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# THE STOKES-SYSTEM IN EXTERIOR DOMAINS: EXISTENCE, UNIQUENESS, AND REGULARITY OF SOLUTIONS IN $L^p$-SPACES.

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## 1. Introduction and main results.

Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with connected $C^2$-boundary $\partial\Omega$. Let $\nu\in]0,\infty[$. The domain $\Omega$ and the real number $\nu$ will be kept fixed throughout. Then, for functions $\mathbf{f}$: $\mathbb{R}^3 \to \mathbb{C}^3$, $\mathbf{b}$: $\partial\Omega \to \mathbb{C}^3$, consider the Stokes-system in the exterior domain $\mathbb{R}^3\backslash\overline{\Omega}$, with right-hand side $\mathbf{f}$:

(1.1) $\quad -\nu\cdot\Delta\mathbf{u} + \nabla\pi = \mathbf{f}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{\Omega},$

under Dirichlet-boundary conditions, with boundary data $\mathbf{b}$:

(1.2) $\quad \mathbf{u}|\partial\Omega = \mathbf{b}.$

This boundary-value problem, which we shall call exterior problem, models slow motion of a viscous, incompressible fluid around a body of shape $\Omega$, under an external force $\mathbf{f}$. If the system of equations in (1.1) is not considered in $\mathbb{R}^3\backslash\bar{\Omega}$, but in the bounded domain $\Omega$, with a function $\mathbf{f}$ mapping $\Omega$ into $\mathbb{C}^3$, then we shall speak of the interior problem. Let us briefly review some previous results on the exterior problem (1.1), (1.2). Using the method of integral equations, Odquist [16] obtains a strong solution under the assumption $\mathbf{f}=\mathbf{0}$. This solution consists of a sum of certain single-layer and double-layer potentials. In [14] Chapter 3, Ladyzhenskaya takes into account the case $\mathbf{f}\neq\mathbf{0}$ by adding a suitable volume potential to Odquist's solution. However, in [14] a $L^p$-theory is stated only for the interior problem. Ladyzhenskaya's results are worked out in detail in [6]. In [22], Temam proves existence of weak solutions under suitable conditions on $\mathbf{f}$, which are satisfied, for example, if $\mathbf{f}\in L^{6/5}(\mathbb{R}^3\backslash\bar{\Omega})^3$ (see [22] p. 31). The theory in [22] covers any space dimension $n\geq3$. In [19] p. 467, Solonnikov studies the solution constructed in [14]. Assuming $\mathbf{f}\in C_0^\infty(\mathbb{R}^3)^3$, he estimates this solution in $L^p$-norms, with constants depending on supp($\mathbf{f}$) in the case $p\geq3/2$. For the proof of his assertions, he refers to [14], suggesting that it suffices to repeat the arguments used there for dealing with the interior case. In fact, a corresponding remark may be found in [14] p. 79. But when attempting to check this statement, we encountered difficulties related to the single-layer potentials which are part of the solution to the exterior problem, but do not appear in the interior case. The main obstacles arising in this way were removed in [5] Section 8, which is concerned with $L^p$-regularity of Ladyzhenskaya's solution to (1.1), (1.2). Still, the theory in [5] Section 8 is not yet satisfying since it requires $\mathbf{f}$ to belong to $L^r(\mathbb{R}^3\backslash\bar{\Omega})^3$ for some $r\in]3/2,\infty[$ - an assumption which does not hold in certain applications (see [10] for example). Taking a different approach, Specovius [20] and Farwig [7] study problem (1.1), (1.2) in weighted Sobolev spaces.

Sohr and Kosono [12] consider problem (1.1), (1.2) for any space dimension $n\geq3$. They first treat the interior problem and the case of the entire $\mathbb{R}^n$. Then, by a localization procedure, they obtain weak and strong solutions to (1.1), (1.2) in $L^p$-spaces. In their proof, they apply results by Giga [9], Bogovski [3], and an assertion on regularity which Temam derived from Agmon, Douglis, Nirenberg [2] (see [22] p. 33). In [10], Sohr and Giga establish a uniqueness result with respect to strong solutions to

(1.1), (1.2) in $L^p$-spaces ([10] Corollary 3.6 (1)). Sohr, Kosono [13], and Simader, Galdi [8] study weak solutions to (1.1), (1.2) for $\mathbf{b}=\mathbf{0}$. Miyakawa [15] obtains a negative result which is closely related to Theorem 1.3 below.

In the work at hand, we shall derive a $L^p$-theory with respect to the exterior problem. This theory concerns uniqueness of strong solutions (Theorem 1.1), as well as existence and $L^p$-estimates of such solutions (Theorem 1.2). We study a more special problem than treated in [12]: we do not consider weak solutions, nor higher regularity; we treat the equation div $\mathbf{u}$ = 0 instead of div $\mathbf{u}$ = $\rho$, and we only consider space dimension n=3. Still, in other respects, our results are stronger than those in [12]. In fact, for a solution $(u,\pi)$ of (1.1), (1.2), we shall evaluate the terms $\|\mathbf{u}\|_s$, $\|D_m\mathbf{u}\|_q$, $\|\pi\|_q$ for a larger range of coefficients s, q than admitted in [12]. Improving on [5], neither the present paper nor reference [12] require $\mathbf{f}$ to belong to $L^r(\mathbb{R}^3\backslash\overline{\Omega})^3$ for some $r\epsilon]3/2,\infty[$. As for uniqueness, when the result from [10] is taken for space dimension n=3, it is a special case of Theorem 1.1.

Our theory will be established in the follolwing way: First, we shall give an existence and regularity result under the assumption $\mathbf{f}\epsilon C_0^\infty(\mathbb{R}^3)^3$ (Lemma 1.1). This assertion will be shown by improving some estimates from [6] and from [5] Section 8, which are based on the method of integral equations. The main tool involved in this approach is a $L^p$-estimate for the Laplacian in bounded domains. Otherwise, only standard results are needed, such as the Calderon-Zygmund theorem (used to obtain estimate (2.5) below), the Hardy-Littlewood-Sobolev-inequality (also used in the proof of (2.5)), and the Schauder theory for compact operators in Banach spaces (applied in [6], Section 6). In our next step, we shall prove uniqueness (Theorem 1.1). For this purpose, we shall need Lemma 1.1, as well as some results on Fourier transforms of $L^p$-functions, with p not necessarily equal to 1 or 2. Thus we have to recur to the theory of tempered distributions, as given in [23] p. 146-153. In addition, we shall use the multiplier theorem from [21] p. 96. Our final result on existence and regularity (Theorem 1.2) is an immediate consequence of Lemma 1.1 and Theorem 1.1. It should be noted that all the proofs in the work at hand, as well as those in [6] and [5] Section 8, could easily be adapted to any space dimension $n \geq 3$.

