhorn [16] for an application related to the Stokes system on bounded domains. It should be noted that we are able to weaken Giga's assumptions on $\partial\Omega$: we only suppose Ω to be C^2 bounded, instead of $C^{2+\alpha}$ for some $\alpha \in [0, 1]$, as required in [6].

The present paper was inspired by [4] and [8]. Reference [4] represents an elaborated form of results that Ladyzhenskaya [7] obtained for the Stokes system *without* resolvent parameter, in interior and exterior domains. Ladyzhenskaya uses the method of integral equations as adapted to the Stokes system by Odquist [9]. McCracken [8] deals with the Stokes system in the half-space in $\mathbb{R}^3$, also basing her proofs on the method of integral equations. Both [4] and [8] treat the case of space dimension three only. We too shall restrict ourselves to this case, since it will be convenient for us to cite from those references.

At this point we should mention an application of the results proved by Giga [6] and McCracken [8]: Borchers and Sohr [2] were able to estimate the resolvent of the Stokes operator for $|\lambda| \to 0$ by using arguments that involve [6] and [8] in an essential way.

Let us now outline our proofs. This will serve us in two ways. First we want to indicate how the method of integral equations works out when adapted to the boundary-value problem (1.1, 1.2). In particular, we are going to state our integral representation of solutions to (1.1, 1.2). Second, we shall reduce Theorems 1.1, 1.2 and 1.4 to certain technical assertions containing the main difficulties inherent in our approach. These assertions will be proved in later sections, with the exception of Theorem 1.5, which is established in [3].

For the rest of this paper, we shall assume $\nu = 1$. If our results are established under this assumption, they immediately follow for any other value of ν.

Let us introduce the fundamental solution to (1.1) (see [8, p. 204; 16, section 12]). To this end put

$$g_1(r) := e^{-r} + r^{-2}(e^{-r} + re^{-r} - 1),$$
$$g_2(r) := e^{-r} + 3r^{-2}(e^{-r} + re^{-r} - 1), \quad (r \in \mathbb{C}\setminus\{0\});$$
$$\tilde{E}^{\lambda}_{jk}(\mathbf{z}) := (4\pi|\mathbf{z}|)^{-1}[\delta_{jk}g_1(\sqrt{\lambda}|\mathbf{z}|) - z_j z_k |\mathbf{z}|^{-2} g_2(\sqrt{\lambda}|\mathbf{z}|)],$$
$$E_{4k}(\mathbf{z}) := (4\pi|\mathbf{z}|^3)^{-1} z_k, \quad (\mathbf{z} \in \mathbb{R}^3\setminus\{0\},\ \lambda \in \mathbb{C}\setminus\{0\},\ 1 \leqslant j, k \leqslant 3).$$

Observe that

$$-\Delta\tilde{E}^{\lambda}_{jk} + \lambda\tilde{E}^{\lambda}_{jk} + D_j E_{4k} = 0,$$
$$\sum_{1 \leqslant l \leqslant 3} D_l \tilde{E}^{\lambda}_{lk} = 0, \quad (1 \leqslant j, k \leqslant 3,\ \lambda \in \mathbb{C}\setminus\{0\}). \tag{1.7}$$

Next we introduce some potential functions. Set for $\mathbf{f} \in C^{\infty}_{0}(\mathbb{R}^3)^3$, $\lambda \in \mathbb{C}\setminus\{0\}$, $1 \leqslant j \leqslant 3$, $\boldsymbol{\phi} \in L^p(\partial\Omega)^3$ for some $p \in]1, \infty[$:

$$v_j(\lambda, \mathbf{f})(\mathbf{x}) := \int_{\mathbb{R}^3} \sum_{1 \leqslant k \leqslant 3} \tilde{E}^{\lambda}_{jk}(\mathbf{x} - \mathbf{y}) f_k(\mathbf{y})\, d\mathbf{y},$$
$$g(\mathbf{f})(\mathbf{x}) := \int_{\mathbb{R}^3} \sum_{1 \leqslant k \leqslant 3} E_{4k}(\mathbf{x} - \mathbf{y}) f_k(\mathbf{y})\, d\mathbf{y}, \quad (\mathbf{x} \in \mathbb{R}^3),$$
$$\tilde{S}^{\lambda}_{jkl} := \delta_{jk} E_{4l} - D_k \tilde{E}^{\lambda}_{jl} - D_j \tilde{E}^{\lambda}_{kl}, \quad (1 \leqslant j, k \leqslant 3),$$
$$\tilde{W}^{\lambda}_{l}(\mathbf{x}, \boldsymbol{\phi}) := \int_{\partial\Omega} \sum_{1 \leqslant j, k \leqslant 3} -\tilde{S}^{\lambda}_{jkl}(\mathbf{x} - \mathbf{y}) \phi_j(\mathbf{y}) n_k(\mathbf{y})\, d\Omega(\mathbf{y}),$$
$$\tilde{\Pi}^{\lambda}(\mathbf{x}, \boldsymbol{\phi}) := \int_{\partial\Omega} \sum_{1 \leqslant j, k \leqslant 3} [2D_j E_{4k}(\mathbf{x} - \mathbf{y}) - \lambda(4\pi)^{-1}|\mathbf{x} - \mathbf{y}|^{-1}\delta_{jk}]$$
$$\times\, \phi_j(\mathbf{y}) n_k(\mathbf{y})\, d\Omega(\mathbf{y}), \quad (\mathbf{x} \in \mathbb{R}^3\setminus\partial\Omega),$$
$$\tilde{\mathcal{T}}^{\lambda}_{l}(\boldsymbol{\phi})(\mathbf{x}) := 2\int_{\partial\Omega} \sum_{1 \leqslant j, k \leqslant 3} \tilde{S}^{\lambda}_{jkl}(\mathbf{x} - \mathbf{y}) \phi_j(\mathbf{y}) n_k(\mathbf{y})\, d\Omega(\mathbf{y}), \quad (\mathbf{x} \in \partial\Omega). \tag{1.8}$$

It will turn out that the integral operator $\tilde{\mathcal{T}}^{\lambda}$ is well defined, and that the following assertions hold:

Lemma 1.1. *Let $\lambda \in \mathbb{C} \backslash]-\infty, 0]$, $p \in]1, \infty[$. For any $\mathbf{a} \in L^p(\partial\Omega)^3$, the equation*

$$-\tfrac{1}{2}(\Phi + \tilde{\mathscr{T}}^\lambda(\Phi)) = \mathbf{a} \tag{1.9}$$

may be solved uniquely in $L^p(\partial\Omega)^3$. There is a constant $\mathscr{A}_2(\lambda, p, \Omega) > 0$ such that

$$\|\phi\|_p \leqslant \mathscr{A}_2(\lambda, p, \Omega) \|\phi + \tilde{\mathscr{T}}^\lambda(\phi)\|_p \text{ for } \phi \in L^p(\partial\Omega)^3. \tag{1.10}$$

If $\alpha \in [0, 1[$, and $\phi + \tilde{\mathscr{T}}^\lambda(\phi) \in C^\alpha(\partial\Omega)^3$, then ϕ belongs to $C^\alpha(\partial\Omega)^3$.

For $\mathbf{f} \in C_0^\infty(\mathbb{R}^3)^3$, the function $\mathbf{v}(\lambda, \mathbf{f})$ belongs to $C^\infty(\mathbb{R}^3)^3$. Thus, according to Lemma 1.1, there is a uniquely determined function $\phi(\lambda, \mathbf{f}) \in C^0(\partial\Omega)^3$ solving integral equation (1.9) for $\mathbf{a} = -\mathbf{v}(\lambda, \mathbf{f})|\partial\Omega$. We define for $\mathbf{f} \in C_0^\infty(\mathbb{R}^3)^3$, $\lambda \in \mathbb{C} \backslash]-\infty, 0]$:

$$\mathbf{u}(\lambda, \mathbf{f}) := [\mathbf{v}(\lambda, \mathbf{f}) + \tilde{\mathbf{W}}^\lambda(\cdot, \phi(\lambda, \mathbf{f}))] | \mathbb{R}^3 \backslash \bar{\Omega},$$

$$\pi(\lambda, \mathbf{f}) := [g(\mathbf{f}) + \tilde{\Pi}^\lambda(\cdot, \phi(\lambda, \mathbf{f}))] | \mathbb{R}^3 \backslash \bar{\Omega}.$$

The pair $(\mathbf{u}(\lambda, \mathbf{f}), \pi(\lambda, \mathbf{f}))$ solves (1.1, 1.2) in the following sense:

Lemma 1.2. *Let $\mathbf{f} \in C_0^\infty(\mathbb{R}^3)^3$, $\lambda \in \mathbb{C} \backslash]-\infty, 0]$. Then the functions $u(\lambda, \mathbf{f})_j$, $\pi(\lambda, \mathbf{f})$ belong to $C^\infty(\mathbb{R}^3 \backslash \bar{\Omega})$ $(1 \leqslant j \leqslant 3)$, and the pair $(\mathbf{u}(\lambda, \mathbf{f}), \pi(\lambda, \mathbf{f}))$ solves (1.1). Furthermore, $\mathbf{u}(\lambda, \mathbf{f})$ may be continuously extended to $\mathbb{R}^3 \backslash \Omega$, and the corresponding extension vanishes identically on $\partial\Omega$.*

Thus, for $\mathbf{f} \in C_0^\infty(\mathbb{R}^3)^3$, a solution to (1.1, 1.2) is given as a sum of a volume potential and a double-layer potential, with the latter being weighted by a function that solves an integral equation on $\partial\Omega$.

