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The Stokes System in an Infinite Cone

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Preface

The book at hand is based on the habilitation thesis of the author. This thesis was inspired and supported by Prof. Erhard Meister (Darmstadt). Without him, this work would never have come into being. Furthermore, the author gratefully acknowledges that it was Prof. von Wahl (Bayreuth) who introduced him – by means of [13] – to the potential theoretic treatment of the Stokes system. In addition, the author wants to thank Prof. Alber (Darmstadt), Prof. Meister and Prof. von Wahl for acting as referees for his thesis, and for pointing out some errors.

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Anyone who glances through this book may wonder why it contains so many formulas. The reason is that we wanted to extensively present the "hard analysis" part of our theory – of course without neglecting functional analytic arguments. In this way, our proofs should be accessible not only for specialists working in the field of singular integrals, but for any reader with a solid mathematical background.

A number of results derived in Chapter 3 to 8 slightly generalize certain facts which are already known, either from the paper [9] by the present author, or from the article [20] by Fabes, Jodeit, Lewis. Since these facts were proved only very shortly in [9] or [20], we thought it worthwhile to develop them in more detail here. The larger part of this book, however, covers new material which was not published before.

Those elements of our theory contained in Chapter 3 to 11 are studied because they are needed as tools, although some of them, such as Theorem 8.1 and 8.2, are of interest for their own sake. It is in Chapter 12 and 13 that we shall put together all our previous results, like pieces of a puzzle, to obtain our main theorems.

Darmstadt, November 1993 — P. Deuring

Contents

Chapter 1

Introduction

In the work at hand, we shall study certain double-layer potentials related to the Stokes system in a right circular infinite cone in $\mathbb{R}^3$. Let us first define these potentials. To this end, set

$$\widetilde{g}_1(r) := e^{-r} + r^{-2}\cdot(r\cdot e^{-r} + e^{-r} - 1), \tag{1.1}$$
$$\widetilde{g}_2(r) := e^{-r} + 3\cdot r^{-2}\cdot(r\cdot e^{-r} + e^{-r} - 1) \quad \text{for } r \in \mathbb{C}\backslash\{0\},$$

and

$$\widetilde{E}^{\lambda}_{jk}(z) := (4\cdot\pi\cdot|z|)^{-1}\cdot\Big(\delta_{jk}\cdot\widetilde{g}_1(\sqrt{\lambda}\cdot|z|) - z_j\cdot z_k\cdot|z|^{-2}\cdot\widetilde{g}_2(\sqrt{\lambda}\cdot|z|)\Big), \tag{1.2}$$
$$E_{4k}(z) := (4\cdot\pi\cdot|z|^3)^{-1}\cdot z_k$$

for $z \in \mathbb{R}^3\backslash\{0\}$, $j, k \in \{1, 2, 3\}$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$. Note that the matrix-valued function $(\widetilde{E}^{\lambda}_{1k}, \widetilde{E}^{\lambda}_{2k}, \widetilde{E}^{\lambda}_{3k}, E_{4k})_{1\le k\le 3}$ is a fundamental solution of the resolvent problem

$$-\Delta u + \lambda\cdot u + \nabla\pi = f, \qquad \operatorname{div} u = 0 \tag{1.3}$$

related to the Stokes system. This fundamental solution was constructed by McCracken [33, p. 204-206]. Let the stress tensor $(\widetilde{\mathcal{D}}^{\lambda}_{jkl})_{j,k,l\in\{1,2,3\}}$ be defined by

$$\widetilde{\mathcal{D}}^{\lambda}_{jkl} := D_j\widetilde{E}^{\lambda}_{kl} + D_k\widetilde{E}^{\lambda}_{jl} - \delta_{jk}\cdot E_{4l} \qquad (j, k, l \in \{1, 2, 3\},\ \lambda \in \mathbb{C}\backslash(-\infty, 0]). \tag{1.4}$$

Take $p \in (1, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\tau \in \{-1, 1\}$, and assume that B is an open set in $\mathbb{R}^3$, with boundary ∂B smooth enough such that an outward unit normal $n^{(B)}$ to B exists. Then we define the double layer potential $\Gamma(\tau, p, \lambda, B) : L^p(\partial B)^3 \mapsto L^p(\partial B)^3$ by

$$\Gamma(\tau, p, \lambda, B)(f)(x) \tag{1.5}$$
$$:= (\tau/2)\cdot f(x) + \Big(\int_{\partial B}\sum_{j,k=1}^{3}\widetilde{\mathcal{D}}^{\lambda}_{jkl}(x-y)\cdot n^{(B)}_k(y)\cdot f_j(y)\,dB(y)\Big)_{1\le l\le 3},$$

for $f \in L^p(\partial B)^3$, $x \in \partial B$. Of course, this definition only makes sense if the integral on the right-hand side of (1.5) exists and, when considered as a function of x, belongs to $L^p(\partial B)^3$. This condition is satisfied if, for example, B is a bounded domain with a smooth boundary; see [8, p. 342-344, Section 3]. Here we shall mainly be interested in the case $B = \mathbb{K}(\varphi)$, where $\mathbb{K}(\varphi)$ denotes an open right circular infinite cone in $\mathbb{R}^3$, with vertex angle $2\cdot\varphi$ ($\varphi \in (0, \pi)$). Without loss of generality, we may assume that the axis of this cone coincides with the positive x_3-axis. This means in particular the vertex of $\mathbb{K}(\varphi)$ is located at the origin. It will turn out that for $B = \mathbb{K}(\varphi)$, the integral on the right-hand side of (1.5) does in fact exist and belongs to $L^p(\partial B)^3$ if considered as a function of x; see Lemma 6.5.

For $z \in \mathbb{R}^3\backslash\{0\}$, $j, k, l \in \{1, 2, 3\}$, define

$$E_{jk}(z) := (8\cdot\pi)^{-1}\cdot\left(\delta_{jk}\cdot|z|^{-1} + z_j\cdot z_k\cdot|z|^{-3}\right), \quad (1.6)$$

$$\overline{E}(z) := (4\cdot\pi)^{-1}\cdot|z|^{-1}.$$

The matrix-valued function $(E_{1k}, E_{2k}, E_{3k}, E_{4k})_{1\le k\le 3}$, with E_{4k} from (1.2), is a fundamental solution of the Stokes system

$$-\Delta u + \nabla\pi = f, \qquad \operatorname{div} u = 0,$$

and $\overline{E}$ is a fundamental solution of the Poisson equation $-\Delta v = f$. We further set

$$\mathcal{D}_{jkl} := D_j E_{kl} + D_k E_{jl} - \delta_{jk}\cdot E_{4l}, \qquad \overline{\mathcal{D}}_k := D_k\overline{E}, \quad (1.7)$$

for $j, k, l \in \{1, 2, 3\}$. Let $p \in (1, \infty)$, $\tau \in \{-1, 1\}$, $B \subset \mathbb{R}^3$, with B open. Then we define the operators

$$\Lambda(\tau, p, B) : L^p(\partial B)^3 \mapsto L^p(\partial B)^3, \qquad \Pi(\tau, p, B) : L^p(\partial B) \mapsto L^p(\partial B)$$

by setting

$$\Lambda(\tau, p, B)(f)(x) \quad (1.8)$$
$$:= (\tau/2)\cdot f(x) + \left(\int_{\partial B}\sum_{j,k=1}^{3}\mathcal{D}_{jkl}(x-y)\cdot n_k^{(B)}(y)\cdot f_j(y)\,dB(y)\right)_{1\le l\le 3},$$

$$\Pi(\tau, p, B)(h)(x) \quad (1.9)$$
$$:= (\tau/2)\cdot h(x) + \int_{\partial B}\sum_{k=1}^{3}\overline{\mathcal{D}}_k(x-y)\cdot n_k^{(B)}(y)\cdot h(y)\,dB(y),$$

for $f \in L^p(\partial B)^3$, $h \in L^p(\partial B)$, $x \in \partial B$. Here again, $n^{(B)}$ denotes the outward unit normal to B. As in the case of (1.5), these definitions are justified if B is a bounded domain with a smooth boundary (see [13, p. 135, Lemma 4.7; p. 144, Lemma 5.1]), or if $B = \mathbb{K}(\varphi)$ (see Lemma 6.2). Moreover, if B is a bounded domain with a Lipschitz boundary, then the integral on the right-hand side of (1.8) and (1.9) may be understood as a principal-value integral, as follows from a result by Coifman, McIntosh, Meyer [4]. We shall discuss and use this fact in Chapter 13.

It is the principal aim of this book to investigate the Fredholm properties of the operator

$\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$. As we shall show, these properties may be derived from certain facts pertaining to invertibility of the operators $\Lambda(\tau, p, \mathbb{K}(\varphi))$ and $\Pi(-\tau, p/(p-1), \mathbb{K}(\varphi))$. Concerning the last two operators, we shall investigate their features by making use of the theory of locally compact abelian groups.

Before elaborating these indications, let us explain why it is of interest to study the behaviour of $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$. To this end, we fix a bounded domain Ω in $\mathbb{R}^3$ with a smooth boundary. Then consider the resolvent problem (1.3) on the domain Ω, under Dirichlet boundary conditions

$$u\,|\,\partial\Omega = 0. \quad (1.10)$$

This boundary value problem will be called "interior problem" (related to (1.3)). The term "exterior problem" is used when the system in (1.3) is to be solved in the exterior domain $\mathbb{R}^3\backslash\overline{\Omega}$. Concerning existence and regularity of solutions to the interior problem, the following theorem holds true:

Theorem 1.1. *Let $p \in (1, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$ and $f \in L^p(\Omega)^3$. Then there is one and only one velocity $u \in W^{2,p}(\Omega)^3$, as well as a pressure $\pi \in W^{1,p}(\Omega)$ – uniquely determined up to an additive constant – such that the pair of functions (u, π) solves the resolvent equation (1.3) in Ω, and u satisfies boundary condition (1.10). Thus, (u, π) is a solution of the interior problem related to (1.3).*

In addition, for $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, there is a constant $C = C(p, \vartheta, \Omega) > 0$ such that

$$\|u\|_p \le C\cdot|\lambda|^{-1}\cdot\|f\|_p, \qquad \|u\|_{2,p} + \|\nabla\pi\|_p \le C\cdot\|f\|_p, \quad (1.11)$$

for $f \in L^p(\Omega)^3$, for λ belonging to the sector $\Sigma_\vartheta := \{\mu \in \mathbb{C}\backslash\{0\} : |\arg\mu| \le \vartheta\}$ of the complex plane, and for the corresponding solution $(u, \pi) \in W^{2,p}(\Omega)^3 \times W^{1,p}(\Omega)$ of the interior problem related to (1.3).

This theorem, which was proved by Solonnikov [44] und Giga [25], is the key to solving the non-linear, time-dependent Navier-Stokes system in Ω by means of semi-group theory in $L^p(\Omega)$. A detailed discussion of this approach may be found in von Wahl [48], [49]. For the most recent results, we refer to Giga, Sohr [26].

Solonnikov [44] reduces Theorem 1.1 to a L^p-theory for the time-dependent Stokes system. Giga [25] proves this theorem by referring to the theory of pseudodifferential operators. In [8], [9] we apply the method of integral equations in order to derive similar results for solutions of (1.3) in the exterior domain $\mathbb{R}^3\backslash\overline{\Omega}$, under boundary condition (1.10). Solonnikov [44] and Giga [25] also deal with the exterior problem, but the access in [8], [9] is more elementary. For other results on solutions to (1.3), we refer to [10], [11], [12], [14], [46], among many other papers.

Let us briefly explain the approach from [8], [9], and then discuss whether this approach may be applied to a non-smoothly bounded domain as well. From these arguments it should become clear why it is interesting to study the operator $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ defined above.

Although in [8], [9], the exterior problem related to (1.3) is investigated, the method used there may be applied to the interior problem as well, and in the latter case, it turns out to be somewhat less complicated. Therefore, we shall continue to consider the interior problem.

In the first step of the approach from [8], [9], the interior problem related to (1.3) is reduced to the homogeneous resolvent equation

$$-\Delta u \; + \; \lambda \cdot u \; + \; \nabla \pi \;\; = \;\; 0, \qquad \operatorname{div} u \;\; = \;\; 0 \qquad \text{in } \Omega, \tag{1.12}$$

under nonhomogeneous Dirichlet boundary conditions:

$$u \,|\, \partial\Omega \;\; = \;\; g. \tag{1.13}$$

In fact, Theorem 1.1 above may be derived from the following result on (1.12), (1.13) (compare [8, p. 338 - 340]):

Theorem 1.2. *For $p \in (1, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $g \in W^{2-1/p,\,p}(\partial\Omega)^3$ with $\int_{\partial\Omega} g \cdot n^{(\Omega)} d\Omega = 0$, there is one and only one velocity $u \in W^{2,p}(\Omega)^3 \cap C^\infty(\Omega)^3$, as well as a pressure $\pi \in W^{1,p}(\Omega) \cap C^\infty(\Omega)$, which is unique up to an additive constant, such that the pair of functions (u, π) solves boundary value problem (1.12), (1.13).*

In addition, for $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, there is a constant $C = C(p, \vartheta, \Omega) > 0$ such that

$$\|u\|_p \;\; \le \;\; C \cdot |\lambda|^{-1/(2\cdot p)} \cdot \|g\|_p, \tag{1.14}$$

$$\|u\|_{2,p} \; + \; \|\nabla\pi\|_p \;\; \le \;\; C \cdot \Big(|\lambda|^{1-1/(2\cdot p)} \cdot \|g\|_p \; + \; \|g\|_{2-1/p,\,p} \Big),$$

for $\lambda \in \Sigma_\vartheta$, $g \in W^{2-1/p,\,p}(\partial\Omega)^3$ with $\int_{\partial\Omega} g \cdot n^{(\Omega)} \, d\Omega = 0$, and for the corresponding solution (u, π) of boundary value problem (1.12), (1.13).

(For the definition of Σ_ϑ, we refer to Theorem 1.1.)

In [8], [9], similar results are proved for the exterior problem by using the method of integral equations. In order to explain this method, as applied to the interior problem, let us first consider Laplace's equation

$$-\Delta v \;\; = \;\; 0 \qquad \text{in } \Omega, \tag{1.15}$$

under Dirichlet boundary conditions

$$v \,|\, \partial\Omega \;\; = \;\; \overline{g}. \tag{1.16}$$

For any $h \in L^p(\partial\Omega)$, the function

$$v(h)(x) := \int_{\partial\Omega} \sum_{k=1}^{3} \overline{\mathcal{D}}_k(x-y) \cdot n_k^{(\Omega)}(y) \cdot h(y) \;\; d\Omega\,(y) \qquad (x \in \Omega)$$

belongs to $C^\infty(\Omega)$ and solves (1.15). (The symbol $n^{(\Omega)}$ denotes the outward unit normal to Ω.) In addition, the function $v(h)$ has a trace on $\partial\Omega$, which we denote, as usual, by $v(h)|\partial\Omega$. This trace function satisfies the equation

$$v(h) \,|\, \partial\Omega \;\; = \;\; \Pi(+1, p, \Omega)(h),$$

with $\Pi(+1, p, \Omega)$ from (1.9). This property of $v(h)$ is called "jump relation", for reasons which will be explained at the beginning of Chapter 9. Thus, if it is possible to find a function $h \in L^p(\partial\Omega)$ solving the integral equation

$$\Pi(+1, p, \Omega)(h) \;\; = \;\; \overline{g}, \tag{1.17}$$

with $\overline{g}$ from boundary condition (1.16), then $v(h)$ is a solution of boundary value problem (1.15), (1.16). In fact, such a solution $h \in L^p(\partial\Omega)$ exists for any $\overline{g} \in L^p(\partial\Omega)$. In this way, boundary value problem (1.15), (1.16) is reduced to the integral equation in (1.17), hence the name "method of integral equations".

Odquist [37] succeeded in adapting this approach to the Stokes system

$$-\Delta u \; + \; \nabla\pi \;\; = \;\; 0, \qquad \operatorname{div} u \;\; = \;\; 0 \qquad \text{in } \Omega, \tag{1.18}$$

under Dirchlet boundary condition (1.13). He arrived at the integral equation

$$\Lambda(+1, p, \Omega)(f) \;\; = \;\; g, \tag{1.19}$$

with g from (1.13). (The operator $\Lambda(+1, p, \Omega)$ was defined in (1.8).) Odquist [37] then proceeded to solve this integral equation in Hölder spaces.

Ladyzhenskaya [30, p. 67-79] referred to Odquist's approach in order to derive a L^p-theory for the Stokes system (1.18). She could show:

Theorem 1.3. *For $g \in W^{2-1/p,\,p}(\partial\Omega)^3$ with $\int_{\partial\Omega} g \cdot n^{(\Omega)} \, d\Omega = 0$, there exists a uniquely determined velocity $u \in W^{2,p}(\Omega)^3$ and a pressure $\pi \in W^{1,p}(\Omega)$, which is unique up to an additive constant, such that the pair of functions (u, π) solves (1.18), (1.13).*

The proof of this result is indicated only shortly in [30, p. 67-79]. More details may be found in [13].

Odquist's method may be carried over to the resolvent problem (1.12), (1.13), and then leads to the integral equation

$$\Gamma(+1, p, \lambda, \Omega)(f) \;\; = \;\; g, \tag{1.20}$$

with $\Gamma(+1, p, \lambda, \Omega)$ from (1.5).

It turns out that the operator $\Gamma(+1, p, \lambda, \Omega) - \Lambda(+1, p, \Omega)$ is compact in $L^p(\partial\Omega)^3$.

Starting from this observation, a L^p-theory for (1.20) may be deduced from Ladyzhenskaya's results related to (1.19); see [8, p. 338-340, 342-344]. This L^p-theory can then be used in order to obtain the first part of Theorem 1.2, which concerns existence and uniqueness in L^p-spaces of solutions to the resolvent problem (1.12), (1.13). This first part of Theorem 1.2 may thus be regarded as a corollary of Theorem 1.3. We refer to [8, p. 339/340, 345-348] for more details.

It is much more difficult to prove estimate (1.14). Proceeding as in [8], [9], we may start by considering the double-layer potential $\Gamma(\tau, p, \lambda, \mathbb{R}^2 \times (0, \infty))$ related to the halfspace $\mathbb{R}^2 \times (0, \infty)$. It holds:

$$\|\Phi\|_p \;\le\; D \cdot \|\Gamma(1, p, \lambda, \mathbb{R}^2 \times (0, \infty))(\Phi)\|_p \tag{1.21}$$

for $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\Phi \in L^p(\mathbb{R}^2)^3$, $\lambda \in \Sigma_\vartheta$, where the constant D only depends on p or ϑ. (See Theorem 1.1 for the definition of Σ_ϑ.)

Inequality (1.21), which was established by McCracken [33, p. 219, Theorem 5.8], is then used in order to estimate the double-layer operator $\Gamma(+1, p, \lambda, \Omega)$ related to the domain Ω. In fact, the relation in (1.21) yields that

$$\|\Phi\|_p \;\le\; C \cdot \|\Gamma(+1, p, \lambda, \Omega)(\Phi)\|_p \tag{1.22}$$

for $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\Phi \in L^p(\partial\Omega)^3$ with $\int_{\partial\Omega} \Phi \cdot n^{(\Omega)}\, d\Omega = 0$, and for $\lambda \in \Sigma_\vartheta$ with $|\lambda| \ge \widetilde{C}$. Here the symbols C, $\widetilde{C}$ denote constants which only depend on p, ϑ or Ω. We refer to [9] for details, which are rather technical. Inequality (1.22), in turn, leads to the resolvent estimate (1.14); see [8, p. 340] for the corresponding arguments in the case of the exterior problem related to (1.3).

Now we weaken our assumptions on the domain Ω and only require that Ω is Lipschitz bounded. Then it is an open question how to obtain a L^p-theory for the resolvent problem (1.3), (1.10). However, based on what is known if $p = 2$, – we shall discuss this case presently – let us guess what might be the best possible result:

Supposition 1.1. *For $p \in (1, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $f \in L^p(\Omega)^2$, there is one and only one velocity*

$$u \in \bigcap_{\epsilon \in (0,1)} W^{1+1/p-\epsilon,\, p}(\Omega)^3,$$

as well as a pressure

$$\pi \in \bigcap_{\epsilon \in (0,1)} W^{1/p-\epsilon,\, p}(\Omega)$$

– uniquely determined up to an additive constant – such that the pair of functions (u, π) solves boundary value problem (1.3), (1.10).

In addition, for $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\epsilon, \delta \in (0, 1)$, there is a constant C – depending on p, ϑ, ϵ, δ or Ω – such that

$$\|u\|_p \;\le\; C \cdot |\lambda|^{-1} \cdot \|f\|_p, \tag{1.23}$$

$$\|u\|_{1+1/p-\epsilon,\,p} + \Big\| \pi - \int_\Omega \pi(x)\, dx \Big\backslash \Big(\int_\Omega dx \Big) \Big\|_{1/p-\epsilon,\,p} \;\le\; C \cdot |\lambda|^{-1/2+1/(2\cdot p)+\delta} \cdot \|f\|_p$$

for $\lambda \in \Sigma_\vartheta$, $f \in L^p(\Omega)^3$, and for the corresponding solution (u, π) of (1.3), (1.10).

In the case $p = 2$, Deuring, von Wahl [15] could show that Supposition 1.1 is true if $\lambda = i \cdot \tau$, with $\tau \in \mathbb{R}$. Their proof is based on the L^2-theory developed by Fabes, Kenig, Verchota [21] and Shen [43]. In the former paper, the Stokes system (1.18) is studied in interior and exterior Lipschitz domains. The latter paper contains a L^2-theory for the time-dependent Stokes system in interior Lipschitz domains. This theory is reduced to L^2-estimates pertaining to the gradient of single-layer potentials related to equation (1.3) with $\lambda = i \cdot \tau$, $\tau \in \mathbb{R}$.

Let us now drop the condition $p = 2$. Then Farwig, Sohr [22, p. 7, Corollary 1.4] treat boundary value problem (1.3), (1.10) under the assumption that the boundary of Ω has conical points of angle $\pi/2 \pm \epsilon$, with $\epsilon > 0$ small. The authors prove that in such a situation, Theorem 1.1 remains true. We further mention a paper by Galdi, Simader, Sohr [24] dealing with weak solutions of (1.3), (1.10), under the assumption that $\partial\Omega$ may be described by local parameters having small Lipschitz constants. However, under less restricting assumptions on Ω, it is not known whether Supposition 1.1 is true if $p \ne 2$. In this situation, it seems to be natural to consider a Lipschitz domain Ω which, on one hand, is sufficiently general in order to cause typical effects of non-smooth boundaries. On the other hand, Ω should be chosen in such a way that its boundary may be described by explicit local parameters. This may simplify the study of boundary value problems in Ω. In this spirit, we assume that all boundary points of Ω are regular, with the exception of a single point x_0. In a neighbourhood of this point x_0, the domain Ω is to coincide with a right circular cone with vertex x_0. Let $2 \cdot \varphi$, for some $\varphi \in (0, \pi)$, be the vertex angle of this cone.

Solutions of partial differential equations in this type of domain are often estimated in weighted norms, with the weight of a point $x \in \Omega$ depending on the distance of x from the vertex of the cone. Results of this kind pertaining to the Stokes system may be found in the paper [31] by Maz'ya, Plamenevski. Concerning other partial differential equations, we refer to the book [32] by Maz'ya, Nasarov, Plamenevski. We did not investigate whether the theory given in [31], [32] proves or disproves Supposition 1.1. Instead we attempted to check our conjecture by recurring to the approach from [8], [9], which reduces estimate (1.11) to McCracken's inequality (1.21), that is, to a result related to solutions in halfspace. Since a solution theory for domains with conical boundary points cannot be solely based on a result pertaining to halfspace, it should be expected an additional estimate is needed, different from (1.21) in so far as the halfspace $\mathbb{R}^2 \times (0, \infty)$ is replaced by the cone $\mathbb{K}(\varphi)$ defined above. More precisely, the following inequality should be proved:

Supposition 1.2. *Let $\tau \in \{-1, 1\}$, $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi)$. Then there is a constant $\widetilde{D}$ only depending on τ, p, ϑ or φ such that*

$$\|h\|_p \quad \le \quad \tilde{D} \cdot \| \Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))(h) \|_p \qquad (1.24)$$

for $\lambda \in \Sigma_\vartheta$, $h \in L^p(\partial\mathbb{K}(\varphi))^3$. *(The set* Σ_ϑ *was introduced in Theorem 1.1.)*

This is the point where the operator $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ arises. As mentioned further above, the study of this operator is a main topic of this book.

If estimate (1.24) were available, we might attempt to proceed in a similar way as in the case of a smoothly bounded domain. First, Supposition 1.2 and McCracken's result (1.21) would lead to inequality (1.22), which – in a second step – may possibly be used in order to derive the resolvent estimate (1.11). Unfortunately, this program cannot be carried through since we shall show that in general, inequality (1.24) is false for $p \neq 2$. This is one of the principal results of this book.

Let us explain this fact – and some others – in more detail. Since we shall study both $\Gamma(+1, p, \lambda, \mathbb{K}(\varphi))$ and $\Gamma(-1, p, \lambda, \mathbb{K}(\varphi))$, we may assume without loss of generality that $\varphi \in (0, \pi/2]$. Then it holds by (6.13), Lemma 6.13, 6.17, 12.2 and 12.10:

Theorem 1.4. *Let* $p \in (1, \infty)$, $\tau \in \{-1, 1\}$. *Then inequality (1.24) is true for any* $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$ *if and only if the operator* $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ *is Fredholm and its adjoint is one-to-one* ($\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\varphi \in (0, \pi/2]$).

The next theorem is a consequence of Lemma 8.5, Corollary 12.5 and 12.6:

Theorem 1.5. *Let* $\tau \in \{-1, 1\}$, $p \in (1, \infty)$. *Then* $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ *is Fredholm for any* $\varphi \in (0, \pi/2]$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$ *if and only if the operators* $\Lambda(\tau, p, \mathbb{K}(\varphi))$ *and* $\Pi(-\tau, p/(p-1), \mathbb{K}(\varphi))$ *are both topological for all* $\varphi \in (0, \pi/2]$.

Note that the last two operators mentioned in Theorem 1.5 do not depend on λ.

The operator $\Pi(-1, p, \mathbb{K}(\pi/4))$ is treated by Fabes, Jodeit, Lewis [20, p. 101/102], who transform this mapping into an operator $I + K_p$ defined by

$$(\gamma + K_p(\gamma))(r, \theta) := \gamma(r, \theta) + \int_0^{2\cdot\pi} \int_0^\infty \widetilde{K_p}(r/s, \theta - \sigma) \cdot \gamma(s, \sigma) \cdot s^{-1}\, ds\, d\sigma$$

with $\widetilde{K_p}$ denoting a suitable kernel function, (r, θ) being a member of $(0, \infty) \times [0, 2\cdot\pi]$, and $\gamma : (0, \infty) \times [0, 2\cdot\pi] \mapsto \mathbb{C}^3$ belonging to an appropriate function space (see Definition 8.1, 8.2, Lemma 8.3). Then in [20, p. 102], the Mellin transform in the variable $r \in (0, \infty)$, and the discrete Fourier transform in the variable $\theta \in [0, 2\cdot\pi]$ are applied to the operator $I + K_p$. In this way, a symbol of the operator $\Pi(-1, p, \mathbb{K}(\pi/4))$ is obtained. By recurring to the theory of locally compact abelian groups, it may be shown this symbol has no zeros if and only if the operator $\Pi(-1, p, \mathbb{K}(\pi/4))$ is topological. Therefore, if we want to find those values of p having the property that the preceding operator is topological, we have to look for numbers $p \in (1, \infty)$ which yield a symbol staying away from zero.

This approach is only briefly indicated in [20, p. 102]. In Chapter 7 and 8, we shall work out the details and shall arrive at the following result (see Corollary 8.3, Theorem 8.1, 8.2):

Theorem 1.6. *On one hand, the operator* $\Pi(\tau, p, \mathbb{K}(\varphi))$ *is topological for* $p \in [2, \infty)$, $\varphi \in (0, \pi/2]$, $\tau \in \{-1, 1\}$.

