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The Stokes-system in exterior domains: L^p-estimates for small values of a resolvent parameter

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1. Introduction and main results

Let Ω be a bounded domain in $\mathbb{R}^3$, with C^2-boundary $\partial\Omega$. Then we consider the Stokes-system in the exterior domain $\mathbb{R}^3\backslash\bar{\Omega}$, with resolvent parameter $\lambda \in \mathbb{C}\backslash]-\infty, 0]$, viscosity $\nu \in]0, \infty[$, and with right-hand side f belonging to $L^p(\mathbb{R}^3\backslash\bar{\Omega})^3$ for some $p \in]1, \infty[$:

$$-\nu \cdot \Delta u + \lambda \cdot u + \nabla\pi = f, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3\backslash\bar{\Omega}. \tag{1.1}$$

Furthermore we prescribe Dirichlet boundary conditions:

$$u|\partial\Omega = 0. \tag{1.2}$$

This boundary-value problem is of interest because it arises in the numerical treatment of the Navier-Stokes system; see [15] for further indications in this respect. Moreover, L^p-estimates of solutions to (1.1), (1.2) are fundamental for treating the full Navier-Stokes system (nonlinear, time-dependent) by functional-analytic means. It is this aspect which motivated the present paper. As is well known (see Giga [9], Deuring [5]), problem (1.1), (1.2) may be solved in the following sense: for $\lambda \in \mathbb{C}\backslash]-\infty, 0]$, $p \in]1, \infty[$, $\nu \in]0, \infty[$, $f \in L^p(\mathbb{R}^3\backslash\bar{\Omega})^3$, the set

$$\begin{aligned} \mathrm{SOL}(\nu, \lambda, p, f) := \{&(u, \pi) \in W^{2,p}(\mathbb{R}^3\backslash\bar{\Omega})^3 \times \tilde{H}^{1,p}(\mathbb{R}^3\backslash\bar{\Omega}) : u \in W_0^{1,p}(\mathbb{R}^3\backslash\bar{\Omega})^3, \\ &-\nu \cdot \Delta u + \lambda \cdot u + \nabla\pi = f, \operatorname{div} u = 0\} \end{aligned}$$

contains a pair (u, π), with u uniquely determined, and with π unique up to an additive constant. Here we used the abbreviation

$$\tilde{H}^{1,p}(\mathbb{R}^3\backslash\bar{\Omega}) := \{g \in W_{\mathrm{loc}}^{1,p}(\mathbb{R}^3\backslash\bar{\Omega}) : \nabla g \in L^p(\mathbb{R}^3\backslash\bar{\Omega})^3\} \quad (p \in]1, \infty[).$$

In the work at hand, we intend to show the ensuing estimate:

Theorem 1.1. Let $\vartheta \in [0, \pi[$, $\nu \in]0, \infty[$, $p \in]1, \infty[$. Set $\delta = 0$ in the case $p < 3/2$, and let $\delta \in]1 - 3/(2 \cdot p), 1]$ in the case $p \geqq 3/2$. Then there are constants $K_1(p, \nu, \Omega)$, $K_2(\vartheta, p, \nu, \Omega)$, $K_3(\vartheta, p, \nu, \delta, \Omega)$ with the properties to follow:

(a) The inequality

$$\|u\|_p \leqq C_1 \cdot |\lambda|^{-1} \cdot \|f\|_p \tag{1.3}$$

holds true for $C_1 = K_2(\vartheta, p, \nu, \Omega)$, $\lambda \in \mathbb{C}\backslash\{0\}$ with $|\arg \lambda| \leqq \vartheta$, $|\lambda| \leqq K_1(p, \nu, \Omega)$, $f \in L^p(\mathbb{R}^3\backslash\bar{\Omega})^3$, $(u, \pi) \in \mathrm{SOL}(\nu, \lambda, p, f)$.

(b) For λ, f, (u, π) as in (a), and for $C_2 = K_3(\vartheta, p, \nu, \delta, \Omega)$, the ensuing inequality is satisfied:

$$\sum_{c \in \mathbb{N}_0^3,\, |c|_* \leqq 2} |\lambda|^{1-|c|_*/2} \cdot \|D^c u\|_p + \|\nabla \pi\|_p \leqq C_2 \cdot (|\lambda| \wedge 1)^{-\delta} \cdot \|f\|_p. \tag{1.4}$$

It is the main feature of Theorem 1.1 that inequalities (1.3), (1.4) are considered for small values of $|\lambda|$. Theorem 1.1 complements some well-known results dealing with L^p-estimates of solutions to (1.1), (1.2) in the case of large $|\lambda|$ (see Solonnikov [13; p. 495], Giga [9; p. 327], Deuring [5; Theorem 1.2]).

In [3], Sohr and Borchers proved for any space dimension $n \geqq 3$ that part (a) of Theorem 1.1 holds for $p > 1$, and part (b) for $1 < p < 3/2$. Their proof, a simplified version of which may be found in Giga, Sohr [10], is based on Bogovski's theory of the boundary value problem $\operatorname{div} u = 0$ in $\mathbb{R}^3\backslash\bar{\Omega}$, $u|\partial\Omega = 0$ (see [2]), on a result by Miyakawa [12] concerning the adjoint of the Stokes-operator, and on L^p-estimates given by Solonnikov in [13; Theorem 5.2]. The latter estimates concern solutions of system (1.1) in bounded domains. References [3] and [10] also give applications of Theorem 1.1 to the Stokes-operator and to the Stokes-semigroup in L^p. As an obvious but prominent example of such applications, we mention that Theorem 1.1(a) implies the Stokes-semigroup in L^p to be bounded. It is perhaps interesting to remark that according to [3; p. 425], Theorem 1.1(b) cannot hold for $\delta = 0$ if $p \geqq 3/2$.

The results of the present paper are new in two respects. First, it is shown that Theorem 1.1(b) holds without the restriction $p < 3/2$. Second, we shall prove Theorem 1.1 by a new approach which yields a direct and rather elementary access to this theorem. This approach starts with the integral representation of solutions to (1.1), (1.2) which was constructed in [5]; see Varnhorn [14] for another proof of such a representation. In [4] and [5], these integrals were estimated in L^p-norms for large values of $|\lambda|$, by a method inspired by McCracken [11], who considered the analogue of problem (1.1), (1.2) in half-space in $\mathbb{R}^3$.

In the following, we shall develop techniques suitable for estimating our integral representation for small values of $|\lambda|$. For this purpose, we shall apply some assertions from [5]–[7], which are based on standard results of analysis (Calderon-Zygmund theorem, some elements of the theory of compact operators in Banach spaces). Furthermore, we shall need a L^p-estimate for the Laplacian on bounded domains, and some assertions on the order of eigenvalues of compact operators in Banach spaces (see [8; (11.4.1)]).

The plan of the paper is as follows: In Section 2, we shall introduce most of the auxiliary functions needed later on. Although the sense of these definitions cannot become clear from Section 2, it is perhaps convenient for references to collect these definitions in one place. In Section 3, we reduce Theorem 1.1 to some technical lemmas, which will be proved in Section 4 and 5.