Now we turn to the more difficult task of estimating the functions $\mathbf{u}(\lambda, \mathbf{f})$, $\pi(\lambda, \mathbf{f})$ in L^p norms. For this purpose we shall show:

Lemma 1.3. (*Estimate of the volume potential*) *Take $\theta \in [0, \pi[$, $p \in]1, \infty[$. Then there is a constant $\mathscr{D}_4(\theta, p, \Omega) > 0$ such that*

$$|\lambda|^{1 - |\mathbf{c}|_*/2} \|D^\mathbf{c}\mathbf{v}(\lambda, \mathbf{f})\|_p + \|\nabla g(\mathbf{f})\|_p \leqslant \mathscr{D}_4(\theta, p, \Omega) \|\mathbf{f}\|_p, \tag{1.11}$$

$$\|\mathbf{v}(\lambda, \mathbf{f})|\partial\Omega\|_p \leqslant \mathscr{D}_4(\theta, p, \Omega) |\lambda|^{-1 + 1/2p} \|\mathbf{f}\|_p \tag{1.12}$$

for $\mathbf{f} \in C_0^\infty(\mathbb{R}^3)^3$, $\lambda \in \mathbb{C} \backslash \{0\}$ with $|\arg(\lambda)| \leqslant \theta$, $\mathbf{c} \in \mathbb{N}_0^3$ with $|\mathbf{c}|_* \leqslant 2$.

Lemma 1.4. (*Estimate of the double-layer potential*) *Let $\theta \in [0, \pi[$, $p \in]1, \infty[$. Then there is a constant $\mathscr{D}_5(\theta, p, \Omega)$ with the properties to follow: take $\lambda \in \mathbb{C} \backslash \{0\}$ with $|\arg(\lambda)| \leqslant \theta$, $\phi \in C^0(\partial\Omega)^3$, $\mathbf{w} \in C^2(\mathbb{R}^3)^3 \cap L^p(\mathbb{R}^3)^3$ with $\operatorname{div} \mathbf{w} = 0$, $\mathbf{c} \in \mathbb{N}_0^3$ with $|\mathbf{c}|_* \leqslant 2$. Assume*

$$\phi + \tilde{\mathscr{T}}^\lambda(\phi) = -2\mathbf{w}|\partial\Omega. \tag{1.13}$$

Then

$$\|\tilde{\mathbf{W}}^\lambda(\cdot, \phi) | \mathbb{R}^3 \backslash \bar{\Omega}\|_p \leqslant \mathscr{D}_5(\theta, p, \Omega)(|\lambda|^{-1/2p} \|\phi\|_p + \|\mathbf{w}\|_p), \tag{1.14}$$

$$\begin{aligned} &|\lambda|^{1 - |\mathbf{c}|_*/2} \|D^\mathbf{c} \tilde{\mathbf{W}}^\lambda(\cdot, \phi) | \mathbb{R}^3 \backslash \bar{\Omega}\|_p + \|\nabla \tilde{\Pi}^\lambda(\cdot, \phi) | \mathbb{R}^3 \backslash \bar{\Omega}\|_p \\ &\leqslant \mathscr{D}_5(\theta, p, \Omega)(|\lambda|^{1 - 1/2p} \|\phi\|_p + \|\phi\|_p + |\lambda| \|\mathbf{w}\|_p + \|\mathbf{w}|\partial\Omega\|_{2 - 1/p, p}). \end{aligned} \tag{1.15}$$

Lemmas 1.1, 1.3 and 1.4 imply that the terms $\|\mathbf{u}(\lambda, \mathbf{f})\|_{2,p}$ and $\|\nabla\pi(\lambda, \mathbf{f})\|_p$ may be estimated against $\|\mathbf{f}\|_p$ times a constant depending on λ, p, Ω $(\lambda \in \mathbb{C} \backslash]-\infty, 0]$,

$p \in]1, \infty[$, $\mathbf{f} \in C_0^\infty(\mathbb{R}^3)^3$). Thus, for λ, p and $\mathbf{f}$ as before, the pair $(\mathbf{u}(\lambda, \mathbf{f}), \pi(\lambda, \mathbf{f}))$ belongs to A^p, $A(p, 1)$ is closed and λ belongs to the resolvent set of $-A(p, 1)$. Furthermore, the existence result from Theorem 1.1 may now be proved by a simple density argument. The assertion on uniqueness stated in Theorem 1.1 will be deduced from our existence result (see section 7).

In order to finish the proof of Theorems 1.2 and 1.4, we must attempt to establish inequality (1.10) with $\mathscr{A}_2(\lambda, p, \Omega)$ replaced by a constant that does not depend on λ. In this respect we can show:

Theorem 1.5. *Let* $\theta \in [0, \pi[$, $p \in]1, \infty[$. *Then there are constants* $\mathscr{D}_6(\theta, p, \Omega)$, $\mathscr{D}_7(\theta, p, \Omega) > 0$ *with*

$$\|\phi\|_p \leqslant \mathscr{D}_7(\theta, p, \Omega) \cdot \|\phi + \tilde{\mathscr{T}}^\lambda(\phi)\|_p$$

for $\phi \in C^0(\partial\Omega)^3$, $\lambda \in \mathbb{C}$ *with* $|\arg(\lambda)| \leqslant \theta$, $|\lambda| \geqslant \mathscr{D}_6(\theta, p, \Omega)$.

The proof of this theorem, which is independent from the rest of our arguments, is given in [3]. It involves a detailed study of the integral operator $\tilde{\mathscr{T}}^\lambda$.

Theorem 1.5 and Lemmas 1.3 and 1.4 imply inequality (1.3). As for inequalities (1.4, 1.6), they are established at this point only for parameters θ, p and E with $E \geqslant \mathscr{D}_6(\theta, p, \Omega)$. However, in the case $E < \mathscr{D}_6(\theta, p, \Omega)$, there exists an upper bound for the set

$$\{\|(A(p, 1) + \lambda I_p)^{-1}\|_p : \lambda \in \mathbb{C}, |\arg(\lambda)| \leqslant \theta, E \leqslant |\lambda| \leqslant \mathscr{D}_6(\theta, p, \Omega)\}.$$

This is implied by standard arguments involving the expansion of a resolvent in a series (see [18, p. 211]), and the Heine–Borel theorem. Now inequality (1.6) follows. As for the estimate in (1.4), it may be deduced from (1.6) by means of Theorem 1.3.

In sections 2–7, we shall prove Lemmas 1.1–1.4, as well as the uniqueness result from Theorem 1.1. Let us indicate here the non-elementary tools which will be needed. In addition to the Helmholtz decomposition (Theorem 1.3), and the expansion of the resolvent in a series, these tools include L^p estimates of the solution of the Dirichlet problem for the Laplacian on *bounded* domains, some properties of compact operators on Banach spaces ([18, p. 283, Theorem 1]), the open mapping theorem, a multiplier theorem as in [15, p. 96], and the Calderon–Zygmund theorem for odd kernels. The last two results will be mainly applied in [3]. The idea used for the proof of equation (5.12) is due to Professor Simader. It helped us remove a vexing difficulty involving the estimate of the gradient of the function that solves a certain Neumann problem.

At this point, the author would like to thank Professor Simader for helpful discussions, Professor Sohr for pointing out the problem, and Professor von Wahl for introducing the author—via [4]—to potential theoretic treatment of the Stokes-system.