On the other hand, there are certain values of $p \in (1, 2)$ *such that* $\Pi(-1, p, \mathbb{K}(\varphi))$ *is not topological for any* $\varphi \in (0, \pi/2]$. *If* $\varphi \in (0, \pi/2]$ *is fixed, then the set of these exceptional values of* p *is countable.*

In Chapter 8, we shall further use the approach from [20, p. 102] in order to deal with the operator $\Lambda(\tau, p, \mathbb{K}(\varphi))$. However, compared with studying the operator $\Pi(\tau, p, \mathbb{K}(\varphi))$, it will be much more difficult to find a symbol of $\Lambda(\tau, p, \mathbb{K}(\varphi))$, and then check whether this symbol has any zeros. Still, we shall be able to obtain a fairly complete result (see Theorem 13.1, 8.1, 8.2):

Theorem 1.7. *The operator* $\Lambda(\tau, p, \mathbb{K}(\varphi))$ *is topological for* $\varphi \in (0, \pi/2]$, $p \in [2, \infty)$, $\tau \in \{-1, 1\}$.

There are reals $p_1, p_2 \in (1, 2)$, $\varphi_1, \varphi_2 \in (0, \pi/2)$ *such that the operators* $\Lambda(1, p_1, \mathbb{K}(\varphi_1))$ *and* $\Lambda(-1, p_2, \mathbb{K}(\varphi_2))$ *are not topological. If* $\varphi \in (0, \pi/2]$ *and* $\tau \in \{-1, 1\}$ *are fixed, there is at most a countable number of values* $p \in (1, 2)$ *such that* $\Lambda(\tau, p, \mathbb{K}(\varphi))$ *is not topological.*

The second part of this theorem will be established by examining the symbol of $\Lambda(\tau, p, \mathbb{K}(\varphi))$. For the proof of the first part, however, we shall additionally need two lemmas by Shen [43, p. 364, Lemma 5.2.11 (ii), p. 369, Lemma 5.3.7, with $\tau = 0$], pertaining to the L^2-theory of the Stokes system (1.18) on domains with bounded Lipschitz boundary.

We further mention that the second part of Theorem 1.7 should imply non-regularity results for solutions to the Stokes system on Lipschitz domains. An example for such a result is provided by Theorem 13.2. It might be worthwhile to look for further consequences.

In the context of this book, Theorem 1.7 is of interest mainly in connection with the study of the operator $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$. In fact, we may conclude from Theorem 1.5, 1.6 and 1.7: The operator $\Gamma(\tau, 2, \lambda, \mathbb{K}(\varphi))$ is Fredholm for any $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\varphi \in (0, \pi/2]$, $\tau \in \{-1, 1\}$. On the other hand, it will be shown the adjoint of $\Gamma(\tau, 2, \lambda, \mathbb{K}(\varphi))$ is one-to-one (Theorem 9.4). Now Theorem 1.4 implies that Supposition 1.2 – that is, inequality (1.24) – is true in the case $p = 2$ (Corollary 13.3). This result in turn leads to existence and regularity of solutions to the resolvent problem (1.12) in the cone $\mathbb{K}(\varphi)$, with boundary data in $L^2(\partial\mathbb{K}(\varphi))^3$ (Corollary 13.4).

Furthermore, by Theorem 1.5, 1.6, 1.7, we arrive at the following negative result concerning the behaviour of the operator $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ in the case $p \neq 2$ (see Corollary 12.9): For $\vartheta \in [0, \pi)$, there are numbers $p_1, p_2 \in (1, 2)$, $p_3 \in (2, \infty)$, $\varphi_1, \varphi_2, \varphi_3 \in (0, \pi/2)$, such that the operators $\Gamma(+1, p_1, \lambda, \mathbb{K}(\varphi_1))$ and $\Gamma(-1, p_j, \lambda, \mathbb{K}(\varphi_j))$ $(j \in \{2, 3\})$ are <u>not</u> Fredholm for $\lambda = r \cdot e^{i \cdot \vartheta}$, with $r \in (0, \infty)$. This implies by Theorem 1.4 that Supposition 1.2 fails in the case $p \neq 2$.

This leaves open the question as to what the preceding results imply for L^p-regularity of solutions to the resolvent problem (1.3), (1.10) in a domain which has one conical boundary point and is smoothly bounded everywhere else. To put it differently, we may ask whether Supposition 1.1 is correct for such a domain. It must be expected that the behaviour of the operator $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ is closely related to regularity of solutions to (1.3), (1.10). This seems likely since in the case of a smoothly bounded domain, it could be shown that estimate (1.21) of the operator $\Gamma(\tau, p, \lambda, \mathbb{R}^2 \times (0, \infty))$ leads to the resolvent estimate (1.11), as we explained above. Now recall that for $p \neq 2$, the operator $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ is not even Fredholm in general. Thus we suspect that inequality (1.23) is not valid for domains with conical boundary points. We further recall the close connection existing in the case of smoothly bounded domains between the Stokes system (1.18) and the resolvent problem (1.3). Thus, since in general the operator $\Lambda(\tau, p, \mathbb{K}(\varphi))$ related to (1.18) is not Fredholm either, Supposition 1.1 may be false even if inequality (1.23) is deleted.

Let us finally mention another aspect of our results: Consider the Helmholtz equation

$$-\Delta + \lambda \cdot u = 0 \quad \text{in } \mathbb{K}(\varphi). \tag{1.25}$$

Up to now, it seemed to be unknown how to estimate solutions of (1.25) uniformly in λ. In Chapter 13, we shall be able to obtain such an estimate for the resolvent problem (1.12) related to the Stokes system (Corollary 13.4), even though the latter problem is more difficult to treat than (1.25), due to the pressure term $\nabla \pi$, and because of the additional equation $\operatorname{div} u = 0$. Moreover, (1.25) is a scalar equation, whereas (1.12) is a system. Therefore, since it works when applied to the Stokes system, our approach should also be an appropriate tool for investigating other differential equations, such as (1.25). Perhaps it may even be adapted to standard domains other than a right circular infinite cone.

Chapter 2

Notations

- δ_{kl}

  Kronecker symbol.

- $\mathbb{N}$, $\mathbb{N}_0$

  the set of positive and nonnegative integers, respectively.

- $\mathbb{Z}, \mathbb{R}, \mathbb{C}$

  the set of integers, real numbers, complex numbers, respectively.

- r^{cc}

  the conjugate of the complex number r.

- $|x|$

  For $x \in \mathbb{C}$, the absolute value of x is denoted by $|x|$. If $n \in \mathbb{N}$, $x \in \mathbb{C}^n$, we set
  $|x| := \Big(\sum_{i=1}^{n} |x_i|^2\Big)^{1/2}$.

- $|a|_*$

  For $n \in \mathbb{N}$, $a \in \mathbb{N}_0^n$, we put $|a|_* := a_1 + \cdots + a_n$.

- $\cdot$ (point)

  Three kinds of multiplications will be denoted by a point:

  multiplication in $\mathbb{C}$ (in particular in $\mathbb{R}$): $a \cdot b$ for $a, b \in \mathbb{C}$;

  inner product in $\mathbb{R}^n$: $x \cdot y$ for $x, y \in \mathbb{R}^n$;

  multiplication of matrices, in particular multiplication of a matrix with a vector: $A \cdot B$, $A \cdot x$ for $A \in \mathbb{C}^{l \times m}$, $B \in \mathbb{C}^{m \times n}$, $x \in \mathbb{C}^m$, $l, m, n \in \mathbb{N}$.

- $a \wedge b$, $a \vee b$

  For $a, b \in \mathbb{R}$, we define $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$.

- $\mathbb{B}_n(x, R)$
  abbreviation for the set $\{ y \in \mathbb{R}^n : |x-y| < R \}$, with $n \in \mathbb{N}$, $x \in \mathbb{R}^n$, $R \in [0, \infty]$ (open ball with centre x and radius R). Note that $\mathbb{B}_n(x, 0) = \emptyset$, $\mathbb{B}_n(x, \infty) = \mathbb{R}^n$.

- χ_A
  characteristic function of $A \subset \mathbb{R}^n$ $(n \in \mathbb{N})$.

- $dist(x, \Omega)$
  For $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, we put $dist(x, \Omega) := \inf\{|x-y| : y \in \Omega\}$.

- E_n
  unit matrix in $\mathbb{C}^{n \times n}$ $(n \in \mathbb{N})$.

- arg
  Let the function $S : [0, 2\cdot\pi) \mapsto \{z \in \mathbb{C} : |z| = 1\}$ be defined by $S(\varphi) := e^{i\cdot\varphi}$ for $\varphi \in [0, 2\cdot\pi)$. Then we set $\arg := S^{-1}$.

- $\Re(z)$, $\Im(z)$
  real and imaginary part of $z \in \mathbb{C}$.

- x^b
  For $n \in \mathbb{N}$, $x \in \mathbb{R}^n$, $b \in \mathbb{N}_0^n$, put $x^b := \prod_{j=1}^{n} x_j^{b_j}$.

- $\partial\Omega$, $\overline{\Omega}$
  boundary and closure of $\Omega \subset \mathbb{R}^n$ with respect to the usual topology of the euclidean space $\mathbb{R}^n$ $(n \in \mathbb{N})$.

- $supp(f)$
  support of a function $f : A \mapsto \mathbb{C}^k$ $(A \subset \mathbb{R}^n,\ n, k \in \mathbb{N})$.

- $|f|_0$
  For $n, \sigma \in \mathbb{N}$, $B \subset \mathbb{R}^n$, and for a function $f : B \mapsto \mathbb{C}^\sigma$, we set
  $|f|_0 := \sup\{ |f(x)| : x \in B \}$.

- $\circ$ (circle)
  Given some sets A, B, C, as well as mappings $f : A \mapsto B$, $g : B \mapsto C$, let the mapping $g \circ f : A \mapsto C$ be defined by $(g \circ f)(x) := g(f(x))$ for $x \in A$.

- $g\,|\,C$
  Given the sets A, B with $C \subset A$, and the mapping $g : A \mapsto B$, we denote by $g\,|\,C$ the restriction of g to C.

- $id(A)$
  identical mapping of a set A.

- $im(F)$, $kern(F)$
  For sets A, B, and for a mapping $F : A \mapsto B$, we put
  $im(F) := \{ F(a) : a \in A \}$, $\quad kern(F) := \{ a \in A : F(a) = 0 \}$.

- $\mathcal{LB}(X, Y)$
  Let X, Y be normed linear spaces. Then we denote by $\mathcal{LB}(X, Y)$ the linear space of all bounded linear operators from X in Y, equipped with the usual norm.

- $index(G)$
  Let X, Y be Banach spaces and $G \in \mathcal{LB}(X, Y)$. Assume that either $dim\ kern(G)$ or $dim\ (Y/im(G))$ is finite, where $Y/im(G)$ denotes the quotient space of Y by $im(G)$. Then we define
  $index(G) := dim\ kern(G) - dim\ (Y/im(G))$.

- F_+-operator
  Let X, Y be Banach spaces, and assume that $G : X \mapsto Y$ is a bounded linear operator having a finite dimensional kernel and a closed range. Then we call G a "F_+-operator", or we say that "G has property F_+", or simply: "G is F_+". Note that any Fredholm operator has property F_+.

- adjoint operator
  The notion of "adjoint operator" is to be understood as in [29, p. 167]

- A topological
  Let U, V be topological spaces and $A : U \mapsto V$ a continuous, bijective mapping. Assume that the inverse mapping A^{-1} of A is continuous too. Then we call A "topological".

- G^σ (product of function spaces)
  Let $\sigma \in \mathbb{N}$. Assume that A is some set, and G is a subspace of the set of all mappings from A into $\mathbb{C}$. Then define
  $G^\sigma := \{ f \text{ mapping from } A \text{ into } \mathbb{C}^\sigma \ : \ f_i \in G \ \text{ for } i \in \{1, \cdots, \sigma\} \}$.

- $\partial/\partial x_i\, h(x)$
  Let $n, \sigma \in \mathbb{N}$, $i \in \{1, \cdots, n\}$, $A \subset \mathbb{R}^n$, $x \in A$ with x_i cluster point of the set $\{ r \in \mathbb{R} : (x_1, \cdots, x_{i-1}, r, x_{i+1}, \cdots, x_n) \in A \}$. Put $e_i := (\delta_{ik})_{1 \le k \le n}$. Let $h : A \mapsto \mathbb{C}^\sigma$ be a function such that the limit
  $\lim_{\epsilon \to 0} \Big(h(x + \epsilon \cdot e_i) - h(x) \Big) / \epsilon$
  exists. Then we denote this limit by $\partial/\partial x_i\, h(x)$.

- $D_l g$, $D^a g$, ∇ , Δ

  Let $\sigma, n \in \mathbb{N}$, $B \subset \mathbb{R}^n$ open, $l \in \{1, \cdots, n\}$, and $g : B \mapsto \mathbb{C}^\sigma$ a function. Assume the limit $\partial/\partial x_l\, g(x)$ exists for any $x \in B$. Then we define the function $D_l g : B \mapsto \mathbb{C}^\sigma$ by setting $D_l g(x) := \partial/\partial x_l\, g(x)$ for $x \in B$.

  Suppose g is m-times differentiable, for some $m \in \mathbb{N}$ with $m \geq 2$. Then the function $D_l^m g$ is introduced by iterating the preceding definition. For $k_1, \cdots, k_n \in \mathbb{N}_0$ with $k_1 + \cdots + k_n \leq m$, the partial derivative $D_1^{k_1} \cdots D_n^{k_n} g$ is again defined by iteration. If $a \in \mathbb{N}_0^n$ with $|a|_* \leq m$, we write $D^a g := D_1^{a_1} \cdots D_n^{a_n} g$. Finally we set

  $$\nabla g := (D_1 g, \cdots, D_n g) \qquad \text{(gradient of } g\text{)},$$

  $$\Delta g := \sum_{i=1}^{n} D_i^2 g \qquad \text{(Laplacian, applied to } g\text{)}.$$

- $C^m(B)$, $C_0^m(B)$

  Let $n \in \mathbb{N}$, $B \subset \mathbb{R}^n$. Then we denote by $C^0(B)$ the space of all continuous functions mapping B into $\mathbb{C}$.

  Let $B \subset \mathbb{R}^n$ be an open set. Then $C^m(B)$ denotes the space of all functions from B into $\mathbb{C}$ which are m-times continuously differentiable $(m \in \mathbb{N})$. In addition, we put

  $$C^\infty(B) := \bigcap_{m \in \mathbb{N}} C^m(B),$$

  and for $m \in \mathbb{N} \cup \{\infty\}$:

  $$C_0^m(B) := \{ f \in C^m(B) \ : \ supp(f) \subset B, \ \ supp(f) \text{ bounded} \}.$$

- f regular

  Let $n, \sigma \in \mathbb{N}$, $G \subset \mathbb{R}^n$ an open set, $f : G \mapsto \mathbb{R}^\sigma$ a function which is continuously differentiable. We call the function f "regular" if the jacobian matrix of f has maximal rank at any point $x \in G$.

- $\partial^a/\partial x^a f(x)$, $\nabla_x f(x)$

  Let $m, n, \sigma \in \mathbb{N}$, $A \subset \mathbb{R}^n$, $f : A \mapsto \mathbb{C}^\sigma$ a function, $x \in A$, $a \in \mathbb{N}_0^n$ with $|a|_* \leq m$. Assume there is some $\epsilon > 0$ such that $\mathbb{B}_n(x, \epsilon) \subset A$ and $f \,|\, \mathbb{B}_n(x, \epsilon) \in C^m(\mathbb{B}_n(x, \epsilon))$. Then the notations $\partial^a/\partial x^a f(x)$ and $\nabla_x f(x)$ are defined by referring to the function $f \,|\, \mathbb{B}_n(x, \epsilon)$ in an obvious way. We shall use these notations when A is not open.

- almost every (a.e.), set of measure zero, measurable, integrable

  Whenever these notions from measure theory will be used without further indications, they will relate either to Lebesgue measure in $\mathbb{R}^n$ $(n \in \mathbb{N})$, or to the usual surface measure on the boundary $\partial\Omega$ of an open, Lipschitz bounded set Ω in $\mathbb{R}^k$ $(k \in \mathbb{N}, \ k \geq 2)$. In each case, it will be clear from context which is the number n or the set Ω we shall refer to.

  By the usual surface measure on $\partial\Omega$, we mean the measure which may be reduced to Lebesgue measure in $\mathbb{R}^{k-1}$ by means of local coordinates.

- identity of measurable functions

  Let $n, k \in \mathbb{N}$, $k \geq 2$, $\Omega \subset \mathbb{R}^k$ open and Lipschitz bounded. Suppose that B is a measurable subset of either $\mathbb{R}^n$ or $\partial\Omega$. Then, as usual, we identify two measurable functions $f, g : B \mapsto \mathbb{C}^\sigma$ $(\sigma \in \mathbb{N})$ if they only differ on a set of measure zero.

- $\int_B g(x)\ dx$

  Let $n, \sigma \in \mathbb{N}$, $B \subset \mathbb{R}^n$ measurable. If either $g : B \mapsto [0, \infty)$ is a measurable function or $g : B \mapsto \mathbb{C}^\sigma$ is an integrable function, then we denote the Lebesgue integral of g over B by $\int_B g(x)\, dx$; compare the definitions in Cohn [3, p. 63, 64/65].

- $\int_B f(x)\ d\Omega(x)$

  Let $k, \sigma \in \mathbb{N}$, $k \geq 2$, $\Omega \subset \mathbb{R}^k$ open and Lipschitz bounded, $B \subset \partial\Omega$ measurable. Assume that either $f : B \mapsto [0, \infty)$ is a measurable function or $f : B \mapsto \mathbb{C}^\sigma$ is an integrable function. Then $\int_B f(x)\, d\Omega(x)$ denotes the surface integral of f over B .

- $\int_B f(x)\ do_x$

  In the case $\Omega = \mathbb{B}_n(y, r)$ $(n \in \mathbb{N}, \ \ y \in \mathbb{R}^n, \ \ r > 0)$, we write $\int_B f(x)\, do_x$ instead of $\int_B f(x)\ d\,\mathbb{B}_n(y, r)\,(x)$.

- $\|f\|_p$, $L^p(B)$

  Let $p \in [1, \infty)$, $\sigma \in \mathbb{N}$. Assume either $n \in \mathbb{N}, B \subset \mathbb{R}^n$ measurable, or $k \in \mathbb{N}$, $k \geq 2$, $\Omega \subset \mathbb{R}^k$ Lipschitz bounded, $B \subset \partial\Omega$ measurable. Furthermore, let $f : B \mapsto \mathbb{C}^\sigma$ be a measurable function. Then we put

  $$\|f\|_p := \Big(\int_B |f(x)|^p\ dx \Big)^{1/p} \qquad \text{and} \qquad \|f\|_p := \Big(\int_B |f(x)|^p\ d\,\Omega\,(x) \Big)^{1/p},$$

  respectively, where we implicitly used the notation $\infty^{1/p} := \infty$. In addition, define

  $$L^p(B) := \{ f : B \mapsto \mathbb{C} \ : \ f \text{ measurable}, \ \ \|f\|_p < \infty \}.$$

  Since we identify functions which coincide almost everywhere, we obtain a norm by restricting $\| \cdot \|_p$ to $L^p(B)$.

- $L^p(B) - \lim_{\epsilon \downarrow 0} f_\epsilon$

  Let $n, \sigma \in \mathbb{N}$, $B \subset \mathbb{R}^n$ measurable, $f_\epsilon \in L^p(B)^\sigma$ for $\epsilon \in (0, \infty)$. Suppose there is some $f \in L^p(B)^\sigma$ with $\|f_\epsilon - f\|_p \to 0$ for $\epsilon \downarrow 0$. Then the limit function f is denoted by $L^p(B) - \lim_{\epsilon \downarrow 0} f_\epsilon$.

- $\hat{f}$, $\overset{\vee}{f}$

Let $n, \sigma \in \mathbb{N}$, $f \in L^1(\mathbb{R}^n)^\sigma \cup L^2(\mathbb{R}^n)^\sigma$. In the case $f \in L^1(\mathbb{R}^n)^\sigma$, we set for $\xi \in \mathbb{R}^n$:

$$\hat{f}(\xi) := (2\cdot\pi)^{-n/2} \cdot \int_{\mathbb{R}^n} e^{-i\cdot\xi\cdot\eta} \cdot f(\eta)\, d\eta,$$

$$\overset{\vee}{f}(\xi) := (2\cdot\pi)^{-n/2} \cdot \int_{\mathbb{R}^n} e^{i\cdot\xi\cdot\eta} \cdot f(\eta)\, d\eta$$

(Fourier transform and inverse Fourier transform of the function f).

If $f \in L^2(\mathbb{R}^n)^\sigma$, we define the Fourier and inverse Fourier transform $\hat{f}$, $\overset{\vee}{f}$ of f in the usual way, that is, by a limit in $L^2(\mathbb{R}^n)^\sigma$, choosing constants as in the case $f \in L^1(\mathbb{R}^n)^\sigma$ so that both definitions coincide if $f \in L^2(\mathbb{R}^n)^\sigma \cap L^1(\mathbb{R}^n)^\sigma$.

- $W^{s,p}(U)$, $\|\ \|_{s,p}$

Let $n \in \mathbb{N}$, $U \subset \mathbb{R}^n$ open, $s \in (0, \infty)$, $p \in (1, \infty)$. Then we write $W^{s,p}(U)$ for the usual Sobolev space over U, of order s and with exponent p; see [1, p. 44/45, 3.1] in the case $s \in \mathbb{N}$, and [23, p. 330 ff, 6.8] in the case $s \in (0, \infty)\backslash\mathbb{N}$. The corresponging norm is denoted by $\|\ \|_{s,p}$.

For $u \in W^{s,p}(\partial\Omega)^3$, we set

$$\|u\|_{s,p} := |(\|u_1\|_{s,p},\ \|u_2\|_{s,p},\ \|u_3\|_{s,p})| \tag{2.1}$$

- $W^{s,p}(\partial\Omega)$, $\|\ \|_{s,p}$

Let $k \in \mathbb{N}$, $k \geq 2$, $\Omega \subset \mathbb{R}^k$ open. Assume in addition that either Ω is Lipschitz bounded and $s \in (0, 1]$, or that Ω is C^2-bounded and $s \in (0, 2]$. Then we use the symbol $W^{s,p}(\partial\Omega)$ for the usual Sobolev space over $\partial\Omega$, of order s and with exponent p; see [23, p. 327, 6.7.2].

There are many equivalent norms which may be introduced on this Sobolev space (see [23, p. 328, 6.7.3, 6.7.4]). In order to choose one of them, it would be necessary to first fix a parametric representation of $\partial\Omega$. However, whenever these norms will arise, that is, in Chapter 1 and 13, such details will be of no interest. Therefore, we assume that for k, s, Ω as above, one particular norm of $W^{s,p}(\partial\Omega)$ has been chosen, and we denote this norm by $\|\ \|_{s,p}$.

If $u \in W^{s,p}(\partial\Omega)^3$, then the expression $\|u\|_{s,p}$ is defined in analogy to (2.1).

- $K \otimes \Phi$

Let $\sigma, \tilde{\sigma} \in \mathbb{N}$, $A, B, \tilde{B} \subset \mathbb{R}^2$ measurable sets, $K : A \times \tilde{B} \mapsto \mathbb{C}^{\sigma\times\tilde{\sigma}}$, $\Phi : B \mapsto \mathbb{C}^{\tilde{\sigma}}$ measurable functions. Suppose that $B \subset \tilde{B}$, and

$$\int_B |K(\xi, \eta) \cdot \Phi(\eta)|\, d\eta < \infty \quad \text{for almost every } \xi \in A.$$

Then we define the function $K \otimes \Phi : A \mapsto \mathbb{C}^\sigma$ by

$$(K \otimes \Phi)(\xi) := \int_B K(\xi, \eta) \cdot \Phi(\eta)\, d\eta \quad \text{for a.e. } \xi \in A.$$

In order to indicate that the functions K and Φ satisfy the preceding conditions, we shall say for short: The function $K \otimes \Phi$ is well defined.

- $K_\epsilon \otimes \Phi$

Let $\sigma, \tilde{\sigma} \in \mathbb{N}$, $B \subset \mathbb{R}^2$ a measurable set, $K : \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{C}^{\sigma\times\tilde{\sigma}}$, $\Phi : B \mapsto \mathbb{C}^{\tilde{\sigma}}$ measurable functions. Let $\epsilon \in (0, \infty)$, and assume that

$$\int_B \chi_{(\epsilon,\infty)}(|\xi - \eta|) \cdot |K(\xi, \eta) \cdot \Phi(\eta)|\, d\eta < \infty \quad \text{for a.e. } \xi \in \mathbb{R}^2. \tag{2.2}$$

Then we introduce the function $K_\epsilon \otimes \Phi : \mathbb{R}^2 \mapsto \mathbb{C}^\sigma$ by setting

$$(K_\epsilon \otimes \Phi)(\xi) := \int_B \chi_{(\epsilon,\infty)}(|\xi - \eta|) \cdot K(\xi, \eta) \cdot \Phi(\eta)\, d\eta \quad \text{for a.e. } \xi \in \mathbb{R}^2.$$

- $K \otimes_p \Phi$

Let $\sigma, \tilde{\sigma}, B, K$ be given as in the preceding definition, and let $p \in (1, \infty)$, $\Phi \in L^p(B)^{\tilde{\sigma}}$. Assume that (2.2) holds for $\epsilon \in (0, \infty)$. In addition, we require that the limit

$L^p(\mathbb{R}^2) - \lim_{\epsilon\downarrow 0} (K_\epsilon \otimes \Phi)$

exists. Then we denote this limit by $K \otimes_p \Phi$. In order to indicate that the functions K and Φ satisfy the requirements just stated, we shall write briefly: The function $K \otimes_p \Phi$ is well defined.

- $K * \Phi$

Let $\sigma, \tilde{\sigma}, B, K, \Phi$ be given as in the definition of $K_\epsilon \otimes \Phi$. Assume

$$\int_B |K(\xi, \xi - \eta) \cdot \Phi(\eta)|\, d\eta < \infty \quad \text{for a.e. } \xi \in \mathbb{R}.$$

Then define the function $K * \Phi : \mathbb{R}^2 \mapsto \mathbb{C}^\sigma$ by

$$(K * \Phi)(\xi) := \int_B K(\xi, \xi - \eta) \cdot \Phi(\eta)\, d\eta \quad \text{for a.e. } \xi \in \mathbb{R}^2.$$

Once more, we shall use a shorter statement in order to point out that the preceding conditions on K und Φ are fulfilled. In fact, we shall write: The function $K * \Phi$ is well defined.

- $K_\epsilon * \Phi$

Let $\sigma, \tilde{\sigma}, B, K$ be given as in the definition of $K_\epsilon \otimes \Phi$. Take $p \in (1, \infty)$, $\Phi \in L^p(B)^{\tilde{\sigma}}$. For any $\epsilon \in (0, \infty)$ and for almost every $\xi \in \mathbb{R}^2$, we assume that

$$\int_B \chi_{(\epsilon,\infty)}(|\xi - \eta|) \cdot |K(\xi, \xi - \eta) \cdot \Phi(\eta)|\, d\eta < \infty. \tag{2.3}$$

Then, for $\epsilon \in (0, \infty)$, we introduce the function $K_\epsilon * \Phi : \mathbb{R}^2 \mapsto \mathbb{C}^\sigma$ by setting

$$(K_\epsilon * \Phi)(\xi) := \int_B \chi_{(\epsilon,\infty)}(|\xi - \eta|) \cdot K(\xi, \xi - \eta) \cdot \Phi(\eta)\, d\eta \quad \text{for a.e. } \xi \in \mathbb{R}^2.$$

- $K *_p \Phi$

  Let $\sigma, \tilde{\sigma}, B, K$ be given as in the definition of $K_\epsilon \otimes \Phi$. Assume that (2.3) is satisfied for any $\epsilon \in (0, \infty)$ and for almost every $\xi \in \mathbb{R}^2$. We require in addition that the limit $L^p(\mathbb{R}^2) - \lim_{\epsilon \downarrow 0} (K_\epsilon * \Phi)$ exists. Then we set

  $$K *_p \Phi := L^p(\mathbb{R}^2) - \lim_{\epsilon \downarrow 0} (K_\epsilon * \Phi),$$

  and we say for short: $K *_p \Phi$ is well defined.