2. Notations. Definition of auxiliary functions

For $a \in \mathbb{N}_0^3$, the term $|a|_*$ is to denote the length $a_1 + a_2 + a_3$ of a. The symbols D^a, D_k, $D_k D_m$, for $a \in \mathbb{N}_0^3$, $1 \leqq k, m \leqq 3$, denote partial derivatives, with obvious meanings. The inner product of $x, y \in \mathbb{R}^3$ will be written as $\langle x, y \rangle$. By B_R, we denote a ball in $\mathbb{R}^3$ with centre 0 and radius $R > 0$. For any set C, the symbol χ_C means the corresponding characteristic function. Let e_1, e_2, e_3 stand for the unit vectors in $\mathbb{R}^3$. The letter n denotes the outward unit normal to Ω. For $r \in]1, \infty[$, let id(r) be defined as the identity map of $L^r(\partial\Omega)^3$. The class of functions INT is to contain any pair (u, π) with the properties to follow:

$$\pi \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^3 \backslash \bar{\Omega}), \quad u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^3 \backslash \bar{\Omega})^3 \cap L^r(\mathbb{R}^3 \backslash \bar{\Omega})^3 \quad \text{for } r \in]3, \infty[,$$

$$D_m u_k, \quad \pi \in L^r(\mathbb{R}^3 \backslash \bar{\Omega}) \quad \text{for } r \in]3/2, \infty[,$$

$$D_m D_j u_k, \ D_m \pi \in L^r(\mathbb{R}^3 \backslash \bar{\Omega}) \quad \text{for } r \in]1, \infty[\quad (1 \leqq j, k, m \leqq 3).$$

Set for $z \in \mathbb{R}^3 \backslash \{0\}$, $\quad 1 \leqq j, k \leqq 3$, $\quad \lambda \in \mathbb{C} \backslash]-\infty, 0]$:

$$E_{jk}(z) := (8 \cdot \pi \cdot |z|)^{-1} \cdot (\delta_{jk} + z_j \cdot z_k \cdot |z|^{-2}), \quad E_{4k}(z) := (4 \cdot \pi \cdot |z|^3)^{-1} \cdot z_k,$$

$$E^1_{jk}(z) := (32 \cdot \pi)^{-1} \cdot (3 \cdot \delta_{jk} \cdot |z| - z_j \cdot z_k \cdot |z|^{-1}),$$

$$\hat{\hat{E}}^\lambda_{jk}(z) := (4 \cdot \pi)^{-1} \cdot \Big(\delta_{jk} \cdot \sqrt{\lambda} \cdot \sum_{m \in \mathbb{N},\, m \geqq 2} (-\sqrt{\lambda} \cdot |z|)^m \cdot (m+2)^2 \cdot 1/(m+3)! + \lambda \cdot z_j \cdot z_k \cdot |z|^{-1} \cdot \sum_{m \in \mathbb{N}} (-\sqrt{\lambda} \cdot |z|)^m \cdot (1 - (m+2)^2) \cdot 1/(m+4)! \Big)$$

$$\hat{E}^\lambda_{jk}(z) := -(6 \cdot \pi)^{-1} \cdot \sqrt{\lambda} \cdot \delta_{jk} + \lambda \cdot E^1_{jk} + \hat{\hat{E}}^\lambda_{jk},$$

$$\tilde{E}^\lambda_{jk} := E_{jk} + \hat{E}^\lambda_{jk}.$$

For $1 \leqq j, k, m \leqq 3$, define S_{jkm}, $\tilde{S}^\lambda_{jkm}$ as the term $\delta_{jk} \cdot E_{4m} - D_j K_{km} - D_k K_{jm}$, with K_{st} equal to E_{st}, and $\tilde{E}^\lambda_{st}$, respectively $(1 \leqq s, t \leqq 3)$. Similarly, for

$1 \leqq j, k, m \leqq 3$, let S^1_{jkm}, $\hat{\hat{S}}^\lambda_{jkm}$, $\hat{S}^\lambda_{jkm}$ be given as $-D_j K_{km} - D_k K_{jm}$, with K_{st} equal to E^1_{st}, $\hat{\hat{E}}^\lambda_{st}$, $\hat{E}^\lambda_{st}$, respectively $(1 \leqq s, t \leqq 3)$.

For $\lambda \in \mathbb{C}\backslash]-\infty, 0]$, $1 \leqq m \leqq 3$, $\Lambda \in C^0(\partial\Omega)^3$, $x \in \mathbb{R}^3\backslash\partial\Omega$, we define $W_m(x, \Lambda)$, $\tilde{W}^\lambda_m(x, \Lambda)$, $F_m(x, \Lambda)$, $J^\lambda_m(x, \Lambda)$ as the integral

$$\int_{\partial\Omega} \sum_{1 \leqq j,k \leqq 3} -L_{jkm}(x-y) \cdot \Lambda_j(y) \cdot n_k(y)\, d\Omega(y), \tag{2.1}$$

with L_{jkm} equal to S_{jkm}, $\tilde{S}^\lambda_{jkm}$, $\delta_{jk} \cdot E_{4m}$, $-D_j\tilde{E}^\lambda_{km} - D_k\tilde{E}^\lambda_{jm}$, respectively $(1 \leqq j, k \leqq 3)$. For λ, m, Λ as before, and for $x \in \mathbb{R}^3$, let $\hat{W}^\lambda_m(x, \Lambda)$, $W^1_m(x, \Lambda)$ denote the integral in (2.1), with L_{jkm} replaced by $\hat{S}^\lambda_{jkm}$, S^1_{jkm}, respectively $(1 \leqq j, k \leqq 3)$. Thus we have for $\lambda \in \mathbb{C}\backslash]-\infty, 0]$, $\Lambda \in C^0(\partial\Omega)^3$:

$$\tilde{W}^\lambda(\cdot, \Lambda) = W(\cdot, \Lambda) + \hat{W}^\lambda(\cdot, \Lambda) = J^\lambda(\cdot, \Lambda) + F(\cdot, \Lambda). \tag{2.2}$$

We further set

$$\begin{aligned} &\Pi(x, \Lambda) := 2 \cdot \int_{\partial\Omega} \sum_{1 \leqq j,k \leqq 3} D_j E_{4k}(x-y) \cdot \Lambda_j(y) \cdot n_k(y)\, d\Omega(y), \\ &\hat{\Pi}^\lambda(x, \Lambda) := -\lambda \cdot (4 \cdot \pi)^{-1} \cdot \int_{\partial\Omega} |x-y|^{-1} \cdot \langle n(y), \Lambda(y) \rangle\, d\Omega(y), \\ &\tilde{\Pi}^\lambda(x, \Lambda) := \Pi(x, \Lambda) + \hat{\Pi}^\lambda(x, \Lambda) \\ &(\lambda \in \mathbb{C}\backslash]-\infty, 0], \Lambda \in C^0(\partial\Omega)^3, x \in \mathbb{R}^3\backslash\partial\Omega). \end{aligned} \tag{2.3}$$