2. Some auxiliary results

The ensuing theorem gives L^p estimates of functions solving the Dirichlet problem for the Laplacian in bounded domains:

Theorem 2.1. *Take* $n \in \mathbb{N}$. *Let* U *be a bounded domain in* $\mathbb{R}^n$ *with* C^2 *boundary. Let* $u \in C^0(\bar{U})$ *with* $u|U \in C^2(U)$, $u|\partial U \in W^{2-1/r, r}(\partial U)$, $\Delta u \in L^r(U)$, *for* $r \in]1, \infty[$. *Take*

$p \in]1, \infty[$. *Then* $u|U$ *belongs to* $W^{2,p}(U)$, *and the estimate*

$$\|u\|_{2,p} \leqslant \mathscr{F}(p, U)(\|\Delta u\|_p + \|u|\partial U\|_{2-1/p,p}) \tag{2.1}$$

holds true, with a constant $\mathscr{F}(p, U)$ *that depends only on* U and p.

Let us remark on the proof of this theorem. In the case $p > \frac{3}{2}$, we refer to [4, Lemma 7.12]. There our assertion was deduced by combining [10, Theorem 10.10] and the maximum principle. Now assume $p \leqslant \frac{3}{2}$. From [4, Lemma 7.12] we know that $u|U \in L^r(U)$ for $r \in]\frac{3}{2}, \infty[$. Since U is bounded, we thus have $u|U \in W^{2,p}(U)$. On the other hand, the Laplacian, considered as an operator from $W^{2,p}(U)$ into $L^p(U)$, is both one to one and onto. The first of these assertions may be shown by the method used in section 7, whereas the second property follows from the first one and from [10, Theorem 10.10]. Thus inequality (2.1) is obtained by combining the open mapping theorem, and the relation $u|U \in W^{2,p}(U)$.

Next we choose local parameters describing $\partial\Omega$. We may take $k(\Omega) \in \mathbb{N}$, $\alpha(\Omega) \in]0, \infty[$, and a tuple $(\overset{t}{\mathbf{u}})_{1 \leqslant t \leqslant k(\Omega)}$ with the ensuing properties: for $1 \leqslant t \leqslant k(\Omega)$, $\overset{t}{\mathbf{u}}$ is a C^2 diffeomorphism mapping $]-\alpha(\Omega), \alpha(\Omega)[^2$ into some open subset of $\partial\Omega$. Furthermore, $\partial\Omega$ equals the union of the sets

$$\overset{t}{\mathbf{u}}(]-\alpha(\Omega)/4, \alpha(\Omega)/4[^2),$$

with $1 \leqslant t \leqslant k(\Omega)$.

Abbreviate $\Delta :=]-\alpha(\Omega), \alpha(\Omega)[^2$. As remarked in [5, section 6.2; 4, Lemma 2.1], we may require that for any $1 \leqslant t \leqslant k(\Omega)$, there is an orthonormal matrix $\mathbf{A}_t \in \mathbb{R}^{3\times 3}$, a vector $\mathbf{C}_t \in \mathbb{R}^3$, and a function $a_t \in C^2(\bar{\Delta})$, such that

$$\overset{t}{\mathbf{u}}(\boldsymbol{\rho}) = \mathbf{A}_t(\boldsymbol{\rho}, a_t(\boldsymbol{\rho})) + \mathbf{C}_t, \quad \text{for } \boldsymbol{\rho} \in \Delta, \quad 1 \leqslant t \leqslant k(\Omega).$$

Define the function $\mathbf{p}_t$ by

$$\mathbf{p}_t(\mathbf{y}) := (((\mathbf{A}_t)^{-1}(\mathbf{y} - \mathbf{C}_t))_j)_{1 \leqslant j \leqslant 2}, \quad (\mathbf{y} \in \mathbb{R}^3, \quad 1 \leqslant t \leqslant k(\Omega)).$$

Note that

$$|\mathbf{p}_t(\mathbf{y}) - \boldsymbol{\rho}| \leqslant |\overset{t}{\mathbf{u}}(\boldsymbol{\rho}) - \mathbf{y}|, \qquad |\boldsymbol{\rho} - \boldsymbol{\eta}| \leqslant |\overset{t}{\mathbf{u}}(\boldsymbol{\rho}) - \overset{t}{\mathbf{u}}(\boldsymbol{\eta})|,$$
$$(\boldsymbol{\rho}, \boldsymbol{\eta} \in \Delta,\ 1 \leqslant t \leqslant k(\Omega),\ \mathbf{y} \in \mathbb{R}^3). \tag{2.2}$$

According to [4, Lemma 2.2], there are constants $K(\Omega)$, $\varepsilon(\Omega) > 0$ with

$$|\langle \mathbf{x} - \mathbf{y}, \mathbf{n}(\mathbf{y}) \rangle| \leqslant K(\Omega)|\mathbf{x} - \mathbf{y}|^2, \tag{2.3}$$

$$|\overset{t}{\mathbf{u}}(\boldsymbol{\rho}) - \overset{t}{\mathbf{u}}(\boldsymbol{\eta})| \leqslant K(\Omega)|\boldsymbol{\rho} - \boldsymbol{\eta}|, \tag{2.4}$$

$$|\mathbf{x} \pm \kappa \mathbf{n}(\mathbf{x}) - \mathbf{y}| \geqslant (\varepsilon(\Omega)|\mathbf{x} - \mathbf{y}|) \wedge \kappa, \tag{2.5}$$

$$\mathbf{x} + \kappa \mathbf{n}(\mathbf{x}) \in \Omega, \quad \mathbf{x} - \kappa \mathbf{n}(\mathbf{x}) \in \mathbb{R}^3 \backslash \bar{\Omega}, \tag{2.6}$$

$(\boldsymbol{\rho}, \boldsymbol{\eta} \in \Delta,\ \mathbf{x}, \mathbf{y} \in \partial\Omega,\ \kappa \in]0, \varepsilon(\Omega)],\ 1 \leqslant t \leqslant k(\Omega))$.

The norms of the spaces $L^p(\partial\Omega)$, $W^{s,p}(\partial\Omega)$ ($p \in]1, \infty[$, $s \in]0, 2[$) are to be defined with respect to the local parameters

$$\overset{t}{\mathbf{u}},\ 1 \leqslant t \leqslant k(\Omega);$$

compare [5, section 6.3.2; 4, p. 73].

Take $\tilde{\varepsilon}(\Omega) \in]0, \varepsilon(\Omega)]$, and define a function $\mathbf{T}_t$ from $\Delta \times]-\tilde{\varepsilon}(\Omega), \tilde{\varepsilon}(\Omega)[$ into S_t by setting

$$\mathbf{T}_t(\boldsymbol{\rho}, \kappa) := \overset{t}{\mathbf{u}}(\boldsymbol{\rho}) + \kappa \mathbf{n} \circ \overset{t}{\mathbf{u}}(\boldsymbol{\rho}),$$
$$(\boldsymbol{\rho} \in \Delta, \quad \kappa \in]-\tilde{\varepsilon}(\Omega), \tilde{\varepsilon}(\Omega)[, \quad 1 \leqslant t \leqslant k(\Omega)).$$

Choosing $\tilde{\varepsilon}(\Omega)$ small enough, and defining S_t in an appropriate way, we may achieve that $\mathbf{T}_t$ is bijective $(1 \leqslant t \leqslant k(\Omega))$, and that any $\mathbf{y} \in \mathbb{R}^3$ with $\operatorname{dist}(y, \partial\Omega) \leqslant \tilde{\varepsilon}(\Omega)$ is contained in one of the sets $S_1, \ldots, S_{k(\Omega)}$.