- Sometimes, for shortness, we shall compress two inequalities into a single one: If A, B, C are reals, and if $A \leq C$ as well as $B \leq C$, then we shall write: $A, B \leq C$.

A rather large number of functions, operators and sets will be introduced in the main body of this book. For the convenience of the reader, we shall list those of them here which will used most frequently:

$A_\tau^{\lambda,\varphi}$, $A_\tau^{\infty,\varphi}$	Definition 5.2
$A(\varphi, \delta)$	Definition 10.1
$A(\tau, p, \varphi, R, S)$	Definition 6.3
$A^*(\tau, p, \varphi, R, S)$	Definition 6.3
$\mathcal{A}^{(v)}(\varphi, \lambda, \delta, R, S, \Phi)$	Definition 10.3
$\mathcal{B}^{(v)}(\varphi, \delta, R, S, \Phi)$	Definition 10.3
$\overline{\mathcal{D}}_k$, $\mathcal{D}_{jkl}$	(1.7)
$\widetilde{\mathcal{D}}_{jkl}^\lambda$	(1.4)
E_{4k}, $\widetilde{E}_{jk}^\lambda$	(1.2)
$\overline{E}$, E_{jk}	(1.6)
$\overline{E}_{jk}^\lambda$	Definition 5.1
f_1, f_2, f_3	Definition 5.1
$F^*(\tau, p, \varphi, R, S)$	Definition 6.3
$\mathcal{F}(R, S)$, $\widetilde{\mathcal{F}}(R)$	Definition 6.2
g_1, g_2	Definition 5.1
$\tilde{g}_1$, $\tilde{g}_2$	(1.1)
$\mathcal{G}_1^\lambda$, $\mathcal{G}_2^\lambda$, $\mathcal{G}_2^\infty$	Definition 5.1
$g^{(\varphi)}$	Chapter 3
$\gamma^{(\varphi,\epsilon)}$	Chapter 3, (3.10)
$G(R, S)$, $\widetilde{G}(R)$	Definition 6.2
$G(\tau, p, \varphi, \epsilon)$	Definition 6.3
$G^*(\tau, p, \varphi, \epsilon)$	Definition 6.3
$\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$	(1.5)
$\Gamma^*(\tau, p, \lambda, \mathbb{K}(\varphi))$	Definition 6.1
$\Gamma^{(inf)}(\tau, p, \lambda, \varphi, R)$	Definition 6.1
$\Gamma^{(ver)}(\tau, p, \lambda, \varphi, R)$	Definition 6.1
$h^{(\varphi)}$	Chapter 3
$H^*(\tau, p, \varphi, R, S)$	Definition 6.3
$J^{(\varphi,\epsilon)}$	Chapter 3, (3.10)
$J(\tau, p, \lambda, \varphi, R, S)$	Definition 6.3

$\mathbb{K}(\varphi)$	Chapter 1, 3
$K_{jl}^{(v)}(\varphi, \lambda, \delta)$	Definition 10.2
$\mathbb{L}(\varphi, \epsilon)$	Chapter 3, (3.10)
$L_j(r, \varphi)$, $L_j(0, \varphi)$	Definition 4.1
$L_{jl}^{(v)}(\varphi, \delta)$	Definition 10.2
$L(\tau, p, \lambda, \varphi, R)$	Definition 6.3
$\Lambda(\tau, p, B)$	(1.8), Chapter 13
$\Lambda^*(\tau, p, B)$	Defin. 6.1, Chapt. 13
$\Lambda^{(ver)}(\tau, p, \varphi, R)$	Definition 6.1
$\Lambda^{*\,(ver)}(\tau, p, \varphi, R)$	Definition 6.1
$M(\tau, p, \lambda, \varphi, \epsilon)$	Definition 6.3
$M^*(\tau, p, \lambda, \varphi, \epsilon)$	Definition 6.3
$\mathcal{M}(\varphi, \lambda)$, $\mathcal{M}(\varphi, \infty)$	Definition 10.2
$n^{(\varphi)}$	Chapter 3
$n^{(\varphi,\epsilon)}$	Chapter 3, (3.10)
P_{jkl}^λ	Definition 5.1
Ψ_ϵ	Definition 5.3
$\Pi(\tau, p, B)$	(1.9)
$\Pi^*(\tau, p, \mathbb{K}(\varphi))$	Definition 6.1
$Q(\partial B)$	Defin. 9.2, Chapt. 13
Q_{jkl}^λ	Definition 5.1
$T(\varphi)$	Definition 5.3
$T^{(\varphi,\epsilon)}$	Chapter 3, (3.10)
$\overline{V}(\partial\mathbb{L}(\varphi, \epsilon))$	Definition 9.2
$V(\partial B)$	Defin. 9.2, Chapt. 13
$\widetilde{V}^\lambda(\partial\mathbb{L}(\varphi, \epsilon))$	Definition 9.2
$\mathcal{X}_l^\lambda$, $\mathcal{X}_l^\infty$	Definition 5.1
X_l^λ, X_l^∞, $\underline{X}_l^\lambda$, $\underline{X}_l^\infty$	Definition 5.2
$\mathcal{Y}_l^\lambda$	Definition 5.1
Y_l^λ, $\underline{Y}_l^\lambda$	Definition 5.2

Chapter 3

Parametric Representations of the Surface of a Right Circular Cone

Let $\varphi \in (0, \pi/2]$, and recall that in Chapter 1, an open right circular infinite cone with vertex angle $2 \cdot \varphi$ was denoted by $\mathbb{K}(\varphi)$. Stated more formally, this means

$$\mathbb{K}(\varphi) = \{ (\xi, \; r + |\xi| \cdot \cot \varphi) \; : \; \xi \in \mathbb{R}^2, \; r \in (0, \infty) \} .$$

In this chapter, we are going to consider two parametric representations of the surface $\partial\mathbb{K}(\varphi)$ of $\mathbb{K}(\varphi)$. In addition, we shall introduce a parametric representation of the surface of a cone-like object having a smooth end instead of a vertex.

First, let us describe $\partial\mathbb{K}(\varphi)$ in cylindrical coordinates: Define the function $g^{(\varphi)} : \mathbb{R}^2 \mapsto \partial\mathbb{K}(\varphi)$ by

$$g^{(\varphi)}(\xi) := (\xi, \; |\xi| \cdot \cot \varphi) \qquad \text{for } \xi \in \mathbb{R}^2 .$$

Note that $g^{(\varphi)}$ is bijective and Lipschitz continuous. Moreover, $g^{(\varphi)} \,|\, \mathbb{R}^2 \backslash \{0\}$ is a regular C^∞-function, and we have for $\xi \in \mathbb{R}^2 \backslash \{0\}$:

$$\det\Big(\partial/\partial\xi_j \, (g^{(\varphi)}(\xi)) \cdot \partial/\partial\xi_k \, (g^{(\varphi)}(\xi)) \Big)_{1 \le j,k \le 2} \quad = \quad \sin^{-2}(\varphi) .$$

This means the constant $\sin^{-1}(\varphi)$ represents the element of surface area related to $g^{(\varphi)}$. For $\sigma \in \mathbb{N}$ and for every function $f : \partial\mathbb{K}(\varphi) \mapsto \mathbb{C}^\sigma$, it follows: f is measurable (integrable) if and only if the function $f \circ g^{(\varphi)} : \mathbb{R}^2 \mapsto \mathbb{C}^\sigma$ is measurable (integrable). Furthermore, if either $f : \partial\mathbb{K}(\varphi) \mapsto [0, \infty)$ is a measurable function or $f : \partial\mathbb{K}(\varphi) \mapsto \mathbb{C}^\sigma$ is an integrable function, then the following equation holds true:

$$\int_{\partial\mathbb{K}(\varphi)} f \; d\,\mathbb{K}(\varphi) \quad = \quad \int_{\mathbb{R}^2} (f \circ g^{(\varphi)})(\xi) \cdot \sin^{-1}(\varphi) \; d\xi . \tag{3.1}$$

For $x \in \partial\mathbb{K}(\varphi) \backslash \{0\}$, let $n^{(\varphi)}(x)$ denote the outward unit normal to $\mathbb{K}(\varphi)$ in the point x. Then we have for $\xi \in \mathbb{R}^2 \backslash \{0\}$:

$$(n^{(\varphi)} \circ g^{(\varphi)})(\xi) \quad = \quad \Big(\cos\varphi \cdot \xi_1 \cdot |\xi|^{-1}, \; \cos\varphi \cdot \xi_2 \cdot |\xi|^{-1}, \; -\sin\varphi \Big) . \tag{3.2}$$

Now let us describe $\partial\mathbb{K}(\varphi)$ by polar coordinates: Define the function $h^{(\varphi)} : \mathbb{R}^2 \mapsto \partial\mathbb{K}(\varphi)$ by

$$h^{(\varphi)}(r,\theta) := |r| \cdot \Big(\cos\theta \cdot \sin\varphi\,,\ \sin\theta \cdot \sin\varphi\,,\ \cos\varphi\Big) \qquad \text{for } (r,\theta) \in \mathbb{R}^2.$$

Then $h^{(\varphi)}$ is onto and Lipschitz continuous. The image set $h^{(\varphi)}([0,\infty) \times [0, 2\cdot\pi))$ coincides with $\mathbb{K}(\varphi)$, and the restriction $h^{(\varphi)}\,|\,(0,\infty) \times (0, 2\cdot\pi)$ is one-to-one and regular. Furthermore, it holds for $\eta \in (0,\infty) \times (0, 2\cdot\pi)$:

$$det\Big(\partial/\partial\eta_j\, h^{(\varphi)}(\eta) \cdot \partial/\partial\eta_k\, h^{(\varphi)}(\eta)\Big)_{1\le j,k\le 2} \;=\; \eta_1^2 \cdot \sin^2(\eta_2).$$

Thus, if the function $\widetilde{J}^{(\varphi)} : (0,\infty) \times (0, 2\cdot\pi) \mapsto (0,\infty)$ is defined by

$$\widetilde{J}^{(\varphi)}(r,\theta) := r \cdot \sin\varphi \qquad \text{for } r \in (0,\infty)\,,\ \theta \in (0, 2\cdot\pi)\,,$$

then $\widetilde{J}$ is the element of surface area related to $h\,|\,(0,\infty) \times (0, 2\cdot\pi)$. From these facts, it follows for $\sigma \in \mathbb{N}$, and for any function $f : \partial\mathbb{K}(\varphi) \mapsto \mathbb{C}^\sigma$: The function f is measurable (integrable) if and only if the mapping $\big(f \circ h^{(\varphi)}\,|\,(0,\infty) \times (0, 2\cdot\pi)\big) \cdot \widetilde{J}^{(\varphi)}$ is measurable (integrable). In the case that either $f : \partial\mathbb{K}(\varphi) \mapsto \mathbb{C}^\sigma$ is integrable or $f : \partial\mathbb{K}(\varphi) \mapsto [0,\infty)$ is measurable, it holds

$$\int_{\partial\mathbb{K}(\varphi)} f\ d\,\mathbb{K}(\varphi) \;=\; \int_0^\infty \int_0^{2\cdot\pi} (f \circ h^{(\varphi)})(r,\theta) \cdot r \cdot \sin\varphi\ d\theta\ dr. \tag{3.3}$$

For the convenience of the reader, we further point out the following equations, which hold for $r, s \in (0,\infty)$, $\theta, \varrho \in \mathbb{R}$:

$$(n^{(\varphi)} \circ h^{(\varphi)})(s,\varrho) \;=\; \Big(\cos\varrho \cdot \cos\varphi\,,\ \sin\varrho \cdot \cos\varphi\,,\ -\sin\varphi\Big);$$

$$\Big(h^{(\varphi)}(r,\theta) - h^{(\varphi)}(s,\varrho)\Big) \cdot (n^{(\varphi)} \circ h^{(\varphi)})(s,\varrho) \tag{3.4}$$

$$= -r \cdot (1 - \cos(\theta - \varrho)) \cdot \cos\varphi \cdot \sin\varphi;$$

$$|\,h^{(\varphi)}(r,\theta) - h^{(\varphi)}(s,\varrho)\,|^2 \tag{3.5}$$

$$= (r-s)^2 + (1 - \cos(\theta - \varrho)) \cdot 2 \cdot r \cdot s \cdot \sin^2(\varphi)$$

$$= r^2 + s^2 - 2 \cdot r \cdot s \cdot (\cos^2(\varphi) + \sin^2(\varphi) \cdot \cos(\theta - \varrho)).$$

Now let us construct a right circular cone with a round end instead of a vertex. For this purpose, we choose a function $\widetilde{\varphi} \in C^\infty(\mathbb{R})$ with $\widetilde{\varphi}\,|\,[0, 1/2) = 0$, $\widetilde{\varphi}\,|\,(1,\infty) = 1$, $\widetilde{\varphi}\,|\,[0,\infty)$ monotone increasing. It follows

$$1 - \int_0^1 \widetilde{\varphi}(s)\ ds \;>\; 0; \qquad 0 \le \widetilde{\varphi}(s) \le 1 \quad \text{for } s \in [0,\infty).$$

Let the function $\widetilde{\Psi} : [0,\infty) \mapsto (0,\infty)$ be defined by

$$\widetilde{\Psi}(r) := \int_0^r \widetilde{\varphi}(s)\ ds + 1 - \int_0^1 \widetilde{\varphi}(s)\ ds \qquad \text{for } r \in [0,\infty).$$

Then we have

$$\widetilde{\Psi}(r) \;=\; 1 - \int_0^1 \widetilde{\varphi}(s)\ ds \;>\; 0 \quad \text{for } r \in [0, 1/2); \qquad \widetilde{\Psi}(r) \;=\; r \quad \text{for } r \in [1,\infty);$$

$$\widetilde{\Psi}(r) \;\ge\; r \quad \text{for } r \in [0,\infty); \qquad \widetilde{\Psi} \in C^\infty([0,\infty)).$$

Let $\overline{\psi} : \mathbb{R}^2 \mapsto (0,\infty)$ be defined by $\overline{\psi}(\eta) := \widetilde{\Psi}(|\eta|)$ for $\eta \in \mathbb{R}^2$. Then it holds for $\eta \in \mathbb{R}^2$:

$$\overline{\psi}(\eta) \;=\; 1 - \int_0^1 \widetilde{\varphi}(s)\ ds \quad \text{in the case } |\eta| \le 1/2; \qquad \overline{\psi}(\eta) \;=\; |\eta| \quad \text{in the case } |\eta| \ge 1;$$

$$\overline{\psi}(\eta) \;\ge\; |\eta|.$$

In addition, we have $\overline{\psi} \in C^\infty(\mathbb{R}^2)$, and the partial derivatives of $\overline{\psi}$ are bounded.

For $\epsilon \in (0,\infty)$, we define the functions $\beta^{(\varphi,\epsilon)} : \mathbb{R}^2 \mapsto \mathbb{R}$, $\gamma^{(\varphi,\epsilon)} : \mathbb{R}^2 \mapsto \mathbb{R}^3$, $T^{(\varphi,\epsilon)} : \mathbb{R}^3 \mapsto \mathbb{R}^3$, $J^{(\varphi,\epsilon)} : \mathbb{R}^2 \mapsto [1,\infty)$ by

$$\beta^{(\varphi,\epsilon)}(\eta) := \epsilon \cdot \cot\varphi \cdot \overline{\psi}((1/\epsilon) \cdot \eta)\,, \qquad \gamma^{(\varphi,\epsilon)}(\eta) := (\eta\,,\ \beta^{(\varphi,\epsilon)}(\eta))\,,$$

$$T^{(\varphi,\epsilon)}(\eta, r) := (\eta\,,\ \beta^{(\varphi,\epsilon)}(\eta) + r)\,,$$

$$J^{(\varphi,\epsilon)}(\eta) := \Big(1 + \sum_{i=1}^{2} (D_i\beta^{(\varphi,\epsilon)}(\eta))^2\Big)^{1/2} \qquad \text{for } \eta \in \mathbb{R}^2,\ r \in \mathbb{R}.$$

Furthermore, we set for $\epsilon \in (0,\infty)$:

$$\mathbb{L}(\varphi,\epsilon) := T^{(\varphi,\epsilon)}(\mathbb{R}^2 \times (0,\infty)).$$

In particular, $\mathbb{L}(\varphi,\epsilon)$ is a domain in $\mathbb{R}^3$. It holds for $\epsilon \in (0,\infty)$, $\eta \in \mathbb{R}^2\backslash\mathbb{B}_2(0,\epsilon)$:

$$\beta^{(\varphi,\epsilon)}(\eta) \;=\; \cot\varphi \cdot |\eta|\,, \qquad \gamma^{(\varphi,\epsilon)}(\eta) \;=\; g^{(\varphi)}(\eta)\,, \qquad J^{(\varphi,\epsilon)}(\eta) \;=\; \sin^{-1}(\varphi). \tag{3.6}$$

Moreover, we have $\beta^{(\varphi,\epsilon)}(\eta) \ge \cot\varphi \cdot |\eta|$ for any $\eta \in \mathbb{R}^2$, $\epsilon \in (0,\infty)$. Hence $\mathbb{L}(\varphi,\epsilon)$ is a subset of $\mathbb{K}(\varphi)$, and above the hyperplane $x_3 = \epsilon$, the sets $\mathbb{L}(\varphi,\epsilon)$ and $\mathbb{K}(\varphi)$ coincide:

$$\mathbb{L}(\varphi,\epsilon) \cap \{\,x \in \mathbb{R}^3 \ :\ x_3 \ge \epsilon\,\} \;=\; \mathbb{K}(\varphi) \cap \{\,x \in \mathbb{R}^3 \ :\ x_3 \ge \epsilon\,\} \qquad \text{for } \epsilon \in (0,\infty).$$

However, the sets $\mathbb{K}(\varphi)$ and $\mathbb{L}(\varphi,\epsilon)$ differ below this hyperplane, since $\mathbb{K}(\varphi)$ has a vertex, whereas $\mathbb{L}(\varphi,\epsilon)$ has a smooth end. We further observe, for $\epsilon \in (0,\infty)$:

$$\gamma^{(\varphi,\epsilon)}(\mathbb{R}^2) \;=\; \partial\mathbb{L}(\varphi,\epsilon)\,; \qquad \gamma^{(\varphi,\epsilon)} \in C^\infty(\mathbb{R}^2)^3;$$

In addition, the function $\gamma^{(\varphi,\epsilon)}$ is one-to-one and regular. Thus it is a parametric representation of the surface $\partial\mathbb{L}(\varphi,\epsilon)$, and the set $\mathbb{L}(\varphi,\epsilon)$ is C^∞-bounded.

Let $n^{(\varphi,\epsilon)}$ denote the outward unit normal to $\mathbb{L}(\varphi,\epsilon)$. Then it holds by (3.6):

$$(n^{(\varphi,\epsilon)} \circ \gamma^{(\varphi,\epsilon)})(\eta) \;=\; (n^{(\varphi)} \circ g^{(\varphi)})(\eta) \qquad \text{for } \epsilon \in (0,\infty),\ \eta \in \mathbb{R}^2\backslash\mathbb{B}_2(0,\epsilon). \tag{3.7}$$

Note that the function $J^{(\varphi,\epsilon)}$ is the element of surface area related to $\gamma^{(\varphi,\epsilon)}$. Thus, for

$\sigma \in \mathbb{N}$, $\epsilon \in (0,\infty)$, and for functions $f : \partial\mathbb{L}(\varphi,\epsilon) \mapsto \mathbb{C}^\sigma$, it holds: f is measurable (integrable) if and only if $\left(f \circ \gamma^{(\varphi,\epsilon)}\right) \cdot J^{(\varphi,\epsilon)}$ is measurable (integrable). In the case that either $f : \partial\mathbb{L}(\varphi,\epsilon) \mapsto \mathbb{C}^\sigma$ is integrable or $f : \partial\mathbb{L}(\varphi,\epsilon) \mapsto [0,\infty)$ is measurable, the following equation holds true:

$$\int_{\partial\mathbb{L}(\varphi,\epsilon)} f \; d\mathbb{L}(\varphi,\epsilon) \quad = \quad \int_{\mathbb{R}^2} (f \circ \gamma^{(\varphi,\epsilon)})(\eta) \cdot J^{(\varphi,\epsilon)}(\eta) \; d\eta . \tag{3.8}$$

Moreover, we observe for $\epsilon \in (0,\infty)$, $\eta \in \mathbb{R}^2$:

$$(n^{(\varphi,\epsilon)} \circ \gamma^{(\varphi,\epsilon)})(\eta) \quad = \quad \left(J^{(\varphi,\epsilon)}(\eta)\right)^{-1} \cdot \left(D_1\beta^{(\varphi,\epsilon)}(\eta),\; D_2\beta^{(\varphi,\epsilon)}(\eta),\; -1\right). \tag{3.9}$$

In the following, we shall often prove assertions which refer to $\mathbb{L}(\varphi,\epsilon)$ with $\epsilon \in (0,\infty)$, as well as to $\mathbb{K}(\varphi)$. Then it will be convenient to use the ensuing notations:

$$\mathbb{L}(\varphi,0) := \mathbb{K}(\varphi), \quad n^{(\varphi,0)} := n^{(\varphi)}, \quad \gamma^{(\varphi,0)} := g^{(\varphi)}, \quad J^{(\varphi,0)}(\eta) := \sin^{-1}(\varphi), \tag{3.10}$$

$$T^{(\varphi,0)}(\eta, r) := \; g^{(\varphi)}(\eta) \; + \; (0,0,r) \qquad \text{for } \eta \in \mathbb{R}^2, \;\; r \in \mathbb{R}.$$

These definitions imply for $\epsilon \in [0,\infty)$:

$$\mathbb{L}(\varphi,\epsilon) \;=\; T^{(\varphi,\epsilon)}(\mathbb{R}^2 \times (0,\infty)), \qquad \mathbb{R}^3\backslash\overline{\mathbb{L}(\varphi,\epsilon)} \;=\; T^{(\varphi,\epsilon)}(\mathbb{R}^2 \times (-\infty,0)). \tag{3.11}$$

Let us now prove some properties of $g^{(\varphi)}$, $\gamma^{(\varphi,\epsilon)}$ and $T^{(\varphi,\epsilon)}$.

Lemma 3.1. *Take* $\varphi, \varphi' \in (0,\pi/2]$, $\xi, \eta \in \mathbb{R}^2$. *Then*

$$\left|(n^{(\varphi)} \circ g^{(\varphi)})(\xi) \cdot \left(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\right)\right| \;\le\; 4 \cdot \cos\varphi \cdot |\xi-\eta|^2 \cdot (|\xi| + |\eta|)^{-1}; \tag{3.12}$$

$$\Big|(n^{(\varphi)} \circ g^{(\varphi)})(\xi) \cdot \left(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\right) - (n^{(\varphi')} \circ g^{(\varphi')})(\xi) \cdot \left(g^{(\varphi')}(\xi) - g^{(\varphi')}(\eta)\right)\Big| \tag{3.13}$$
$$\le\; 4 \cdot |\varphi - \varphi'| \cdot \left(|\xi| - |\eta|\right)^2 \cdot (|\xi| + |\eta|)^{-1}.$$

The first inequality stated in the lemma will play an essential role when we shall prove continuity of certain potential operators such as those introduced in (1.5), (1.8); see Lemma 5.9, 5.11, 6.1, 6.2, 6.5 for some results of this kind. As for the second inequality, it will be used in Lemma 6.14, which, in turn, leads to the homotopy results stated in Lemma 6.17.

Proof of Lemma 3.1: It holds for $\widetilde{\varphi} \in \{\varphi, \varphi'\}$ (see (3.2)):

$$(n^{(\widetilde{\varphi})} \circ g^{(\widetilde{\varphi})})(\xi) \cdot \left(g^{(\widetilde{\varphi})}(\xi) - g^{(\widetilde{\varphi})}(\eta)\right) \;=\; \cos\widetilde{\varphi} \cdot |\xi|^{-1} \cdot (|\xi| \cdot |\eta| - \xi \cdot \eta), \tag{3.14}$$

where $0 \;\le\; |\xi| \cdot |\eta| - \xi \cdot \eta$, due to the Cauchy-Schwarz inequality. Furthermore, we observe that

$$|\xi - \eta|^2 \;=\; \left(|\xi| - |\eta|\right)^2 \;+\; 2 \cdot \left(|\xi| \cdot |\eta| - \xi \cdot \eta\right) \;\ge\; 2 \cdot \left(|\xi| \cdot |\eta| - \xi \cdot \eta\right).$$

Combining the two preceding inequalities, we obtain:

$$\left|(n^{(\varphi)} \circ g^{(\varphi)})(\xi) \cdot \left(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\right)\right| \;\le\; (1/2) \cdot \cos\varphi \cdot |\xi|^{-1} \cdot |\xi - \eta|^2; \tag{3.15}$$

$$\Big|(n^{(\varphi)} \circ g^{(\varphi)})(\xi) \cdot \left(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\right) - (n^{(\varphi')} \circ g^{(\varphi')})(\xi) \cdot \left(g^{(\varphi')}(\xi) - g^{(\varphi')}(\eta)\right)\Big| \tag{3.16}$$
$$\le\; (1/2) \cdot \left|\cos\varphi - \cos\varphi'\right| \cdot |\xi|^{-1} \cdot |\xi - \eta|^2 \;\le\; (1/2) \cdot |\varphi - \varphi'| \cdot |\xi|^{-1} \cdot |\xi - \eta|^2.$$

If $|\eta| \le 2 \cdot |\xi|$, then we have $2 \cdot |\xi| \ge (|\xi| + |\eta|)/2$. Hence in this case, inequality (3.12) follows from (3.15), and the estimate in (3.13) is a consequence of (3.16).

Now assume that $|\eta| \ge 2 \cdot |\xi|$. Then it holds $|\xi - \eta| \;\ge\; (1/4) \cdot (|\xi| + |\eta|)$ so that

$$|\xi|^{-1} \cdot \left(|\xi| \cdot |\eta| - \xi \cdot \eta\right) \;=\; |\eta| - \left(|\xi|^{-1} \cdot \xi\right) \cdot \eta \;\le\; |\eta| \;\le\; |\xi| + |\eta| \tag{3.17}$$
$$\le\; 4 \cdot |\xi - \eta|^2 \cdot (|\xi| + |\eta|)^{-1},$$

where the first of the preceding inequalities follows from the Cauchy-Schwarz inequality. Combining (3.14) and (3.17) yields (3.12) and (3.13) in the case $|\eta| \ge 2 \cdot |\xi|$.

Lemma 3.2. *For* $\varphi \in (0,\pi/2]$, $\epsilon \in (0,\infty)$, $\xi, \eta \in \mathbb{R}^2$, *it holds:*

$$\gamma^{(\varphi,\epsilon)}(\eta) \;=\; \epsilon \cdot \gamma^{(\varphi,1)}\left((1/\epsilon) \cdot \eta\right); \qquad J^{(\varphi,\epsilon)}(\eta) \;=\; J^{(\varphi,1)}\left((1/\epsilon) \cdot \eta\right),$$

$$(n^{(\varphi,\epsilon)} \circ \gamma^{(\varphi,\epsilon)})(\eta) \;=\; (n^{(\varphi,1)} \circ \gamma^{(\varphi,1)})\left((1/\epsilon) \cdot \eta\right).$$

Set $C_1 := \max\left\{1, \sum_{i=1}^{2} |D_i\overline{\psi}|_0\right\}$. *Then, for* $\varphi \in (0,\pi/2]$, $\epsilon \in (0,\infty)$, $\xi, \eta \in \mathbb{R}^2$, *the following inequalities are satisfied:*

$$|J^{(\varphi,\epsilon)}|_0 \;\le\; C_1 \cdot \sin^{-1}(\varphi), \qquad |\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta)| \;\le\; C_1 \cdot \sin^{-1}(\varphi) \cdot |\xi - \eta|.$$

Proof: The lemma follows by some simple computations.

The next lemma gives an estimate analogous to (3.12), but related to $\mathbb{L}(\varphi,\epsilon)$ with $\epsilon > 0$.