Next, for $1 \leqq m \leqq 3$, $\Lambda \in \cup\{L^r(\partial\Omega)^3 : r \in]1, \infty[\}$, $\lambda \in \mathbb{C}\backslash]-\infty, 0]$, $x \in \partial\Omega$, we define $T_m(\Lambda)(x)$, $\tilde{T}^\lambda_m(\Lambda)(x)$, $T^1_m(\Lambda)(x)$, $\hat{\hat{T}}^\lambda_m(\Lambda)(x)$ as the integral in (2.1), with L_{jkm} equal to $-2 \cdot S_{jkm}$, $-2 \cdot \tilde{S}^\lambda_{jkm}$, $-2 \cdot S^1_{jkm}$, $-2 \cdot \hat{\hat{S}}^\lambda_{jkm}$, respectively $(1 \leqq j, k \leqq 3)$. Note that

$$\tilde{T}^\lambda = T + \lambda \cdot T^1 + \hat{\hat{T}}^\lambda \quad \text{for } \lambda \in \mathbb{C}\backslash]-\infty, 0]. \tag{2.4}$$

For m, Λ, x as in the definition of $T_m(\Lambda)(x)$, let $T^*_m(\Lambda)(x)$ be defined as the integral in (2.1), with $L_{jkm} = -2 \cdot S_{jkm}$, and with $n_k(y)$ replaced by $n_k(x)$ $(1 \leqq j, k \leqq 3, y \in \partial\Omega)$.

For $r \in]1, \infty[$, set $\mathcal{T}_r := T|L^r(\partial\Omega)^3$, $\mathcal{T}^*_r := T^*|L^r(\partial\Omega)^3$, $\tilde{\mathcal{T}}^\lambda_r := \tilde{T}^\lambda|L^r(\partial\Omega)^3$, $F(r) := \{\Lambda + T(\Lambda) : \Lambda \in L^r(\partial\Omega)^3\}$.

For $x \in \mathbb{R}^3$, $1 \leqq m \leqq 3$, set $\varphi_{(m)}(x) := (\delta_{jm})_{1 \leqq j \leqq 3}$, $\varphi_{(4)}(x) := (x_3, 0, -x_1)$, $\varphi_{(5)}(x) := (x_2, -x_1, 0)$, $\varphi_{(6)}(x) := (0, x_3, -x_2)$.

Since $\varphi_{(k)}|\partial\Omega$ is continuous $(1 \leqq k \leqq 3)$, the linear hull induced by the set $\{\varphi_{(1)}|\partial\Omega, \ldots, \varphi_{(6)}|\partial\Omega\}$ in $L^r(\partial\Omega)^3$ is independent of $r \in]1, \infty[$. We denote this linear hull by N. Finally, set

$$\begin{aligned} &\tilde{u}^\lambda_m(f) := \sum_{1 \leqq k \leqq 3} \tilde{E}^\lambda_{km} * f_k, \quad u_m(f) := \sum_{1 \leqq k \leqq 3} E_{km} * f_k, \\ &\pi(f) := \sum_{1 \leqq k \leqq 3} E_{4k} * f_k \end{aligned}$$

$(f \in C^\infty_0(\mathbb{R}^3)^3$, $\lambda \in \mathbb{C}\backslash]-\infty, 0]$, $1 \leqq m \leqq 3)$, where "$*$" means convolution.

We remark that all the preceding operators are well defined. This is not self-evident in the case of $T(\Lambda)$ and $T^*(\Lambda)$; see [7; Lemma 4.7, 5.1] for more details.

3. Outline of proofs

Without loss of generality, we may restrict ourselves to proving Theorem 1.1 for the case $\nu = 1$. Therefore we shall assume this condition for the rest of this paper.

We fix $\vartheta \in [0, \pi[$, $p \in]1, \infty[$, $f \in C_0^\infty(\mathbb{R}^3)^3$. The letter E is to denote constants only depending on ϑ, p, or Ω. We shall write $E(\gamma)$ for constants additionally depending on $\gamma \in]0, \infty[$.

Let us first remark on the integrals which—as mentioned in Section 1—allow us to represent solutions to (1.1), (1.2). Our starting point is the fundamental solution to the Stokes-system (1.1), which is given by the matrix-valued function $((\tilde{E}^\lambda_{jk})_{1 \leq j,k \leq 3}, (E_{4k})_{1 \leq k \leq 3})$, with $\tilde{E}^\lambda_{rs}$, E_{4s} introduced at the beginning of Section 2 ($\lambda \in \mathbb{C} \backslash]-\infty, 0]$, $1 \leqq r, s \leqq 3$). This solution was constructed by McCracken [11]. Using this fundamental solution, we defined in Section 2 the volume potentials $\tilde{u}^\lambda(f)$, $\pi(f)$, and the single-layer potentials $\tilde{W}(\cdot, \Lambda)$, $\tilde{\Pi}^\lambda(\cdot, \Lambda)$ ($\lambda \in \mathbb{C} \backslash]-\infty, 0]$, $\Lambda \in C^0(\partial\Omega)^3$). The function $\tilde{W}^\lambda(\cdot, \Lambda) | \mathbb{R}^3 \backslash \bar{\Omega}$ may be continuously extended to $\mathbb{R}^3 \backslash \Omega$. Denoting this extension by $\tilde{W}^\lambda(\cdot, \Lambda)_{ex}$, we have (see [5; Section 3])

$$\tilde{W}^\lambda(x, \Lambda)_{ex} = (-1/2) \cdot (\Lambda + \tilde{T}^\lambda(\Lambda))(x) \quad \text{for } x \in \partial\Omega, \quad \Lambda \in C^0(\partial\Omega)^3. \tag{3.1}$$

According to [5; Lemma 1.1], for $\lambda \in \mathbb{C} \backslash]-\infty, 0]$, there exists a uniquely determined function $\Gamma = \Gamma(\lambda, f) \in C^0(\partial\Omega)^3$ solving the integral equation

$$(-1/2) \cdot (\Gamma + \tilde{T}^\lambda(\Gamma)) = -\tilde{u}^\lambda(f) | \partial\Omega. \tag{3.2}$$

According to [5; Section 1], the tuple (u, π), with

$$u := [\tilde{u}^\lambda(f) + \tilde{W}^\lambda(\cdot, \Gamma(\lambda, f))] | \mathbb{R}^3 \backslash \bar{\Omega}, \quad \pi := [\pi(f) + \tilde{\Pi}^\lambda(\cdot, \Gamma(\lambda, f))] | \mathbb{R}^3 \backslash \bar{\Omega},$$

is an element of $\mathrm{SOL}(1, \lambda, p, f)$, with u uniquely determined, and π unique up to an additive constant. This is the looked-for integral representation of solutions to (1.1), (1.2). By [5; Lemma 1.3], we have

$$\sum_{c \in \mathbb{N}_0^3,\, |c|_* \leqq 2} |\lambda|^{1 - |c|_*/2} \cdot \|D^c \tilde{u}^\lambda(f)\|_p + \|\pi(f)\|_p \leqq E \cdot \|f\|_p$$
$$\text{for } \lambda \in \mathbb{C} \backslash \{0\} \quad \text{with } |\arg \lambda| \leqq \vartheta. \tag{3.3}$$