3. Proof of Lemmas 1.1 and 1.2

We begin with some definitions:

$$g_1(r) := -\sum_{l=0}^{\infty} (-r)^l (l+2)^2/(l+3)!,$$

$$g_2(r) := \sum_{l=0}^{\infty} (-r)^l [1-(l+2)^2]/(l+4)!, \quad (r \in \mathbb{C});$$

$$E_{jk}(\mathbf{z}) := (8\pi)^{-1}(\delta_{jk}|\mathbf{z}|^{-1} + z_j z_k |\mathbf{z}|^{-3}),$$

$$\mathring{E}^{\lambda}_{jk}(\mathbf{z}) := (4\pi)^{-1}[\delta_{jk}\sqrt{\lambda}\, g_1(\sqrt{\lambda}|\mathbf{z}|) + \lambda |\mathbf{z}|^{-1} z_j z_k\, g_2(\sqrt{\lambda}|\mathbf{z}|)],$$

$$(\lambda \in \mathbb{C}\setminus\{0\},\ 1 \leqslant j, k \leqslant 3,\ z \in \mathbb{R}^3 \setminus \{0\}).$$

Note that

$$\tilde{E}^{\lambda}_{jk} = E_{jk} + \mathring{E}^{\lambda}_{jk}, \quad (1 \leqslant j, k \leqslant 3,\ \lambda \in \mathbb{C}\setminus\{0\}). \tag{3.1}$$

This decomposition of $\tilde{E}^{\lambda}_{jk}$ is due to Varnhorn [16]. Take $\theta \in [0, \pi[$. Then, after some simple but tedious computations, we obtain a constant $\mathscr{D}_1(\theta) > 0$ with the ensuing properties:

$$|D^{\mathbf{a}} \tilde{E}^{\lambda}_{jk}(\mathbf{z})| \leqslant \mathscr{D}_1(\theta) |\lambda|^{-\gamma} |\mathbf{z}|^{-1-2\gamma-|\mathbf{a}|_*}, \tag{3.2}$$

$$|D^{\mathbf{b}} \mathring{E}^{\lambda}_{jk}(\mathbf{z})| \leqslant \mathscr{D}_1(\theta) |\lambda|^{\gamma} |\mathbf{z}|^{-1-|\mathbf{b}|_*+2\gamma}, \tag{3.3}$$

$$|\mathring{E}^{\lambda}_{jk}(\mathbf{z})| \leqslant \mathscr{D}_1(\theta) |\lambda|^{\gamma/2} |\mathbf{z}|^{-1+\gamma}, \tag{3.4}$$

$(1 \leqslant j,\ k \leqslant 3,\ \mathbf{z} \in \mathbb{R}^3 \setminus \{0\},\ \gamma \in [0, 1],\ \theta \in [0, \pi[,\ \lambda \in \mathbb{C}\setminus\{0\}$ with $|\arg(\lambda)| \leqslant \theta$, $\mathbf{a}, \mathbf{b} \in \mathbb{N}_0^3$ with $|\mathbf{a}|_* \leqslant 3$, $1 \leqslant |\mathbf{b}|_* \leqslant 3$).

Now fix $\lambda \in \mathbb{C} \setminus]-\infty, 0]$. Let

$$\tilde{\mathscr{T}}^{\lambda,*}_l(\boldsymbol{\phi})(\mathbf{x}) := (-2) \int_{\partial\Omega} \sum_{1 \leqslant j,k \leqslant 3} \tilde{S}^{\lambda}_{jkl}(\mathbf{x}-\mathbf{y}) n_k(\mathbf{x}) \phi_j(\mathbf{y})\, d\Omega(\mathbf{y}),$$

$$(\boldsymbol{\phi} \in L^p(\partial\Omega)^3 \text{ for some } p \in]1, \infty[,\ \mathbf{x} \in \partial\Omega,\ 1 \leqslant l \leqslant 3). \tag{3.5}$$

Define $\mathscr{T}_l(\boldsymbol{\phi})(\mathbf{x})$ and $\mathring{\mathscr{T}}^{\lambda}_l(\boldsymbol{\phi})(\mathbf{x})$ as the right-hand side of (1.8), with $\tilde{S}^{\lambda}_{jkl}$ replaced by $D_j E_{kl} + D_k E_{jl} - \delta_{jk} E_{4l}$, $D_j \mathring{E}^{\lambda}_{kl} + D_k \mathring{E}^{\lambda}_{jl}$, respectively $(1 \leqslant j, k, l \leqslant 3)$. Let $\mathscr{T}^*(\boldsymbol{\phi})(\mathbf{x})$, $\mathring{\mathscr{T}}^{\lambda,*}(\boldsymbol{\phi})(\mathbf{x})$ be given by modifying the right-hand side of (3.5) in the same way ($\boldsymbol{\phi}$, l, $\mathbf{x}$ as in (3.5)). By [4, Lemmas 4.7, 5.2] the operators $\mathscr{T}$, $\mathscr{T}^*$ are well defined. Setting $\lambda = 1$ in (3.3), we see that the kernel $D_m \mathring{E}^{\lambda}_{nl}$ $(1 \leqslant m, n, l \leqslant 3)$ has a singularity of order zero. This implies that the definitions of $\mathring{\mathscr{T}}^{\lambda}$, $\mathring{\mathscr{T}}^{\lambda,*}$ make sense. Thus, because of (3.1), the operators $\tilde{\mathscr{T}}^{\lambda}$, $\tilde{\mathscr{T}}^{\lambda,*}$ are well defined, and $\tilde{\mathscr{T}}^{\lambda} = \mathscr{T} + \mathring{\mathscr{T}}^{\lambda}$, with an analogous equation for $\tilde{\mathscr{T}}^{\lambda,*}$. Due to the weak singularity of $D_m \mathring{E}^{\lambda}_{nl}$ $(1 \leqslant m, n, l \leqslant 3)$, the preceding equations imply that the operators $\tilde{\mathscr{T}}^{\lambda}$, $\tilde{\mathscr{T}}^{\lambda,*}$ behave as well as $\mathscr{T}$, $\mathscr{T}^*$, respectively. But the mappings $\mathscr{T}$, $\mathscr{T}^*$ were already studied in [4, sections 4, 5]. Therefore, when stating some assertion about $\tilde{\mathscr{T}}^{\lambda}$ or $\tilde{\mathscr{T}}^{\lambda,*}$, we shall refer to a corresponding result in [4], related to $\mathscr{T}$ or $\mathscr{T}^*$, and shall not bother to deal with $\mathring{\mathscr{T}}^{\lambda}$, $\mathring{\mathscr{T}}^{\lambda,*}$. The only exception will occur in section 6, where it will be somewhat more difficult to estimate the expression $\| \mathring{\mathscr{T}}^{\lambda}(\boldsymbol{\phi}) \|_{2-1/p,p}$.

By [4, Lemma 5.4] $\tilde{\mathscr{T}}^{\lambda}(\boldsymbol{\phi})$ is a C^{α} function if $\boldsymbol{\phi} \in C^{\alpha}(\partial\Omega)^3$ $(\alpha \in [0, 1[)$. Thus, the last remark in Lemma 1.1 holds true. Now fix $p \in]1, \infty[$. Then for $\boldsymbol{\phi} \in L^p(\partial\Omega)^3$, the functions $\tilde{\mathscr{T}}^{\lambda}(\boldsymbol{\phi})$, $\tilde{\mathscr{T}}^{\lambda,*}(\boldsymbol{\phi})$ belong to $L^p(\partial\Omega)^3$ (see [4, Lemmas 4.7, 5.2]). Thus the mappings $\tilde{\mathscr{T}}^{\lambda}$, $\tilde{\mathscr{T}}^{\lambda,*}$ induce operators T^{λ}_p, $T^{\lambda,*}_p$ from $L^p(\partial\Omega)^3$ into $L^p(\partial\Omega)^3$. These operators are linear and compact, with $T^{\lambda,*}_p$ the dual operator of T^{λ}_p ([4, Satz 5.1]). Thus Lemma 1.1 follows from [18, p. 283, Theorem 1], provided we can show that the operator $\mathrm{id}(L^p(\partial\Omega)^3) + T^{\lambda,*}_p$ is one to one. In order to prove the last assertion, take $\boldsymbol{\phi} \in L^p(\partial\Omega)^3$ with $\boldsymbol{\phi} + \mathrm{T}^{\lambda,*}_p(\boldsymbol{\phi}) = 0$. Then $\boldsymbol{\phi}$ is continuous. Define for $\kappa \in [-\varepsilon(\Omega), \varepsilon(\Omega)]$, $1 \leqslant j \leqslant 3$:

$$V^{\kappa}_j(\mathbf{x}) := \sum_{1 \leqslant k \leqslant 3} \int_{\partial\Omega} \tilde{E}^{\lambda}_{jk}(\mathbf{x} - \mathbf{y} + \kappa \mathbf{n}(\mathbf{y})) \phi_k(\mathbf{y}) \, d\Omega(\mathbf{y}), \quad (\mathbf{x} \in \mathbb{R}^3),$$

$$Q^{\kappa}(\mathbf{x}) := \sum_{1 \leqslant k \leqslant 3} \int_{\partial\Omega} E_{4k}(\mathbf{x} - \mathbf{y} + \kappa \mathbf{n}(\mathbf{y})) \phi_k(\mathbf{y}) \, d\Omega(\mathbf{y}),$$

$(\mathbf{x} \in \mathbb{R}^3 \backslash \Omega, \ \mathbb{R}^3 \backslash \partial\Omega, \ \bar{\Omega}$ if $\kappa > 0$, $\kappa = 0$, $\kappa < 0$ respectively).