Our existence and regularity results (Theorem 1.2) cannot be improved in an essential way. This is shown by some counterexamples in [5] p. 56-63, and by the negative result stated in Theorem 1.3.

We further remark that integral representations of solutions to (1.1), (1.2) may be useful for numerical purposes. We refer to Hebeker [11] for an example, although the representation used in [11] is somewhat different from ours.

Before stating our theorems, let us introduce some notations. Partial derivatives are denoted by $D_m$ ($m \in \mathbb{N}$), with obvious meaning. $C_0^\infty(\mathbb{R}^3)$ is defined as the set of $C^\infty$-functions on $\mathbb{R}^3$ with compact support. We write $\mathbf{n}$ for the outward unit normal to $\Omega$. For $x,y \in \mathbb{R}^3$, the term $\langle x,y \rangle$ means the inner product of $x$ and $y$. The symbol $\mathbf{I}$ stands for the identity mapping of $\mathbb{R}^3$. By $B_r(x)$, we denote a ball in $\mathbb{R}^3$ with centre $x \in \mathbb{R}^3$, and radius $r>0$. For an open subset $U$ of $\mathbb{R}^3$, and for $k \in \mathbb{N}$, $p \in ]1,\infty[$, the set $W^{k,p}_{loc}(U)$ is to contain any function from $L^p_{loc}(U)$ which has weak derivatives of all orders up to $k$, with these derivatives belonging to $L^p_{loc}(U)$. $S(\mathbb{R}^3)$ denotes the set of rapidly decreasing functions on $\mathbb{R}^3$ (see [23] p. 146). For $f \in L^2(\mathbb{R}^3)$, we write $\hat{f}$ for the Fourier-transform and $\check{f}$ for the inverse Fourier transform of $f$. We shall use the notation $\hat{T}$ for the Fourier transform of a tempered distribution $T$ in $\mathbb{R}^3$ ([23] p. 150/151). Define

$$SLI := \left\{ g \in L^1_{loc}(\mathbb{R}^3) \ : \ \int_{\mathbb{R}^3} |g(x)| \cdot (1+|x|^2)^{-k}\, dx < \infty \quad \text{for some } k \in \mathbb{N} \right\},$$

("**slo**wly increasing measures" in the sense of [23] p.150, Example 3). Note that any function $f \in L^r(\mathbb{R}^3)$ belongs to SLI ($r \in [1,\infty[$). For $f \in SLI$, the term $T_f$ is to denote the corresponding tempered distribution ([23] p. 150). Let $f \in C^\infty(\mathbb{R}^3)$ be slowly increasing in the sense of [23], p. 150, Definition 2, and let $T$ be a tempered distribution in $\mathbb{R}^3$. Then we write $f \cdot T$ for the product of $f$ and $T$ ([23] p. 151, Definition 3). The set

$$BOU := \bigcap \left\{ W^{2-1/s,s}(\partial\Omega)^3 \ : \ s \in [1,\infty[ \right\}$$

("smooth functions on the **bou**ndary") is a subset of $C^\alpha(\partial\Omega)^3$, for $\alpha \in [0,1]$ (see[1] 5.4, 7.56).

Let INT ("**int**egrable functions") denote the set of all tuples $(\mathbf{u},\pi)$ with
$\mathbf{u} \in W^{2,1}_{loc}(\mathbb{R}^3\backslash\overline{\Omega})^3 \cap L^s(\mathbb{R}^3\backslash\overline{\Omega})^3$ for $s\in]3,\infty[$, $\pi \in W^{1,1}_{loc}(\mathbb{R}^3\backslash\overline{\Omega})$,
$D_k u_j,\ \pi \in \bigcap \{L^s(\mathbb{R}^3\backslash\overline{\Omega}) : s\in]3/2,\infty[\}$,
$D_m D_k u_j,\ D_k\pi \in \bigcap \{L^s(\mathbb{R}^3\backslash\overline{\Omega}) : s\in]1,\infty[\}$, for $1\le j,k,m\le 3$.

The set UNI ("**uni**queness") is to contain any pair $(u,\pi)$ satisfying the properties to follow:
$\mathbf{u}\in W^{2,1}_{loc}(\mathbb{R}^3\backslash\overline{\Omega})^3$, $\pi\in W^{1,1}_{loc}(\mathbb{R}^3\backslash\overline{\Omega})$; there is some $p\in]1,\infty[$ with
$D_k D_m \mathbf{u}|B_T(0)\backslash\overline{\Omega} \in L^p(B_T(0)\backslash\overline{\Omega})^3$ for $T>0$, $1\le k,m\le 3$, $\nabla\pi \in L^p(\mathbb{R}^3\backslash\overline{\Omega})^3$;
there exists $q\in]1,\infty[$, $R>0$ with $\overline{\Omega}\subseteq B_R(0)$, $\mathbf{u}|\mathbb{R}^3\backslash B_R(0) \in L^q(\mathbb{R}^3\backslash B_R(0))^3$.

The last condition implies that $\mathbf{u}(x)$ decays for large values of $|x|$. Of course a condition of this type is necessary to obtain uniqueness of the velocity part $\mathbf{u}$ of the solution to (1.1), (1.2).

Note that if $(\mathbf{u},\pi)\in$UNI, then the function $\mathbf{u}$ has a trace on $\partial\Omega$. In fact, as a consequence of [1] 5.4, we know for $(\mathbf{u},\pi)\in$UNI: $\mathbf{u}\in W^{2,p}_{loc}(\mathbb{R}^3\backslash\overline{\Omega})^3$, $\pi\in W^{1,p}_{loc}(\mathbb{R}^3\backslash\overline{\Omega})$ for some $p\in]1,\infty[$. Thus, by a lemma in [18], and by the definition of the set UNI, it follows that

(1.3) for any $(\mathbf{u},\pi)\in$UNI there is some $p\in]1,\infty[$ with
$\mathbf{u}|B_T(0)\backslash\overline{\Omega} \in W^{2,p}(B_T(0)\backslash\overline{\Omega})^3$, $\pi|B_T(0)\backslash\overline{\Omega} \in W^{1,p}(B_T(0)\backslash\overline{\Omega})$ for $T>0$.

Of course the assertion in (1.3) implies the trace property stated above.