Lemma 3.3. *Define* $C_2 := \sum_{j,k=1}^{2} |D_j D_k \overline{\psi}|_0$. *Let* $\xi, \eta \in \mathbb{R}^2$, $\epsilon \in (0,\infty)$, $\varphi \in (0,\pi/2]$. *Then*

$$\left|\left(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta)\right) \cdot (n^{(\varphi,\epsilon)} \circ \gamma^{(\varphi,\epsilon)})(\xi)\right| \;\le\; C_2 \cdot (1/\epsilon) \cdot \cot\varphi \cdot |\xi - \eta|^2.$$

Proof: We obtain by (3.9), after twice applying the mean value theorem:

$$\Big(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta)\Big) \cdot \big(n^{(\varphi,\epsilon)} \circ \gamma^{(\varphi,\epsilon)}\big)(\xi)$$

$$= \Big(J^{(\varphi,\epsilon)}(\xi)\Big)^{-1} \cdot \sum_{j,k=1}^{2} (\xi-\eta)_j \cdot (\xi-\eta)_k \cdot \int_0^1 \int_0^1 (1-\vartheta) \cdot D_j\, D_k\, \beta^{(\varphi,\epsilon)}\big(\xi + \tau \cdot (\vartheta - 1) \cdot (\xi - \eta)\big)\ d\tau\ d\vartheta .$$

Since $J^{(\varphi,\epsilon)}(\xi) \ge 1$, the lemma follows by the preceding equation and Lemma 3.2.

Corollary 3.1. *There is some constant* $C_3 > 0$ *such that for* $\varphi \in (0, \pi/2]$, $\epsilon \in [0, \infty)$, $\xi, \eta \in \mathbb{R}^2 \backslash \{0\}$, *the following inequality holds true (recall the notations from (3.10)):*

$$\Big| \Big(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta)\Big) \cdot \big(n^{(\varphi,\epsilon)} \circ \gamma^{(\varphi,\epsilon)}\big)(\eta) \Big|$$

$$\le C_3 \cdot \sin^{-1}(\varphi) \cdot (|\xi| + |\eta|)^{-1} \cdot |\xi - \eta|^2 .$$

Proof: Define $C_3 := 4 \cdot \max\{1, C_1, C_2\}$. Let $\varphi, \epsilon, \xi, \eta$ be given as in the lemma. In the case $\epsilon = 0$, the above inequality follows from Lemma 3.1. Therefore, let us require $\epsilon > 0$. For brevity we set

$$L := \Big| \Big(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta)\Big) \cdot \big(n^{(\varphi,\epsilon)} \circ \gamma^{(\varphi,\epsilon)}\big)(\eta) \Big| .$$

We distinguish four cases:

1st case: $2 \cdot \epsilon > |\xi|$, $2 \cdot \epsilon > |\eta|$. Then $|\xi| + |\eta| \le 4 \cdot \epsilon$, so that Lemma 3.3 implies:

$$L \le 4 \cdot C_2 \cdot \cot\varphi \cdot (|\xi| + |\eta|)^{-1} \cdot |\xi - \eta|^2 ,$$

2nd case: $|\xi| \ge \epsilon$, $|\eta| \ge \epsilon$. Now it follows from (3.6) and Lemma 3.1:

$$L \le 4 \cdot \cos\varphi \cdot (|\xi| + |\eta|)^{-1} \cdot |\xi - \eta|^2 .$$

3rd case: $|\xi| \ge 2 \cdot \epsilon$, $|\eta| < \epsilon$. Then $|\xi - \eta| \ge (1/4) \cdot (|\xi| + |\eta|)$, and we may conclude from Lemma 3.2:

$$L \le \big|\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta)\big| \le \sin^{-1}(\varphi) \cdot C_1 \cdot |\xi - \eta|$$

$$\le \sin^{-1}(\varphi) \cdot 4 \cdot C_1 \cdot (|\xi| + |\eta|)^{-1} \cdot |\xi - \eta|^2 .$$

4th case: $|\eta| \ge 2 \cdot \epsilon$, $|\xi| < \epsilon$. Use arguments as in the preceding case.

Lemma 3.4. *For* $\varphi \in (0, \pi/2]$, $\epsilon \in [0, \infty)$, $\xi, \eta \in \mathbb{R}^2$, $r \in \mathbb{R}$, *it holds (recall (3.10)):*

$$\big|\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)\big| \ge |\xi - \eta| . \tag{3.18}$$

$$\big|\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)\big| \tag{3.19}$$

$$\ge (1/4) \cdot \sin\varphi \cdot \big(\big|\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta)\big| + |r|\big) .$$

Proof: Inequality (3.18) is obvious. For the proof of (3.19), take $\varphi \in (0, \pi/2]$, $r \in \mathbb{R}$, $\epsilon \in [0, \infty)$. In the case $\epsilon = 0$, we put $\beta^{(\varphi,\epsilon)}(\eta) := \cot\varphi \cdot |\eta|$ for $\eta \in \mathbb{R}^2$. Then we have for any $\epsilon \in [0, \infty)$:

$$\gamma^{(\varphi,\epsilon)}(\eta) = \big(\eta,\ \beta^{(\varphi,\epsilon)}(\eta)\big) \quad \text{for } \eta \in \mathbb{R}^2 .$$

Take $\xi, \eta \in \mathbb{R}^2$. In the case $\epsilon > 0$, we find:

$$\big|\beta^{(\varphi,\epsilon)}(\xi) - \beta^{(\varphi,\epsilon)}(\eta)\big| = \cot\varphi \cdot \epsilon \cdot \big|\tilde\Psi(|\xi|/\epsilon) - \tilde\Psi(|\eta|/\epsilon)\big|$$

$$= \cot\varphi \cdot \big|\,|\xi| - |\eta|\,\big| \cdot \Big| \int_0^1 \tilde\varphi\Big(|\eta|/\epsilon + \vartheta \cdot (|\xi| - |\eta|)/\epsilon\Big)\ d\vartheta \Big| \le \cot\varphi \cdot |\xi - \eta| .$$

Hence, being obvious for $\epsilon = 0$, the inequality

$$\big|\beta^{(\varphi,\epsilon)}(\xi) - \beta^{(\varphi,\epsilon)}(\eta)\big| \le \cot\varphi \cdot |\xi - \eta| \tag{3.20}$$

holds if $\epsilon > 0$ and if $\epsilon = 0$. Now we distinguish two cases:

1st case: $\big|\beta^{(\varphi,\epsilon)}(\xi) - \beta^{(\varphi,\epsilon)}(\eta)\big| \le (1/2) \cdot |r|$. Then

$$\big|\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)\big| \ge \big|\beta^{(\varphi,\epsilon)}(\xi) - \beta^{(\varphi,\epsilon)}(\eta) + r\big| \ge (1/2) \cdot |r| .$$

2nd case: $\big|\beta^{(\varphi,\epsilon)}(\xi) - \beta^{(\varphi,\epsilon)}(\eta)\big| > (1/2) \cdot |r|$. This assumption implies

$$\big|\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)\big| \ge |\xi - \eta| \ge \tan\varphi \cdot \big|\beta^{(\varphi,\epsilon)}(\xi) - \beta^{(\varphi,\epsilon)}(\eta)\big|$$

$$\ge (1/2) \cdot \tan\varphi \cdot |r| ,$$

where the first inequality follows from (3.18), and the second one from (3.20). Therefore, it holds in both cases:

$$\big|\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)\big| \ge (1/2) \cdot (1 \wedge \tan\varphi) \cdot |r| . \tag{3.21}$$

On the other hand, by first using (3.18), and then (3.20), we find:

$$\big|\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)\big| \ge |\xi - \eta| \tag{3.22}$$

$$= \big(1 + \cot^2(\varphi)\big)^{-1/2} \cdot \Big(|\xi - \eta|^2 + \cot^2(\varphi) \cdot |\xi - \eta|^2\Big)^{1/2}$$

$$\ge \sin\varphi \cdot \Big(|\xi - \eta|^2 + \big|\beta^{(\varphi,\epsilon)}(\xi) - \beta^{(\varphi,\epsilon)}(\eta)\big|^2\Big)^{1/2}$$

$$= \sin\varphi \cdot \big|\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta)\big| .$$

Now inequality (3.19) follows from (3.21) and (3.22).

Lemma 3.5. *Let* $\varphi \in (0, \pi/2]$, $\epsilon \in [0, \infty)$. *Then* $T^{(\varphi,\epsilon)}$ *is a topological mapping of the set* $\mathbb{R}^3$ *onto itself, and* $T^{(\varphi,\epsilon)} \,|\, (\mathbb{R}^2\backslash\{0\}) \times \mathbb{R}$ *is a regular* C^∞*-function satisfying the equation*

$$\int_{\mathbb{R}^3} f(x)\, dx \;=\; \int_{\mathbb{R}} \int_{\mathbb{R}^2} (f \circ T^{(\varphi,\epsilon)})(\eta, r)\, d\eta\, dr \qquad \text{for } f \in L^1(\mathbb{R}^3). \tag{3.23}$$

Proof: Define the function $F : \mathbb{R}^3 \mapsto \mathbb{R}^3$ by setting for $x \in \mathbb{R}^3$:

$$F(x) := \Big(x_1,\, x_2,\, x_3 - \cot\varphi \cdot |(x_1, x_2)| \Big) \qquad \text{if } \epsilon = 0,$$

$$F(x) := \Big(x_1,\, x_2,\, x_3 - \beta^{(\varphi,\epsilon)}(x_1, x_2) \Big) \qquad \text{if } \epsilon > 0.$$

Then F is a left and right inverse of $T^{(\varphi,\epsilon)}$. Since F and $T^{(\varphi,\epsilon)}$ are continuous, we conclude that $T^{(\varphi,\epsilon)}$ is topological. Obviously, $T^{(\varphi,\epsilon)} \,|\, (\mathbb{R}^2\backslash\{0\}) \times \mathbb{R}$ is a C^∞-function. It is easy to compute that the jacobian determinant of $T^{(\varphi,\epsilon)} \,|\, (\mathbb{R}^2\backslash\{0\}) \times \mathbb{R}$ equals 1 at every point $(\eta, r) \in (\mathbb{R}^2\backslash\{0\}) \times \mathbb{R}$. On the other hand, $T^{(\varphi,\epsilon)}$ was shown to be bijective, so equation (3.23) now follows by the substitution rule.

Chapter 4

L^p-Estimates for a Riesz Potential on the Surface of a Cone

Definition 4.1. *For* $\varphi \in (0, \pi/2]$, $r \in [0, \infty[$, $j \in \{1, 2, 3\}$, $(\xi, \eta) \in \mathbb{R}^2$ *with* $\xi \neq \eta$, *we set*

$$\begin{aligned} &L_j(r, \varphi)(\xi, \eta) \\ &\quad := -(4 \cdot \pi)^{-1} \cdot \big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r) \big)_j \cdot \big| g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r) \big|^{-3} \\ &\quad = \overline{\mathcal{D}}_j \big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r) \big). \end{aligned}$$

In this chapter, we intend to show that for $r \in \mathbb{R}$, $j \in \{1, 2, 3\}$, $\Phi \in L^p(\mathbb{R}^2)$, the function $L_j(r, \varphi) \otimes_p \Phi$ is well defined. In addition, we are going to estimate the L^p-norm of this function, and we shall carefully study its dependence on φ, in view of later homotopy arguments. The basic idea of our proof consists in expanding $L_j(r, \varphi)$ into an infinite sum so that the function $L_j(r, \varphi) \otimes_p \Phi$ is transformed into such a sum as well. To each summand of the latter series, we shall apply the Calderón-Zygmund theorem. For this purpose, we shall need a number of technical lemmas, some of which will be useful at a later point once more. For example, Lemma 4.3 will be needed in the present chapter as well as in the proof of Lemma 11.2.

Note that the results from [4] cannot be applied to potential functions defined on the surface $\partial\mathbb{K}(\varphi)$ because $\partial\mathbb{K}(\varphi)$ is unbounded.

Theorem 4.1. *Let* $p \in (1, \infty)$. *Put*

$$C_{4,1}(p) := 8 \;+\; 2 \cdot \max\{(p-1),\; (p-1)^{-1}\}, \qquad C_4(p) := 24 \cdot C_{4,1}(p),$$

$$C_5(p) := 8 \cdot C_{4,1}(p) \cdot \int_0^1 (1-t)^{-3/4} \cdot |1 - \sqrt{2} \cdot t|^{-1/4}\, dt.$$

Let $\delta \in (0, \pi/16)$, $\Phi \in L^p(\mathbb{R}^2)$ *with* $\Phi(\eta) \ge 0$ *for* $\eta \in \mathbb{R}^2$. *Define*
$A := \{ (r \cdot \cos\phi,\ r \cdot \sin\phi) \ :\ r \in (0,\infty),\ \phi \in (-\delta, \delta) \}$.
Then the following inequalities hold true:

$$\Big(\int_A \Big(\int_A |\xi - \eta|^{-1} \cdot (|\xi| + |\eta|)^{-1} \cdot \Phi(\eta)\ d\eta \Big)^p\ d\xi \Big)^{1/p} \le C_4(p) \cdot \sin^{1/2}(\delta) \cdot \Big(\int_A |\Phi(\eta)|^p\ d\eta \Big)^{1/p}, \tag{4.1}$$

$$\Big(\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} |\xi - \eta|^{-1} \cdot (|\xi| + |\eta|)^{-1} \cdot \Phi(\eta)\ d\eta \Big)^p\ d\xi \Big)^{1/p} \le C_5(p) \cdot \|\Phi\|_p . \tag{4.2}$$

Note that the integrals on the left-hand side of (4.1) and (4.2) are singular, with non-negative kernels. However, due to the special form of these kernels, these integrals exist and the estimates stated in (4.1) and (4.2) are valid, as will be proved below.

Proof of Theorem 4.1: Set $q := (1 - 1/p)^{-1}$, $\gamma := \min\{1/p,\ 1 - 1/p\}$. Then we first note the ensuing inequality, which holds for $s \in (0, \infty)$:

$$\int_0^\infty (s^2 + r^2)^{-1/4} \cdot |s - r|^{-1/2} \cdot (s + r)^{-1} \cdot r^{1-\gamma\cdot q}\ dr \tag{4.3}$$
$$= \int_0^\infty (s^2 + t^2 \cdot s^2)^{-1/4} \cdot |s - t \cdot s|^{-1/2} \cdot (s + t \cdot s)^{-1} \cdot (t \cdot s)^{1-\gamma\cdot q} \cdot s\ dt$$
$$= s^{-\gamma\cdot q} \cdot \int_0^\infty (1 + t^2)^{-1/4} \cdot |1 - t|^{-1/2} \cdot (1 + t)^{-1} \cdot t^{1-\gamma\cdot q}\ dt$$
$$\le s^{-\gamma\cdot q} \cdot 2 \cdot \Big(\int_0^2 |1 - t|^{-1/2}\ dt \ +\ \int_2^\infty t^{-1-\gamma\cdot q}\ dt \Big) \ =\ s^{-\gamma\cdot q} \cdot C_{4,1}(p) .$$

The preceding inequality is valid since by the choice of γ, we have $1 - \gamma \cdot q \ge 0$. The last equation follows from the definition of $C_{4,1}(p)$ and from the fact that $\gamma \cdot q = 1/(p-1)$ or $\gamma \cdot q = 1$.

By the choice of γ, we have in addition: $1 - \gamma \cdot p \ge 0$, and $\gamma \cdot p = 1$ or $\gamma \cdot p = (p-1)$. Thus, a similar computation as in (4.3) yields:

$$\int_0^\infty (s^2 + r^2)^{-1/4} \cdot |s - r|^{-1/2} \cdot (s + r)^{-1} \cdot s^{1-\gamma\cdot p}\ ds \ \le\ r^{-\gamma\cdot p} \cdot C_{4,1}(p) . \tag{4.4}$$

Inequalities (4.3) and (4.4) will be of use in order to deal with the right-hand side of the following estimate, which follows from Hölder's inequality:

$$\Big(\int_A \Big(\int_A |\xi - \eta|^{-1} \cdot (|\xi| + |\eta|)^{-1} \cdot \Phi(\eta)\, d\eta \Big)^p d\xi \Big)^{1/p} \tag{4.5}$$
$$= \Big(\int_{-\delta}^{\delta} \int_0^\infty s \cdot \Big(\int_{-\delta}^{\delta} \int_0^\infty r \cdot \Big| (s \cdot \cos\phi,\ s \cdot \sin\phi) - (r \cdot \cos\vartheta,\ r \cdot \sin\vartheta) \Big|^{-1} \cdot (s + r)^{-1} \cdot \Phi(r \cdot \cos\vartheta,\ r \cdot \sin\vartheta)\ dr\, d\vartheta \Big)^p\ ds\, d\phi \Big)^{1/p}$$
$$\le \Big(\int_{-\delta}^{\delta} \int_0^\infty s \cdot \Big(\int_{-\delta}^{\delta} \int_0^\infty r^{1-\gamma\cdot q} \cdot \Big| (s \cdot \cos\phi,\ s \cdot \sin\phi) - (r \cdot \cos\vartheta,\ r \cdot \sin\vartheta) \Big|^{-1} \cdot (s + r)^{-1}\, dr\, d\vartheta \Big)^{p-1}$$
$$\cdot \int_{-\delta}^{\delta} \int_0^\infty r^{1+\gamma\cdot p} \cdot \Big| (s \cdot \cos\phi,\ s \cdot \sin\phi) \ -\ (r \cdot \cos\vartheta,\ r \cdot \sin\vartheta) \Big|^{-1} \cdot (s + r)^{-1} \cdot \Phi(r \cdot \cos\vartheta,\ r \cdot \sin\vartheta)^p\ dr\, d\vartheta\, ds\, d\phi \Big)^{1/p} .$$

In order to estimate the right-hand side of (4.5), we first observe for $s \in (0,\infty)$, $\phi \in (-\delta, \delta)$:

$$\int_{-\delta}^{\delta} \int_0^\infty r^{1-\gamma\cdot q} \cdot \Big| (s \cdot \cos\phi,\ s \cdot \sin\phi) - (r \cdot \cos\vartheta,\ r \cdot \sin\vartheta) \Big|^{-1} \cdot (s+r)^{-1}\, dr\, d\vartheta$$
$$= \int_{-\delta}^{\delta} \int_0^\infty r^{1-\gamma\cdot q} \cdot \Big| 2 \cdot (s^2+r^2) \cdot \sin^2((\phi-\vartheta)/2) \ +\ (s-r)^2 \cdot \cos(\phi-\vartheta) \Big|^{-1/2} \cdot (s+r)^{-1}\, dr\, d\vartheta$$
$$\le \int_{-2\cdot\delta}^{2\cdot\delta} \int_0^\infty r^{1-\gamma\cdot q} \cdot (s^2+r^2)^{-1/4} \cdot |\sin(\sigma/2)|^{-1/2} \cdot |s - r|^{-1/2} \cdot \cos^{-1/4}(\sigma) \cdot (s+r)^{-1}\, dr\, d\sigma .$$

The last inequality follows by first performing the substitution $\vartheta = \phi - \sigma$, then increasing the domain of integration $(-\delta, \delta)$ to $(-2\cdot\delta,\ 2\cdot\delta)$, and finally applying the inequality $(a + b)^{-1/2} \le a^{-1/4} \cdot b^{-1/4}$ $(a,\ b \ge 0)$. Next we note, for s, φ as before:

$$\int_{-\delta}^{\delta} \int_0^\infty r^{1-\gamma\cdot q} \cdot \Big| (s \cdot \cos\phi,\ s \cdot \sin\phi) - (r \cdot \cos\vartheta,\ r \cdot \sin\vartheta) \Big|^{-1} \cdot (s+r)^{-1}\, dr\, d\vartheta \tag{4.6}$$
$$\le C_{4,1}(p) \cdot s^{-\gamma\cdot q} \cdot \int_{-2\cdot\delta}^{2\cdot\delta} |\sin(\vartheta/2)|^{-1/2} \cdot \cos^{-1/4}(\vartheta)\ d\vartheta$$
$$\le 3 \cdot C_{4,1}(p) \cdot s^{-\gamma\cdot q} \cdot \int_{-2\cdot\delta}^{2\cdot\delta} |\sin(\vartheta/2)|^{-1/2} \cdot \cos(\vartheta/2)\ d\vartheta$$
$$= 6 \cdot C_{4,1}(p) \cdot s^{-\gamma\cdot q} \cdot \int_{-\sin\delta}^{\sin\delta} |t|^{-1/2}\ dt \ =\ C_4(p) \cdot \sin^{1/2}(\delta) \cdot s^{-\gamma\cdot q}.$$

The first of the preceding inequalities follows from (4.3). As for the second one, observe that $|\vartheta| \le \pi/8$ for $\vartheta \in (-2\cdot\delta,\ 2\cdot\delta)$, hence $\cos(\vartheta/2) \ge \cos\vartheta \ge 1/2$, so that
$\cos^{-1/4}(\vartheta) \ \le\ 2^{1/4} \ \le\ 2^{5/4} \cdot \cos(\vartheta/2) \ \le\ 3 \cdot \cos(\vartheta/2)$.

Using (4.4) instead of (4.3), we obtain by analogous computations:

$$\int_{-\delta}^{\delta}\int_0^\infty s^{1-\gamma\cdot p}\cdot\Big|(s\cdot\cos\phi,\,s\cdot\sin\phi)-(r\cdot\cos\vartheta,\,r\cdot\sin\vartheta)\Big|^{-1}\cdot(s+r)^{-1}\,ds\,d\phi \tag{4.7}$$
$$\le\ C_4(p)\cdot\sin^{1/2}(\delta)\cdot r^{-\gamma\cdot p}\qquad \text{für } r\in(0,\infty)\,,\ \ \vartheta\in(-\delta,\,\delta)\,.$$

Collecting our results, we find:

$$\Big(\int_A\Big(\int_A|\xi-\eta|^{-1}\cdot(|\xi|+|\eta|)^{-1}\cdot\Phi(\eta)\,d\eta\Big)^p\,d\xi\Big)^{1/p} \tag{4.8}$$
$$\le\ \Big(\int_{-\delta}^{\delta}\int_0^\infty s\cdot\big(C_4(p)\cdot\sin^{1/2}(\delta)\cdot s^{-\gamma\cdot q}\big)^{p-1}$$
$$\cdot\int_{-\delta}^{\delta}\int_0^\infty r^{1+\gamma\cdot p}\cdot\Big|(s\cdot\cos\phi,\,s\cdot\sin\phi)-(r\cdot\cos\vartheta,\,r\cdot\sin\vartheta)\Big|^{-1}$$
$$\cdot(s+r)^{-1}\cdot\Phi(r\cdot\cos\vartheta,\,r\cdot\sin\vartheta)^p\,dr\,d\vartheta\,ds\,d\phi\Big)^{1/p}$$
$$\le\ \Big(C_4(p)\cdot\sin^{1/2}(\delta)\Big)^{1-1/p}$$
$$\cdot\Big(\int_{-\delta}^{\delta}\int_0^\infty r^{1+\gamma\cdot p}\cdot\int_{-\delta}^{\delta}\int_0^\infty s^{1-\gamma\cdot p}\cdot\Big|(s\cdot\cos\phi,\,s\cdot\sin\phi)$$
$$-(r\cdot\cos\vartheta,\,r\cdot\sin\vartheta)\Big|^{-1}\cdot(s+r)^{-1}\cdot\Phi(r\cdot\cos\vartheta,\,r\cdot\sin\vartheta)^p\,ds\,d\phi\,dr\,d\vartheta\Big)^{1/p}$$
$$\le\ C_4(p)\cdot\sin^{1/2}(\delta)\cdot\Big(\int_{-\delta}^{\delta}\int_0^\infty r\cdot\Phi(r\cdot\cos\vartheta,\,r\cdot\sin\vartheta)^p\,dr\,d\vartheta\Big)^{1/p}$$
$$=\ C_4(p)\cdot\sin^{1/2}(\delta)\cdot\Big(\int_A|\Phi(\eta)|^p\,d\eta\Big)^{1/p},$$

where the first inequality is valid due to (4.5), (4.6). The third one follows from (4.7).

In order to prove (4.2), we shall modify inequality (4.6). In fact, it holds for $\vartheta\in[-\pi,\pi]$, $r\in(0,\infty)$:

$$\int_{-\pi}^{\pi}\int_0^\infty s^{1-\gamma\cdot p}\cdot\Big|(s\cdot\cos\phi,\,s\cdot\sin\phi)-(r\cdot\cos\vartheta,\,r\cdot\sin\vartheta)\Big|^{-1}\cdot(s+r)^{-1}\,ds\,d\phi$$
$$=\ \int_{-\pi}^{\pi}\int_0^\infty s^{1-\gamma\cdot p}\cdot\Big(2\cdot(s^2+r^2)\cdot\sin^2((\phi-\vartheta)/2)\ +\ (s-r)^2\cdot\cos(\phi-\vartheta)\Big)^{-1/2}$$
$$\cdot(s+r)^{-1}\,ds\,d\phi$$
$$\le\ \int_{-\pi}^{\pi}\int_0^\infty s^{1-\gamma\cdot p}\cdot(s^2+r^2)^{-1/4}\cdot|\sin((\phi-\vartheta)/2)|^{-1/2}\cdot|s-r|^{-1/2}$$
$$\cdot|\cos(\phi-\vartheta)|^{-1/4}\cdot(s+r)^{-1}\,ds\,d\phi$$
$$\le\ C_{4,1}(p)\cdot r^{-\gamma\cdot p}\cdot\int_{-\pi}^{\pi}|\sin((\phi-\vartheta)/2)|^{-1/2}$$
$$\cdot\Big|\cos^2((\phi-\vartheta)/2)\ -\ \sin^2((\phi-\vartheta)/2)\Big|^{-1/4}\,d\phi\,,$$

where we used (4.4). Thus, resuming the preceding estimate, we find:

$$\int_{-\pi}^{\pi}\int_0^\infty s^{1-\gamma\cdot p}\cdot\Big|(s\cdot\cos\phi,\,s\cdot\sin\phi)-(r\cdot\cos\vartheta,\,r\cdot\sin\vartheta)\Big|^{-1}\cdot(s+r)^{-1}\,ds\,d\phi$$
$$\le\ C_{4,1}(p)\cdot r^{-\gamma\cdot p}\cdot 2\cdot\int_{-\pi/2}^{\pi/2}|\sin(\tau-\vartheta/2)|^{-1/2}$$
$$\cdot\Big|\cos^2(\tau-\vartheta/2)\ -\ \sin^2(\tau-\vartheta/2)\Big|^{-1/4}\,d\tau$$
$$=\ C_{4,1}(p)\cdot r^{-\gamma\cdot p}\cdot 2\cdot\int_{-\pi/2-\vartheta/2}^{\pi/2-\vartheta/2}|\sin\sigma|^{-1/2}\cdot|\cos^2(\sigma)\ -\ \sin^2(\sigma)|^{-1/4}\,d\sigma$$
$$=\ C_{4,1}(p)\cdot r^{-\gamma\cdot p}\cdot 8\cdot\int_0^{\pi/2}\sin^{-1/2}(\sigma)\cdot|\cos^2(\sigma)\ -\ \sin^2(\sigma)|^{-1/4}\,d\sigma\,,$$

where we first increased the domain of definition $(-\pi/2-\vartheta/2\,,\ \pi/2-\vartheta/2)$ to $(-\pi,\pi)$, and then performed some simple substitutions. We may continue our estimate as follows:

$$\int_{-\pi}^{\pi}\int_0^\infty s^{1-\gamma\cdot p}\cdot\Big|(s\cdot\cos\phi,\,s\cdot\sin\phi)\ -\ (r\cdot\cos\vartheta,\,r\cdot\sin\vartheta)\Big|^{-1}\cdot(s+r)^{-1}\,ds\,d\phi \tag{4.9}$$
$$=\ 8\cdot C_{4,1}(p)\cdot r^{-\gamma\cdot p}\cdot\int_0^{\pi/2}\sin\sigma\cdot(1-\cos^2(\sigma))^{-3/4}\cdot|-1\ +\ 2\cdot\cos^2(\sigma)|^{-1/4}\,d\sigma$$
$$=\ 8\cdot C_{4,1}(p)\cdot r^{-\gamma\cdot p}\cdot\int_0^1(1-t^2)^{-3/4}\cdot|1-2\cdot t^2|^{-1/4}\,dt$$
$$=\ 8\cdot C_{4,1}(p)\cdot r^{-\gamma\cdot p}\cdot\int_0^1(1-t)^{-3/4}\cdot(1+t)^{-3/4}\cdot|1-\sqrt{2}\cdot t|^{-1/4}\cdot(1+\sqrt{2}\cdot t)^{-1/4}\,dt$$
$$\le\ 8\cdot C_{4,1}(p)\cdot r^{-\gamma\cdot p}\cdot\int_0^1(1-t)^{-3/4}\cdot|1-\sqrt{2}\cdot t|^{-1/4}\quad=\quad C_5(p)\cdot r^{-\gamma\cdot p}\,.$$

A corresponding computation, based on (4.3) instead of (4.4), yields for $s\in(0,\infty)$, $\phi\in[-\pi,\pi]$:

$$\int_{-\pi}^{\pi}\int_0^\infty r^{1-\gamma\cdot q}\cdot\Big|(s\cdot cos\,\phi,\,s\cdot sin\,\phi)\ -\ (r\cdot cos\,\vartheta,\,r\cdot sin\,\vartheta)\Big|^{-1}\cdot(s+r)^{-1}\,dr\,d\vartheta \tag{4.10}$$
$$\le\ C_5(p)\cdot s^{-\gamma\cdot q}\,.$$

Now, by proceding as in (4.5) and (4.8), but referring to (4.9) and (4.10) instead of (4.6), (4.7), we finally arrive at (4.2).