The proof of Theorem 1.1 thus reduces to an estimate of the double-layer potentials $\tilde{W}^\lambda(\cdot, \Gamma(\lambda, f)) | \mathbb{R}^3 \backslash \bar{\Omega}$, $\tilde{\Pi}^\lambda(\cdot, \Gamma(\lambda, f)) | \mathbb{R}^3 \backslash \bar{\Omega}$. Concerning these functions, we shall be able to show:

Theorem 3.1. There is a constant $K = K(\vartheta, p, \Omega) \in]0, 1]$ such that the ensuing estimates hold for $\lambda \in \mathbb{C}$ with $0 < |\lambda| < K$, $|\arg \lambda| \leqq \vartheta$, and for δ as in Theorem 1.1:

$$\|\tilde{W}^\lambda(\cdot, \Gamma(\lambda, f))|\mathbb{R}^3\backslash\bar{\Omega}\|_p \leqq E \cdot |\lambda|^{-1} \cdot \|f\|_p; \tag{3.4}$$

$$\sum_{c \in \mathbb{N}_0^3, |c|_* \leqq 2} |\lambda|^{1-|c|_*/2} \cdot \|D^c \tilde{W}^\lambda(\cdot, \Gamma(\lambda, f))|\mathbb{R}^3\backslash\bar{\Omega}\|_p + \|\tilde{\Pi}^\lambda(\cdot, \Gamma(\lambda, f))|\mathbb{R}^3\backslash\bar{\Omega}\|_p \leqq E(\delta) \cdot |\lambda|^{-\delta} \cdot \|f\|_p. \tag{3.5}$$

Theorem 1.1 then follows from (3.3) and Theorem 3.1.

Let us indicate how to proceed in the proof of Theorem 3.1. First we recall that the operators $\mathscr{T}_r$, $\tilde{\mathscr{T}}_r^\lambda$ map the space $L^r(\partial\Omega)^3$ into itself ([7; Lemma 5.2, 5.3; 5; Section 3]), and they are compact ([7; Satz 5.1; 5; Section 3]), for $r \in]1, \infty[$. The operator $\mathrm{id}(p) + \tilde{\mathscr{T}}_p^\lambda$ is continuously invertible ([5; Lemma 1.1]), for $\lambda \in \mathbb{C}\backslash]-\infty, 0]$. However, the mapping $\mathrm{id}(p) + \mathscr{T}_p$ is not invertible. In fact, the kernel of the latter operator is given by the space N defined in Section 2 (see [7; Lemma 6.10]). We shall show that $L^p(\partial\Omega)^3$ is the direct sum of N and the space $F(p)$. The latter space was also defined in Section 2. Furthermore, it will be proved that $(\mathrm{id}(p) + \mathscr{T}_p)|F(p)$ and $T^1|N$ are continuously invertible. These facts are stated in a precise form in the following lemmas:

Lemma 3.1. The equation $L^p(\partial\Omega)^3 = N + F(p)$ holds in the sense of a direct sum. For $\Phi \in F(p)$, the inequality $\|\Phi\|_p \leqq E \cdot \|\Phi + T(\Phi)\|_p$ is satisfied.

Lemma 3.2. $\|\Psi\|_p \leqq E \cdot \|T^1(\Psi)\|_p$ for $\Psi \in N$.

Lemma 3.1 amounts to saying that the eigenvalue 1 of the operator $\mathscr{T}_p$ has order 1 (see [8; (11.4.1)]). From Lemma 3.1 and 3.2, we shall be able to conclude that the operators $(\mathrm{id}(p) + \mathscr{T}_p)|F(p)$ and $T^1|N$ are orthogonal in a certain sense:

Lemma 3.3. $\|\Phi + T(\Phi)\|_p + \|T^1(\Psi)\|_p \leqq E \cdot \|\Phi + T(\Phi) + T^1(\Psi)\|_p$ for $\Phi \in F(p)$, $\Psi \in \mathrm{N}$.

For $\Psi \in N$, the expression $\Psi + \tilde{T}^\lambda(\Psi)$ equals $\lambda \cdot T^1(\Psi) + \hat{\tilde{T}}^\lambda(\Psi)$ (see (2.4)). On the other hand, the expression $\|\hat{\tilde{T}}^\lambda(\Psi)\|_p \cdot |\lambda|^{-3/2}$ remains bounded for $|\lambda| \to 0$, so that for small values of $|\lambda|$, $\hat{\tilde{T}}^\lambda$ may be considered as a perturbation of $\lambda \cdot T^1$. Thus, as an easy conclusion from Lemma 3.1–3.3 and (2.4), we obtain the following estimate, which is the key to our proof of Theorem 3.1:

Corollary 3.1. There exists a constant $K = K(\vartheta, p, \Omega) \in]0, 1]$ such that

$$\|\Phi\|_p + |\lambda| \cdot \|\Psi\|_p \leqq E \cdot \|\Phi + \Psi + \tilde{T}^\lambda(\Phi + \Psi)\|_p$$

for $\lambda \in \mathbb{C}$ with $|\arg \lambda| \leqq \vartheta$, $0 < |\lambda| \leqq K$, $\Phi \in F(p)$, $\Psi \in N$.

Now we fix $\lambda \in \mathbb{C}$ with $|\arg \lambda| \leqq \vartheta$, $0 < |\lambda| \leqq K(\vartheta, p, \Omega)$, with $K(\vartheta, p, \Omega)$ from Corollary 3.1. Furthermore, we take δ as in Theorem 1.1.

Due to Lemma 3.1, we may write $\Gamma(\lambda, f)$ as a sum $\Phi(\lambda, f) + \Psi(\lambda, f)$, with $\Phi(\lambda, f) \in F(p)$, $\Psi(\lambda, f) \in N$. We shall suppress the subscripts λ, f in the following. Due to Corollary 1.1, it makes sense to estimate $\tilde{W}^\lambda(\cdot, \Phi)|\mathbb{R}^3\backslash\bar{\Omega}$ and $\tilde{W}^\lambda(\cdot, \Psi)|\mathbb{R}^3\backslash\bar{\Omega}$ separately. In fact, we shall show in Section 5:

Lemma 3.4.
$\|\tilde{W}^\lambda(\cdot, \Phi)|\mathbb{R}^3\backslash\bar{\Omega}\|_p \leqq E(\delta) \cdot |\lambda|^{\delta - 1} \cdot (\|\Phi\|_p + |\lambda| \cdot \|\Psi\|_p + \|\tilde{u}^\lambda(f)|\Omega\|_p)$.

Lemma 3.5. $\|W^\lambda(\cdot, \Psi)|\mathbb{R}^3\backslash\bar{\Omega}\|_p \leqq E(\delta) \cdot |\lambda|^\delta \cdot \|\Psi\|_p$.

Inequality (3.4) will then be derived from Lemma 3.4, 3.5, Corollary 3.1, and from an appropriate estimate of $\tilde{u}^\lambda(f)|\Omega$.