For functions $\mathbf{f}$: $\mathbb{R}^3\backslash\overline{\Omega} \to \mathbb{C}^3$, $\mathbf{b}$: $\partial\Omega \to \mathbb{C}^3$, we define SOL($\mathbf{f}$,$\mathbf{b}$) ("**sol**utions of (1.1), (1.2)") as the set of all tuples $(\mathbf{u},\pi) \in W^{2,1}_{loc}(\mathbb{R}^3\backslash\overline{\Omega})^3\times W^{1,1}_{loc}(\mathbb{R}^3\backslash\overline{\Omega})$ such that $\mathbf{u}|B_T(0)\backslash\overline{\Omega}$ belongs to $W^{1,1}(B_T(0)\backslash\overline{\Omega})^3$ for some $T>0$ with $\overline{\Omega}\subseteq B_T(0)$ (that is, $\mathbf{u}$ has a trace on $\partial\Omega$), and such that $(\mathbf{u},\pi)$ solves (1.1) with right-hand side $\mathbf{f}$, and (1.2) with right-hand side $\mathbf{b}$.

For $r\in]1,\infty[$, $t\in[1, 3/2[$ with $t\le r$, define $I_0(r,t)$ as the set of all numbers $s\in]1,\infty[$ such that $(1/t - 2/3)^{-1} \le s \le (1/r - 2/3)^{-1}$, if $r<3/2$ and $t>1$; $3 < s \le (1/r - 2/3)^{-1}$, if $r<3/2$ and $t=1$; $(1/t - 2/3)^{-1} \le s$, if $r\ge 3/2$ and $t>1$; $3<s$, if $r\ge 3/2$ and $t=1$.

Furthermore, for r, t as above, let $I_1(r,t)$ be the set of all those numbers $q\in]1,\infty[$ which satisfy the inequality $(1/t - 1/3)^{-1} \le q \le (1/r - 1/3)^{-1}$, if $r<3$

and $t>1$; $3/2 < q \le (1/r - 1/3)^{-1}$, if $r<3$ and $t=1$; $(1/t - 1/3)^{-1} \le q$, if $r\ge 3$ and $t>1$; $3/2<q$, if $r\ge 3$ and $t=1$.

Now we are in a position to state our results:

Lemma 1.1: Let $r \in ]1,\infty[$. Then there is a solution operator for (1.1), (1.2) which maps any pair $(\mathbf{f},\mathbf{b}) \in C_0^\infty(\mathbb{R}^3)^3 \times BOU$ to a tuple $\big(\mathbf{U}(\mathbf{f},\mathbf{b},r),\ V(\mathbf{f},\mathbf{b},r)\big)$ belonging to $INT \cap SOL(\mathbf{f}|\mathbb{R}^3\backslash\overline{\Omega},\ \mathbf{b})$. This operator may be chosen in such a way that the following estimate holds:

Take $t\in[1,3/2[$ with $t\le r$, $s\in I_0(r,t)$, $q\in I_1(r,t)$. Then there are constants $K_1(r,s,t,\nu,\Omega)$, $K_2(r,q,t,\nu,\Omega)$, $K_3(r,t,\nu,\Omega) > 0$ with

$$\|\mathbf{u}\|_s \cdot K_1(r,s,t,\nu,\Omega)^{-1} + (\|D_k\mathbf{u}\|_q + \|\pi\|_q)\cdot K_2(r,q,t,\nu,\Omega)^{-1} + (\|D_mD_k\mathbf{u}\|_r + \|D_k\pi\|_r)\cdot K_3(r,t,\nu,\Omega)^{-1} \le \|\mathbf{f}\|_r + \|\mathbf{f}\|_t + \|\mathbf{b}\|_{2-1/r,r} \tag{1.4}$$

for $\mathbf{f}\in C_0^\infty(\mathbb{R}^3)^3$, $\mathbf{b}\in BOU$, $\mathbf{u} = \mathbf{U}(\mathbf{f},\mathbf{b},r)$, $\pi = V(\mathbf{f},\mathbf{b},r)$, $1\le k,m\le 3$.

Theorem1.1 (Uniqueness): Let $\mathbf{f}$, $\mathbf{b}$ be functions mapping $\mathbb{R}^3\backslash\overline{\Omega}$ and $\partial\Omega$, respectively, into $\mathbb{C}^3$. For $j\in\{1,2\}$, let $(\mathbf{u}(j),\ \pi(j))$ belong to $UNI \cap SOL(\mathbf{f},\mathbf{b})$. Then it follows: $\mathbf{u}(1) = \mathbf{u}(2)$, $\nabla\pi(1) = \nabla\pi(2)$.

If in addition there are numbers $S\in]0,\infty[$, $r(1),\ r(2) \in ]1,\infty[$ with $\overline{\Omega}\subseteq B_S(0)$, $\pi(j)\,|\mathbb{R}^3\backslash B_S(0) \in L^{r(j)}(\mathbb{R}^3\backslash B_S(0))$ for $j\in\{1,2\}$, then $\pi(1) = \pi(2)$.

Theorem 1.2 (Existence and regularity): Let $\mathbf{f} \in L^r(\mathbb{R}^3\backslash\overline{\Omega})^3 \cap L^t(\mathbb{R}^3\backslash\overline{\Omega})^3$, $\mathbf{b}\in W^{2-1/r,r}(\partial\Omega)^3$ for some $r\in]1,\infty[$, $t\in[1,\ 3/2[$ with $t\le r$. Then there exists a pair of functions $(\mathbf{X}(\mathbf{f},\mathbf{b}),\ Y(\mathbf{f},\mathbf{b})) \in UNI \cap SOL(\mathbf{f},\mathbf{b})$. According to Theorem 1.1, this pair is uniquely determined.

Let $r\in]1,\infty[$, $t\in[1,\ 3/2[$ with $t\le r$. Take $s\in I_0(r,t)$, $q\in I_1(r,t)$. Then inequality (1.4) holds for $\mathbf{f} \in L^r(\mathbb{R}^3\backslash\overline{\Omega})^3 \cap L^t(\mathbb{R}^3\backslash\overline{\Omega})^3$, $\mathbf{b}\in W^{2-1/r,r}(\partial\Omega)^3$, $\mathbf{u} = \mathbf{X}(\mathbf{f},\mathbf{b})$, $\pi = Y(\mathbf{f},\mathbf{b})$, $1\le k,m\le 3$, with the same constants $K_1(r,s,t,\nu,\Omega)$, $K_2(r,q,t,\nu,\Omega)$, $K_3(r,t,\nu,\Omega)$ as in Lemma 1.1.

Theorem 1.3: Assume $\Omega = B_1(0)$, $\nu = 1$. Take $p\in]3/2,\infty[$. Then there is no constant $K>0$ with the property to follow:

The inequality $\|D_k D_m \mathbf{u}\|_p \le K \cdot \|\mathbf{f}\|_p$ holds for $\mathbf{f} \in \bigcap \{ L^s(\mathbb{R}^3 \setminus \bar{\Omega})^3 \, : \, s \in [1,\infty] \}$, $(\mathbf{u},\pi) \in SOL(\mathbf{f},\mathbf{0}) \cap INT$, $1 \le k,m \le 3$.