For $\kappa \in]-\varepsilon(\Omega), 0[$, we see from (2.6) that $\mathbf{V}^{\kappa}$, Q^{κ} are C^{∞} functions on $\bar{\Omega}$. In the case $\kappa \in]0, \varepsilon(\Omega)[$, the functions $\mathbf{V}^{\kappa}$, Q^{κ} are smooth on $\mathbb{R}^3 \backslash \Omega$. From (3.2) and [4, Lemma 3.1] we obtain the following equation, for $U \in \{\bar{\Omega}, \mathbb{R}^3 \backslash \Omega\}$, for $\kappa \in]-\varepsilon(\Omega), 0]$ in the case $U = \bar{\Omega}$, and for $\kappa \in]0, \varepsilon(\Omega)]$ otherwise:

$$\lambda \sum_{1 \leqslant j \leqslant 3} \int_U |V^{\kappa}_j|^2 \, d\mathbf{x} + \frac{1}{2} \sum_{1 \leqslant j,k \leqslant 3} \int_U |D_j V^{\kappa}_k + D_k V^{\kappa}_j|^2 \, d\mathbf{x}$$

$$= \sum_{1 \leqslant j,k \leqslant 3} \int_{\partial\Omega} (D_k V^{\kappa}_j + D_j V^{\kappa}_k - \delta_{jk} Q^{\kappa}) \overline{V^{\kappa}_j} n_k \, d\Omega(\mathbf{x}). \tag{3.6}$$

However, the right-hand side in (3.6) tends to

$$\sum_{1 \leqslant j \leqslant 3} \int_{\partial\Omega} \frac{1}{2} (\boldsymbol{\phi} + \tilde{\mathscr{T}}^{\lambda,*}(\boldsymbol{\phi}))_j V^0_j \, d\mathbf{x} \tag{3.7}$$

for $\kappa \to 0$, $\kappa < 0$ (see [4, pp. 170–172]), and thus to zero. Using (3.6), we may conclude: $\mathbf{V}^0 | \Omega = 0$. This means that $\mathbf{V}^0 | \partial\Omega = 0$, since $\mathbf{V}^0$ is continuous on $\mathbb{R}^3$ ([4, Lemma 6.1]). Furthermore, the right-hand side of (3.6) converges for $\kappa \to 0$, $\kappa > 0$, and the corresponding limit is given by the expression in (3.7), with $\boldsymbol{\phi} + \tilde{\mathscr{T}}^{\lambda,*}(\boldsymbol{\phi})$ replaced by $-\boldsymbol{\phi} + \tilde{\mathscr{T}}^{\lambda,*}(\boldsymbol{\phi})$. This modified form of (3.7) is also vanishing since $\mathbf{V}^0 | \partial\Omega = 0$. Now we obtain from (3.6): $\mathbf{V}^0 | \mathbb{R}^3 \backslash \bar{\Omega} = 0$. Equation (1.7) then implies: $Q^0 | \mathbb{R}^3 \backslash \bar{\Omega} = 0$. It follows as in [4, Lemma 4.8]: $-\boldsymbol{\phi} + \tilde{\mathscr{T}}^{\lambda,*}(\boldsymbol{\phi}) = 0$. Recalling the assumptions on $\boldsymbol{\phi}$, we obtain $\boldsymbol{\phi} = 0$; and Lemma 1.1 is proved.

Next, by referring to [4, Satz 4.1] we see that for any $\psi \in C^0(\partial\Omega)^3$, the functions $\mathbf{W}(.,\psi)|\Omega$, $\mathbf{W}(.,\psi)|\mathbb{R}^3\backslash\bar{\Omega}$ may be continuously extended to $\bar{\Omega}$, and $\mathbb{R}^3\backslash\Omega$, respectively. Denoting the corresponding extensions by $\mathbf{W}(.,\psi)_{\text{in}}$, $\mathbf{W}(.,\psi)_{\text{ex}}$, we have in addition:

$$\mathbf{W}(.,\psi)_{\text{in}}|\partial\Omega = -\tfrac{1}{2}(-\psi + \tilde{\mathscr{T}}^{\lambda}(\psi)), \tag{3.8}$$

$$\mathbf{W}(.,\psi)_{\text{ex}}|\partial\Omega = -\tfrac{1}{2}(\psi + \tilde{\mathscr{T}}^{\lambda}(\psi)), \quad (\psi \in C^0(\partial\Omega)^3). \tag{3.9}$$

Lemma 1.2 is now implied by (3.9).

4. Proof of Lemma 1.3

First we remark that inequality (1.11) is established in [8, p. 107/8] via Fourier transforms and the multiplier theorem from [15, p. 96]. A similar proof is given in [6, p. 303/4]. However, (1.11) could be shown just as well by applying the Calderon–Zygmund theorem and the simple form of Young's inequality given in [1, 4.30]. As for inequality (1.12), it turns out that it cannot be obtained from (1.11) by standard trace estimates. Thus, it must be proved directly.

To this end, take $\theta \in [0, \pi[$, $p \in]1, \infty[$. We shall keep these parameters fixed for the rest of this paper. Constants that only depend on θ, p or Ω will be denoted by $\mathscr{D}$. Furthermore, we fix $\lambda \in \mathbb{C}\backslash\{0\}$ with $|\arg(\lambda)| \leqslant \theta$. For brevity we shall write $q := (1 - 1/p)^{-1}$.

Take $\mathbf{f} \in C_0^\infty(\mathbb{R}^3)^3$, $1 \leqslant t \leqslant k(\Omega)$, $1 \leqslant j, k \leqslant 3$. Then inequality (1.12) is proved if we can show that

$$J := \left(\int_\Delta \left| \int_{\mathbb{R}^3} \tilde{E}^\lambda_{jk}(\overset{t}{\mathbf{u}}(\rho) - \mathbf{y}) f(\mathbf{y})\, d\mathbf{y} \right|^p d\rho \right)^{1/p} \leqslant \mathscr{D} |\lambda|^{-1+1/2p} \|\mathbf{f}\|_p. \tag{4.1}$$

To this end, choose some real number κ such that κ is greater than $1 - 1/p$, and less than the minimum of $3(1 - 1/p)$ and $1 - 2/3p$. Then we have by Hölder's inequality:

$$J \leqslant \left[\int_\Delta \left(\int_{\mathbb{R}^3} |\tilde{E}^\lambda_{jk}(\overset{t}{u}(\rho) - \mathbf{y})|^{\kappa q}\, d\mathbf{y} \right)^{p/q} \right.$$
$$\left. \times \int_{\mathbb{R}^3} \tilde{E}^\lambda_{jk}(\overset{t}{\mathbf{u}}(\rho) - \mathbf{y})|^{(1-\kappa)p} |f_k(\mathbf{y})|^p\, d\mathbf{y}\, d\rho \right]^{1/p}. \tag{4.2}$$

Take $\mathbf{y} \in \mathbb{R}^3$. After breaking up the domain of integration Δ into the parts

$$D(\mathbf{y}) := \{\rho \in \Delta : |\mathbf{p}_t(\mathbf{y}) - \rho| \leqslant |\lambda|^{-1/2}\}, \text{ and } \Delta\backslash D(\mathbf{y}),$$

we obtain from (2.2) and (3.2):

$$\int_\Delta |\tilde{E}^\lambda_{jk}(\overset{t}{\mathbf{u}}(\rho) - \mathbf{y})|^{(1-\kappa)p}\, d\rho \leqslant \mathscr{D} |\lambda|^{(1-\kappa)p/2 - 1}. \tag{4.3}$$

Next take $\rho \in \Delta$. Then we have

$$\int_{\mathbb{R}^3} |\tilde{E}^\lambda_{jk}(\overset{t}{\mathbf{u}}(\rho) - \mathbf{y})|^{\kappa q}\, d\mathbf{y} \leqslant \mathscr{D} |\lambda|^{\kappa q/2 - 3/2}, \tag{4.4}$$

as may be seen by splitting the domain of integration $\mathbb{R}^3$ into the set containing any $\mathbf{y} \in \mathbb{R}^3$ with

$$|\mathbf{y} - \overset{t}{\mathbf{u}}(\boldsymbol{\rho})| \leqslant |\lambda|^{-1/2},$$

and its complement, and by applying (2.2, 3.2). Inequality (4.1) now follows from (4.2–4.4).