Concerning the proof of (3.5), it will be possible to evaluate the expressions $\tilde{W}^\lambda(\cdot, \Psi)|\mathbb{R}^3\backslash\bar{\Omega}$, $\tilde{\Pi}^\lambda(\cdot, \Psi)|\mathbb{R}^3\backslash\bar{\Omega}$ rather quickly, since Ψ connects nicely with the kernels $\tilde{S}^\lambda_{jkm}$, E_{4m}. As for $\tilde{W}^\lambda(\cdot, \Phi)|\mathbb{R}^3\backslash\bar{\Omega}$, $\tilde{\Pi}^\lambda(\cdot, \Phi)|\mathbb{R}^3\backslash\bar{\Omega}$, we mention that the function $F(\cdot, \Phi)|\mathbb{R}^3\backslash\bar{\Omega}$—defined in Section 2—belongs to $L^r(\mathbb{R}^3\backslash\bar{\Omega})^3$, for any $r \in]1, \infty[$ (see [5; Section 5]). Furthermore, we have $\Phi \in W^{2-1/r,r}(\partial\Omega)^3$ for $r \in]1, \infty[$ (see [5; Section 6]). Thus we may state the following inequality, which our estimates of $D_k D_m \tilde{W}^\lambda(\cdot, \Phi)|\mathbb{R}^3\backslash\bar{\Omega}$, $\nabla\tilde{\Pi}^\lambda(\cdot, \Phi)|\mathbb{R}^3\backslash\bar{\Omega}$ are based on:

$$\begin{aligned}\sum_{c \in \mathbb{N}_0^3,\, |c|_* \leqq 2} & |\lambda|^{1 - |c|_*/2} \cdot \|D^c \tilde{W}^\lambda(\cdot, \Phi)|\mathbb{R}^3\backslash\bar{\Omega}\|_p + \|\nabla\tilde{\Pi}^\lambda(\cdot, \Phi)|\mathbb{R}^3\backslash\bar{\Omega}\|_p \\ &\leqq E \cdot (\|\Phi + \tilde{T}^\lambda(\Phi)\|_{2-1/p,p} + |\lambda| \cdot \|F(\cdot, \Phi)|\mathbb{R}^3\backslash\bar{\Omega}\|_p + \|\Phi\|_{2-1/p,p} \\ &\quad + |\lambda| \cdot \|W^\lambda(\cdot, \Phi)|\mathbb{R}^3\backslash\bar{\Omega}\|_p). \end{aligned} \tag{3.6}$$

Let us remark on the proof of (3.6). Choose some $S > 0$ with $\bar{\Omega} \subset B_S$. Then $v := \tilde{W}^\lambda(\cdot, \Phi)|B_S\backslash\bar{\Omega}$ solves the Laplace equation with right-hand side $R := \nabla\tilde{\Pi}^\lambda(\cdot, \Phi) + \lambda \cdot \tilde{W}^\lambda(\cdot, \Phi)|B_S\backslash\bar{\Omega}$. Furthermore, v satisfies on $\partial\Omega$ the boundary-condition $v|\partial\Omega = -\tilde{u}^\lambda(f)|\partial\Omega$ (see (3.1) and (3.2)). If S is large enough, then $\|v|\partial B_S\|_{2-1/p,p}$ may be trivially estimated by $E \cdot \|\Phi\|_p$ (use (5.1)). Now L^p-estimates for the Laplacian, as stated in [5; Theorem 2.1], yield the inequality

$$\|D_k D_m \tilde{W}^\lambda(\cdot, \Phi)|B_S\backslash\bar{\Omega}\|_p \leqq E \cdot (\|R\|_p + \|\Phi\|_p + \|\tilde{u}^\lambda(f)|\partial\Omega\|_{2-1/p,p}),$$

for $1 \leqq k, m \leqq 3$. If S is large enough, $\|D_m D_k \tilde{W}^\lambda(\cdot, \Phi)|\mathbb{R}^3\backslash B_S\|_p$ may be evaluated against $E \cdot \|\Phi\|_p$ in a straightforward way (use (5.1)). Furthermore,

$\|\nabla\Pi(\cdot,\Phi)|\mathbb{R}^3\backslash\bar{\Omega}\|_p$ is bounded by $\|\Phi\|_{2-1/p,p}$, as was shown in [5; Section 6] by another application of L^p-estimates for the Laplacian on bounded domains. The proof of (3.6) is completed by recalling (2.3) and by noting that $\nabla\hat{\Pi}^\lambda(\cdot,\Phi) = \lambda \cdot F(\cdot,\Phi)$.

In Section 5, we shall evaluate the right-hand side of (3.6).

4. Proof of Lemma 3.1–3.3

We begin by establishing Lemma 3.1. According to [7; Satz 5.1], the operators $\mathscr{T}_r$, $\mathscr{T}_r^*$ are compact, and $\mathscr{T}_r$ is dual to $\mathscr{T}^*_{1/(1-1/r)}$ ($r \in]1,\infty[$). As noted in Section 3, the kernel N of $\mathrm{id}(r) + \mathscr{T}_r$ does not depend on $r \in]1,\infty[$. In this situation, the standard theory of compact operators in Banach spaces, as presented in [8; XI, 2–5] for example, yields that the assertions of Lemma 3.1 hold if they are true for $p = 2$, that is, if

$$(\mathrm{id}(2) + \mathscr{T}_2^*)^{-2}(\{0\}) \subset (\mathrm{id}(2) + \mathscr{T}_2^*)^{-1}(\{0\}). \tag{4.1}$$

To establish (4.1), we assume there is some $\Lambda \in (\mathrm{id}(2) + \mathscr{T}_2^*)^{-2}(\{0\})$ with $\varrho := (\mathrm{id}(2) + \mathscr{T}_2^*)\Lambda \neq 0$. Set

$$V_m(x) := \int_{\partial\Omega} \sum_{1 \leqq j \leqq 3} E_{jm}(x-y) \cdot \varrho_j(y)\, d\Omega(y) \quad \text{for } x \in \partial\Omega,\ 1 \leqq m \leqq 3.$$

Denote the inner product in $L^2(\partial\Omega)^3$ by $\langle\!\langle\ ,\ \rangle\!\rangle$. Then we have $\langle\!\langle (\mathrm{id}(2) + \mathscr{T}_2^*)\varrho, V \rangle\!\rangle = 0$. By [7; Lemma 6.4, 6.6, Satz 6.1] it follows that $(\mathrm{id}(2) + \mathscr{T}_2)V = 0$, so $\langle\!\langle (\mathrm{id}(2) + \mathscr{T}_2^*)\mu, V \rangle\!\rangle = 0$ for any $\mu \in L^2(\partial\Omega)^3$. By the preceding equation, we obtain

$$\begin{aligned}\langle\!\langle (-\mathrm{id}(2) + \mathscr{T}_2^*)\varrho, V \rangle\!\rangle &= -2 \cdot \langle\!\langle \varrho, V \rangle\!\rangle \\ &= -2 \cdot \langle\!\langle (\mathrm{id}(2) + \mathscr{T}_2^*)\Lambda, V \rangle\!\rangle = 0.\end{aligned}$$

Since $(\mathrm{id}(2) + \mathscr{T}_2^*)\varrho = 0$, it follows by [7; Lemma 6.4, 6.6, 6.9]: $\varrho = 0$, a contradiction.