## 2. Proof of Lemma 1.1.

First we shall define some functions which will be used for constructing a solution to (1.1), (1.2). Set

$$E_{jk}(x) := (8\cdot\pi\cdot\nu\cdot|x|)^{-1} \cdot (\delta_{jk} + x_j \cdot x_k \cdot |x|^{-2}), \qquad E_{4k}(x) := (4\cdot\pi\cdot|x|^3)^{-1}\cdot x_k$$
$(x \in \mathbb{R}^3 \setminus \{0\}, \quad 1 \le j,k \le 3)$;

$$u_m(x) := \int_{\mathbb{R}^3} \sum_{1\le j\le 3} E_{jm}(x-y)\cdot f_j(y)\, dy, \qquad \pi(f)(x) := \int_{\mathbb{R}^3} \sum_{1\le j\le 3} E_{j4}(x-y)\cdot f_j(y)\, dy$$
$(x \in \mathbb{R}^3, \quad \mathbf{f} \in C_0^\infty(\mathbb{R}^3)^3, \quad 1 \le m \le 3)$;

$$V_m(z,\Phi) := \int_{\partial\Omega} \sum_{1\le j\le 3} E_{jm}(x-y)\cdot \Phi_j(y)\, d\Omega(y),$$

$$Q(x,\Phi) := \int_{\partial\Omega} \sum_{1\le j\le 3} E_{4j}(x-y)\cdot \Phi_j(y)\, d\Omega(y) \qquad (z \in \mathbb{R}^3, \ x \in \mathbb{R}^3\setminus\partial\Omega, \ \Phi \in C^0(\partial\Omega)^3, \ 1 \le m \le 3);$$

$$S_{jkm} := \delta_{jk}\cdot E_{4m} - D_j E_{km} - D_k E_{jm} \qquad (1 \le j,k,m \le 3);$$

$$W_m(x,\Phi) := \int_{\partial\Omega} \sum_{1\le j,k\le 3} - S_{jkm}(x-y)\cdot \Phi_j(y)\cdot n_k(y)\, d\Omega(y),$$

$$\Pi(x,\Phi) := \int_{\partial\Omega} \sum_{1\le j,k\le 3} 2\cdot\nu\cdot D_j E_{4k}(x-y)\cdot \Phi_j(y)\cdot n_k(y)\, d\Omega(y)$$

$(x \in \mathbb{R}^3\setminus\partial\Omega, \quad \Phi \in C^0(\partial\Omega)^3, \quad 1 \le m \le 3)$.

The integral appearing on the right-hand side of the definition of $W_m(x,\Phi)$ exists for $x \in \partial\Omega$ too (see [6] Lemma 4.7). However, this integral, considered as a function of $x \in \mathbb{R}^3$, does not belong to $C^0(\mathbb{R}^3)$. For this reason, we use a different notation in the case $x \in \partial\Omega$, setting

$$(2.1) \qquad \mathcal{T}_m(\Phi)(x) := 2\cdot \int_{\partial\Omega} \sum_{1\le j,k\le 3} S_{jkm}(x-y)\cdot \Phi_j(y)\cdot n_k(y)\, d\Omega(y)$$
$(\Phi \in C^0(\partial\Omega)^3, \ x \in \partial\Omega, \ 1 \le m \le 3)$.

Setting

(2.2) $\mathbf{W}(x,\Phi)_{ex} := \mathbf{W}(x,\Phi)$ for $x \in \mathbb{R}^3 \setminus \overline{\Omega}$,

$\mathbf{W}(x,\Phi)_{ex} := (-1/2)\cdot(\Phi + \mathcal{T}(\Phi))(x)$ for $x \in \partial\Omega$,

we obtain a function $\mathbf{W}(\cdot,\Phi)_{ex}$ from $\mathbb{R}^3 \setminus \Omega$ into $\mathbb{C}^3$ which is continuous ([6] Satz 4.1). For $\Phi \in C^0(\partial\Omega)^3$, $1 \le m \le 3$, $x \in \partial\Omega$, define $\mathcal{T}^*_m(\Phi)(x)$ by replacing $n_k(y)$ with $n_k(x)$ on the right-hand side of (2.1). Choose functions $\Psi_{(1)}, \ldots, \Psi_{(6)} \in C^0(\partial\Omega)^3$ forming a basis of the subspace $\{\Phi \in L^p(\partial\Omega)^3 : \Phi + \mathcal{T}^*(\Phi) = \mathbf{0}\}$ of $L^p(\partial\Omega)^3$, for $p \in ]1,\infty[$; see [6] Lemma 6.10. Define a matrix $M \in \mathbb{C}^{6\times 6}$ by setting

$$M := \left( \int_{\partial\Omega} \langle \mathbf{V}(x,\Psi_{(m)}), \Psi_{(k)}(x) \rangle \, d\Omega(x) \right)_{1\le k,m\le 6}.$$

M is regular; see [6] Lemma 6.12. For $\mathbf{b} \in C^0(\partial\Omega)^3$, set

$$\mathbf{c}(\mathbf{b}) := M^{-1} \cdot \left( \int_{\partial\Omega} \langle \mathbf{b}(x), \Psi_{(k)}(x) \rangle \, d\Omega(x) \right)_{1\le k\le 6}.$$

We further define for $\Phi$, $\mathbf{b} \in C^0(\partial\Omega)^3$:

(2.3) $$\mathbf{v}(\Phi,\mathbf{b}) := \mathbf{W}(\cdot,\Phi)_{ex} + \sum_{1\le m\le 6} c_m(\mathbf{b}) \cdot \mathbf{V}(\cdot,\Psi_{(m)}),$$

$$g(\Phi,\mathbf{b}) := \left( \Pi(\cdot,\Phi) + \sum_{1\le m\le 6} c_m(\mathbf{b}) \cdot Q(\cdot,\Psi_{(m)}) \right) \Big| \mathbb{R}^3 \setminus \overline{\Omega}.$$

For $\mathbf{f} \in C_0^\infty(\mathbb{R}^3)^3$, the mappings $\mathbf{u}(\mathbf{f})$, $\pi(\mathbf{f})$ are $C^\infty$-functions on $\mathbb{R}^3$, and the pair $(\mathbf{u}(\mathbf{f}), \pi(\mathbf{f}))$ is a solution in $\mathbb{R}^3$ of the systems of equations in (1.1) (see [6] (1.12), Lemma 1.10). If $\Phi$, $\mathbf{b} \in C^0(\partial\Omega)^3$, then $\mathbf{v}(\Phi,\mathbf{b})$ is continuous on $\mathbb{R}^3 \setminus \Omega$ ([6] Satz 4.1, Lemma 6.1), and $\mathbf{v}(\Phi,\mathbf{b}) | \mathbb{R}^3 \setminus \overline{\Omega}$, $g(\Phi,\mathbf{b})$ are $C^\infty$-functions on $\mathbb{R}^3 \setminus \overline{\Omega}$. Moreover, according to [6] Satz 6.4, the pair $\left(\mathbf{v}(\Phi,\mathbf{b}) | \mathbb{R}^3 \setminus \overline{\Omega}, g(\Phi,\mathbf{b})\right)$ solves (1.1) for $\mathbf{f} = \mathbf{0}$.