Next we shall consider a number of technical results which will be used in order to expand the function $L_j(r,\varphi)\otimes_p\Phi$ into a series.

Lemma 4.1. *Let* $j \in \{1, 2\}$, $\xi, \eta \in \mathbb{R}^2$ *with* $\xi \neq 0$. *Then*

$$\left|(\xi+\eta)_j \cdot (|\xi|+|\eta|)^{-1} - \xi_j \cdot |\xi|^{-1}\right| \;\leq\; 2 \cdot |\xi - \eta| \cdot (|\xi|+|\eta|)^{-1}.$$

Proof: The lemma readily follows from the equation

$$(\xi+\eta)_j \cdot (|\xi|+|\eta|)^{-1} - \xi_j \cdot |\xi|^{-1}$$
$$= \Big((\eta-\xi)_j \cdot |\xi| + \xi_j \cdot (|\xi| - |\eta|) \Big) \cdot \Big((|\xi|+|\eta|) \cdot |\xi| \Big)^{-1}.$$

Lemma 4.2. *Let* $\xi, \eta \in \mathbb{R}^2$ *with* $|\xi| = 1$. *Then* $\left| \Big(\sum_{i=1}^{2} \eta_i \cdot \xi_i \Big)^2 - |\eta|^2 \right| \leq |\eta|^2$.

Proof: Observe that

$$\Big(\sum_{i=1}^{2} \eta_i \cdot \xi_i\Big)^2 - |\eta|^2 \;=\; \sum_{i=1}^{2} \eta_i^2 \cdot \xi_i^2 + 2 \cdot \eta_1 \cdot \eta_2 \cdot \xi_1 \cdot \xi_2 - \sum_{i=1}^{2} \eta_i^2$$
$$= \sum_{i=1}^{2} \eta_i^2 \cdot (\xi_i^2 - 1) + 2 \cdot \eta_1 \cdot \eta_2 \cdot \xi_1 \cdot \xi_2 \;=\; -\Big(\eta \cdot (-\xi_2, \xi_1)\Big)^2.$$

Now the lemma may be derived from the Cauchy-Schwarz inequality.

Lemma 4.3. *Let* $\xi, \eta \in \mathbb{R}^2$ *with* $\xi \neq 0$. *It follows:*

$$\left| (|\xi| - |\eta|)^2 - \Big(\sum_{i=1}^{2} (\xi-\eta)_i \cdot \xi_i / |\xi| \Big)^2 \right| \;\leq\; 16 \cdot |\xi - \eta|^3 \cdot (|\xi| + |\eta|)^{-1}.$$

In addition, take $v \in \mathbb{N}$. *Then*

$$\left| \Big((|\xi| - |\eta|)^2 - |\xi - \eta|^2 \Big)^v - \Big(\Big(\sum_{i=1}^{2} (\xi-\eta)_i \cdot \xi_i/|\xi| \Big)^2 - |\xi-\eta|^2 \Big)^v \right|$$
$$\leq\; 16 \cdot v \cdot (|\xi| + |\eta|)^{-1} \cdot |\xi - \eta|^{2 \cdot v + 1}.$$

Proof: We start with the following observation:

$$(|\xi| - |\eta|)^2 - \Big(\sum_{i=1}^{2} (\xi-\eta)_i \cdot \xi_i / |\xi| \Big)^2$$
$$= \Big(\sum_{i=1}^{2} (\xi-\eta)_i \cdot (\xi+\eta)_i \cdot (|\xi|+|\eta|)^{-1} \Big)^2 - \Big(\sum_{i=1}^{2} (\xi-\eta)_i \cdot \xi_i/|\xi| \Big)^2$$

$$= \Big(\sum_{i=1}^{2} (\xi-\eta)_i \cdot (\xi+\eta)_i \cdot (|\xi|+|\eta|)^{-1} \;+\; \sum_{i=1}^{2} (\xi-\eta)_i \cdot \xi_i/|\xi| \Big)$$
$$\cdot \Big(\sum_{i=1}^{2} (\xi-\eta)_i \cdot \Big((\xi+\eta)_i \cdot (|\xi|+|\eta|)^{-1} - \xi_i/|\xi| \Big) \Big).$$

Applying Lemma 4.1 to the right-hand side of the preceding equation, we see that

$$\left| (|\xi| - |\eta|)^2 - \Big(\sum_{i=1}^{2} (\xi-\eta)_i \cdot \xi_i/|\xi| \Big)^2 \right|$$
$$\leq \Big(\sum_{i=1}^{2} |\xi - \eta| \cdot 2 \Big) \cdot \Big(\sum_{i=1}^{2} 2 \cdot |\xi-\eta|^2 \cdot (|\xi|+|\eta|)^{-1} \Big) \;\leq\; 16 \cdot |\xi - \eta|^3 \cdot (|\xi| + |\eta|)^{-1}.$$

Thus the first part of the lemma is proved. As for the second part, we use a well known formula for expanding the expression $(a-b)^v$ $(a, b \in \mathbb{R})$. It follows:

$$\left| \Big((|\xi| - |\eta|)^2 - |\xi - \eta|^2 \Big)^v - \Big(\Big(\sum_{i=1}^{2} (\xi-\eta)_i \cdot \xi_i/|\xi| \Big)^2 - |\xi-\eta|^2 \Big)^v \right|$$
$$\leq \left| \Big((|\xi| - |\eta|)^2 - \Big(\sum_{i=1}^{2} (\xi-\eta)_i \cdot \xi_i/|\xi| \Big)^2 \Big) \right.$$
$$\left. \cdot \sum_{k=0}^{v-1} \Big((|\xi| - |\eta|)^2 - |\xi - \eta|^2 \Big)^k \cdot \Big(\Big(\sum_{i=1}^{2} (\xi-\eta)_i \cdot \xi_i/|\xi| \Big)^2 - |\xi-\eta|^2 \Big)^{v-1-k} \right|$$
$$\leq\; 16 \cdot v \cdot |\xi - \eta|^3 \cdot (|\xi| + |\eta|)^{-1} \cdot |\xi - \eta|^{(v-1)\cdot 2},$$

where the last inequality is implied by Lemma 4.2 and the first part of Lemma 4.3.

Lemma 4.4. *Let* $\xi, \eta \in \mathbb{R}^2$ *with* $|\xi| = 1$, $v \in \mathbb{N}$. *It holds:*

$$\left| \Big(\Big(\sum_{i=1}^{2} \eta_i \cdot \xi_i \Big)^2 - |\eta|^2 \Big)^v - (-\eta_2^2)^v \right| \;\leq\; 3 \cdot v \cdot |\xi_2| \cdot |\eta|^{2 \cdot v}.$$

Proof: Note that

$$\left| \Big(\sum_{i=1}^{2} \eta_i \cdot \xi_i \Big)^2 - \eta_1^2 \right| \;=\; \left| \eta_1^2 \cdot \xi_1^2 + \eta_2^2 \cdot \xi_2^2 + 2 \cdot \eta_1 \cdot \eta_2 \cdot \xi_1 \cdot \xi_2 - \eta_1^2 \right| \tag{4.11}$$
$$= \left| \eta_1^2 \cdot (-\xi_2^2) + \eta_2^2 \cdot \xi_2^2 + 2 \cdot \eta_1 \cdot \eta_2 \cdot \xi_1 \cdot \xi_2 \right| \;\leq\; 3 \cdot |\eta|^2 \cdot |\xi_2|.$$

Now, by using the formula for expanding the expression $(a-b)^v$ $(a, b \geq 0)$, and by referring to (4.11) and Lemma 4.2, we arrive at the estimate

$$\left| \Big(\Big(\sum_{i=1}^{2} \eta_i \cdot \xi_i \Big)^2 - |\eta|^2 \Big)^v - (-\eta_2^2)^v \right|$$

$$\leq \left|\Big(\sum_{i=1}^{2} \eta_i \cdot \xi_i\Big)^2 - |\eta|^2 + \eta_2^2\right| \cdot \sum_{k=0}^{\upsilon-1} \left|\Big(\sum_{i=1}^{2} \eta_i \cdot \xi_i\Big)^2 - |\eta|^2\right|^k \cdot \eta_2^{2\cdot(\upsilon-1-k)}$$

$$\leq 3 \cdot |\eta|^2 \cdot |\xi_2| \cdot \upsilon \cdot |\eta|^{2\cdot\upsilon-2}.$$

Lemma 4.5. *For* $\varphi_0 \in (0, \pi/2]$, $\varphi, \varphi' \in [\varphi_0, \pi/2]$, $\upsilon \in \mathbb{N}$, *it holds:*

$$\left|\sin^3(\varphi) \cdot \cos^{2\cdot\upsilon}(\varphi) - \sin^3(\varphi') \cdot \cos^{2\cdot\upsilon}(\varphi')\right| \tag{4.12}$$

$$\leq |\varphi - \varphi'| \cdot 3 \cdot (\upsilon + 1) \cdot \cos^{2\cdot\upsilon-1}(\varphi_0).$$

Proof: The lemma is a simple consequence of the mean value theorem.

Lemma 4.6. *Let* $\varphi \in (0, \pi/2]$, $\xi, \eta \in \mathbb{R}^2$ *mit* $\xi \neq \eta$, $r \in \mathbb{R}$ *with*

$$|r| \leq \min\{\, |\xi - \eta|/2, \quad (8 \cdot \cot\varphi)^{-1} \cdot |\xi - \eta| \,\} \qquad \textit{in the case } \varphi < \pi/2,$$

and with $|r| \leq |\xi - \eta|/2$ *in the case* $\varphi = \pi/2$. *Then the series*

$$\sum_{\upsilon=0}^{\infty} \binom{-3/2}{\upsilon} \cdot \Big(\big(\cot\varphi \cdot (|\xi| - |\eta|) + r\big)^2 - \cot^2(\varphi) \cdot |\xi - \eta|^2 \Big)^{\upsilon} \cdot \sin^{2\cdot\upsilon+3}(\varphi) \cdot |\xi - \eta|^{-2\cdot\upsilon-3}$$

$$= \sin^3(\varphi) \cdot \sum_{\upsilon=0}^{\infty} \binom{-3/2}{\upsilon} \cdot \cos^{2\cdot\upsilon}(\varphi) \cdot \Big(\big(|\xi| - |\eta| + r \cdot \tan\varphi\big)^2 - |\xi - \eta|^2 \Big)^{\upsilon} \cdot |\xi - \eta|^{-2\cdot\upsilon-3}$$

converges absolutely and equals $\left| g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0, 0, r) \right|^{-3}$.

Proof: First note that

$$\left| g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0, 0, r) \right|^{-3} = \Big(|\xi - \eta|^2 + \big(\cot\varphi \cdot (|\xi| - |\eta|) + r\big)^2 \Big)^{-3/2}$$

$$= \Big(\sin^2(\varphi) \cdot |\xi - \eta|^2 + \big(\cot\varphi \cdot (|\xi| - |\eta|) + r\big)^2 - \cot^2(\varphi) \cdot |\xi - \eta|^2 \Big)^{-3/2}.$$

On the other hand, since $\big||\xi| - |\eta|\big| \leq |\xi - \eta|$, and because of our assumptions on r, we find:

$$\left| (|\xi| - |\eta|)^2 - |\xi - \eta|^2 \right| \leq |\xi - \eta|^2; \qquad r^2 \leq |\xi - \eta|^2/4;$$

$$\left| 2 \cdot r \cdot \cot\varphi \cdot (|\xi| - |\eta|) \right| \leq 2 \cdot |r| \cdot \cot\varphi \cdot |\xi - \eta| \leq (1/4) \cdot |\xi - \eta|^2.$$

Combining these estimates yields

$$\left| \big(\cot\varphi \cdot (|\xi| - |\eta|) + r\big)^2 - \cot^2(\varphi) \cdot |\xi - \eta|^2 \right|$$

$$= \left| \cot^2(\varphi) \cdot \big((|\xi| - |\eta|)^2 - |\xi - \eta|^2 \big) + 2 \cdot r \cdot \cot\varphi \cdot (|\xi| - |\eta|) + r^2 \right|$$

$$\leq \cot^2(\varphi) \cdot |\xi - \eta|^2 + |\xi - \eta|^2/4 + |\xi - \eta|^2/4$$

$$< (1 + \cot^2(\varphi)) \cdot |\xi - \eta|^2 = \sin^{-2}(\varphi) \cdot |\xi - \eta|^2.$$

Now the lemma follows by using some well known facts concerning the series expansion of the function $x^{-3/2}$.

Theorem 4.2. *Take* $p \in (1, \infty)$, *and let* $K : \mathbb{R}^2\backslash\{0\} \times \mathbb{R}^2\backslash\{0\} \mapsto \mathbb{C}$ *be a measurable function. Assume that for* $\xi \in \mathbb{R}^2\backslash\{0\}$, *the function* $K(\xi, \cdot)$ *is odd and homogeneous of order* -2. *In addition, suppose that*

$$\sup\Big\{ |K(\xi, \eta)| \,:\, \xi \in \mathbb{R}^2\backslash\{0\} \Big\} < \infty \qquad \textit{for } \eta \in \mathbb{R}^2\backslash\{0\} \textit{ with } |\eta| = 1;$$

$$\int_{\partial\mathbb{B}_2(0;1)} \sup\Big\{ |K(\xi, \eta)| \,:\, \xi \in \mathbb{R}^2\backslash\{0\} \Big\}\, do_\eta < \infty.$$

Then the function $K *_p \Phi$ *is well defined for any* $\Phi \in L^p(\mathbb{R}^2)$.

Let $E \in (0, \infty)$. *Then there is a constant* $C_6(p, E) > 0$ *such that*

$$\|K_\epsilon * \Phi\|_p, \quad \|K *_p \Phi\|_p \leq C_6(p, E) \cdot \|\Phi\|_p$$

for $\Phi \in L^p(\mathbb{R}^2)$, $\epsilon \in (0, \infty)$, *and for all measurable functions* $K : \mathbb{R}^2\backslash\{0\} \times \mathbb{R}^2\backslash\{0\} \mapsto \mathbb{C}$ *satisfying the properties to follow:* $K(\xi, \cdot)$ *is odd and homogeneous of order* -2 *for* $\xi \in \mathbb{R}^2$,

$$\sup\Big\{ |K(\xi, \eta)| \,:\, \xi \in \mathbb{R}^2\backslash\{0\} \Big\} < \infty \qquad \textit{for } \eta \in \mathbb{R}^2\backslash\{0\} \textit{ with } |\eta| = 1,$$

$$\int_{\partial\mathbb{B}_2(0;1)} \sup\Big\{ |K(\xi, \eta)| \,:\, \xi \in \mathbb{R}^2\backslash\{0\} \Big\}\, do_\eta \leq E.$$

This theorem is a special case of the Calderón-Zygmund theorem. For a proof, we refer to [36, p. 82-93].

Lemma 4.7. *Let* $l \in \{1, 2\}$, $\varphi \in (0, \pi/2]$. *For* $\xi, \eta \in \mathbb{R}^2\backslash\{0\}$, *put*

$$\widetilde{K}_l(\varphi)(\xi, \eta) := (4 \cdot \pi)^{-1} \cdot \eta_l \cdot \sin^3(\varphi) \cdot \sum_{\upsilon=0}^{\infty} \binom{-3/2}{\upsilon} \cdot \cos^{2\cdot\upsilon}(\varphi) \tag{4.13}$$

$$\cdot \Big(\big(\sum_{i=1}^{2} \eta_i \cdot \xi_i/|\xi|\big)^2 - |\eta|^2 \Big)^v \cdot |\eta|^{-2\cdot v-3}.$$

For any $\xi, \eta \in \mathbb{R}^2\backslash\{0\}$, *the preceding series converges absolutely. Hence, this infinite sum defines a measurable function* $\widetilde{K}_l(\varphi)$ *from* $\mathbb{R}^2\backslash\{0\} \times \mathbb{R}^2\backslash\{0\}$ *into* $\mathbb{R}$.

Let $p \in (1,\infty)$, $\Phi \in L^p(\mathbb{R}^2)$, $\epsilon \in (0,\infty)$, $E \in \mathbb{R}$ *with*

$$E \geq (1/2) \cdot \sum_{v=0}^{\infty} \Big| \binom{-3/2}{v} \Big| \cdot \cos^{2\cdot v}(\varphi).$$

Then the function $\widetilde{K}_l(\varphi) *_p \Phi$ *is well defined, and it holds*

$$\|(\widetilde{K}_l(\varphi))_\epsilon * \Phi\|_p, \quad \|\widetilde{K}_l(\varphi) *_p \Phi\|_p \quad \leq \quad C_6(p,E) \cdot \|\Phi\|_p.$$

Proof: According to Lemma 4.2, we have for $\xi, \eta \in \mathbb{R}^2\backslash\{0\}$:

$$\Big| \big(\sum_{i=1}^{2} \eta_i \cdot \xi_i/|\xi|\big)^2 - |\eta|^2 \Big| \quad \leq \quad |\eta|^2.$$

Thus, recalling the series expansion of the function $x^{-3/2}$, we may conclude that the infinite sum on the right-hand side of (4.13) converges absolutely for any $\xi, \eta \in \mathbb{R}^2\backslash\{0\}$. In addition, it follows for $\xi, \eta \in \mathbb{R}^2\backslash\{0\}$ with $|\eta| = 1$:

$$|\widetilde{K}_l(\varphi)(\xi,\eta)| \quad \leq \quad (4\cdot\pi)^{-1} \cdot \sum_{v=0}^{\infty} \Big| \binom{-3/2}{v} \Big| \cdot \cos^{2\cdot v}(\varphi).$$

Obviously, $\widetilde{K}_l(\varphi)$ is measurable, and $\widetilde{K}_l(\varphi)(\xi,\cdot)$ is odd and homogeneous of order -2, for $\xi \in \mathbb{R}^2\backslash\{0\}$. Now the lemma follows from Theorem 4.2.

Theorem 4.3. *For* $\xi, \eta \in \mathbb{R}^2\backslash\{0\}$, $l \in \{1,2\}$, $\varphi, \varphi' \in (0, \pi/2]$, $\gamma \in (0,1]$, *we set*

$$G_l(\varphi,\gamma)(\xi,\eta) \quad := \quad (4\cdot\pi)^{-1} \cdot \chi_{(0,\gamma]}(|\xi_2|/|\xi|) \cdot \eta_l$$
$$\cdot \Big\{ \Big(|\eta|^2 + \cot^2(\varphi) \cdot \big(\sum_{i=1}^{2} \eta_i \cdot \xi_i/|\xi|\big)^2 \Big)^{-3/2} - \Big(|\eta|^2 + \cot^2(\varphi) \cdot \eta_1^2 \Big)^{-3/2} \Big\}.$$

For l, φ, γ *as before, and for* $p \in (1,\infty)$, $\Phi \in L^p(\mathbb{R}^2)$, *the functions* $L_l(0,\varphi) \otimes_p \Phi$ *and* $G_l(\varphi,\gamma) *_p \Phi$ *are well defined.*

If $p \in (1,\infty)$, $\varphi_0 \in (0, \pi/2]$, *there is a constant* $C_7(p,\varphi_0) > 0$, *such that for* $\varphi, \varphi' \in [\varphi_0, \pi/2]$, $\epsilon \in (0,\infty)$, $l \in \{1,2\}$, $\Phi \in L^p(\mathbb{R}^2)$, $\gamma \in (0,1]$, *it holds:*

$$\|(L_l(0,\varphi))_\epsilon \otimes \Phi\|_p, \quad \|L_l(0,\varphi) \otimes_p \Phi\|_p \quad \leq \quad C_7(p,\varphi_0) \cdot \|\Phi\|_p, \tag{4.14}$$

$$\|(G_l(\varphi,\gamma))_\epsilon * \Phi\|_p, \quad \|G_l(\varphi,\gamma) *_p \Phi\|_p \quad \leq \quad C_7(p,\varphi_0) \cdot \gamma \cdot \|\Phi\|_p, \tag{4.15}$$

$$\|(L_l(0,\varphi))_\epsilon \otimes \Phi - (L_l(0,\varphi'))_\epsilon \otimes \Phi\|_p, \quad \|L_l(0,\varphi) \otimes_p \Phi - L_l(0,\varphi') \otimes_p \Phi\|_p \tag{4.16}$$
$$\leq \quad C_7(p,\varphi_0) \cdot |\varphi - \varphi'| \cdot \|\Phi\|_p.$$

Proof: Fix $p \in (1,\infty)$, $\varphi_0 \in (0, \pi/2]$. We first define the constants appearing in (4.14) – (4.16). To this end, set

$$C_{7,1}(\varphi_0) := \pi^{-1} \cdot 12 \cdot \sum_{v=0}^{\infty} \Big| \binom{-3/2}{v} \Big| \cdot v \cdot (v+1) \cdot \cos^{2\cdot v-1}(\varphi_0),$$

$$C_{7,2}(p,\varphi_0) := C_6(p, 2\cdot\pi\cdot C_{7,1}(\varphi_0)), \qquad C_{7,3}(p,\varphi_0) := C_5(p) \cdot C_{7,1}(\varphi_0),$$

$$C_7(p,\varphi_0) := C_{7,3}(p,\varphi_0) + C_{7,2}(p,\varphi_0).$$

Since $\cos\varphi_0 \in [0,1)$, the series appearing in the definition of $C_{7,1}(\varphi_0)$ is convergent.

Now let $\varphi, \varphi' \in [\varphi_0, \pi/2]$, $\epsilon \in (0,\infty)$, $l \in \{1,2\}$, $\Phi \in L^p(\mathbb{R}^2)$, $\gamma \in (0,1]$. Define the functions $\widetilde{K}_l(\varphi)$, $\widetilde{K}_l(\varphi')$ as in Lemma 4.7. Then we obtain for $\xi, \eta \in \mathbb{R}^2\backslash\{0\}$:

$$(\widetilde{K}_l(\varphi) - \widetilde{K}_l(\varphi'))(\xi,\eta) \tag{4.17}$$
$$= (4\cdot\pi)^{-1} \cdot \eta_l \cdot \sum_{v=1}^{\infty} \binom{-3/2}{v} \cdot \Big(\sin^3(\varphi) \cdot \cos^{2\cdot v}(\varphi) - \sin^3(\varphi') \cdot \cos^{2\cdot v}(\varphi') \Big)$$
$$\cdot \Big(\big(\sum_{i=1}^{2} \eta_i \cdot \xi_i/|\xi|\big)^2 - |\eta|^2 \Big)^v \cdot |\eta|^{-2\cdot v-3}.$$

By referring to Lemma 4.2, and using the series expansion of the function $x^{-3/2}$, we get for $\xi, \eta \in \mathbb{R}^2\backslash\{0\}$:

$$G_l(\varphi,\gamma)(\xi,\eta) \quad = \quad (4\cdot\pi)^{-1} \cdot \chi_{(0,\gamma]}(|\xi_2|/|\xi|) \cdot \eta_l \tag{4.18}$$
$$\cdot \Bigg[\Big(\sin^{-2}(\varphi) \cdot |\eta|^2 + \cot^2(\varphi) \cdot \Big(\big(\sum_{i=1}^{2} \eta_i \cdot \xi_i/|\xi|\big)^2 - |\eta|^2 \Big) \Big)^{-3/2}$$
$$- \Big(\sin^{-2}(\varphi) \cdot |\eta|^2 - \cot^2(\varphi) \cdot \eta_2^2 \Big)^{-3/2} \Bigg]$$
$$= (4\cdot\pi)^{-1} \cdot \chi_{(0,\gamma]}(|\xi_2|/|\xi|) \cdot \eta_l \cdot \sin^3(\varphi) \cdot \sum_{v=0}^{\infty} \binom{-3/2}{v} \cdot \cos^{2\cdot v}(\varphi)$$
$$\cdot |\eta|^{-2\cdot v-3} \cdot \Big(\Big(\big(\sum_{i=1}^{2} \eta_i \cdot \xi_i/|\xi|\big)^2 - |\eta|^2 \Big)^v - (-\eta_2^2)^v \Big),$$

with the preceding sums being absolutely convergent.

Obviously, if $K \in \{G_l(\varphi,\gamma), \widetilde{K}_l(\varphi) - \widetilde{K}_l(\varphi')\}$, then K is measurable, and $K(\xi,\cdot)$ is odd and homogeneous of order -2, for any $\xi \in \mathbb{R}^2\backslash\{0\}$. Moreover, combining (4.17), (4.18), Lemma 4.2, 4.4 and 4.5, we obtain for $\xi \in \mathbb{R}^2\backslash\{0\}$, $\eta \in \mathbb{R}^2$ with $|\eta| = 1$:

$$|G_l(\varphi,\gamma)(\xi,\eta)| \;\le\; C_{7,1}(\varphi_0)\cdot\gamma\,, \qquad |(\widetilde{K}_l(\varphi)-\widetilde{K}_l(\varphi'))(\xi,\eta)| \;\le\; C_{7,1}(\varphi_0)\cdot|\varphi-\varphi'|\,.$$

It follows

$$\int_{\partial\mathbb{B}_2(0;1)} \sup\Big\{\,|G_l(\varphi,\gamma)(\xi,\eta)| \;:\; \xi\in\mathbb{R}^2\backslash\{0\}\Big\}\,do_\eta \;\le\; 2\cdot\pi\cdot C_{7,1}(\varphi_0)\cdot\gamma\,;$$

$$\int_{\partial\mathbb{B}_2(0;1)} \sup\Big\{\,|(\widetilde{K}_l(\varphi)-\widetilde{K}_l(\varphi'))(\xi,\eta)| \;:\; \xi\in\mathbb{R}^2\backslash\{0\}\Big\}\,do_\eta$$

$$\le\; 2\cdot\pi\cdot C_{7,1}(\varphi_0)\cdot|\varphi-\varphi'|\,.$$

Now Theorem 4.2 yields: The function $G_l(\varphi,\gamma) *_p \Phi$ is well defined; estimate (4.15) is valid, and it holds

$$\|(\widetilde{K}_l(\varphi)-\widetilde{K}_l(\varphi'))_\epsilon * \Phi\|_p\,, \qquad \|(\widetilde{K}_l(\varphi)-\widetilde{K}_l(\varphi')) *_p \Phi\|_p \tag{4.19}$$

$$\le\; C_{7,2}(p,\varphi_0)\cdot\|\Phi\|_p\cdot|\varphi-\varphi'|\,.$$

Furthermore, we note a consequence of Lemma 4.7, namely:

$$\|(\widetilde{K}_l(\varphi))_\epsilon * \Phi\|_p\,, \quad \|\widetilde{K}_l(\varphi) *_p \Phi\|_p \;\le\; C_{7,2}(p,\varphi_0)\cdot\|\Phi\|_p\,. \tag{4.20}$$

For the proof of (4.14) and (4.16), we split $L_l(0,\varphi)$ into two parts. One of these is the function $\widetilde{K}_l(\varphi)$, the other one will be defined now: For $\xi,\eta\in\mathbb{R}^2$ with $\xi\neq\eta$, $\xi\neq 0$, and for $\tilde\varphi\in\{\varphi,\varphi'\}$, set

$$F_l(\tilde\varphi)(\xi,\eta) \;:=\; (4\cdot\pi)^{-1}\cdot\sin^3(\tilde\varphi)\cdot(\xi-\eta)_l\cdot\sum_{v=0}^{\infty}\binom{-3/2}{v}\cdot\cos^{2\cdot v}(\tilde\varphi)\cdot|\xi-\eta|^{-2\cdot v-3}$$

$$\cdot\left(\Big((|\xi|-|\eta|)^2-|\xi-\eta|^2\Big)^v - \Big(\Big(\sum_{i=1}^{2}(\xi-\eta)_i\cdot\xi_i/|\xi|\Big)^2-|\xi-\eta|^2\Big)^v\right).$$

Due to Lemma 4.2, the preceding sum converges absolutely. Moreover, Lemma 4.3 yields for $\xi,\eta\in\mathbb{R}^2$ with $\xi\neq\eta$, $\xi\neq 0$, and for $\tilde\varphi\in\{\varphi,\varphi'\}$:

$$|F_l(\tilde\varphi)(\xi,\eta)|$$

$$\le\; (4\cdot\pi)^{-1}\cdot\sin^3(\tilde\varphi)\cdot\sum_{v=0}^{\infty}\left|\binom{-3/2}{v}\right|\cdot 16\cdot v\cdot\cos^{2\cdot v}(\tilde\varphi)\cdot|\xi-\eta|^{-1}\cdot(|\xi|+|\eta|)^{-1}$$

$$\le\; C_{7,1}(\varphi_0)\cdot|\xi-\eta|^{-1}\cdot(|\xi|+|\eta|)^{-1}\,.$$

By Theorem 4.1 it follows for $\tilde\varphi\in\{\varphi,\varphi'\}$:

$$\left(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}|F_l(\tilde\varphi)(\xi,\eta)\cdot\Phi(\eta)|\;d\eta\Big)^p\,d\xi\right)^{1/p} \;\le\; C_{7,3}(p,\varphi_0)\cdot\|\Phi\|_p\,. \tag{4.21}$$

Now Lebesgue's theorem on dominated convergence implies that the function $F_l(\tilde\varphi)\otimes_p\Phi$ is well defined $(\tilde\varphi\in\{\varphi,\varphi'\})$.