Next, we intend to prove Lemma 3.2. For this purpose, take $\Psi \in N$ with $T^1(\Psi) = 0$. We have to show that Ψ vanishes identically. Choose $\alpha_1, \ldots, \alpha_6 \in \mathbb{C}$ with $\psi = \sum_{1 \leqq m \leqq 6} \alpha_m \cdot \varphi_{(m)} | \partial\Omega$. Since

$$D_j\varphi_{(m),k} + D_k\varphi_{(m),j} = 0, \quad \Delta E^1_{jk} = E_{jk}, \sum_{1 \leqq r \leqq 3} D_r E^1_{rk} = 0 \tag{4.2}$$
$$(1 \leqq j, k \leqq 3,\ 1 \leqq m \leqq 6),$$

we have $T^1(\Psi)(x) = 2 \cdot u(\chi_\Omega \cdot \Lambda)(x)$ for $x \in \partial\Omega$, where we used the abbreviation $\Lambda := \sum_{1 \leqq m \leqq 6} \alpha_m \cdot \varphi_{(m)}$. It follows from the choice of Ψ: $u(\chi_\Omega \cdot \Lambda)|\partial\Omega = 0$.

The tuple $(u(\chi_\Omega \cdot \Lambda)|\mathbb{R}^3\backslash\bar{\Omega}, \pi(\chi_\Omega \cdot \Lambda)|\mathbb{R}^3\backslash\bar{\Omega})$ belongs to the set of functions INT, introduced in Section 2, and it solves the Stokes-system (1.1), with $f = 0$ and $\lambda = 0$; see [7; Satz 1.4]. Now it is implied by the assertion on uniqueness in [6; Satz 8.1]:

$$u(\chi_\Omega \cdot \Lambda)(x) = 0, \quad \pi(\chi_\Omega \cdot \Lambda)(x) = 0 \text{ for } x \in \mathbb{R}^3\backslash\bar{\Omega}.$$

Using the equation $\operatorname{div} \Lambda = 0$, we obtain after some calculations, for $1 \leqq m \leqq 3$, $x \in \mathbb{R}^3\backslash\bar{\Omega}$:

$$0 = D_m \pi(\chi_\Omega \cdot \Lambda)(x) = (-4 \cdot \pi)^{-1} \cdot \int_{\partial\Omega} (x-y)_m \cdot |x-y|^{-3} \cdot \langle n(y), \Lambda(y) \rangle \, d\Omega(y).$$

By [7; Satz 4.1], it follows for $x \in \partial\Omega$:

$$0 = \langle n(x), \Lambda(x) \rangle + \int_{\partial\Omega} \langle n(x), x-y \rangle \cdot |x-y|^{-3} \cdot \langle n(y), \Lambda(y) \rangle \, d\Omega(y).$$

It is well known that the preceding equation implies $\langle n(x), \Lambda(x) \rangle = 0$ for $x \in \partial\Omega$; see the remark at the end of [4; Section 2].

Now we want to exploit the relation $u(\chi_\Omega \cdot \Lambda)\mathbb{R}^3\backslash\bar{\Omega} = 0$. By splitting up the kernels E_{jk} in a suitable way, the function $u(\chi_\Omega \cdot \Lambda)|\mathbb{R}^3\backslash\bar{\Omega}$ may be represented as the sum of the gradient of a certain function, plus a rest. Due to the relations $\operatorname{div} \Lambda = 0$, $\langle n, \Lambda|\partial\Omega \rangle = 0$, the first of these summands disappears after a partial integration. We end up with the relation

$$0 = u_m(\chi_\Omega \cdot \Lambda)(x) = (4 \cdot \pi)^{-1} \cdot \int_\Omega |x-y|^{-1} \cdot \Lambda_m(y) \, dy$$
$$(1 \leqq m \leqq 3, \quad x \in \mathbb{R}^3\backslash\bar{\Omega}). \tag{4.3}$$

After differentiating the right-hand side of (4.3) in x_j and x_k $(1 \leqq j, k \leqq 3)$, then performing a partial integration, and finally taking a sum over suitable indices k, m, so as to be able to apply the first equation in (4.2), we arrive at the relation

$$0 = \int_{\partial\Omega} (x-y)_j \cdot |x-y|^{-3} \cdot (n_m \cdot \Lambda_k + n_k \cdot \Lambda_m)(y) \, d\Omega(y)$$
$$(1 \leqq j, k, m \leqq 3, \quad x \in \mathbb{R}^3\backslash\bar{\Omega}).$$

The same argument previously applied for showing the equation $\langle n, \Lambda|\partial\Omega \rangle = 0$ now yields: $(n_k \cdot \Lambda_m + n_m \cdot \Lambda_k)(x) = 0$ $(x \in \partial\Omega)$. Hence we have $T(\Psi) = 0$, so Ψ is vanishing identically.

Turning to the proof of Lemma 3.3, we take $\Phi \in F(p)$, $\Psi \in N$ with $\Phi + T(\Phi) + T^1(\Psi) = 0$. It must be shown that both functions Φ and Ψ are zero. Let $\alpha_1, \ldots, \alpha_6 \in \mathbb{C}$ be such that Ψ is the restriction of $\Lambda :=$

$\sum_{1 \leqq m \leqq 6} \alpha_m \cdot \varphi_{(m)}$ to $\partial\Omega$. As a consequence of (4.2), we have

$$W^1(\cdot, \Psi)|\mathbb{R}^3\backslash\bar{\Omega} = -u(\chi_\Omega \cdot \Lambda)|\mathbb{R}^3\backslash\bar{\Omega}. \tag{4.4}$$

Setting $U := (W^1(\cdot, \Psi) + W(\cdot, \Phi))|\mathbb{R}^3\backslash\bar{\Omega}$, $P := (-\pi(\chi_\Omega \cdot \Lambda) + \Pi(\cdot, \Phi))|\mathbb{R}^3\backslash\bar{\Omega}$, it follows:

$$(U, P) \in \mathrm{INT}, \quad -\Delta U + \nabla P = 0, \quad \operatorname{div} U = 0. \tag{4.5}$$