Take $\mathbf{b} \in BOU$, $\mathbf{f} \in C_0^\infty(\mathbb{R}^3)^3$. By [6] Satz 6.4, there exists a function $\Phi \in C^0(\partial\Omega)^3$ with

(2.4) $\mathbf{v}(\Phi, \mathbf{b}-\mathbf{u}(\mathbf{f})) | \partial\Omega = \mathbf{b}-\mathbf{u}(\mathbf{f})$.

For any $\Phi \in C^0(\partial\Omega)^3$ satisfying equation (2.4), the tuple $(\mathbf{R}(\Phi,\mathbf{f},\mathbf{b}), S(\Phi,\mathbf{f},\mathbf{b}))$, defined by

$$\mathbf{R}(\Phi,\mathbf{f},\mathbf{b}) := \left(\mathbf{u}(\mathbf{f}) - \mathbf{v}(\Phi, \mathbf{b}-\mathbf{u}(\mathbf{f}))\right) \Big| \mathbb{R}^3 \setminus \overline{\Omega}, \qquad S(\Phi,\mathbf{f},\mathbf{b}) := \pi(\mathbf{f}) + g(\Phi, \mathbf{b}-\mathbf{u}(\mathbf{f})),$$

belongs to $SOL(\mathbf{f} | \mathbb{R}^3 \setminus \overline{\Omega}, \mathbf{b})$. This is the type of solutions constructed by Ladyzhenskaya. The key step in her approach consists in solving the integral equation given implicitly by (2.4).

Let us now study the regularity properties of the solutions constructed before. For this purpose, we shall use the symbol $M(\rho_1,...,\rho_n)$ to denote constants only depending on $\nu$, $\Omega$, $\rho_1$, ..., $\rho_n$, where $n\in\mathbb{N}$, and $\rho_1$, ..., $\rho_n$ $\in$ $]0,\infty[$.

By referring to the Hardy-Littlewood-Sobolev inequality (see [21] p. 119), to the Calderon-Zygmund theorem, and to [6] Lemma 1.9, 1.10, we obtain the following inequalities:

(2.5) $\|\mathbf{u}(\mathbf{f})\|_s \leq M(s)\cdot\|\mathbf{f}\|_{(1/s\,+\,2/3)^{-1}}$,

$\|D_k\mathbf{u}(\mathbf{f})\|_q + \|\pi(\mathbf{f})\|_q \leq M(q)\cdot\|\mathbf{f}\|_{(1/q\,+\,1/3)^{-1}}$,

$\|D_mD_k\mathbf{u}(\mathbf{f})\|_r + \|D_k\pi(\mathbf{f})\|_r \leq M(r)\cdot\|\mathbf{f}\|_r$

$(s\in]3,\infty[,\ q\in]3/2,\infty[,\ r\in]1,\infty[,\ 1\leq k,m\leq 3,\ \mathbf{f}\in C_0^\infty(\mathbb{R}^3)^3)$.

The preceding relations imply for $r\in]1,\infty[$, $t\in[1,\ 3/2[$ with $t\leq r$:

(2.6) $\|\mathbf{u}(\mathbf{f})|\partial\Omega\|_{2-1/r,r} \leq M(r,t)\cdot(\|\mathbf{f}\|_r + \|\mathbf{f}\|_t)$ for $\mathbf{f}\in C_0^\infty(\mathbb{R}^3)^3$.

Now take $\boldsymbol{\Phi}\in BOU$, $\mathbf{b}\in C^0(\partial\Omega)^3$. Abbreviate for $r\in]1,\infty[$:

$K(r) := \|\boldsymbol{\Phi}\|_{2-1/r,r} + \|\mathbf{b}\|_r$.

Then, for $r\in]1,\infty[$, we may evaluate the functions $\mathbf{v}(\boldsymbol{\Phi},\mathbf{b})$, $g(\boldsymbol{\Phi},\mathbf{b})$ as follows:

(2.7) $\|\mathbf{v}(\boldsymbol{\Phi},\mathbf{b})\|_s \leq M(r,s)\cdot K(r)$ for $s\in]3,\ (1/r - 2/3)^{-1}]$, if $r<3/2$, and for $s\in]3,\infty[$ else;

$\|D_k\big(\mathbf{v}(\boldsymbol{\Phi},\mathbf{b})|\mathbb{R}^3\backslash\overline{\Omega}\big)\|_q + \|g(\boldsymbol{\Phi},\mathbf{b})\|_q \leq M(r,q)\cdot K(r)$ for $1\leq k\leq r$, $q\in]3/2,\ (1/r - 1/3)^{-1}]$, if $r<3$, and for $q\in]3/2,\infty[$ else;

$\|D_mD_k\big(\mathbf{v}(\boldsymbol{\Phi},\mathbf{b})|\mathbb{R}^3\backslash\overline{\Omega}\big)\|_r + \|D_kg(\boldsymbol{\Phi},\mathbf{b})\|_r \leq M(r)\cdot K(r)$ for $1\leq k\leq 3$.

Let us remark on the proof of (2.7). First we note a property of the single-layer potentials $\mathbb{V}(\cdot,\boldsymbol{\Psi}_{(m)})$ and $Q(\cdot,\boldsymbol{\Psi}_{(m)})$, namely:

(2.8) $\mathbb{V}(\cdot,\boldsymbol{\Psi}_{(m)})|\partial\Omega \in W^{2-1/r,r}(\partial\Omega)^3$, $Q(\cdot,\boldsymbol{\Psi}_{(m)})|B_R(0)\backslash\overline{\Omega} \in W^{1,r}(B_R(0)\backslash\overline{\Omega})$ for $R>0$, $r\in]1,\infty[$, $1\leq m\leq 6$.

The relations in (2.8) represent a difficulty particular to the exterior problem. For their proof, we refer to [5] p. 8-38. Note that the restriction $r>3/2$ in [5] Lemma 8.4 may be removed by replacing the reference to [6] Lemma 7.12 by another one to [4] Theorem 2.1 . These two references concern $L^p$-estimates for the Laplacian on bounded domains; they were both derived from [17] Theorem 10.10.