5. Estimate of the double-layer potential: inequality (1.14)

Let $\mathbf{w}$, $\boldsymbol{\phi}$ be given as in Lemma 1.4. This means in particular that equation (1.13) holds. Define for $\kappa \in [-\varepsilon(\Omega), \varepsilon(\Omega)]$, $1 \leqslant l \leqslant 3$:

$$W_l^\kappa(\mathbf{x}) := \int_{\partial\Omega} \sum_{1 \leqslant j,k \leqslant 3} \tilde{S}_{jkl}^\lambda(\mathbf{x} - \mathbf{y} + \kappa \mathbf{n}(\mathbf{y}))\phi_j(\mathbf{y}) n_k(\mathbf{y})\, d\Omega(\mathbf{y}), \tag{5.1}$$

with $\mathbf{x} \in \bar{\Omega}$, $\mathbb{R}^3 \backslash \partial\Omega$, $\mathbb{R}^3 \backslash \Omega$, if $\kappa < 0$, $\kappa = 0$, $\kappa > 0$, respectively. In particular, we have $\mathbf{W}^0 = \mathbf{W}^\lambda(\cdot, \boldsymbol{\phi})$. In the following we shall prefer the shorter notation $\mathbf{W}^0$. For $\mathbf{x}$, κ and l as in (5.1), let the expressions $\mathbf{J}_l^\kappa(\mathbf{x})$ and $\mathbf{F}_l^\kappa(\mathbf{x})$ be given by the right-hand side of (5.1), with $\tilde{S}_{jkl}^\lambda$ replaced by $D_j \tilde{E}_{kl}^\lambda + D_k \tilde{E}_{jl}^\lambda$ and $-\delta_{jk} E_{4l}$, respectively ($1 \leqslant j, k \leqslant 3$). Note that the functions $\mathbf{W}^{-\varepsilon}$, $\mathbf{J}^{-\varepsilon}$ and $\mathbf{F}^{-\varepsilon}$ belong to $C^\infty(\bar{\Omega})^3$, and $\mathbf{W}^\varepsilon$, $\mathbf{J}^\varepsilon$ and $\mathbf{F}^\varepsilon$ to $C^\infty(\mathbb{R}^3 \backslash \Omega)^3$, for $\varepsilon \in]0, \varepsilon(\Omega)]$ (see (2.6)), whereas $\mathbf{W}^0$, $\mathbf{J}^0$ and $\mathbf{F}^0$ are C^∞ functions on $\mathbb{R}^3 \backslash \partial\Omega$. For any $\kappa \in [-\varepsilon(\Omega), \varepsilon(\Omega)]$, $\mathbf{W}^\kappa$ is equal to the sum $\mathbf{J}^\kappa + \mathbf{F}^\kappa$. Let us show that

$$\|\mathbf{J}^0\|_p \leqslant \mathscr{D}|\lambda|^{-1/2p}\|\boldsymbol{\phi}\|_p. \tag{5.2}$$

To this end, choose $R > 0$ with $\bar{\Omega} \subset B_R$, and with $|\mathbf{x} - \mathbf{y}| \geqslant \frac{1}{2}|\mathbf{x}|$ for $\mathbf{x} \in \mathbb{R}^3 \backslash B_R$, $\mathbf{y} \in \bar{\Omega}$. Then, applying inequality (3.2) with $\gamma = 1/2p$, we obtain an upper bound $\mathscr{D}|\lambda|^{-1/2p}\|\boldsymbol{\phi}\|_p$ for $\|\mathbf{J}^0 | \mathbb{R}^3 \backslash B_R\|_p$. The remark at the end of section 2 shows that an analogous estimate holds for $\|\mathbf{J}^0 | B_R \backslash S\|_p$, where S denotes the union of the sets S_t ($1 \leqslant t \leqslant k(\Omega)$). This leaves us to evaluate $\|\mathbf{J}^0 | S_t\|_p$, for $1 \leqslant t \leqslant k(\Omega)$. Since the function $\mathbf{T}_t$ (see section 2) is a diffeomorphism, we may instead estimate the term $\|\mathbf{J}^0 \circ \mathbf{T}_t\|_p$ ($1 \leqslant t \leqslant k(\Omega)$). To this end, we fix parameters $1 \leqslant s, t \leqslant k(\Omega)$, $1 \leqslant j, k, m, n \leqslant 3$, and abbreviate

$$K(\boldsymbol{\rho}, \varepsilon, \boldsymbol{\eta}) := |D_n \tilde{E}_{mk}^\lambda(\overset{t}{\mathbf{u}}(\boldsymbol{\rho}) + \varepsilon \mathbf{n} \circ \overset{t}{\mathbf{u}}(\boldsymbol{\rho}) - \overset{s}{\mathbf{u}}(\boldsymbol{\eta}))|, \quad (\boldsymbol{\rho}, \boldsymbol{\eta} \in \Delta,\ \varepsilon \in] -\tilde{\varepsilon}(\Omega), \tilde{\varepsilon}(\Omega)[).$$

If we can show that

$$J := \left[\int_{]-\tilde{\varepsilon}(\Omega), \tilde{\varepsilon}(\Omega)[} \int_\Delta \left(\int_\Delta K(\boldsymbol{\rho}, \varepsilon, \boldsymbol{\eta}) |\phi_j \circ \overset{s}{\mathbf{u}}(\boldsymbol{\eta})|\, d\boldsymbol{\eta}\right)^p d\boldsymbol{\rho}\, d\varepsilon\right]^{1/p} \leqslant \mathscr{D}|\lambda|^{-1/2p}\|\boldsymbol{\phi}\|_p, \tag{5.3}$$

then the term $\|\mathbf{J}^0 \circ \mathbf{T}_t\|_p$ is bounded by an expression as on the right-hand side of (5.3). In order to establish (5.3), we split the domain of integration $] -\varepsilon(\Omega), \varepsilon(\Omega)[$ into the part

$$I_1 :=] -\varepsilon(\Omega), \varepsilon(\Omega)[\cap]0, |\lambda|^{-1/2}[,$$

and its complement I_2.

Hölder's inequality yields an upper bound $J_1 + J_2$ of J, with

$$J_j := \left[\int_{I_j} \int_\Delta \left(\int_\Delta K(\boldsymbol{\rho}, \varepsilon, \boldsymbol{\eta})\, d\boldsymbol{\eta} \right)^{p-1} \times \int_\Delta K(\boldsymbol{\rho}, \varepsilon, \boldsymbol{\eta}) |\phi_j \circ \overset{s}{u}(\boldsymbol{\eta})|^p \, d\boldsymbol{\eta}\, d\boldsymbol{\rho}\, d\varepsilon \right]^{1/p}, \quad (j \in \{1, 2\}). \tag{5.4}$$

Using techniques as in the proof of (4.3), we obtain for $\boldsymbol{\rho}, \boldsymbol{\eta} \in \Delta$, $\varepsilon \in] - \tilde{\varepsilon}(\Omega), \tilde{\varepsilon}(\Omega)[$, $\mu \in \{1, 3\}$:

$$\int_\Delta K(\boldsymbol{\rho}, \varepsilon, \tilde{\boldsymbol{\eta}})\, d\tilde{\boldsymbol{\eta}}, \int_\Delta K(\tilde{\boldsymbol{\rho}}, \varepsilon, \boldsymbol{\eta})\, d\tilde{\boldsymbol{\rho}} \leqslant \mathscr{D} \varepsilon^{-\mu/2p} |\lambda|^{-\mu/4p}. \tag{5.5}$$

Now insert (5.5) into (5.4), with $\mu = 1$, $\mu = 3$ in the cases $j = 1$, $j = 2$, respectively. After integrating in ε, we see that J_1, J_2 are bounded by an expression as on the right-hand side of (5.2). This proves (5.3), and by extension (5.2).