Concerning the first of these relations, we note that the tuple $(W^1(\cdot, \Psi), -\pi(\chi_\Omega \cdot \Lambda))|\mathbb{R}^3\backslash\bar{\Omega}$ belongs to INT, as follows from (4.4) and [7; Satz 1.4]. As for $(W(\cdot, \Phi), \Pi(\cdot, \Phi))|\mathbb{R}^3\backslash\bar{\Omega}$, this tuple is an element of INT too. This may be shown by somewhat modifying the arguments used in the proof of (3.6), provided we know that $\Phi \in W^{2-1/r,r}(\partial\Omega)^3$ for $r \in]1, \infty[$. But the latter relation follows from the equation $\Phi + T(\Phi) = -T^1(\Psi)$, and from [7; Lemma 7.8]. The second and third relation in (4.5) are a consequence of [7; Lemma 1.7]. According to [7; Satz 4.1], the function U may be continuously extended to $\mathbb{R}^3\backslash\Omega$. Let $\tilde{U}$ denote the corresponding extension. Then, by [7; Satz 4.1], and by our assumptions on Φ and Ψ, we have $\tilde{U}|\partial\Omega = 0$. Since $(U, P) \in \mathrm{INT}$, we may apply the uniqueness assertion from [6; Satz 8.1] to obtain $U = 0$, that is,

$$\begin{aligned} 0 = &\int_{\partial\Omega} \sum_{1 \leqq j,k \leqq 3} S_{jkm}(x-y) \cdot \Phi_j(y) \cdot n_k(y)\, d\Omega(y) \\ &- \int_\Omega \sum_{1 \leqq j \leqq 3} E_{jm}(x-y) \cdot \Lambda_j(y)\, dy \quad (x \in \mathbb{R}^3\backslash\bar{\Omega},\ 1 \leqq m \leqq 3). \end{aligned}$$

Choosing $R(\Omega) \in]0, \infty[$ large enough, it follows by a Taylor expansion, for $1 \leqq m \leqq 3$, $x \in \mathbb{R}^3$ with $|x| \geqq R(\Omega)$:

$$\begin{aligned} 0 = &\sum_{1 \leqq j,k \leqq 3} \sum_{a \in \mathbb{N}_0^3} \{(1/(a+e_k)!) \cdot \alpha(j,k,a) \cdot D^{a+e_k} E_{jm}(x) \\ &- (1/a!) \cdot \beta(j,k,a) \cdot D^a S_{jkm}(x)\} \\ &+ \sum_{1 \leqq j \leqq m} E_{jm}(x) \cdot \int_\Omega \Lambda_j(y)\, dy, \end{aligned} \tag{4.6}$$

with

$$\alpha(j,k,a) := \int_\Omega y^{a+e_k} \cdot \Lambda_j(y)\, dy,$$

$$\beta(j,k,a) := \int_{\partial\Omega} y^a \cdot \Phi_j(y) \cdot n_k(y)\, d\Omega(y),$$

for $1 \leqq j, k \leqq 3$, $a \in \mathbb{N}_0^3$. (4.6) implies for $1 \leqq m \leqq 3$, $x \in \mathbb{R}^3 \backslash \{0\}$:

$$\int_\Omega \Lambda_j(y)\, dy = 0 = \sum_{1 \leqq j,k \leqq 3} (\alpha(j, k, 0) \cdot D_k E_{jm}(x) - \beta(j, k, 0) \cdot S_{jkm}(x)). \tag{4.7}$$

Now fix $m \in \{1, 2, 3\}$. By writing out the expressions $D_k E_{jm}$, S_{jkm} ($1 \leqq j, k \leqq 3$, $x \in \mathbb{R}^3 \backslash \{0\}$), and collecting those terms which do not depend on x_m, we may conclude from the second equation in (4.7): $\alpha(k, m, 0) = \alpha(m, k, 0)$ ($1 \leqq k \leqq 3$). This result, combined with the first equation in (4.7), implies:

$$\int_\Omega \langle \varphi_{(m)}(x), \Lambda(x) \rangle\, dx = 0 \quad \text{for } 1 \leqq m \leqq 6,$$

so $\Lambda|\Omega$—and hence Ψ—must vanish identically. This means $\Phi + T(\Phi) = 0$. Due to Lemma 3.1 and the relation $\Phi \in F(p)$, it follows $\Phi = 0$.

5. Proof of Lemma 3.4, 3.5, and of Theorem 3.1

Take δ as in Theorem 1.1. Fix $\lambda \in \mathbb{C}$ with $|\arg \lambda| \leqq \vartheta$, $0 < |\lambda| \leqq K(\vartheta, p, \Omega)$, with $K(\vartheta, p, \Omega)$ introduced in Corollary 3.1. Note that ϑ, p, f were fixed at the beginning of Section 3.

We recall some estimates of $\tilde{E}^\lambda_{jk}$, $\hat{E}^\lambda_{jk}$ already used in [5] (see [5; Section 3]):

$$|D^a \tilde{E}^\lambda_{jk}(x)| \leqq E \cdot |\lambda|^{-\gamma} \cdot |x|^{-1-2\cdot\gamma-|a|_*}, \tag{5.1}$$

$$|D^b \hat{E}^\lambda_{jk}(x)| \leqq E \cdot |\lambda|^{\gamma} \cdot |x|^{-1+2\cdot\gamma-|b|_*} \tag{5.2}$$

$$(x \in \mathbb{R}^3 \backslash \{0\},\quad \gamma \in [0, 1],\quad a, b \in \mathbb{N}_0^3 \text{ with } |a|_* \leqq 3,\ 1 \leqq |b|_* \leqq 3).$$

Choose $\Gamma = \Gamma(\lambda, f)$ as in (3.2). Write Γ as a sum $\Phi + \Psi$, with $\Phi \in F(p)$, $\Psi \in N$ (see Lemma 3.1). Take $\alpha_1, \ldots, \alpha_6 \in \mathbb{C}$ with $\Psi = \sum_{1 \leqq m \leqq 6} \alpha_m \cdot \varphi_{(m)} | \partial\Omega$, and set $\Lambda := \sum_{1 \leqq m \leqq 6} \alpha_m \cdot \varphi_{(m)}$. By (4.2) the ensuing equation holds:

$$\tilde{W}_\lambda(\cdot, \Psi) | \mathbb{R}^3 \backslash \bar{\Omega} = -\lambda \cdot \tilde{u}^\lambda(\chi_\Omega \cdot \Lambda) | \mathbb{R}^3 \backslash \bar{\Omega}. \tag{5.3}$$

Choose $S(\Omega) > 0$ so large that $|x - y| \geqq |x|/2$ for $x \in \mathbb{R}^3 \backslash B_{S(\Omega)}$, $y \in \bar{\Omega}$. Applying (5.1), we may separately estimate the terms $\|\tilde{u}^\lambda(\chi_\Omega \cdot \Lambda) | \mathbb{R}^3 \backslash B_{S(\Omega)}\|_p$, and $\|\tilde{u}^\lambda(\chi_\Omega \cdot \Lambda) | B_{S(\Omega)} \backslash \bar{\Omega}\|_p$, to obtain the following inequality:

$$\|\tilde{W}^\lambda(\cdot, \Psi) | \mathbb{R}^3 \backslash \bar{\Omega}\|_p \leqq E(\delta) \cdot |\lambda|^\delta \cdot \|\Lambda|\Omega\|_p \tag{5.4}$$

According to [8; (5.9.1)], we have $\|\Lambda|\Omega\|_p \leqq E \cdot \|\Psi\|_p$. Thus Lemma 3.5 follows from (5.4).

Turning to the proof of Lemma 3.4, we first note:

$$\|\tilde{u}^{\lambda}(f)|\bar{\Omega}\|_{2,p} \leqq E(\delta) \cdot |\lambda|^{-\delta} \cdot \|f\|_{p}. \tag{5.5}$$

In fact, referring to the definition of $\tilde{u}^{\lambda}(f)$, we may split up the domain of integration $\mathbb{R}^3$ into the parts B_1, and $\mathbb{R}^3 \backslash B_1$. Then (5.5) may be derived from (5.1).