Now fix some number $R>0$ with $|x-y| \geq |x|/2$ for $x \in \mathbb{R}^3 \backslash B_R(0)$, $y \in \overline{\Omega}$. Then

(2.9) $\quad \|\Pi(\cdot,\bullet) \mid B_R(0) \backslash \overline{\Omega}\|_r \leq M(r) \cdot \|\bullet\|_{2-1/r,r} \qquad (r \in ]1,\infty[)$.

For $r \in ]3/2, \infty[$, this inequality is given by [6] Lemma 7.15. Its proof is based on $L^p$-estimates of the Laplacian on bounded domains, as stated in [6] Lemma 7.12. In the case $r \leq 3/2$, this proof also carries through, provided we again replace the reference [6] Lemma 7.12 by [4] Theorem 2.1.

With (2.8) and (2.9) at our disposal, we immediately arrive at the inequality

(2.10) $\quad \|g(\bullet,\mathbf{b}) \mid B_R(0) \backslash \overline{\Omega}\|_{1,r} \leq M(r) \cdot K(r) \qquad (r \in ]1,\infty[)$.

Recalling that the tuple $\left(\mathbf{v}(\bullet,\mathbf{b}) \mid \mathbb{R}^3 \backslash \overline{\Omega},\ g(\bullet,\mathbf{b})\right)$ solves (1.1) with $\mathbf{f} = \mathbf{0}$, we may conclude from (2.10), from the definitions in (2.2), (2.3), and from [4] Theorem 2.1 ($L^p$-estimates for the Laplacian):

(2.11) $\quad \|\mathbf{v}(\bullet,\mathbf{b}) \mid B_R(0) \backslash \overline{\Omega}\|_{2,r} \leq M(r) \cdot \left( \|\bullet + \mathcal{T}(\bullet)\|_{2-1/r,r} + K(r) \right) \leq M(r) \cdot K(r)$.

with the last inequality implied by [6] Lemma 7.7. Now the estimates in (2.7), with $\mathbf{v}(\bullet,\mathbf{b})$, $g(\bullet,\mathbf{b})$ replaced by $\mathbf{v}(\bullet,\mathbf{b}) \mid B_R(0) \backslash \overline{\Omega}$, $g(\bullet,\mathbf{b}) \mid B_R(0) \backslash \overline{\Omega}$, follow from (2.10), (2.11), and [1] 5.4 (Sobolev's lemma). Thus (2.7) is proved if we can evaluate the restrictions of $\mathbf{v}(\bullet,\mathbf{b})$, $g(\bullet,\mathbf{b})$ to $\mathbb{R}^3 \backslash B_R(0)$. But this is an easy task. The corresponding calculations may be found in the proof of [5] Lemma 8.6.

For $\bullet$, $\mathbf{b}$, $\mathbf{f}$ from BOU, $C^0(\partial\Omega)^3$, $C_0^\infty(\mathbb{R}^3)^3$, respectively, it follows from (2.5), (2.7) that $\left(\mathbf{R}(\bullet,\mathbf{f},\mathbf{b}),\ S(\bullet,\mathbf{f},\mathbf{b})\right) \in$ INT.

Let us now fix $r \in ]1,\infty[$. By combining (2.8) with [6] Satz 6.4, 7.2, Lemma 6.1, 7.8, we may construct a constant $M(r)>0$ with the following properties:

For $\mathbf{b} \in$ BOU, $\mathbf{f} \in C_0^\infty(\mathbb{R}^3)^3$, there is a function $\bullet = \bullet(\mathbf{b},\mathbf{f},r) \in$ BOU satisfying equation (2.4), as well as the inequality

(2.12) $\quad \|\bullet(\mathbf{b},\mathbf{f},r)\|_{2-1/r,r} \leq M(r) \cdot \|\mathbf{b} - \mathbf{u}(\mathbf{f})\|_{2-1/r,r}$.

Define for $\mathbf{b}\in BOU$, $\mathbf{f}\in C_0^\infty(\mathbb{R}^3)^3$:

$\mathbf{U}(\mathbf{f},\mathbf{b},r) := \mathbb{R}\big(\Phi(\mathbf{f},\mathbf{b},r),\ \mathbf{f},\ \mathbf{b}\big)$, $\quad V(\mathbf{f},\mathbf{b},r) := S\big(\Phi(\mathbf{f},\mathbf{b},r),\ \mathbf{f},\ \mathbf{b}\big)$.

Then Lemma 1.1 follows from (2.5), (2.6), (2.7), and (2.12).

## 3. Uniqueness

Let $\mathbf{f}$, $\mathbf{b}$ be functions mapping $\mathbb{R}^3\backslash\overline{\Omega}$, $\partial\Omega$, respectively, into $\mathbb{C}^3$. For $j\in\{1,2\}$, assume the pair $\big(\mathbf{u}(j),\ \pi(j)\big)$ belongs to UNI and to $SOL(\mathbf{f},\mathbf{b})$. According to the definition of UNI, we may choose $R\in]0,\infty[$, $p(j)\in]1,\infty[$ with

(3.1) $\quad \overline{\Omega}\subseteq B_R(0), \qquad \mathbf{u}(j)|\mathbb{R}^3\backslash B_R(0) \in L^{p(j)}(\mathbb{R}^3\backslash B_R(0))^3 \qquad (j\in\{1,2\}).$

Furthermore, by (1.5) there is some number $r(j)\in]1,\infty[$ such that

$\mathbf{u}(j)|B_T(0)\backslash\overline{\Omega} \in W^{2,r(j)}(B_T(0)\backslash\overline{\Omega})^3, \qquad \pi(j)|B_T(0)\backslash\overline{\Omega} \in W^{1,r(j)}(B_T(0)\backslash\overline{\Omega})$

for $T>0$; $\qquad \nabla\pi(j) \in L^{r(j)}(\mathbb{R}^3\backslash\overline{\Omega})^3 \qquad (j\in\{1,2\})$.

Thus it is clear that the zero extensions of the functions $u(j)_k|\mathbb{R}^3\backslash B_R(0)$, $D_k\pi(j)$ belong to SLI ($1\le j\le 2$, $1\le k\le 3$). But this property is also true for $\pi(j)|\mathbb{R}^3\backslash B_R(0)$. To see this, approximate $\pi(j)$ by smooth functions, and then apply the mean-value theorem.