Next we remark on some technical facts concerning the potentials $\mathbf{W}^\kappa$, $\mathbf{J}^\kappa$ and $\mathbf{F}^\kappa$ ($\kappa \in [- \varepsilon(\Omega), \varepsilon(\Omega)]$). As we observed at the end of section 3, the function $\mathbf{W}^0 | \Omega$ may be continuously extended to $\bar{\Omega}$. Thus it belongs to $L_r(\Omega)^3$, for $r \in]1, \infty[$. We further note the following convergence result, which is a special case of [4, Lemma 4.8]:

$$\langle \mathbf{n}(\mathbf{x}), \mathbf{F}^0(\mathbf{x} + \varepsilon \mathbf{n}(\mathbf{x})) \rangle \to S(\mathbf{x}) \tag{5.6}$$

for $\varepsilon \to 0$, $\varepsilon > 0$, uniformly in $\mathbf{x} \in \partial\Omega$, with

$$S(\mathbf{x}) := -\tfrac{1}{2} \langle \mathbf{n}(\mathbf{x}), \boldsymbol{\phi}(\mathbf{x}) \rangle - (4\pi)^{-1} \int_{\partial\Omega} \langle \mathbf{n}(\mathbf{x}), \mathbf{x} - \mathbf{y} \rangle |\mathbf{x} - \mathbf{y}|^{-3} \langle \mathbf{n}(\mathbf{y}), \boldsymbol{\phi}(\mathbf{y}) \rangle \, d\Omega(\mathbf{y}), \quad (\mathbf{x} \in \partial\Omega). \tag{5.7}$$

Note that due to (2.3), the integral in (5.7) exists. By (1.13), (5.6) and (3.9) we have

$$\langle \mathbf{n}(\mathbf{x}), \mathbf{J}^0(\mathbf{x} + \varepsilon \mathbf{n}(\mathbf{x})) \rangle \to \langle \mathbf{n}(\mathbf{x}), \mathbf{w}(\mathbf{x}) \rangle - S(\mathbf{x}), \quad (\varepsilon \to 0, \varepsilon > 0), \tag{5.8}$$

uniformly in $\mathbf{x} \in \partial\Omega$. Furthermore, by slightly modifying the proof of [4, Lemma 6.4], we obtain the ensuing convergence results, which also hold uniformly in $\mathbf{x} \in \partial\Omega$:

$$\begin{aligned} &\mathbf{W}^0(\mathbf{x} \pm \varepsilon \mathbf{n}(\mathbf{x})) - \mathbf{W}^{\pm\varepsilon}(\mathbf{x}), \\ &\mathbf{J}^0(\mathbf{x} \pm \varepsilon \mathbf{n}(\mathbf{x})) - \mathbf{J}^{\pm\varepsilon}(\mathbf{x}), \\ &\mathbf{F}^0(\mathbf{x} \pm \varepsilon \mathbf{n}(\mathbf{x})) - \mathbf{F}^{\pm\varepsilon}(\mathbf{x}) \to 0 \text{ for } \varepsilon \to 0, \varepsilon > 0. \end{aligned} \tag{5.9}$$

By using the transformation $\mathbf{T}_t$ $(1 \leqslant t \leqslant k(\Omega))$ in a similar way as in the proof of (5.2), we may show the relations

$$\mathbf{F}^\varepsilon | \mathbb{R}^3 \backslash \bar{\Omega} \in L^r(\mathbb{R}^3 \backslash \bar{\Omega})^3 \quad (\varepsilon \in [0, \varepsilon(\Omega)[, r \in]\tfrac{3}{2}, \infty[),$$

and

$$\begin{aligned} &\| \mathbf{F}^\varepsilon - \mathbf{F}^0 | \mathbb{R}^3 \backslash \bar{\Omega} \|_2, \| \mathbf{J}^\varepsilon - \mathbf{J}^0 | \mathbb{R}^3 \backslash \bar{\Omega} \|_2, \\ &\| \mathbf{W}^{-\varepsilon} - \mathbf{W}^0 | \Omega \|_2 \to 0, \quad (\varepsilon \to 0, \varepsilon > 0). \end{aligned} \tag{5.10}$$

Now we intend to show that $\mathbf{F}^0 | \mathbb{R}^3 \backslash \bar{\Omega}$ belongs to $L^p(\mathbb{R}^3 \backslash \bar{\Omega})^3$, a fact that is not evident in the case $p \leqslant \frac{3}{2}$. We first conclude from (1.13), for $x \in \mathbb{R}^3 \backslash \bar{\Omega}$, $1 \leqslant j \leqslant 3$:

$$-8\pi F_j^0(\mathbf{x}) = \int_{\partial\Omega} (x - y)_j |\mathbf{x} - \mathbf{y}|^{-3} \cdot \langle \mathbf{n}(\mathbf{y}), \boldsymbol{\phi}(\mathbf{y}) - \mathscr{T}^\lambda(\boldsymbol{\phi})(\mathbf{y}) - 2\mathbf{w}(\mathbf{y}) \rangle \, d\Omega(\mathbf{y}).$$

By combining this result with (5.9, with minus signs), (3.8), Gauss' theorem, and with the relations $\mathbf{W}^{-\varepsilon} \in C^1(\bar{\Omega})^3$, $\operatorname{div} \mathbf{W}^{-\varepsilon} = 0$, $(\varepsilon \in (0, \varepsilon(\Omega)])$, we are led to the ensuing equation:

$$- 8\pi F_j^0(\mathbf{x}) = \lim_{\varepsilon \to 0,\, \varepsilon > 0} \int_\Omega \sum_{1 \leqslant k \leqslant 3} \frac{\partial}{\partial y_k} [(x-y)_j |\mathbf{x} - \mathbf{y}|^{-3}]$$
$$\times [2W_k^{-\varepsilon}(\mathbf{y}) - 2w_k(\mathbf{y})]\, d\Omega(\mathbf{y}), \quad (\mathbf{x} \in \mathbb{R}^3 \backslash \bar{\Omega},\ 1 \leqslant j \leqslant 3).$$

This equation, along with the Calderon–Zygmund theorem and the last relation in (5.10), yield the property of $\mathbf{F}^0 | \mathbb{R}^3 \backslash \bar{\Omega}$, which we sought.

As our next aim, we want to establish the inequality

$$\| \mathbf{F}^0 | \mathbb{R}^3 \backslash \bar{\Omega} \|_p \leqslant \mathscr{D}(\| \mathbf{J}^0 \|_p + \| \mathbf{w} \|_p). \tag{5.11}$$

To this end, take a function $\varphi \in C^1(\mathbb{R}^3 \backslash \bar{\Omega})$ with compact support. Combining (5.10, 5.6, 5.9 (with plus signs), 5.8), Gauss' theorem, and the relations $\mathbf{J}^\varepsilon$, $\mathbf{F}^\varepsilon \in C^1(\mathbb{R}^3 \backslash \bar{\Omega})^3$, $\operatorname{div} \mathbf{F}^\varepsilon = 0$ $(\varepsilon \in]0, \varepsilon(\Omega)])$, we arrive at the following equation:

$$\int_{\mathbb{R}^3 \backslash \bar{\Omega}} \langle \mathbf{F}^0, \nabla \varphi \rangle \, d\mathbf{x} = \lim_{\varepsilon \to 0,\, \varepsilon > 0} \int_{\mathbb{R}^3 \backslash \bar{\Omega}} \langle \mathbf{F}^\varepsilon, \nabla \varphi \rangle \, d\mathbf{x}$$
$$= \lim_{\varepsilon \to 0,\, \varepsilon > 0} \int_{\partial \Omega} \langle \mathbf{J}^\varepsilon - \mathbf{w}, \varphi \mathbf{n} \rangle \, d\Omega = \int_{\mathbb{R}^3 \backslash \bar{\Omega}} \langle \mathbf{w} - \mathbf{J}^0, \nabla \varphi \rangle \, d\mathbf{x}. \tag{5.12}$$

Those functions in $C^1(\mathbb{R}^3 \backslash \bar{\Omega})^3$ having compact support form a dense subset of $\tilde{H}^{1,q}(\mathbb{R}^3 \backslash \bar{\Omega})$ with respect to the semi-norm $\| \nabla \cdot \|_p$; see [12]. Since $\mathbf{J}^0 | \mathbb{R}^3 \backslash \bar{\Omega}$, $\mathbf{F}^0 | \mathbb{R}^3 \backslash \bar{\Omega}$ are L^p functions, as proved above, we may now conclude that (5.12) also holds for $\varphi \in \tilde{H}^{1,q}(\mathbb{R}^3 \backslash \bar{\Omega})$. Let $\mathbf{f} \in L^p(\mathbb{R}^3 \backslash \bar{\Omega})^3$, and take $g \in \tilde{H}^{1,q}(\mathbb{R}^3 \backslash \bar{\Omega})$ with $\mathbf{f} = P_p \mathbf{f} + \nabla g$ (see Theorem 1.3). Then we have by (5.12) and Theorem 1.3:

$$\left| \int_{\mathbb{R}^3 \backslash \bar{\Omega}} \langle \mathbf{F}^0, \mathbf{f} \rangle \, d\mathbf{x} \right| \leqslant \mathscr{A}_1(p, \Omega) \| \mathbf{f} \|_q (\| \mathbf{J}^0 \|_p + \| \mathbf{w} \|_p).$$

This proves (5.11). The relations in (5.2) and (5.11) yield (1.14), as well as the following inequality:

$$\| \mathbf{F}^0 | \mathbb{R}^3 \backslash \bar{\Omega} \|_p \leqslant \mathscr{D}(|\lambda|^{-1/2p} \| \mathbf{\Phi} \|_p + \| \mathbf{w} \|_p). \tag{5.13}$$