According to Lemma 4.6, we have for $\xi,\eta\in\mathbb{R}^2\backslash\{0\}$ with $\xi\neq\eta$:

$$L_l(0,\tilde\varphi)(\xi,\eta) \;=\; -F_l(\tilde\varphi)(\xi,\eta) \;-\; \widetilde{K}_l(\tilde\varphi)(\xi,\xi-\eta) \quad \text{for } \tilde\varphi\in\{\varphi,\varphi'\}\,. \tag{4.22}$$

This implies on one hand: The function $L_l(0,\tilde\varphi)\otimes_p\Phi$ is well defined $(\tilde\varphi\in\{\varphi,\varphi'\})$. On the other hand, by combining (4.20), (4.21) and (4.22), we obtain (4.14).

In order to prove (4.16), we observe for $\xi,\eta\in\mathbb{R}^2$ with $\xi\neq 0$, $\xi\neq\eta$:

$$|F_l(\varphi)(\xi,\eta) \;-\; F_l(\varphi')(\xi,\eta)|$$

$$\le\; (4\cdot\pi)^{-1}\cdot|\xi-\eta|^{-1}\cdot(|\xi|+|\eta|)^{-1}\cdot\sum_{v=1}^{\infty}\left|\binom{-3/2}{v}\right|\cdot 16\cdot v$$

$$\cdot\,|\sin^3(\varphi)\cdot\cos^{2\cdot v}(\varphi) \;-\;\sin^3(\varphi')\cdot\cos^{2\cdot v}(\varphi')|$$

$$\le\; C_{7,1}(\varphi_0)\cdot|\varphi-\varphi'|\cdot|\xi-\eta|^{-1}\cdot(|\xi|+|\eta|)^{-1}\,.$$

For the first of these inequalities, we refer to Lemma 4.3, for the second one, to Lemma 4.5. From the preceding estimate and from Theorem 4.1, we get:

$$\left(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}|(F_l(\varphi)(\xi,\eta) \;-\; F_l(\varphi')(\xi,\eta))\cdot\Phi(\eta)|\;d\eta\Big)^p\,d\xi\right)^{1/p} \tag{4.23}$$

$$\le\; C_{7,3}(p,\varphi_0)\cdot\|\Phi\|_p\cdot|\varphi-\varphi'|\,.$$

Inequality (4.16) now follows from (4.19), (4.22) and (4.23).

Corollary 4.1. *For $p\in(1,\infty)$, $\Phi\in L^p(\mathbb{R}^2)$, $\varphi\in(0,\pi/2]$, the function $L_3(0,\varphi)\otimes_p\Phi$ is well defined. If $p\in(1,\infty)$ and $\varphi_0\in(0,\pi/2]$, then there is a constant $C_8(p,\varphi_0)>0$ so that for $\varphi,\varphi'\in[\varphi_0,\pi/2]$, $\Phi\in L^p(\mathbb{R}^2)$, $\epsilon\in(0,\infty)$, the ensuing inequalities are valid:*

$$\|(L_3(0,\varphi))_\epsilon\otimes\Phi\|_p\,, \quad \|L_3(0,\varphi)\otimes_p\Phi\|_p \;\le\; C_8(p,\varphi_0)\cdot\|\Phi\|_p\,; \tag{4.24}$$

$$\|(L_3(0,\varphi))_\epsilon\otimes\Phi \;-\; (L_3(0,\varphi'))_\epsilon\otimes\Phi\|_p\,, \quad \|L_3(0,\varphi)\otimes_p\Phi \;-\; L_3(0,\varphi')\otimes_p\Phi\|_p \tag{4.25}$$

$$\le\; C_8(p,\varphi_0)\cdot\|\Phi\|_p\cdot|\varphi-\varphi'|\,.$$

Proof: Fix $\varphi_0\in(0,\pi/2]$, $p\in(1,\infty)$. In order to define the constant $C_8(p,\varphi_0)$, we put

$$C_{8,1}(p,\varphi_0) := \pi^{-1}\cdot\cot\varphi_0\cdot C_5(p)\,; \quad C_{8,2}(p,\varphi_0) := C_{8,1}(p,\varphi_0) \;+\; 2\cdot\cot\varphi_0\cdot C_7(p,\varphi_0)\,;$$

$$C_{8,3}(\varphi_0) := \pi^{-1}\cdot\cot\varphi_0\cdot\left(2\cdot\sin^{-2}(\varphi_0) \;+\; 3\cdot\sum_{v=1}^{\infty}\left|\binom{-3/2}{v}\right|\cdot(v+1)\cdot\cos^{2\cdot v-1}(\varphi_0)\right).$$

Since $0\le\cos\varphi_0<1$, the preceding series is convergent. We further define

$C_{8,4}(p, \varphi_0) := C_5(p) \cdot C_{8,3}(\varphi_0)$;

$C_{8,5}(p, \varphi_0) := C_{8,4}(p, \varphi_0) \quad + \quad 2 \cdot (\cot \varphi_0 + 2 \cdot \sin^{-2}(\varphi_0)) \cdot C_7(p, \varphi_0)$;

$C_8(p, \varphi_0) := \max \{ C_{8,2}(p, \varphi_0), \ C_{8,5}(p, \varphi_0) \}$.

For $\xi, \eta \in \mathbb{R}^2$ with $\xi \neq 0$, we have

$$|\xi| - |\eta| = (|\xi|^2 - |\eta|^2) \cdot (|\xi| + |\eta|)^{-1} = \sum_{i=1}^{2} (\xi - \eta)_i \cdot (\xi + \eta)_i \cdot (|\xi| + |\eta|)^{-1} \tag{4.26}$$

$$= \sum_{i=1}^{2} (\xi - \eta)_i \cdot \Big((\xi + \eta)_i \cdot (|\xi| + |\eta|)^{-1} - \xi_i/|\xi| \Big) + \sum_{i=1}^{2} (\xi - \eta)_i \cdot \xi_i/|\xi| .$$

We shall split $L_j(0, \varphi)$ into a sum of two parts, each of which can be estimated in a suitable way. To this end, we set for $\xi, \eta \in \mathbb{R}^2$ with $\xi \neq \eta, \xi \neq 0$, and for $\tilde{\varphi} \in \{\varphi, \varphi'\}$:

$$\bar{F}(\tilde{\varphi})(\xi, \eta) := (4 \cdot \pi)^{-1} \cdot \sum_{i=1}^{2} (\xi - \eta)_i \cdot \Big((\xi + \eta)_i \cdot (|\xi| + |\eta|)^{-1} - \xi_i/|\xi| \Big) \cdot \big| g^{(\tilde{\varphi})}(\xi) - g^{(\tilde{\varphi})}(\eta) \big|^{-3} .$$

Then it holds by (4.26), for $\xi, \eta, \tilde{\varphi}$ as before:

$$L_3(0, \tilde{\varphi})(\xi, \eta) = - \cot \tilde{\varphi} \cdot \bar{F}(\tilde{\varphi})(\xi, \eta) + \cot \tilde{\varphi} \cdot \sum_{l=1}^{2} (\xi_l/|\xi|) \cdot L_l(0, \tilde{\varphi})(\xi, \eta) . \tag{4.27}$$

Lemma 4.1 yields, for $\xi, \eta, \tilde{\varphi}$ as before:

$$|\bar{F}(\tilde{\varphi})(\xi, \eta)| \leq \pi^{-1} \cdot |\xi - \eta|^2 \cdot (|\xi| + |\eta|)^{-1} \cdot \big| g^{(\tilde{\varphi})}(\xi) - g^{(\tilde{\varphi})}(\eta) \big|^{-3} \tag{4.28}$$

$$\leq \pi^{-1} \cdot |\xi - \eta|^{-1} \cdot (|\xi| + |\eta|)^{-1} .$$

It follows from Theorem 4.1, for $\tilde{\varphi} \in \{\varphi, \varphi'\}$, $\Phi \in L^p(\mathbb{R}^2)$:

$$\Big(\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} |\cot \tilde{\varphi} \cdot \bar{F}(\tilde{\varphi})(\xi, \eta) \cdot \Phi(\eta)| \, d\eta \Big)^p \, d\xi \Big)^{1/p} \leq C_{8,1}(p, \varphi_0) \cdot \|\Phi\|_p . \tag{4.29}$$

This means in particular: The function $\bar{F}(\tilde{\varphi}) \otimes_p \Phi$ is well defined ($\tilde{\varphi} \in \{\varphi, \varphi'\}$, $\Phi \in L^p(\mathbb{R}^2)$). In addition, inequality (4.24) may be deduced from (4.27), (4.29) and Theorem 4.3.

For $\xi, \eta \in \mathbb{R}^2$ with $\xi \neq \eta$, $\xi \neq 0$, it holds by Lemma 4.6:

$$|\cot \varphi \cdot \bar{F}(\varphi)(\xi, \eta) - \cot \varphi' \cdot \bar{F}(\varphi')(\xi, \eta)|$$

$$= \Big| (\cot \varphi - \cot \varphi') \cdot \bar{F}(\varphi)(\xi, \eta)$$
$$+ \cot \varphi' \cdot (4 \cdot \pi)^{-1} \cdot \Big(\sum_{i=1}^{2} (\xi - \eta)_i \cdot \Big((\xi + \eta)_i \cdot (|\xi| + |\eta|)^{-1} - \xi_i/|\xi| \Big) \Big)$$
$$\cdot \sum_{v=0}^{\infty} \binom{-3/2}{v} \cdot \Big(\sin^3(\varphi) \cdot \cos^{2 \cdot v}(\varphi) - \sin^3(\varphi') \cdot \cos^{2 \cdot v}(\varphi') \Big)$$

$$\cdot \Big((|\xi| - |\eta|)^2 - |\xi - \eta|^2 \Big)^v \cdot |\xi - \eta|^{-2 \cdot v - 3} \Big| .$$

Now apply (4.28) and Lemma 4.1, 4.5, to obtain:

$$|\cot \varphi \cdot \bar{F}(\varphi)(\xi, \eta) - \cot \varphi' \cdot \bar{F}(\varphi')(\xi, \eta)|$$

$$\leq |\cot \varphi - \cot \varphi'| \cdot \pi^{-1} \cdot |\xi - \eta|^{-1} \cdot (|\xi| + |\eta|)^{-1}$$
$$+ \pi^{-1} \cdot \cot \varphi_0 \cdot \sum_{v=1}^{\infty} \Big| \binom{-3/2}{v} \Big| \cdot (v+1) \cdot 3 \cdot \cos^{2 \cdot v - 1}(\varphi_0) \cdot |\varphi - \varphi'| \cdot |\xi - \eta|^{-1} \cdot (|\xi| + |\eta|)^{-1}$$

$$\leq C_{8,3}(p, \varphi_0) \cdot |\varphi - \varphi'| \cdot |\xi - \eta|^{-1} \cdot (|\xi| + |\eta|)^{-1} .$$

Using this estimate, and recalling Theorem 4.1, we find for $\Phi \in L^p(\mathbb{R}^2)$:

$$\Big(\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} \Big| (\cot \varphi \cdot \bar{F}(\varphi)(\xi, \eta) - \cot \varphi' \cdot \bar{F}(\varphi')(\xi, \eta)) \cdot \Phi(\eta) \Big| \, d\eta \Big)^p \, d\xi \Big)^{1/p} \tag{4.30}$$

$$\leq C_{8,4}(p, \varphi_0) \cdot |\varphi - \varphi'| \cdot \|\Phi\|_p .$$

Combining (4.27), (4.30) and Theorem 4.3 leads to (4.25).

Corollary 4.2. *For $p \in (1, \infty)$, $\Phi \in L^p(\mathbb{R}^2)$, $\varphi \in (0, \pi/2]$, $j \in \{1, 2, 3\}$, the function $L_j(0, \varphi) \otimes_p \Phi$ is well defined.*

If $p \in (1, \infty)$, $\varphi_0 \in (0, \pi/2]$, there is a constant $C_9(p, \varphi_0) > 0$ such that

$$\| (L_j(0, \varphi))_\epsilon \otimes \Phi \|_p , \quad \|L_j(0, \varphi) \otimes_p \Phi\|_p \leq C_9(p, \varphi_0) \cdot \|\Phi\|_p ,$$

$$\| (L_j(0, \varphi))_\epsilon \otimes \Phi - (L_j(0, \varphi'))_\epsilon \otimes \Phi \|_p , \quad \|L_j(0, \varphi) \otimes_p \Phi - L_j(0, \varphi') \otimes_p \Phi\|_p$$
$$\leq C_9(p, \varphi_0) \cdot \|\Phi\|_p \cdot |\varphi - \varphi'|$$

for $\varphi, \varphi' \in [\varphi_0, \pi/2]$, $\Phi \in L^p(\mathbb{R}^2)$, $\epsilon \in (0, \infty)$, $j \in \{1, 2, 3\}$.

Proof: This corollary combines Theorem 4.3 and Corollary 4.1.

Lemma 4.8. *For $\xi \in \mathbb{R}^2$, $r \in [0, \infty)$, it holds:* $\int_{\mathbb{R}^2} r \cdot (|\xi - \eta| + r)^{-3} \, d\eta \leq 3 \cdot \pi$.

Proof: In the case $r = 0$, this estimate is trivial. Hence we may assume $r \neq 0$. Then we find:

$$\int_{\mathbb{R}^2} r \cdot (|\xi - \eta| + r)^{-3} \, d\eta$$

$$= \int_{\mathbb{B}_2(\xi,r)} r\cdot(|\xi-\eta|+r)^{-3}\ d\eta \quad + \quad \int_{\mathbb{R}^2\setminus\mathbb{B}_2(\xi,r)} r\cdot(|\xi-\eta|+r)^{-3}\ d\eta$$

$$\leq r^{-2}\cdot\int_{\mathbb{B}_2(\xi,r)} d\eta \quad + \quad r\cdot\int_{\mathbb{R}^2\setminus\mathbb{B}_2(\xi,r)} |\xi-\eta|^{-3}\ d\eta \quad = \quad 3\cdot\pi\,.$$

Lemma 4.9 (Young's inequality). *Let* $K:\mathbb{R}^2\times\mathbb{R}^2\mapsto\mathbb{C}$, $\Phi:\mathbb{R}^2\mapsto\mathbb{C}$ *be measurable functions. Take* $p\in(1,\infty)$. *Then*

$$\Big(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}|K(\xi,\eta)\cdot\Phi(\eta)|\ d\eta\Big)^p\ d\xi\Big)^{1/p} \tag{4.31}$$

$$\leq \sup\Big\{\Big(\int_{\mathbb{R}^2}|K(\xi,\eta)|\ d\eta\Big)^{1-1/p}\ :\ \xi\in\mathbb{R}^2\Big\}$$

$$\cdot\sup\Big\{\Big(\int_{\mathbb{R}^2}|K(\xi,\eta)|\ d\xi\Big)^{1/p}\ :\ \eta\in\mathbb{R}^2\Big\}\cdot\|\Phi\|_p\,.$$

In particular, it holds for measurable functions $M:\mathbb{R}^2\setminus\{0\}\mapsto\mathbb{C}$, $\Phi:\mathbb{R}^2\mapsto\mathbb{C}$:

$$\Big(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}|M(\xi-\eta)\cdot\Phi(\eta)|\ d\eta\Big)^p\ d\xi\Big)^{1/p} \quad\leq\quad \|M\|_1\cdot\|\Phi\|_p\,.$$

Of course, these estimates are nontrivial only if their right-hand side is finite.

Proof of Lemma 4.9: According to Hölder's inequality, the left-hand side of (4.31) is bounded by

$$\Big(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}|K(\xi,\eta)|\ d\eta\Big)^{p-1}\cdot\Big(\int_{\mathbb{R}^2}|K(\xi,\eta)|\cdot|\Phi(\eta)|^p\ d\eta\Big)\ d\xi\Big)^{1/p}.$$

Now the lemma follows by Fubini's theorem.

Lemma 4.10. *For* $p\in(1,\infty)$, $\Phi\in L^p(\mathbb{R}^2)$, $r\in\mathbb{R}$, *it holds:*

$$\Big(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}|r|\cdot(|\xi-\eta|+|r|)^{-3}\cdot|\Phi(\eta)|\ d\eta\Big)^p\ d\xi\Big)^{1/p} \quad\leq\quad 3\cdot\pi\cdot\|\Phi\|_p\,.$$

Proof: Use Lemma 4.8 and 4.9.

Corollary 4.3. *Take* $\varphi\in(0,\pi/2]$. *Then there exists some number* $C_{10}(\varphi)>0$ *such that*

$$\Big(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}|r|\cdot|g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)+(0,0,r)|^{-3}\cdot|\Phi(\eta)|\ d\eta\Big)^p\ d\xi\Big)^{1/p} \tag{4.32}$$

$$\leq C_{10}(\varphi)\cdot\|\Phi\|_p \qquad \textit{for } p\in(1,\infty),\ \Phi\in L^p(\mathbb{R}^2),\ r\in\mathbb{R}\,.$$

Proof: According to Lemma 3.4, the left-hand side of (4.32) is bounded by

$$64\cdot\sin^{-3}(\varphi)\cdot\Big(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}|r|\cdot(|\xi-\eta|+|r|)^{-3}\cdot|\Phi(\eta)|\ d\eta\Big)^p\ d\xi\Big)^{1/p}.$$

Hence, inequality (4.32) is a consequence of Lemma 4.10.

Lemma 4.11. *Let* $\xi,\eta\in\mathbb{R}^2$ *with* $|\xi-\eta|\geq 1$, $\xi\neq 0$, $\varphi\in(0,\pi/2]$, $r\in\mathbb{R}$ *with* $|r|\leq\min\{(8\cdot\cot\varphi)^{-1},\,1/2\}$ *in the case* $\varphi<\pi/2$, *and with* $|r|\leq 1/2$ *else. Then*

$$\Big|2\cdot r\cdot\cot\varphi\cdot\Big(\sum_{i=1}^{2}(\xi-\eta)_i\cdot\xi_i/|\xi|\Big)+r^2\Big| \quad\leq\quad |\xi-\eta|/2 \quad\leq\quad |\xi-\eta|^2/2\,.$$

Proof: In the case $\varphi=\pi/2$, the preceding estimate is obvious. Thus we may require $\varphi<\pi/2$. But then it follows by our assumptions on r, and by the Cauchy-Schwarz inequality:

$$\Big|2\cdot r\cdot\cot\varphi\cdot\Big(\sum_{i=1}^{2}(\xi-\eta)_i\cdot\xi_i/|\xi|\Big)\Big| \quad\leq\quad |\xi-\eta|/4\,.$$

Since $r^2\leq 1/4\leq|\xi-\eta|/4$, the claim follows.

Lemma 4.12. *Let* ξ,η,φ,r *be given as in Lemma 4.11, and let* $v\in\mathbb{N}$. *Then*

$$\Big|\Big(\big(\cot\varphi\cdot(|\xi|-|\eta|)+r\big)^2-\cot^2(\varphi)\cdot|\xi-\eta|^2\Big)^v$$

$$-\Big(\Big(\cot\varphi\cdot\sum_{i=1}^{2}(\xi-\eta)_i\cdot\xi_i/|\xi|+r\Big)^2-\cot^2(\varphi)\cdot|\xi-\eta|^2\Big)^v\Big|$$

$$\leq 16\cdot\sin^{-2}(\varphi)\cdot v\cdot(1/2+\cot^2(\varphi))^{v-1}\cdot\Big((|\xi|+|\eta|)^{-1}\cdot|\xi-\eta|^{2\cdot v+1}+|\xi-\eta|^{2\cdot v-1}\Big).$$

Proof: First we conclude from Lemma 4.2 and 4.11:

$$\Big|\Big(\cot\varphi\cdot\sum_{i=1}^{2}(\xi-\eta)_i\cdot\xi_i/|\xi|+r\Big)^2-\cot^2(\varphi)\cdot|\xi-\eta|^2\Big|$$

$$=\Big|\cot^2(\varphi)\cdot\Big(\sum_{i=1}^{2}(\xi-\eta)_i\cdot\xi_i/|\xi|\Big)^2-\cot^2(\varphi)\cdot|\xi-\eta|^2$$

$$+ \; 2\cdot r\cdot\cot\varphi\cdot\Big(\sum_{i=1}^{2}(\xi-\eta)_i\cdot\xi_i/|\xi|\Big) \; + \; r^2\Big|$$

$$\le \; \cot^2(\varphi)\cdot|\xi-\eta|^2 \; + \; |\xi-\eta|^2/2 \;\; \le \;\; (1/2+\cot^2(\varphi))\cdot|\xi-\eta|^2 .$$

Furthermore, applying our assumptions on r in a similar way as in the proof of Lemma 4.11, we obtain:

$$\Big|\Big(\cot\varphi\cdot(|\xi|-|\eta|) \; + \; r\Big)^2 \; - \; \cot^2(\varphi)\cdot|\xi-\eta|^2\Big|$$

$$\le \; \Big|\cot^2(\varphi)\cdot|\xi-\eta|^2 \; - \; \cot^2(\varphi)\cdot(|\xi|-|\eta|)^2\Big| \; + \; \Big|2\cdot r\cdot\cot\varphi\cdot(|\xi|-|\eta|) \; + \; r^2\Big|$$

$$\le \; (1/2+\cot^2(\varphi))\cdot|\xi-\eta|^2 .$$

In addition, we find that

$$\Big|\Big(\cot\varphi\cdot(|\xi|-|\eta|) \; + \; r\Big)^2 \; - \; \Big(\cot\varphi\cdot\sum_{i=1}^{2}(\xi-\eta)_i\cdot\xi_i/|\xi| \; + \; r\Big)^2\Big|$$

$$\le \; \cot^2(\varphi)\cdot\Big|(|\xi|-|\eta|)^2 \; - \; \Big(\sum_{i=1}^{2}(\xi-\eta)_i\cdot\xi_i/|\xi|\Big)^2\Big|$$

$$+ \; 2\cdot|r|\cdot\cot\varphi\cdot\Big||\xi| \; - \; |\eta| \; - \; \sum_{i=1}^{2}(\xi-\eta)_i\cdot\xi_i/|\xi|\Big|$$

$$\le \; \cot^2(\varphi)\cdot 16\cdot|\xi-\eta|^3\cdot(|\xi|+|\eta|)^{-1} \; + \; 6\cdot|r|\cdot\cot\varphi\cdot|\xi-\eta|$$

$$\le \; \sin^{-2}(\varphi)\cdot 16\cdot\Big(|\xi-\eta|^3\cdot(|\xi|+|\eta|)^{-1} \; + \; |\xi-\eta|\Big) .$$

Note that the second of the preceding inequalities follows from Lemma 4.3. In the third one, we again used our assumptions on r. Collecting our estimates, and using the abbreviations

$$a \; := \; \Big(\cot\varphi\cdot(|\xi|-|\eta|) \; + \; r\Big)^2 \; - \; \cot^2(\varphi)\cdot|\xi-\eta|^2 ,$$

$$b \; := \; \Big(\cot\varphi\cdot\sum_{i=1}^{2}(\xi-\eta)_i\cdot\xi_i/|\xi| \; + \; r\Big)^2 \; - \; \cot^2(\varphi)\cdot|\xi-\eta|^2 ,$$

we finally arrive at the inequality

$$|(a-b)^v| \;\; = \;\; |a-b|\cdot\Big|\sum_{k=0}^{v-1} a^k\cdot b^{v-1-k}\Big|$$

$$\le \; 16\cdot\sin^{-2}(\varphi)\cdot v\cdot(1/2+\cot^2(\varphi))^{v-1}\cdot|\xi-\eta|^{2\cdot v-2} \cdot\Big(|\xi-\eta|^3\cdot(|\xi|+|\eta|)^{-1} \; + \; |\xi-\eta|\Big)$$

Theorem 4.4. *For* $l\in\{1,2\}$, $\varphi\in(0,\pi/2]$, $r\in\mathbb{R}\backslash\{0\}$, $p\in(1,\infty)$, $\Phi\in L^p(\mathbb{R}^2)$ *the function* $L_l(r,\varphi)\otimes\Phi$ *is well defined.*

If $p\in(1,\infty)$, $\varphi\in(0,\pi/2]$, *then there is a constant* $C_{11}(p,\varphi)>0$ *such that*

$$\|L_l(r,\varphi)\otimes\Phi\|_p \;\; \le \;\; C_{11}(p,\varphi)\cdot\|\Phi\|_p \tag{4.33}$$

for $r\in\mathbb{R}\backslash\{0\}$, $\Phi\in L^p(\mathbb{R}^2)$, $l\in\{1,2\}$.

Proof: Take $p\in(1,\infty)$, $\varphi\in(0,\pi/2]$. As in similar situations before, we first define the constant $C_{11}(p,\varphi)$. In this way, it will be easier to check the ensuing estimates. So we set

$$C_{11,1}(\varphi) := 4\cdot\pi^{-1}\cdot\sin^{-2}(\varphi);$$

$$C_{11,2}(\varphi) := (4/\pi)\cdot\sum_{v=1}^{\infty}\Big|\binom{-3/2}{v}\Big|\cdot v\cdot(1/2+\cot^2(\varphi))^{v-1}\cdot\sin^{2\cdot v+1}(\varphi);$$

$$C_{11,3}(p,\varphi) := (C_5(p)+2\cdot\pi)\cdot C_{11,2}(\varphi);$$

$$C_{11,4}(\varphi) := (1/2)\cdot\sum_{v=0}^{\infty}\Big|\binom{-3/2}{v}\Big|\cdot\cos^{2\cdot v}(\varphi);$$

$$C_{11,5}(p,\varphi) := C_6(p,\,C_{11,4}(\varphi));$$

$$C_{11,6}(\varphi) := (4\cdot\pi)^{-1}\cdot\sum_{v=0}^{\infty}\Big|\binom{-3/2}{v}\Big|\cdot(1/2+\cot^2(\varphi))^{v}\cdot\sin^{2\cdot v+3}(\varphi);$$

$$C_{11,7}(p,\varphi) := C_{11,3}(p,\varphi) \; + \; C_{11,5}(p,\varphi) \; + \; 2\cdot\pi\cdot C_{11,6}(\varphi);$$

$$C_{11,8}(p,\varphi) := C_{11,1}(\varphi)\cdot\pi\cdot\Big((1/2)\cdot\min\{(8\cdot\cot\varphi)^{-1},\; 1/2\}\Big)^{-2} \; + \; C_{11,7}(p,\varphi)$$

in the case $\varphi<\pi/2$;

$C_{11,8}(p,\varphi) := C_{11,1}(\varphi)\cdot\pi\cdot 16 \; + \; C_{11,7}(p,\varphi)$ in the case $\varphi=\pi/2$;

$$C_{11}(p,\varphi) \; := \; 3\cdot C_{11,8}(p,\varphi) \; + \; \pi\cdot C_{11,1}(\varphi).$$

Note that $(1/2+\cot^2(\varphi))\cdot\sin^2(\varphi) < 1$ so that the preceding infinite sums do in fact converge.