Now set $\mathbf{u} := \mathbf{u}(1) - \mathbf{u}(2)$, $\pi := \pi(1) - \pi(2)$, $r := \min\{r(1),\ r(2)\}$. This means

(3.2) $\quad \mathbf{u}|B_T(0)\backslash\overline{\Omega} \in W^{2,r}(B_T(0)\backslash\overline{\Omega})^3, \qquad \pi|B_T(0)\backslash\overline{\Omega} \in W^{1,r}(B_T(0)\backslash\overline{\Omega}) \qquad$ for $T>0$;

(3.3) $\quad -\nu\cdot\Delta\mathbf{u} + \nabla\pi = \mathbf{0}, \qquad \operatorname{div}\mathbf{u} = 0, \qquad \mathbf{u}|\partial\Omega = \mathbf{0}.$

Note that the zero extensions of $u_k|\mathbb{R}^3\backslash B_R(0)$, $\pi|\mathbb{R}^3\backslash B_R(0)$, $D_k\pi$ ($1\le k\le 3$) belong to SLI. Next we want to show that

(3.4) $\quad \nabla\pi|\mathbb{R}^3\backslash B_{R+1}(0) \in L^s(\mathbb{R}^3\backslash B_{R+1}(0))^3 \qquad$ for $s\in]1,r]$.

To this end, choose $\varphi\in C^\infty(\mathbb{R}^3)$ with $0\le\varphi\le 1$, $\varphi|\mathbb{R}^3\backslash B_{R+1/2}(0) = 1$, $\varphi|B_R(0) = 0$. The zero extensions of the functions $u_k\cdot\varphi$, $\pi\cdot\varphi$, $\langle\mathbf{u},\nabla\varphi\rangle$, $-\nu\cdot(\langle\nabla u_k,\ \nabla\varphi\rangle + u_k\cdot\Delta\varphi) + \pi\cdot D_k\varphi$ to $\mathbb{R}^3$ will be denoted by $v_k$, $\Pi$, $g$, $h_k$, respectively ($1\le k\le 3$). Observe that

$\mathbf{v}\in W^{2,r}_{loc}(\mathbb{R}^3)^3, \qquad \Pi\in W^{1,r}_{loc}(\mathbb{R}^3), \qquad \mathbf{h}\in\bigcap\{L^s(\mathbb{R}^3)^3 : s\in[1,r]\},$

$g\in\bigcap\{W^{1,s}(\mathbb{R}^3) : s\in[1,r]\},$

(3.5) $-\nu \cdot \Delta \mathbf{v} + \nabla \Pi = \mathbf{h}, \qquad \operatorname{div} \mathbf{v} = g \qquad \text{in } \mathbb{R}^3.$

In addition, the functions $v_k$, $\Pi$, $h_k$, $g$, $D_k\Pi$, $D_k g$ are contained in SLI, for $1 \le k \le 3$. Because of (3.5), this property also holds for $\Delta v_k$ and $\operatorname{div} \mathbf{v}$ $(1 \le k \le 3)$. Now we obtain from (3.5):

$$-\nu \cdot \hat{T}_{\Delta v_k} + \hat{T}_{D_k \Pi} = \hat{T}_{h_k} \quad (1 \le k \le 3), \qquad \hat{T}_{\operatorname{div} \mathbf{v}} = \hat{T}_g.$$

By an argument based on [23] p. 147 (5), we may conclude from the preceding equations:

(3.6) $$\nu \cdot |I|^2 \cdot \hat{T}_{v_k} + i \cdot I_k \cdot \hat{T}_{\Pi} = \hat{T}_{h_k} \quad (1 \le k \le 3), \qquad \sum_{1 \le m \le 3} i \cdot I_m \cdot \hat{T}_{v_m} = \hat{T}_g.$$

(3.6) yields after some computations:

$$|I|^2 \cdot \hat{T}_{D_m \Pi} = \nu \cdot |I|^2 \cdot \hat{T}_{D_m g} + \sum_{1 \le k \le 3} I_m \cdot I_k \cdot \hat{T}_{h_k} \quad (1 \le m \le 3).$$

It follows by [23] p. 149/150:

(3.7) $$\int_{\mathbb{R}^3} \{ (D_m \Pi - \nu \cdot D_m g) \cdot \Delta \hat{\Psi} + \sum_{1 \le k \le 3} h_k \cdot (I_m \cdot I_k \cdot \Psi)^{\wedge} \} \, dx = 0$$

for $\Psi \in S(\mathbb{R}^3)$, $1 \le m \le 3$.

Now fix $q \in ]1,r]$, $m \in \{1,2,3\}$. By [21] p. 96 there exists a bounded operator $P_k$ mapping the space $L^q(\mathbb{R}^3)$ into itself, and satisfying the equation

(3.8) $$P_k(\zeta) = [ I_m \cdot I_k \cdot |I|^{-2} \cdot \hat{\zeta} ]^{\sim}, \qquad \text{for } \zeta \in L^q(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \qquad (1 \le k \le 3).$$

From (3.7) and (3.8) we obtain

(3.9) $$\int_{\mathbb{R}^3} \{ D_m \Pi - \nu \cdot D_m g - \sum_{1 \le k \le 3} P_k(h_k) \} \cdot \Delta \hat{\Psi} \, dx = 0 \qquad (\Psi \in S(\mathbb{R}^3)).$$

Since the Fourier-transform maps $S(\mathbb{R}^3)$ onto itself, and because $C_0^\infty(\mathbb{R}^3)$ is a subset of $S(\mathbb{R}^3)$, we may replace $\Psi$ in (3.9) by any function $\zeta \in C_0^\infty(\mathbb{R}^3)$. Thus the mapping

$$F := D_m \Pi - \nu \cdot D_m g - \sum_{1 \le k \le 3} P_k(h_k)$$

is a weak solution of the equation $\Delta u = 0$ in $\mathbb{R}^3$. This means that for any $\varepsilon > 0$, the corresponding mollifier $F_\varepsilon$ (see [1] 2.18) satisfies the equation $\Delta F_\varepsilon = 0$. Since

(3.10) $$D_m \Pi \in L^{r(1)}(\mathbb{R}^3) + L^{r(2)}(\mathbb{R}^3), \qquad \nu \cdot D_m g + \sum_{1 \le k \le 3} P_k(h_k) \in L^q(\mathbb{R}^3),$$

we may conclude that $F_\varepsilon$ is bounded ($\varepsilon>0$). Now Liouville's theorem yields $F_\varepsilon = c(\varepsilon)$ for some $c(\varepsilon)\in\mathbb{C}$ ($\varepsilon>0$). Because of (3.10), this means $F_\varepsilon = 0$ for $\varepsilon>0$, so we have $F = 0$. Thus we may conclude from the second relation in (3.10): $D_m\Pi \in L^q(\mathbb{R}^3)$. This proves (3.4).