6. Estimate of the double-layer potential: inequality (1.15)

Take $\mathbf{w}$, $\mathbf{\Phi}$ as in Lemma 1.4. Observe that $\mathbf{\Phi} \circ \overset{t}{\mathbf{u}} \in C^1(\Delta)$ $(1 \leqslant t \leqslant k(\Omega))$, as may be seen from [4, Lemma 7.5]. Furthermore, $\mathbf{\Phi}$ satisfies the inequality

$$\| \mathbf{\Phi} \|_{2-1/p,p} \leqslant \mathscr{D}(|\lambda|^{1-1/2p} \| \mathbf{\Phi} \|_p + \| \mathbf{\Phi} \|_p + \| \mathbf{w} | \partial \Omega \|_{2-1/p,p}). \tag{6.1}$$

In fact, by referring to [4, Lemma 7.8], this assertion may be reduced to the estimate

$$\| \mathring{\mathscr{T}}^\lambda(\mathbf{\Phi}) \|_{2-1/p,p} \leqslant \mathscr{D} |\lambda|^{1-1/2p} \| \mathbf{\Phi} \|_p. \tag{6.2}$$

The proof of this inequality in turn amounts to evaluating the triple integral

$$\left[\int_\Delta \int_\Delta \left(\int_\Delta | D^{\mathbf{a}} \mathring{\mathbf{E}}^\lambda_{lm}(\overset{s}{\mathbf{u}}(\boldsymbol{\rho}) - \overset{t}{\mathbf{u}}(\boldsymbol{\eta})) - D^{\mathbf{a}} \mathring{\mathbf{E}}^\lambda_{lm}(\overset{s}{\mathbf{u}}(\tilde{\boldsymbol{\rho}}) - \overset{t}{\mathbf{u}}(\boldsymbol{\eta}))| \, |\mathbf{\Phi}_j \circ \overset{t}{\mathbf{u}}(\boldsymbol{\eta})| \, d\boldsymbol{\eta} \right)^p \right.$$
$$\left. \times |\boldsymbol{\rho} - \tilde{\boldsymbol{\rho}}|^{-1-p} d\tilde{\boldsymbol{\rho}} \, d\boldsymbol{\rho} \right]^{1/p},$$

where $\mathbf{a} \in \mathbb{N}_0^3$ with $|\mathbf{a}|_* = 2$, $1 \leqslant l$, $m \leqslant 3$, $1 \leqslant s$, $t \leqslant k(\Omega)$. The preceding term is bounded by an expression as on the right-hand side of (6.2). This may be shown by applying (3.3) and Hölder's inequality, using a similar technique as in the proof of (1.12). The details are somewhat tricky, but we cannot enter into them here. Note that for the parameter p, we only assumed $p \in]1, \infty[$. Thus inequality (6.1) holds for any $r \in]1, \infty[$, implying the relation $\boldsymbol{\Phi} \in W^{2-1/r,r}(\partial\Omega)^3$ for $r \in]1, \infty[$. Now define

$$\mathring{\Pi}^\lambda(\mathbf{x}) := -(4\pi)^{-1}\lambda \int_{\partial\Omega} |\mathbf{x} - \mathbf{y}|^{-1} \langle \mathbf{n}(\mathbf{y}), \boldsymbol{\Phi}(\mathbf{y}) \rangle \, d\Omega(\mathbf{y}), \quad (\mathbf{x} \in \mathbb{R}^3 \backslash \bar{\Omega}),$$

$\Pi := \tilde{\Pi}^\lambda(\cdot, \boldsymbol{\Phi}) - \mathring{\Pi}^\lambda$, with $\tilde{\Pi}^\lambda(\cdot, \boldsymbol{\Phi})$ introduced in section 1. Then we have $\nabla \mathring{\Pi}^\lambda = -\lambda \mathbf{F}^0$, with $\mathbf{F}^0$ introduced in section 5. Thus the term $|\lambda| \, \| \nabla \mathring{\Pi}^\lambda \|_p$ is bounded by the right-hand side in (5.13). In order to evaluate $\| \nabla \Pi \|_p$, take R as in the proof of (5.2). It is clear that $\| \nabla \Pi | \mathbb{R}^3 \backslash B_R \|_p$ is less than $\mathscr{D} \| \boldsymbol{\Phi} \|_p$. In the case $p > \frac{3}{2}$, [4, Lemma 7.15] implies that $\| \nabla \Pi | B_R \backslash \bar{\Omega} \|_p$ is bounded by $\mathscr{D} \| \boldsymbol{\Phi} \|_{2-1/p,p}$. However, the proof of [4, Lemma 7.15] also carries through in the case $p \leqslant \frac{3}{2}$, provided we do not refer to [4, Lemma 7.12], but to Theorem 2.1. Note that $\boldsymbol{\Phi} \in W^{2-1/r,r}(\partial\Omega)^3$ for $r \in]1, \infty[$, as we remarked above. Abbreviate

$$K := \| \boldsymbol{\Phi} \|_p + |\lambda|^{1-1/2p} \| \boldsymbol{\Phi} \|_p + |\lambda| \, \| w \|_p + \| w | \partial\Omega \|_{2-1/p,p}.$$

It now follows with (6.1):

$$\| \nabla \tilde{\Pi}^\lambda(\cdot, \boldsymbol{\Phi}) | \mathbb{R}^3 \backslash \bar{\Omega} \|_p \leqslant \mathscr{D} K. \tag{6.3}$$

Since the pair $(\mathbf{W}^0 | \mathbb{R}^3 \backslash \bar{\Omega}, \tilde{\Pi}^\lambda(\cdot, \boldsymbol{\Phi}) | \mathbb{R}^3 \backslash \bar{\Omega})$ solves (1.1) with $\mathbf{f} = \mathbf{0}$, we may estimate $\| \mathbf{W}^0 | B_R \backslash \bar{\Omega} \|_{2,p}$ by applying (3.9, 1.14, 6.3), and Theorem 2.1. In this way we obtain an upper bound $\mathscr{D} K$ for the preceding term. The proof of (1.15) is completed by the inequality

$$\| D_m \mathbf{W}^0 | B_R \backslash \bar{\Omega} \|_p \leqslant \mathscr{D} |\lambda|^{-1/2} K, \quad (1 \leqslant m \leqslant 3),$$

which follows by combining (1.14), the preceding estimate of $\| \mathbf{W}^0 | B_R \backslash \bar{\Omega} \|_{2,p}$, and [1, 4.14]. Note that it is easy to evaluate $\| D^{\mathbf{c}} \mathbf{W}^0 | \mathbb{R}^3 \backslash B_R \|_p$ for $\mathbf{c} \in \mathbb{N}_0^3$ with $1 \leqslant |\mathbf{c}|_* \leqslant 2$.

7. Uniqueness

Let $r \in]1, \infty[$, $(\mathbf{X}, Y) \in A^r$, with $(\mathbf{X}, Y)$ satisfying (1.1) for $\mathbf{f} = \mathbf{0}$. Then we have to show $\mathbf{X} = \mathbf{0}$, $\nabla Y = \mathbf{0}$. For this purpose, take $\boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^3)^3$. As we know by Lemmas 1.1–1.4, the pair $\mathbf{u}(\lambda, \boldsymbol{\varphi})$, $\pi(\lambda, \boldsymbol{\varphi})$ belongs to A^s, for $s \in]1, \infty[$, and solves (1.1). (The functions $\mathbf{u}(\lambda, \boldsymbol{\varphi})$, $\pi(\lambda, \boldsymbol{\varphi})$ were introduced in section 1.) Applying Theorem 1.3 and partial integration, we find:

$$\int_{\mathbb{R}^3 \backslash \bar{\Omega}} \langle \mathbf{X}, \boldsymbol{\varphi} \rangle \, d\mathbf{x} = \int_{\mathbb{R}^3 \backslash \bar{\Omega}} \langle -\Delta \mathbf{X} + \lambda \mathbf{X} + \nabla Y, \mathbf{u}(\lambda, \boldsymbol{\varphi}) \rangle \, d\mathbf{x} = 0.$$

The preceding equation implies $\mathbf{X} = \mathbf{0}$, $\nabla Y = 0$.

Note added in proof. After this paper was finished, we learned of another article treating problem (1.1), (1.2) via the method of integral equations. The paper in question is

Varnhorn, W., 'An explicit potential theory for the Stokes resolvent boundary value problems in three dimensions', Preprint, TH Darmstadt, 1989.

The author constructs solutions of the Dirichlet and Neumann problem in interior and exterior domains. In his introduction, he explains the role which problem (1.1), (1.2) plays in the numerical treatment of Navier–Stokes system. L^p estimates of solutions are not considered.

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