Let us now evaluate $\tilde{W}^{\lambda}(\cdot, \Phi)|\mathbb{R}^3\backslash\bar{\Omega}$. Since Ψ belongs to N, that is, $\Psi + T(\Psi) = 0$, it follows from (3.2):

$$(-1/2) \cdot (\Phi + \tilde{T}^{\lambda}(\Phi)) = (-\tilde{u}^{\lambda}(f) - \hat{W}^{\lambda}(\cdot, \Psi))|\partial\Omega. \tag{5.6}$$

By applying Gauss' theorem as in [4; Section 5], we may now conclude from (5.6):

$$F_k(x, \Phi) = (4 \cdot \pi)^{-1} \cdot \sum_{1 \leqq j \leqq 3} \int_{\Omega} \partial^2/\partial x_j \, \partial x_k(|x-y|^{-1}) \cdot (\tilde{u}_j^{\lambda}(f)(y) - \hat{W}_j^{\lambda}(y, \Psi) + \tilde{W}_j^{\lambda}(y, \Phi)) \, dy$$

for $x \in \mathbb{R}^3\backslash\bar{\Omega}$, $1 \leqq k \leqq 3$. The Calderon-Zygmund theorem now yields:

$$\|F(\cdot, \Phi)|\mathbb{R}^3\backslash\bar{\Omega}\|_p \leqq E \cdot \|(\tilde{u}^{\lambda}(f) - \hat{W}^{\lambda}(\cdot, \Psi) + \tilde{W}^{\lambda}(\cdot, \Phi))|\Omega\|_p.$$

On the other hand, it follows from (5.2):

$$\|\hat{W}^{\lambda}(\cdot, \Psi)|\Omega\|_{2,p} \leqq E \cdot |\lambda| \cdot \|\Psi\|_p. \tag{5.7}$$

The term $\|\tilde{W}^{\lambda}(\cdot, \Psi)|\Omega\|_{\mathrm{p}}$ is bounded by $E \cdot \|\Phi\|_p$, as may be derived from (5.1). Combining the preceding estimates, we arrive at the inequality

$$\|F(\cdot, \Phi)|\mathbb{R}^3\backslash\bar{\Omega}\|_p \leqq E \cdot (|\lambda| \cdot \|\Psi\|_p + \|\Phi\|_p + \|\tilde{u}^{\lambda}(f)|\Omega\|_p). \tag{5.8}$$

Now applying (5.1) once more, this time in order to estimate $J^{\lambda}(\cdot, \Phi)$ over $\mathbb{R}^3\backslash B_{S(\Omega)}$, as well as over $B_{S(\Omega)}\backslash\bar{\Omega}$, we find after some computations:

$$\|J^{\lambda}(\cdot, \Phi)|\mathbb{R}^3\backslash\bar{\Omega}\|_p \leqq E(\delta) \cdot |\lambda|^{-1+\delta} \cdot \|\Phi\|_p. \tag{5.9}$$

By combining (5.8), (5.9), (2.2), we obtain Lemma 3.4.

Now (3.4) follows from Lemma 3.4, 3.5, Corollary 3.1, and from (3.2), (5.5).

This leaves us to derive (3.5) from (3.6). Recalling (5.6), (5.7), we may conclude from [5; (6.1)]:

$$\|\Phi\|_{2-1/p,p} \leqq E \cdot (|\lambda| \cdot \|\Psi\|_p + \|\Phi\|_p + \|\tilde{u}^{\lambda}(f)|\partial\Omega\|_{2-1/p,p}). \tag{5.10}$$

It now follows from (3.6), (5.5)–(5.8), (5.10), and from Lemma 3.4 that the left-hand side in (3.6) is bounded by $E(\delta) \cdot |\lambda|^{-\delta} \cdot \|f\|_p$.

By (3.3) and (5.3), it is easy to show:

$$\sum_{c \in \mathbb{N}_0^3,\, |c|_* \leqq 2} |\lambda|^{1-|c|_*/2} \cdot \|D^c \tilde{W}^\lambda(\cdot, \Psi)|\mathbb{R}^3 \backslash \bar{\Omega}\|_p \leqq E \cdot |\lambda| \cdot \|\Psi\|_p.$$

Concerning the term $\|\nabla \tilde{\Pi}^\lambda(\cdot, \Psi)|\mathbb{R}^3 \backslash \bar{\Omega}\|_p$, we recall (2.3), and note that $\nabla \Pi(\cdot, \Psi)|\mathbb{R}^3 \backslash \bar{\Omega} = 0$, $\nabla \hat{\Pi}^\lambda(\cdot, \Psi) = |\lambda| \cdot F(\cdot, \Psi)$.

But $\|F(\cdot, \Psi)|\mathbb{R}^3 \backslash \bar{\Omega}\|_p$ may be dealt with by first applying Gauss' theorem, as used in the proof of (5.8), and then recurring to the Calderon-Zygmund theorem. In this way we arrive at the estimate $\|\nabla \tilde{\Pi}^\lambda(\cdot, \Psi)|\mathbb{R}^3 \backslash \bar{\Omega}\|_p \leqq E \cdot |\lambda| \cdot \|\Psi\|_p$.

Inequality (3.5) now follows from (5.5) and Corollary 3.1.

It should be mentioned that inequality (3.4) even holds with the factor $|\lambda|^{-1}$ replaced by $|\lambda|^{-\kappa}$, for certain values of $\kappa \in (0, 1)$. This may be seen by checking the calculations leading to estimates (5.4) and (5.9). However, for shortness, we did not elaborate this fact.

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Abstract

We consider the resolvent problem for the Stokes-system in an exterior domain:

$$-\nu \cdot \Delta u + \lambda \cdot u + \nabla \pi = f, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \backslash \bar{\Omega},$$

with $\nu \in]0, \infty[$, $\lambda \in \mathbb{C}\backslash]-\infty, 0]$, Ω bounded domain in $\mathbb{R}^3$, with C^2-boundary $\partial\Omega$. In addition, Dirichlet boundary conditions $u|\partial\Omega = 0$ are prescribed. Using the method of integral equations, we estimate solutions (u, π) in L^p-norms, for small values of $|\lambda|$.

Zusammenfassung

Wir betrachten das Resolventenproblem zum Stokes-System im Außenraum:

$$-\nu \cdot \Delta u + \lambda \cdot u + \nabla\pi = f, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3\backslash\bar{\Omega},$$

Dabei ist Ω ein beschränktes Gebiet in $\mathbb{R}^3$, mit C^2-Berandung $\partial\Omega$. Ferner ist $\nu \in]0, \infty[$ und $\lambda \in \mathbb{C}\backslash]-\infty, 0]$. Zusätzlich verlangen wir, daß u die Randbedingung $u|\partial\Omega = 0$ erfüllt. Lösungen (u, π) dieses Problems schätzen wir in L^p-Normen ab, wobei vorausgesetzt wird, daß $|\lambda|$ klein ist.

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