Now take $\mathbf{\Phi}\in C_0^\infty(\mathbb{R}^3)^3$. According to Lemma 1.1, there is a pair $(\mathbf{w},\Psi)\in\mathrm{INT}$ with

(3.11) $\quad -\nu\cdot\Delta\mathbf{w} + \nabla\Psi = \mathbf{\Phi}|\mathbb{R}^3\backslash\bar{\Omega}, \qquad \operatorname{div}\mathbf{w} = 0, \qquad \mathbf{w}|\partial\Omega = \mathbf{0}.$

For $k\in\mathbb{N}$, choose $\zeta_k\in C_0^\infty(\mathbb{R}^3)$ with $\zeta_k|B_k(0) = 1$, $\zeta_k|\mathbb{R}^3\backslash B_{2\cdot k}(0) = 0$, $0\le\zeta_k\le 1$. Set $\mathbf{w}_k := \mathbf{w}\cdot\zeta_k$, $\Psi := \Psi\cdot\zeta_k$ ($k\in\mathbb{N}$).

Recalling (3.1), (3.11), and the relation $(\mathbf{w},\Psi)\in\mathrm{INT}$, we may conclude:

$$\int_{\mathbb{R}^3\backslash\bar{\Omega}} \langle \mathbf{u},\mathbf{\Phi}\rangle\, dx = \lim_{k\to\infty} \int_{\mathbb{R}^3\backslash\bar{\Omega}} \langle \mathbf{u}, -\nu\cdot\Delta\mathbf{w}_k + \nabla\Psi_k\rangle\, dx.$$

For $k\in\mathbb{N}$, it follows from (3.2), (3.3):

$$\int_{\mathbb{R}^3\backslash\bar{\Omega}} \langle \mathbf{u}, -\nu\cdot\Delta\mathbf{w}_k + \nabla\Psi_k\rangle\, dx = -\int_{\mathbb{R}^3\backslash\bar{\Omega}} \langle \nabla\pi,\mathbf{w}_k\rangle\, dx.$$

Since in general the relation $\mathbf{w}\in L^r(\mathbb{R}^3\backslash\bar{\Omega})^3$ holds for $r>3$ only, we have to recur to (3.4) in order to obtain

(3.12) $\quad \int_{\mathbb{R}^3\backslash\bar{\Omega}} \langle \nabla\pi,\mathbf{w}_k\rangle\, dx \;\to\; \int_{\mathbb{R}^3\backslash\bar{\Omega}} \langle \nabla\pi,\mathbf{w}\rangle\, dx \qquad (k\to\infty).$

But the right-hand side in (3.12) vanishes, as follows from (3.2), (3.4), (3.11). Combining the preceding results, we arrive at the equation

$$\int_{\mathbb{R}^3\backslash\bar{\Omega}} \langle \mathbf{u},\mathbf{\Phi}\rangle\, dx = \mathbf{0} \quad \text{for } \mathbf{\Phi}\in C_0^\infty(\mathbb{R}^3),$$

which implies $\mathbf{u} = \mathbf{0}$. Theorem 1.1 now follows easily.

## 4. Proof of Theorem 1.3.

In this section, we assume $\Omega = B_1(0)$, $\nu = 1$. For $n\in\mathbb{N}$, $x\in\mathbb{R}^3$, set

(4.1) $\quad \mathbf{f}_n(x) := 3\cdot\left((3+n)^2 - 9\right)^{-1}\cdot(1,0,0), \qquad \text{if } x\in B_{3+n}(0)\backslash B_3(0);$
$\quad \mathbf{f}_n(x) := 0$ else.

Using notations as in Section 2, we observe:

$\left(\mathbf{u}(\mathbf{f}_n)|\mathbb{R}^3\setminus\overline{\Omega},\ \pi(\mathbf{f}_n)|\mathbb{R}^3\setminus\overline{\Omega}\right) \in \mathrm{INT} \quad (n\in\mathbb{N}) \quad$ (see (2.5));

$-\nu\cdot\Delta\mathbf{u}(\mathbf{f}_n) + \nabla\pi(\mathbf{f}_n) = \mathbf{f}_n, \quad \operatorname{div}\mathbf{u}(\mathbf{f}_n) = 0 \quad \text{in } \mathbb{R}^3 \quad (n\in\mathbb{N}).$

The definition of $\mathbf{f}_n$ was chosen in such a way that $\mathbf{u}(\mathbf{f}_n)(x) = (1,0,0)$ for $x\in\partial\Omega$, $n\in\mathbb{N}$.

From Lemma 1.1 we know there is a pair $(\mathbf{V},P)\in\mathrm{INT}$ with

$-\nu\cdot\Delta\mathbf{V} + \nabla P = 0, \quad \operatorname{div}\mathbf{V} = \mathbf{0}, \quad \mathbf{V}|\partial\Omega = (-1,0,0).$

Now set for $n\in\mathbb{N}$

(4.2) $\mathbf{w}_n := \mathbf{u}(\mathbf{f}_n) + \mathbf{V}, \quad \Pi_n := \pi(\mathbf{f}_n) + P.$

Then we have $(\mathbf{w}_n,\Pi_n) \in \mathrm{INT} \cap \mathrm{SOL}(\,\mathbf{f}_n|\mathbb{R}^3\setminus\overline{\Omega},\ \mathbf{0}\,)$.

Next fix $p\in]3/2,\infty[$, and assume there is some number $K\in]0,\infty[$ such that $\|D_kD_m\mathbf{u}\|_p \le K\cdot\|\mathbf{g}\|_p$ for $1\le k,m\le 3$, $\mathbf{g} \in \bigcap\{L^r(\mathbb{R}^3\setminus\overline{\Omega})^3 : r\in[1,\infty]\}$, and for $(\mathbf{u},\pi)\in\mathrm{INT}\cap\mathrm{SOL}(\mathbf{g},\mathbf{0})$. Then, recalling (2.5), (4.1), (4.2), we obtain for $n\in\mathbb{N}$, $1\le k,m\le 3$:

$$\|D_kD_m\mathbf{V}\|_p \le \|D_kD_m\mathbf{w}_n\|_p + \|D_kD_m\mathbf{u}(\mathbf{f}_n)\|_p \le (K+M(p))\cdot\|\mathbf{f}_n\|_p$$
$$\le c(p)\cdot n^{(3/p)-2},$$

with a constant $c(p)>0$ which is independent of n. Letting n tend to infinity, it follows $D_kD_m\mathbf{V} = \mathbf{0}$ $(1\le k,m\le 3)$. Since $(\mathbf{V},P)\in\mathrm{INT}$, the preceding equation implies $\mathbf{V} = \mathbf{0}$. This is a contradiction, since $\mathbf{V}$ is continuous on $\mathbb{R}^3\setminus\Omega$, with $\mathbf{V}|\partial\Omega = (-1,0,0)$.

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