Now let us take $l\in\{1,2\}$, $\Phi\in L^p(\mathbb{R}^2)$. Recalling Lemma 3.4 and the definition of $C_{11,1}(\varphi)$, we find that

$$|L_l(r,\varphi)(\xi,\eta)| \;\; \le \;\; C_{11,1}(\varphi)\cdot(|\xi-\eta| \; + \; |r|)^{-2} \tag{4.34}$$

for $\xi,\eta\in\mathbb{R}^2$, $r\in\mathbb{R}\backslash\{0\}$. Therefore, it is obvious that the function $L_l(r,\varphi)\otimes\Phi$ is well defined for $r\in\mathbb{R}\backslash\{0\}$.

Let $r\in\mathbb{R}\backslash\{0\}$ with $|r|\le\min\{(8\cdot\cot\varphi)^{-1},\; 1/2\}$ in the case $\varphi<\pi/2$, and with $|r|\le 1/2$ in the case $\varphi=\pi/2$, respectively. Then we define for $\xi,\eta\in\mathbb{R}^2$:

$$\begin{aligned} F(\xi,\,\eta) \quad := \quad & (4\cdot\pi)^{-1}\cdot(\xi-\eta)_l\cdot\sum_{\upsilon=0}^{\infty}\binom{-3/2}{\upsilon}\cdot\Big(\sin^{-2}(\varphi)\cdot|\xi-\eta|^2\Big)^{-\upsilon-3/2} \\ & \cdot\Bigg(\Big(\Big(\cot\varphi\cdot(|\xi|-|\eta|)\,+\,r\Big)^2\,-\,\cot^2(\varphi)\cdot|\xi-\eta|^2\Big)^{\upsilon} \\ & \quad -\,\Big(\Big(\cot\varphi\cdot\sum_{i=1}^{2}(\xi-\eta)_i\cdot\xi_i/|\xi|\,+\,r\Big)^2\,-\,\cot^2(\varphi)\cdot|\xi-\eta|^2\Big)^{\upsilon}\Bigg), \end{aligned}$$

$$\begin{aligned} G(\xi,\,\eta) \quad := \quad & (4\cdot\pi)^{-1}\cdot(\xi-\eta)_l\cdot\sum_{\upsilon=0}^{\infty}\binom{-3/2}{\upsilon}\cdot\Big(\sin^{-2}(\varphi)\cdot|\xi-\eta|^2\Big)^{-\upsilon-3/2} \\ & \cdot\Big(\Big(\cot\varphi\cdot\sum_{i=1}^{2}(\xi-\eta)_i\cdot\xi_i/|\xi|\,+\,r\Big)^2\,-\,\cot^2(\varphi)\cdot|\xi-\eta|^2\Big)^{\upsilon}. \end{aligned}$$

Lemma 4.6 yields that

$$L_l(r,\,\varphi)(\xi,\,\eta) \quad = \quad -\,F(\xi,\,\eta)\,-\,G(\xi,\,\eta)\,. \tag{4.35}$$

In the following, we shall evaluate $(F)_1\otimes\Phi$ and $(G)_1\otimes\Phi$ so that by the preceding equation, we shall obtain an estimate for $(L_l(r,\,\varphi))_1\otimes\Phi$. Then a scaling argument will lead to the inequality (4.33). In order to carry out this program, we first observe that

$$\begin{aligned} & \Big(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}\chi_{(1,\,\infty)}(|\xi-\eta|)\cdot|F(\xi,\,\eta)\cdot\Phi(\eta)|\;d\eta\Big)^p\;d\xi\Big)^{1/p} \\ & \le\;\Big(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}\chi_{(1,\,\infty)}(|\xi-\eta|)\cdot(4/\pi)\cdot\sum_{\upsilon=1}^{\infty}\Big|\binom{-3/2}{\upsilon}\Big|\cdot\upsilon\cdot\big(1/2+\cot^2(\varphi)\big)^{\upsilon-1} \\ & \qquad\cdot\sin^{2\cdot\upsilon+1}(\varphi)\cdot\Big(|\xi-\eta|^{-1}\cdot(|\xi|+|\eta|)^{-1}\;+\;|\xi-\eta|^{-3}\Big)\cdot\Phi(\eta)\;d\eta\Big)^p\;d\xi\Big)^{1/p} \\ & \le\;C_{11,3}(p,\,\varphi)\cdot\|\Phi\|_p\,. \end{aligned} \tag{4.36}$$

The first of these inequalities is a consequence of Lemma 4.12, the second one follows from Theorem 4.1 and Lemma 4.9 (Young's inequality). Note that

$$\int_{\mathbb{R}^2}\chi_{(1,\,\infty)}(|\xi-\eta|)\;\cdot\;|\xi-\eta|^{-3}\;d\eta\quad=\quad 2\cdot\pi\,.$$

For $\xi,\,\eta\in\mathbb{R}^2$ with $\xi\neq\eta$, $\xi\neq 0$, we put

$$\begin{aligned} H(\xi,\,\eta) \quad := \quad & (4\cdot\pi)^{-1}\cdot(\xi-\eta)_l\cdot\sum_{\upsilon=0}^{\infty}\binom{-3/2}{\upsilon}\cdot\sin^{2\cdot\upsilon+3}\cdot|\xi-\eta|^{-2\cdot\upsilon-3} \\ & \cdot\sum_{k=0}^{\upsilon-1}\binom{\upsilon}{k}\cdot\Big(\Big(\cot\varphi\cdot\sum_{i=1}^{2}(\xi-\eta)_i\cdot\xi_i/|\xi|\Big)^2\,-\,\cot^2(\varphi)\cdot|\xi-\eta|^2\Big)^{k} \\ & \qquad\cdot\Big(2\cdot r\cdot\cot\varphi\cdot\Big(\sum_{i=1}^{2}(\xi-\eta)_i\cdot\xi_i/|\xi|\Big)\,+\,r^2\Big)^{\upsilon-k}. \end{aligned}$$

Then we have for $\xi,\,\eta\in\mathbb{R}^2$ with $\xi\neq\eta$, $\xi\neq 0$:

$$G(\xi,\,\eta)\quad=\quad\widetilde{K_l}(\varphi)\,(\xi,\,\xi-\eta)\;+\;H(\xi,\,\eta)\,, \tag{4.37}$$

where $\widetilde{K_l}(\varphi)$ was defined in Lemma 4.7. This lemma yields that

$$\|\,(\widetilde{K_l}(\varphi))_1*\Phi\,\|_p\quad\le\quad C_{11,5}(p,\varphi)\cdot\|\Phi\|_p\,. \tag{4.38}$$

In order to estimate $(H)_1\otimes\Phi$, we observe for $\xi,\,\eta\in\mathbb{R}^2$ with $|\xi-\eta|\ge 1$, $\xi\neq 0$:

$$\begin{aligned} |H(\xi,\,\eta)|\quad\le\quad & (4\cdot\pi)^{-1}\cdot\sum_{\upsilon=0}^{\infty}\Big|\binom{-3/2}{\upsilon}\Big|\cdot\sum_{k=0}^{\upsilon-1}\binom{\upsilon}{k}\cdot\cot^{2\cdot k}(\varphi) \\ & \cdot|\xi-\eta|^{2\cdot k}\cdot\big(\,|\xi-\eta|\,/\,2\,\big)^{\upsilon-k}\cdot\sin^{2\cdot\upsilon+3}(\varphi)\cdot|\xi-\eta|^{-2\cdot\upsilon-2} \\ \le\;\; & (4\cdot\pi)^{-1}\cdot\sum_{\upsilon=0}^{\infty}\Big|\binom{-3/2}{\upsilon}\Big|\cdot\sum_{k=0}^{\upsilon-1}\binom{\upsilon}{k}\cdot\cot^{2\cdot k}(\varphi)\cdot 2^{-\upsilon+k}\cdot|\xi-\eta|^{-3}\cdot\sin^{2\cdot\upsilon+3}(\varphi) \\ \le\;\; & C_{11,6}(\varphi)\cdot|\xi-\eta|^{-3}. \end{aligned}$$

The first of the preceding inequalities is implied by Lemma 4.2 and 4.11. In the second one, we used the assumption $|\xi-\eta|\ge 1$. Now we apply Young's inequality (Lemma 4.9) in the same way as we did in (4.36), and we arrive at the following estimate:

$$\begin{aligned} & \Big(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}\chi_{(1,\,\infty)}(|\xi-\eta|)\cdot|H(\xi,\,\eta)\cdot\Phi(\eta)|\;d\eta\Big)^p\;d\xi\Big)^{1/p} \\ & \le\;2\cdot\pi\cdot C_{11,6}(\varphi)\cdot\|\Phi\|_p\,. \end{aligned} \tag{4.39}$$

Combining (4.35) - (4.39), we may conclude for $r\in\mathbb{R}\backslash\{0\}$, $\Phi\in L^p(\mathbb{R}^2)^3$, provided $|r|\le 1/2$ in the case $\varphi=\pi/2$, and $|r|\;\le\;\min\{\,(8\cdot\cot\varphi)^{-1},\;1/2\,\}$ in the case $\varphi<\pi/2$:

$$\|\,(L_l(r,\,\varphi))_1\otimes\Phi\,\|_p\quad\le\quad C_{11,7}(p,\,\varphi)\cdot\|\Phi\|_p\,. \tag{4.40}$$

Let us now extend (4.40) to any $r\in\mathbb{R}\backslash\{0\}$. By an easy computation, we find for $s,\,s'\in\mathbb{R}\backslash\{0\}$, $\xi,\,\eta\in\mathbb{R}^2$ with $\xi\neq\eta$:

$$L_l(s',\,\varphi)\,(s\cdot\xi,\,s\cdot\eta)\quad=\quad s^{-2}\cdot L_l(\,s'/s,\,\varphi)(\xi,\,\eta)\,. \tag{4.41}$$

Put $R:=(1/2)\cdot\min\{\,(8\cdot\cot\varphi)^{-1},\;1/2\,\}$ in the case $\varphi<\pi/2$, and $R:=1/4$ in the case $\varphi=\pi/2$. Then it holds by (4.41), for $\Phi\in L^p(\mathbb{R}^2)$, $r\in\mathbb{R}\backslash\{0\}$:

$$\begin{aligned} \|\,(L_l(r,\,\varphi))_1\otimes\Phi\,\|_p\quad=\quad & \Big(\int_{\mathbb{R}^2}(r/R)^2\cdot\Big|\int_{\mathbb{R}^2}(r/R)^2\cdot\chi_{(1,\infty)}\big((r/R)\cdot|\xi-\eta|\big) \\ & \cdot L_l(r,\,\varphi)\big((r/R)\cdot(\xi,\,\eta)\big)\cdot\Phi\big((r/R)\cdot(\xi,\,\eta)\big)\;d\eta\Big|^p\;d\xi\Big)^{1/p} \\ =\;\; & (r/R)^{2/p} \\ & \cdot\Big(\int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\chi_{(R/r,\,\infty)}(|\xi-\eta|)\cdot L_l(R,\,\varphi)(\xi,\,\eta)\cdot\Phi((r/R)\cdot\eta)\;d\eta\Big|^p\;d\xi\Big)^{1/p} \end{aligned}$$

$$\leq (r/R)^{2/p} \cdot \Bigg(\Big(\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} \chi_{(0,1)}(|\xi-\eta|) \cdot \big| L_l(R,\varphi)(\xi,\eta) \cdot \Phi((r/R)\cdot\eta) \big| \, d\eta \Big)^p \, d\xi \Big)^{1/p}$$

$$+ \Big(\int_{\mathbb{R}^2} \Big| \int_{\mathbb{R}^2} \chi_{(1,\infty)}(|\xi-\eta|) \cdot L_l(R,\varphi)(\xi,\eta) \cdot \Phi((r/R)\cdot\eta) \, d\eta \Big|^p \, d\xi \Big)^{1/p} \Bigg)$$

Now we apply (4.40), (4.34), and then Young's inequality (Lemma 4.9). In this way we get:

$$\| (L_l(r,\varphi))_1 \otimes \Phi \|_p \leq (r/R)^{2/p} \tag{4.42}$$

$$\cdot \Bigg(C_{11,1}(\varphi) \cdot R^{-2} \cdot \Big(\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} \chi_{(0,1)}(|\xi-\eta|) \cdot |\Phi((r/R)\cdot\eta)| \, d\eta \Big)^p \, d\xi \Big)^{1/p}$$

$$+ C_{11,7}(p,\varphi) \cdot \Big(\int_{\mathbb{R}^2} |\Phi((r/R)\cdot\eta)|^p \, d\eta \Big)^{1/p} \Bigg)$$

$$\leq (r/R)^{2/p} \cdot C_{11,8}(p,\varphi) \cdot \Big(\int_{\mathbb{R}^2} |\Phi((r/R)\cdot\eta)|^p \, d\eta \Big)^{1/p} = C_{11,8}(p,\varphi) \cdot \|\Phi\|_p .$$

To finish our estimates, we compute for $r \in \mathbb{R}\backslash\{0\}$, $\Phi \in L^p(\mathbb{R}^2)$:

$$\Big(\int_{\mathbb{R}^2} \Big| \int_{\mathbb{R}^2} \chi_{(0,1)}(|\xi-\eta|) \cdot L_l(r,\varphi)(\xi,\eta) \cdot \Phi(\eta) \, d\eta \Big|^p \, d\xi \Big)^{1/p}$$

$$\leq \| (L_l(r,\varphi))_1 \otimes \Phi \|_p + \|L_l(r,\varphi) \otimes \Phi\|_p$$

$$\leq C_{11,8}(p,\varphi) \cdot \|\Phi\|_p + r^{2/p} \cdot \Big(\int_{\mathbb{R}^2} \Big| \int_{\mathbb{R}^2} L_l(1,\varphi)(\xi,\eta) \cdot \Phi(r\cdot\eta) \, d\eta \Big|^p \, d\xi \Big)^{1/p}$$

$$\leq C_{11,8}(p,\varphi) \cdot \|\Phi\|_p$$

$$+ r^{2/p} \cdot \Big(\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} \chi_{(0,1)}(|\xi-\eta|) \cdot |L_l(1,\varphi)(\xi,\eta) \cdot \Phi(r\cdot\eta)| \, d\eta \Big)^p \, d\xi \Big)^{1/p}$$

$$+ r^{2/p} \cdot \Big(\int_{\mathbb{R}^2} \Big| \int_{\mathbb{R}^2} \chi_{(1,\infty)}(|\xi-\eta|) \cdot L_l(1,\varphi)(\xi,\eta) \cdot \Phi(r\cdot\eta) \, d\eta \Big|^p \, d\xi \Big)^{1/p},$$

where the second inequality is implied by (4.42) and (4.41). Now we first refer to (4.34), (4.42), and then again to Young's inequality (Lemma 4.9), to obtain:

$$\Big(\int_{\mathbb{R}^2} \Big| \int_{\mathbb{R}^2} \chi_{(0,1)}(|\xi-\eta|) \cdot L_l(r,\varphi)(\xi,\eta) \cdot \Phi(\eta) \, d\eta \Big|^p \, d\xi \Big)^{1/p} \tag{4.43}$$

$$\leq C_{11,8}(p,\varphi) \cdot \|\Phi\|_p$$

$$+ r^{2/p} \cdot C_{11,1}(\varphi) \cdot \Big(\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} \chi_{(0,1)}(|\xi-\eta|) \cdot |\Phi(r\cdot\eta)| \, d\eta \Big)^p \, d\xi \Big)^{1/p}$$

$$+ r^{2/p} \cdot C_{11,8}(p,\varphi) \cdot \Big(\int_{\mathbb{R}^2} |\Phi(r\cdot\eta)|^p \, d\eta \Big)^{1/p}$$

$$\leq 2 \cdot C_{11,8}(p,\varphi) \cdot \|\Phi\|_p + r^{2/p} \cdot \pi \cdot C_{11,1}(\varphi) \cdot \Big(\int_{\mathbb{R}^2} |\Phi(r\cdot\eta)|^p \, d\eta \Big)^{1/p}$$

$$= \big(2 \cdot C_{11,8}(p,\varphi) + \pi \cdot C_{11,1}(\varphi) \big) \cdot \|\Phi\|_p .$$

By combining (4.42) and (4.43), we arrive at inequality (4.33).

Corollary 4.4. *For* $\varphi \in (0, \pi/2]$, $r \in \mathbb{R}\backslash\{0\}$, $p \in (1,\infty)$, $\Phi \in L^p(\mathbb{R}^2)$, *the function* $L_3(r,\varphi) \otimes \Phi$ *is well defined.*

If $p \in (1,\infty)$, $\varphi \in (0,\pi/2]$, *then there exists a number* $C_{12}(p,\varphi) > 0$ *with*

$$\|L_3(r,\varphi) \otimes \Phi\|_p \leq C_{12}(p,\varphi) \cdot \|\Phi\|_p \quad \textit{for } r \in \mathbb{R}\backslash\{0\},\ \Phi \in L^p(\mathbb{R}^2).$$

Proof: Let $p \in (1,\infty)$, $\varphi \in (0,\pi/2]$, $r \in \mathbb{R}\backslash\{0\}$. Then we have for $\xi, \eta \in \mathbb{R}^2$:

$$|\xi| - |\eta| + r$$

$$= \sum_{i=1}^{2} (\xi-\eta)_i \cdot \big((\xi+\eta)_i \cdot (|\xi|+|\eta|)^{-1} - \xi_i/|\xi| \big) + \sum_{i=1}^{2} (\xi-\eta)_i \cdot \xi_i/|\xi| + r .$$

This equation suggests the following definitions, for $\xi, \eta \in \mathbb{R}^2$:

$$F^{(1)}(\xi,\eta) := (4\cdot\pi)^{-1} \cdot \cot\varphi \cdot \sum_{i=1}^{2} (\xi-\eta)_i \cdot \big((\xi+\eta)_i \cdot (|\xi|+|\eta|)^{-1} - \xi_i/|\xi| \big)$$
$$\cdot \big| g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r) \big|^{-3},$$

$$F^{(2)}(\xi,\eta) := (4\cdot\pi)^{-1} \cdot \cot\varphi \cdot r \cdot \big| g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r) \big|^{-3}.$$

Thus we may write:

$$L_3(r,\varphi)(\xi,\eta) \tag{4.44}$$

$$= -F^{(1)}(\xi,\eta) + \cot\varphi \cdot \sum_{i=1}^{2} (\xi_i/|\xi|) \cdot L_i(r,\varphi)(\xi,\eta) - F^{(2)}(\xi,\eta).$$

Lemma 4.1 and inequality (3.18) yield:

$$|F^{(1)}(\xi,\eta)| \leq \pi^{-1} \cdot \cot\varphi \cdot |\xi-\eta|^{-1} \cdot (|\xi|+|\eta|)^{-1} \quad \text{for } \xi, \eta \in \mathbb{R}^2 \text{ with } \xi \neq \eta .$$

Hence we may apply Theorem 4.1 to get:

$$\Big(\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} |F^{(1)}(\xi,\eta) \cdot \Phi(\eta)| \, d\eta \Big)^p \, d\xi \Big)^{1/p} \leq \pi^{-1} \cdot \cot\varphi \cdot C_5(p) \cdot \|\Phi\|_p . \tag{4.45}$$

The corollary now follows from (4.44), (4.45), Korollar 4.3 and Theorem 4.4.

Corollary 4.5. *For* $j \in \{1, 2, 3\}$, $\varphi \in (0, \pi/2]$, $r \in \mathbb{R}\backslash\{0\}$, $p \in (1, \infty)$, $\Phi \in L^p(\mathbb{R}^2)$, *the function* $L_j(r, \varphi) \otimes \Phi$ *is well defined.*

If $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, *then there is a constant* $C_{13}(p, \varphi) > 0$ *such that*

$$\|L_j(r, \varphi) \otimes \Phi\|_p \leq C_{13}(p, \varphi) \cdot \|\Phi\|_p \qquad \textit{for } j \in \{1, 2, 3\},\ r \in \mathbb{R},\ \Phi \in L^p(\mathbb{R}^2).$$

Proof: Combine Theorem 4.4, Corollary 4.2 and 4.4.

Chapter 5

Estimate of a Fundamental Solution to the Resolvent Problem (1.3). Some Multiplier Transformations

In this chapter, we shall establish some technical results related to the fundamental solution $\widetilde{E}^{\lambda}_{jk}$ (see (1.2)) of the resolvent problem (1.3). In addition, we shall study the stress tensor $\widetilde{\mathcal{D}}^{\lambda}_{jkl}$ defined by means of $\widetilde{E}^{\lambda}_{jk}$; see (1.4). The main problem will consist in checking how certain integral operators related to $\widetilde{E}^{\lambda}_{jk}$ or $\widetilde{\mathcal{D}}^{\lambda}_{jkl}$ depend on the parameter λ. In order to control this parameter, we shall represent the kernels $\widetilde{E}^{\lambda}_{jk}$ and $\widetilde{\mathcal{D}}^{\lambda}_{jkl}$ as a sum of certain functions, thus obtaining a corresponding sum of integral operators, each of which may be estimated in an appropriate way. These studies will be continued in Chapter 11. In the present chapter, we shall mainly collect results which were already applied in [8] or [9], but – for lack of space – were proved there only shortly or not at all. Here they will be accounted for in more detail.

We begin by defining certain functions which will enter into the sums representing $\widetilde{E}^{\lambda}_{jk}$ and $\widetilde{\mathcal{D}}^{\lambda}_{jkl}$. Most of these definitions were previously introduced in [8, p. 338; p. 342], or [9, p. 324; p. 329; Addendum].

Definition 5.1. *For* $r \in \mathbb{C}$, *put*

$$g_1(r) := -\sum_{l=0}^{\infty} (-r)^l \cdot (l+2)^2 \cdot \big((l+3)!\big)^{-1},$$

$$g_2(r) := \sum_{l=0}^{\infty} (-r)^l \cdot \big(1-(l+2)^2\big) \cdot \big((l+4)!\big)^{-1},$$

In addition, define for $r \in \mathbb{C}\backslash\{0\}$:

$$f_1(r) := 3 \cdot e^{-r} + r \cdot e^{-r} + 6 \cdot r^{-2} \cdot (e^{-r} + r \cdot e^{-r} - 1),$$

$$f_2(r) := 1 + 2 \cdot e^{-r} + 6 \cdot r^{-2} \cdot (e^{-r} + r \cdot e^{-r} - 1),$$

$$f_3(r) := 6 \cdot e^{-r} + r \cdot e^{-r} + 15 \cdot r^{-2} \cdot (e^{-r} + r \cdot e^{-r} - 1).$$

These functions g_1, g_2, f_1, f_2, f_3 *will appear in the next definitions. In fact, let* $j, k, l \in \{1, 2, 3\}$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $z \in \mathbb{R}^3\backslash\{0\}$. *Then we set*

$$\overline{E}^{\lambda}_{jk}(z) := (4\cdot\pi)^{-1} \cdot \delta_{jk} \cdot \sqrt{\lambda} \cdot g_1(\sqrt{\lambda}\cdot|z|) + (4\cdot\pi)^{-1} \cdot \lambda \cdot |z|^{-1} \cdot z_j \cdot z_k \cdot g_2(\sqrt{\lambda}\cdot|z|);$$

$$P^{\lambda}_{jkl}(z) := (4\cdot\pi)^{-1} \cdot \delta_{kl} \cdot z_j \cdot |z|^{-3} \cdot f_1(\sqrt{\lambda}\cdot|z|) + (4\cdot\pi)^{-1} \cdot \delta_{jk} \cdot z_l \cdot |z|^{-3} \cdot f_2(\sqrt{\lambda}\cdot|z|);$$

$$Q^{\lambda}_{jkl}(z) := (4\cdot\pi)^{-1} \cdot \delta_{jl} \cdot z_k \cdot |z|^{-3} \cdot f_1(\sqrt{\lambda}\cdot|z|) - (2\cdot\pi)^{-1} \cdot z_j \cdot z_k \cdot z_l \cdot |z|^{-5} \cdot f_3(\sqrt{\lambda}\cdot|z|);$$

$$\mathcal{Y}^{\lambda}_j(z) := (4\cdot\pi)^{-1} \cdot z_j \cdot |z|^{-3} \cdot f_1(\sqrt{\lambda}\cdot|z|);$$

$$\mathcal{X}^{\lambda}_j(z) := (4\cdot\pi)^{-1} \cdot z_j \cdot |z|^{-3} \cdot f_2(\sqrt{\lambda}\cdot|z|); \qquad \mathcal{X}^{\infty}_j := (4\cdot\pi)^{-1} \cdot z_j \cdot |z|^{-3}.$$

The first of these functions will be used to write $\widetilde{E}^{\lambda}_{jk}$ *as a sum of two parts (Lemma 5.2); the others will lead to a representation of* $\widetilde{\mathcal{D}}^{\lambda}_{jkl}$ *as a sum; see Lemma 5.7. It will turn out that* $\overline{E}^{\lambda}_{jk}$ *is the weakly singular part of* $\widetilde{E}^{\lambda}_{jk}$ *(Lemma 5.4).*

Sometimes it will be convenient to use the following two functions, defined for $r \in (0, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$:

$$\mathcal{G}^{\lambda}_1(r) := (4\cdot\pi)^{-1} \cdot r^{-3/2} \cdot f_1(\sqrt{\lambda}\cdot r^{1/2});$$

$$\mathcal{G}^{\lambda}_2(r) := (4\cdot\pi)^{-1} \cdot r^{-3/2} \cdot f_2(\sqrt{\lambda}\cdot r^{1/2}); \qquad \mathcal{G}^{\infty}_2 := (4\cdot\pi)^{-1} \cdot r^{-3/2}.$$

The functions we shall introduce next will play a role when certain integral operators are written in terms of local coordinates, and when the Fourier transform in $\mathbb{R}^2$ is applied to these modified operators. The ensuing definitions should be compared to those in [9, p. 325/326], which refer to the case $\varphi = \pi/2$ (halfspace).

Definition 5.2. *For* $j \in \{1, 2\}$, $\xi \in \mathbb{R}^2\backslash\{0\}$, *set*

$$X^{\infty}_j(\xi) := (4\cdot\pi)^{-1} \cdot \xi_j \cdot |\xi|^{-3}; \qquad \underline{X}^{\infty}_j(\xi) := -i \cdot (4\cdot\pi)^{-1} \cdot \xi_j \cdot |\xi|^{-1}.$$

If $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\xi \in \mathbb{R}^2\backslash\{0\}$, $j \in \{1, 2\}$, *put*

$$Y^{\lambda}_j(\xi) := (4\cdot\pi)^{-1} \cdot \xi_j \cdot |\xi|^{-3} \cdot f_1(\sqrt{\lambda}\cdot|\xi|),$$

$$X^{\lambda}_j(\xi) := (4\cdot\pi)^{-1} \cdot \xi_j \cdot |\xi|^{-3} \cdot f_2(\sqrt{\lambda}\cdot|\xi|);$$

$$\underline{Y}^{\lambda}_j(\xi) := i \cdot \xi_j \cdot \left(4\cdot\pi\cdot\lambda\cdot(\lambda + |\xi|^2)^{1/2}\right)^{-1} \cdot \left(\lambda + 2\cdot|\xi|^2 - 2\cdot|\xi|\cdot(\lambda+|\xi|^2)^{1/2}\right);$$

$$\underline{X}^{\lambda}_j(\xi) := i \cdot \xi_j \cdot \left(2\cdot\pi\cdot\lambda\cdot(\lambda + |\xi|^2)^{1/2}\right)^{-1} \cdot \left(\lambda + |\xi|^2 - |\xi|\cdot(\lambda+|\xi|^2)^{1/2}\right) - i\cdot\xi_j\cdot(4\cdot\pi\cdot|\xi|)^{-1}.$$

Furthermore, define for $\varphi \in (0, \pi/2]$, $\tau \in \{-1, 1\}$:

$$A^{\infty,\varphi}_{\tau} := \begin{pmatrix} \tau/2 - 2\cdot\pi\cdot\sin\varphi\cdot\cos\varphi\cdot\underline{X}^{\infty}_1 & 0 & 2\cdot\pi\cdot\sin^2(\varphi)\cdot\underline{X}^{\infty}_1 \\ -2\cdot\pi\cdot\cos\varphi\cdot\underline{X}^{\infty}_2 & \tau/2 & 2\cdot\pi\cdot\sin\varphi\cdot\underline{X}^{\infty}_2 \\ -2\cdot\pi\cdot\cos^2(\varphi)\cdot\underline{X}^{\infty}_1 & 0 & \tau/2 + 2\cdot\pi\cdot\sin\varphi\cdot\cos\varphi\cdot\underline{X}^{\infty}_1 \end{pmatrix}.$$

Finally, we set for $\varphi \in (0, \pi/2]$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$:

$$A^{\lambda,\varphi}_{\tau} := \begin{pmatrix} \tau/2 - 2\cdot\pi\cdot\sin\varphi\cdot\cos\varphi\cdot(\underline{X}^{\lambda}_1 + \underline{Y}^{\lambda}_1) & -2\cdot\pi\cdot\cos\varphi\cdot\underline{Y}^{\lambda}_2 & 2\cdot\pi\cdot(\sin^2(\varphi)\cdot\underline{X}^{\lambda}_1 - \cos^2(\varphi)\cdot\underline{Y}^{\lambda}_1) \\ -2\cdot\pi\cdot\cos\varphi\cdot\underline{X}^{\lambda}_2 & \tau/2 & 2\cdot\pi\cdot\sin\varphi\cdot\underline{X}^{\lambda}_2 \\ 2\cdot\pi\cdot(\sin^2(\varphi)\cdot\underline{Y}^{\lambda}_1 - \cos^2(\varphi)\cdot\underline{X}^{\lambda}_1) & 2\cdot\pi\cdot\sin\varphi\cdot\underline{Y}^{\lambda}_2 & \tau/2 + 2\cdot\pi\cdot\sin\varphi\cdot\cos\varphi\cdot(\underline{X}^{\lambda}_1 + \underline{Y}^{\lambda}_1) \end{pmatrix}.$$

Next we shall fix a cut-off function Φ_h in $\mathbb{R}^2$ which tends to the constant value 1 for h tending to infinity, as will be made precise in Lemma 5.1 below. In addition, we are going to define a simple elongation $T(\varphi)$ in $\mathbb{R}^2$, which will arise when certain integral operators on $\partial\mathbb{K}(\varphi)$ are transformed by the techniques used in Chapter 10.

Definition 5.3. *Fix a function* $\widetilde{j}$ *belonging to* $C^{\infty}([0, \infty))$, *and satisfying the following properties:* $im(\widetilde{j}) \subset \mathbb{R}$, $\widetilde{j}\,|\,[0, 1] = 1$, $\widetilde{j}\,|\,[2, \infty) = 0$, $\widetilde{j}$ *monotone decreasing.*

For $h \in (0, \infty)$, *let* $\Psi_h : \mathbb{R}^2 \mapsto \mathbb{R}$ *be defined by* $\Psi_h := \widetilde{j}(|\xi|/h)$ *for* $\xi \in \mathbb{R}^2$.

If $\varphi \in (0, \pi/2]$, *then we define the function* $T(\varphi) : \mathbb{R}^2 \mapsto \mathbb{R}^2$ *by*

$T(\varphi)(\eta) := (\sin^{-1}(\varphi)\cdot\eta_1,\ \eta_2)$ *for* $\eta \in \mathbb{R}^2$.

Now we shall investigate the functions introduced before. We begin by an estimate of Ψ_h :

Lemma 5.1. *Let* $h \in (0, \infty)$. *Then*

$\Psi_h \in C_0^\infty(\mathbb{R}^2)$, $\quad 0 \le \Psi_h \le 1$, $\quad supp(\Psi_h) \subset \mathbb{B}_2(0, 2\cdot h)$, $\quad \Psi_h \,|\, \mathbb{B}_2(0, h) = 1$,

$supp(\, D^a\Psi_h \,) \subset \overline{\mathbb{B}_2(0, 2\cdot h)} \setminus \mathbb{B}_2(0, h)$ $\quad$ *for* $a \in \mathbb{N}_0^2$ *with* $a \ne 0$.

In particular, it holds $\Psi_h(\xi) \to 1 \;\; (h \to \infty)$ *for* $\xi \in \mathbb{R}^2$.

There is a constant $C_{14} > 0$ *such that the ensuing estimate is fulfilled for* $h \in (0, \infty)$, $a \in \mathbb{N}_0^2$ *with* $|a|_* \le 4$, $\xi \in \mathbb{R}^2$, $\sigma \in [0, 1]$:

$$|\, D^a\Psi_h(\xi) \,| \quad \le \quad C_{14} \cdot |\xi|^{-\sigma\cdot|a|_*} \cdot h^{-(1-\sigma)\cdot|a|_*} .$$

The proof of this lemma is simple and will be omitted.

Lemma 5.2. *For* $j, k \in \{1, 2, 3\}$, $\lambda \in \mathbb{C} \backslash]-\infty, 0]$, *we have*

$$\widetilde{E}_{jk}^\lambda \;=\; E_{jk} \;+\; \overline{E}_{jk}^\lambda . \tag{5.1}$$

This result, already mentioned in [8, (3.1)], may be proved by a simple but tedious computation, which we shall also omit.

Lemma 5.3. *For* $k \in \mathbb{N}$, $a, b \in \mathbb{N}_0^k$, $x \in \mathbb{R}^k \backslash \{0\}$, $m \in \{1, 2, 3\}$, *the following estimate is valid:*

$$|\, \partial^a/\partial x^a (\, x^b \cdot |x|^{-m} \,) \,| \quad \le \quad 2^{|b|_*} \cdot 4^{2\cdot|a|_*+1} \cdot |a|_*! \cdot |x|^{-|a|_*+|b|_*-m} .$$

This lemma may be shown by induction with respect to $|a|_*$. However, we do not write out the details, which are rather simple.

In the ensuing lemma, as well as in Lemma 5.8 and 5.16 further below, we shall present some estimates which are basic for our later arguments. However, the proof of all three of these lemmas is thoroughly tedious, and we shall restrict ourselves to some indications. We point out that most of the results contained in the next lemma were already mentioned in [8, (3.2) – (3.4)].

Lemma 5.4. *Take* $\vartheta \in [0, \pi)$. *Then there is a constant* $C_{15}(\vartheta) > 0$ *with the properties to follow:*

Let $x \in \mathbb{R}^3 \backslash \{0\}$, $j, k \in \{1, 2, 3\}$, $\lambda \in \mathbb{C} \backslash \{0\}$ *with* $|\arg\lambda| \le \vartheta$, $a, b \in \mathbb{N}_0^3$ *with* $|a|_* \le 3$, $1 \le |b|_* \le 3$, $\gamma \in [0, 1]$. *Then*

$$|\, D^a \widetilde{E}_{jk}^\lambda(x) \,| \quad \le \quad C_{15}(\vartheta) \cdot |\lambda|^{-\gamma} \cdot |x|^{-1-2\cdot\gamma-|a|_*} , \tag{5.2}$$

$$|\, D^b \overline{E}_{jk}^\lambda(x) \,| \quad \le \quad C_{15}(\vartheta) \cdot |\lambda|^{\gamma} \cdot |x|^{-1+2\cdot\gamma-|b|_*} , \tag{5.3}$$

$$|\, \overline{E}_{jk}^\lambda(x) \,| \quad \le \quad C_{15}(\vartheta) \cdot |\lambda|^{\gamma/2} \cdot |x|^{-1+\gamma} . \tag{5.4}$$

In addition, it holds for $x \in \mathbb{R}^3 \backslash \{0\}$, $j, k, l \in \{1, 2, 3\}$, $\sigma, \sigma' \in \mathbb{C} \backslash \{0\}$ *with* $|\sigma| = |\sigma'|$, $|\arg\sigma| \le \vartheta$, $|\arg\sigma'| \le \vartheta$:

$$|\, D_l \widetilde{E}_{jk}^\sigma(x) \;-\; D_l \widetilde{E}_{jk}^{\sigma'}(x) \,| \quad = \quad |\, D_l \overline{E}_{jk}^\sigma(x) \;-\; D_l \overline{E}_{jk}^{\sigma'}(x) \,| \tag{5.5}$$

$$\le \; C_{15}(\vartheta) \cdot \min\Big\{ |\sigma|^{-1} \cdot |x|^{-4} \cdot |\arg\sigma \;-\; \arg\sigma'\,| , \quad |\sigma| \cdot |\arg\sigma \;-\; \arg\sigma'\,| \Big\} .$$

Note that in any estimate of $\widetilde{E}_{jk}^\lambda(x)$ and $\overline{E}_{jk}^\lambda(x)$, the exponents of $|\lambda|$ and $|x|$ are connected.

Proof of Lemma 5.4: We first observe that $|\, e^{-z} \,| \le e^{-|z|\cdot\cos(\vartheta/2)}$ for $z \in \mathbb{C} \backslash \{0\}$ with $|\arg z| \le \vartheta/2$. It follows there is a constant $\mathfrak{C}_1 > 0$ such that

$$|\, \widetilde{g}_j^{(v)}(|z|) \,| \quad \le \quad \mathfrak{C}_1 \cdot |z|^{-2-v}$$

for $z \in \mathbb{R}^3 \backslash \{0\}$ with $|\arg\lambda| \le \vartheta$, $v \in \{0, 1, 2, 3\}$, $j \in \{1, 2\}$, where $\widetilde{g}_j^{(v)}$ denotes the derivative of order v of the function $\widetilde{g}_j$ introduced in (1.1). Hence, by (1.2) and Lemma 5.3, we may construct a constant $\mathfrak{C}_2$ with

$$|\, D^a \widetilde{E}_{jk}^\lambda(x) \,| \quad \le \quad \mathfrak{C}_2 \cdot |\lambda|^{-1} \cdot |x|^{-3-|a|_*} ,$$

and

$$|\, D_l \widetilde{E}_{jk}^\sigma(x) \;-\; D_l \widetilde{E}_{jk}^{\sigma'}(x) \,| \quad \le \quad \mathfrak{C}_2 \cdot |\sigma|^{-1} \cdot |\arg\sigma \;-\; \arg\sigma'\,| \cdot |x|^{-4}$$

for $x, j, k, l, \lambda, a, \sigma, \sigma'$ as in the lemma. We further note an immediate consequence of Lemma 5.3, namely

$$|\, D^a E_{jk}(x) \,| \quad \le \quad \pi^{-1} \cdot 4^5 \cdot 60 \cdot |x|^{-1-|a|_*} . \tag{5.6}$$

Recalling Lemma 5.2, we are now able to evaluate the left-hand side of (5.2) – (5.5) if $|\sqrt{\lambda} \cdot x| \ge 1$ and $|\sqrt{\sigma} \cdot x| \ge 1$. In order to deal with the case that the two preceding assumptions are not valid, we use Lemma 5.3 to obtain a numerical constant $\mathfrak{C}_3 > 0$ with

$$|\, D^b \overline{E}_{jk}^\lambda \,| \quad \le \quad \mathfrak{C}_3 \cdot |\lambda| \cdot |x|^{1-|a|_*} , \qquad |\, \overline{E}_{jk}^\lambda(x) \,| \quad \le \quad \mathfrak{C}_3 \cdot |\lambda|^{1/2} , \tag{5.7}$$

$$|\, D_l \overline{E}_{jk}^\sigma(x) \;-\; D_l \overline{E}_{jk}^{\sigma'}(x) \,| \quad \le \quad \mathfrak{C}_3 \cdot |\sigma| \cdot |\arg\sigma \;-\; \arg\sigma'\,| \tag{5.8}$$

for $x, j, k, l, \lambda, b, \sigma, \sigma'$ as in the lemma. Thus, if $|\sqrt{\lambda} \cdot x| \le 1$ and $|\sqrt{\sigma} \cdot x| \le 1$, the left-hand side of (5.2) - (5.5) may be estimated by means of (5.6) - (5.8) and Lemma 5.2.

Lemma 5.5 *Let* $p \in (1,\infty)$, $\varphi \in (0, \pi/2]$. *For* $l \in \{1,2\}$, $\Phi \in L^p(\mathbb{R}^2)$, *the function* $(X_l^\infty \circ T(\varphi)) *_p \Phi$ *is well defined.*

There is a constant $C_{16}(p,\varphi) > 0$ *so that for* $\Phi \in L^p(\mathbb{R}^2)$, $l \in \{1,2\}$, $\epsilon \in (0,\infty)$, *it holds:*

$$\| (X_l^\infty \circ T(\varphi))_\epsilon * \Phi \|_p, \quad \| (X_l^\infty \circ T(\varphi)) *_p \Phi \|_p \quad \le \quad C_{16}(p,\varphi) \cdot \|\Phi\|_p .$$

Note that in the case $\varphi = \pi/2$, *the function* $T(\varphi)$ *coincides with the identity mapping of* $\mathbb{R}^2$.

Proof: Let $\varphi \in (0,\pi/2]$, $l \in \{1,2\}$. Then we have for $\eta \in \mathbb{R}^2\backslash\{0\}$:

$$(X_l^\infty \circ T(\varphi))(\eta) \;=\; (4\cdot\pi)^{-1} \cdot \left(\sin^{-1}(\varphi)\cdot\eta_1,\ \eta_2\right)_l \cdot \left(\sin^{-2}(\varphi)\cdot\eta_1^2 + \eta_2^2\right)^{-3/2}.$$

Hence we obtain for $\eta \in \mathbb{R}^2\backslash\{0\}$, $t \in (0,\infty)$:

$$(X_l^\infty \circ T(\varphi))(t\cdot\eta) \;=\; t^{-2}\cdot (X_l^\infty \circ T(\varphi))(\eta),$$

$$(X_l^\infty \circ T(\varphi))(-\eta) \;=\; -(X_l^\infty \circ T(\varphi))(\eta).$$

Thus the lemma follows by applying the Calderón-Zygmund inequality; see [36, p. 85; p. 89, Theorem 2].

Lemma 5.6. *If* $\Phi \in L^2(\mathbb{R}^2)$, $l \in \{1,2\}$, *then* $(X_l^\infty *_2 \Phi)^\wedge \;=\; 2\cdot\pi\cdot \underline{X}_l^\infty \cdot \hat{\Phi}$.

This lemma states a standard result on Riesz kernels. For a proof, we refer to [36, p. 101 - 103].

The following equations should be compared to [9, p. 329 below; Addendum].

Lemma 5.7 *Let* $j,k,l \in \{1,2,3\}$, $\lambda \in \mathbb{C}\backslash\{0\}$. *Then*

$$\mathcal{D}_{jkl}(z) \;=\; -3\cdot(4\cdot\pi)^{-1}\cdot z_j\cdot z_k\cdot z_l\cdot|z|^{-5} \quad \text{for } z \in \mathbb{R}^3\backslash\{0\}; \qquad \overline{\mathcal{D}}_k \;=\; -\mathcal{X}_k^\infty; \tag{5.9}$$

$$\widetilde{\mathcal{D}}_{jkl}^\lambda \;=\; \mathcal{D}_{jkl} + D_j\overline{E}_{kl}^\lambda + D_k\overline{E}_{jl}^\lambda; \tag{5.10}$$

$$\widetilde{\mathcal{D}}_{jkl}^\lambda \;=\; -P_{jkl}^\lambda - Q_{jkl}^\lambda; \tag{5.11}$$

$$P_{jkl}^\lambda \;=\; \delta_{kl}\cdot\mathcal{Y}_j^\lambda + \delta_{jk}\cdot\mathcal{X}_l^\lambda. \tag{5.12}$$

Furthermore, it holds for $v \in \{1,2\}$, $z \in \mathbb{R}^3\backslash\{0\}$, $\sigma \in \{\lambda,\infty\}$:

$$\mathcal{Y}_v^\sigma(z) \;=\; z_v\cdot\mathcal{G}_1^\sigma(|z|^2), \qquad \mathcal{X}_v^\sigma(z) \;=\; z_v\cdot\mathcal{G}_2^\sigma(|z|^2).$$

Proof: An easy calculation shows that (5.9) is valid. Equation (5.10) is a consequence of (5.1). Concerning the proof of (5.11), we note for $r \in \mathbb{C}\backslash\{0\}$:

$$g_1(r) \;=\; r^{-1}\cdot(e^{-r}-1/2) \;+\; r^{-2}\cdot e^{-r} \;+\; r^{-3}\cdot(e^{-r}-1),$$

$$g_2(r) \;=\; -3\cdot r^{-4}\cdot(e^{-r}-1) \;-\; 3\cdot r^{-3}\cdot e^{-r} \;-\; r^{-2}\cdot(e^{-r}+1/2).$$

These equations are inserted into the definition of $\overline{E}_{jl}^\lambda$, $\overline{E}_{kl}^\lambda$. Then, after computing $D_k\overline{E}_{jl}^\lambda$, $D_j\overline{E}_{kl}^\lambda$, we put our results into (5.10). After a lengthy calculation, which uses (5.9), we arrive at (5.11). The last two equations stated in the lemma are obvious.

After Lemma 5.4, we now present the second lemma containing estimates which will be essential for all what follows. The proof of this lemma will be kept short since it is rather easy. Note that inequalities similar to those in the next lemma were stated in [9, (3.1), (3.2)].

Lemma 5.8. *Let* $\vartheta \in [0,\pi)$. *Then there is a constant* $C_{17}(\vartheta) > 0$ *with the following properties:*

For $\lambda \in \mathbb{C}\backslash\{0\}$ *with* $|\arg\lambda| \le \vartheta$, $x \in \mathbb{R}^3\backslash\{0\}$, $j,k,l \in \{1,2,3\}$, $a \in \mathbb{N}_0^3$ *with* $|a|_* \le 3$, $\gamma \in [0,1]$, $\sigma \in \{1,2\}$, $r \in (0,\infty)$, *it holds:*

$$|D^a\mathcal{Y}_j^\lambda(x)| \;+\; |D^a(\mathcal{X}_l^\lambda - \mathcal{X}_l^\infty)(x)| \;\le\; C_{17}(\vartheta)\cdot|\lambda|^{-1}\cdot|x|^{-4-|a|_*}; \tag{5.13}$$

$$|D^a\mathcal{Y}_j^\lambda(x)| \;+\; |D^a\mathcal{X}_l^\lambda(x)| \tag{5.14}$$

$$\le\; C_{17}(\vartheta)\cdot\Big(\sum_{v=0}^{|a|_*}|\lambda|^{1-v/2}\Big)^{\gamma}\cdot\Big(\sum_{v=0}^{|a|_*}|\lambda|^{-v/2}\Big)^{1-\gamma}\cdot|x|^{-2+2\cdot\gamma-|a|_*};$$

$$|\widetilde{\mathcal{D}}_{jkl}^\lambda(x)| \;\le\; C_{17}(\vartheta)\cdot|x|^{-2}; \tag{5.15}$$

$$|(\mathcal{G}_1^\lambda)'(r^2)| \;+\; |(\mathcal{G}_2^\lambda - \mathcal{G}_2^\infty)'(r^2)| \;\le\; C_{17}(\vartheta)\cdot|\lambda|^{-1}\cdot(|\lambda|\wedge 1)^{-1/2}\cdot r^{-7}; \tag{5.16}$$

$$|(\mathcal{G}_\sigma^\lambda)'(r^2)| \;\le\; C_{17}(\vartheta)\cdot(|\lambda|\wedge 1)^{-1/2}\cdot r^{-5}; \tag{5.17}$$

$$|(\mathcal{G}_\sigma^\lambda)'(r^2)| \;\le\; C_{17}(\vartheta)\cdot|\lambda|\cdot(|\lambda|\wedge 1)^{-1/2}\cdot r^{-3}. \tag{5.18}$$

Proof: By Lemma 5.4 it is obvious how to construct a constant $\mathfrak{C}_1 > 0$ with

$|\widetilde{\mathcal{D}}_{jkl}^\lambda(x)| \le \mathfrak{C}_1\cdot|x|^{-2}$ for λ, j, k, l as in the lemma.

Next we observe there is some $\mathfrak{C}_2 > 0$ such that for $z \in \mathbb{C}\backslash\{0\}$ with $|\arg z| \le \vartheta/2$, $\mu \in \{1,2\}$, $v \in \{1,2,3\}$, it holds:

$$|f_1(z)|,\ |f_2(z)-1| \;\le\; \mathfrak{C}_2\cdot|z|^{-2}; \qquad |f_\mu^{(v)}(z)| \;\le\; \mathfrak{C}_2\cdot|z|^{-2-v}.$$

Here the symbol $f_\mu^{(v)}$ denotes the v-th order derivative of f_μ. Thus, using Lemma 5.3, we may construct a constant $\mathfrak{C}_3 > 0$ with

$$|D^\alpha \mathcal{Y}_j^\lambda(x)| + |D^\alpha(\mathcal{X}_l^\lambda - \mathcal{X}_l^\infty)(x)| \leq \mathfrak{C}_3 \cdot |\lambda|^{-1} \cdot |x|^{-4-|\alpha|},$$

$$|(\mathcal{G}_1^\lambda)'(r^2)| + |(\mathcal{G}_2^\lambda - \mathcal{G}_2^\infty)'(r^2)| \leq \mathfrak{C}_3 \cdot |\lambda|^{-1} \cdot (|\lambda| \wedge 1)^{-1/2} \cdot r^{-7},$$

for $\lambda,\ x,\ j,\ l,\ r$ as in the lemma. Next we point out an expansion of f_1 and f_2 into a power series, which holds for $z \in \mathbb{C}\backslash\{0\}$:

$$f_1(z) = z^2 \cdot \sum_{n=0}^{\infty} (-z)^n \cdot \Big(1/(n+1)! + 3/(n+2)! - 6/(n+3)! + 6/(n+4)!\Big), \tag{5.19}$$

$$f_2(z) = z^2 \cdot \sum_{n=0}^{\infty} (-z)^n \cdot \Big(1/(n+2)! + 6/(n+4)! - 6/(n+3)!\Big).$$

It follows for $\mu \in \{1, 2\}$, $v \in \{1, 2, 3\}$, $r \in (0, \infty)$ with $r \cdot |\sqrt{\lambda}| \leq 1$:

$$|f_\mu^{(v)}(\sqrt{\lambda} \cdot r)| \leq \mathfrak{C} \cdot (|\lambda|^{1/2} \cdot r)^{2-v} \quad \text{in the case } v \leq 2,$$

$$|f_\mu^{(v)}(\sqrt{\lambda} \cdot r)| \leq \mathfrak{C} \quad \text{in the case } v = 3.$$

Now, applying Lemma 5.3 again, we may find a constant $\mathfrak{C}_4 > 0$ with

$$|D^\alpha \mathcal{Y}_j^\lambda(x)| + |D^\alpha \mathcal{X}_l^\lambda(x)| \leq \mathfrak{C}_4 \cdot \Big(\sum_{v=0}^{|\alpha|_*} |\lambda|^{1-v/2}\Big) \cdot |x|^{-|\alpha|_*},$$

$$|(\mathcal{G}_\sigma^\lambda)'(r^2)| \leq \mathfrak{C}_4 \cdot |\lambda| \cdot (|\lambda| \wedge 1)^{-1/2} \cdot r^{-3}$$

for $\lambda,\ x,\ j,\ l,\ \sigma,\ r$ as in the lemma, but with the further assumptions $|\sqrt{\lambda} \cdot x| \leq 1$ and $|\sqrt{\lambda}| \cdot r \leq 1$.

With these estimates available, it should be obvious how to construct a constant $C_{17}(\vartheta)$ with properties as stated in the lemma.

Lemma 5.9. *Let* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$. *Then there is a constant* $C_{18}(\vartheta, \varphi) > 0$ *with the properties to follow:*

For $\lambda \in \mathbb{C}\backslash\{0\}$ *with* $|\arg\lambda| \leq \vartheta$, $r \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^2\backslash\{0\}$ *with* $\xi \neq \eta$, $j, l \in \{1, 2, 3\}$, *it holds*

$$\Big|\sum_{k=1}^{3} (n_k^{(\varphi)} \circ g^{(\varphi)})(\eta) \cdot Q_{jkl}^\lambda\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r)\big)\Big| \tag{5.20}$$

$$\leq C_{18}(\vartheta, \varphi) \cdot \Big(|\xi - \eta|^{-1} \cdot (|\xi| + |\eta|)^{-1} + |r| \cdot \big|g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r)\big|^{-3}\Big);$$

and

$$\Big|\sum_{k=1}^{3} (n_k^{(\varphi)} \circ g^{(\varphi)})(\eta) \cdot Q_{jkl}^\lambda\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big)\Big| \tag{5.21}$$

$$\leq C_{18}(\vartheta, \varphi) \cdot |\lambda|^{-1} \cdot |\xi - \eta|^{-3} \cdot (|\xi| + |\eta|)^{-1}.$$

Proof: Recalling Definition 5.1, we note for $\lambda,\ \xi,\ \eta,\ j,\ l$ as in the lemma:

$$\Big|\sum_{k=1}^{3} (n_k^{(\varphi)} \circ g^{(\varphi)})(\eta) \cdot Q_{jkl}^\lambda\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r)\big)\Big| \tag{5.22}$$

$$\leq \pi^{-1} \cdot \Big|(n_k^{(\varphi)} \circ g^{(\varphi)})(\eta) \cdot \Big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r)\Big)\Big|$$

$$\cdot \big|g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r)\big|^{-3}$$

$$\cdot \sum_{v \in \{1,3\}} \Big|f_v\Big(\sqrt{\lambda} \cdot \big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r)\big)\Big)\Big|$$

Next we point out that $f_3(s) = -3/2 + f_4(s)$ for $s \in \mathbb{C}\backslash\{0\}$, with

$$f_4(s) := \sum_{n=0}^{\infty} (-s)^{n+2} \cdot \Big(1/(n+1)! + 6/(n+2)! - 15/(n+3)! + 15/(n+4)!\Big).$$

Moreover, we recall equation (5.19), where f_1 was also represented by an infinite sum. Thus there is a numerical constant $\mathfrak{C}_1 > 0$ with

$$|f_1(z)| + |f_3(z) + 3/2| \leq \mathfrak{C}_1 \cdot |z|^2 \quad \text{for } z \in \mathbb{C}\backslash\{0\} \text{ with } |z| \leq 1.$$

We further remark there is a constant $\mathfrak{C}_2 > 0$ such that

$$|f_1(z)| + |f_2(z)| \leq \mathfrak{C}_2 \cdot |z|^{-2} \quad \text{for } z \in \mathbb{C}\backslash\{0\} \text{ with } |\arg\lambda| \leq \vartheta/2.$$

Now, by combining these estimates with (5.22), (3.19) and Lemma 3.1, and by distinguishing between the case

$$\big|\sqrt{\lambda} \cdot \big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r)\big)\big| \leq 1$$

and its complement, we may easily construct a suitable constant $C_{18}(\vartheta, \varphi)$.

Now we are going to exploit the previous results in order to estimate some integral operators.

Lemma 5.10. *For* $p \in (1, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\varphi \in (0, \pi/2]$, $\Phi \in L^p(\mathbb{R}^2)$, $j \in \{1, 2, 3\}$, $k \in \{1, 2\}$, $r \in \mathbb{R}$, *and for* $H \in \{\mathcal{X}_j^\lambda, \mathcal{Y}_j^\lambda\}$, *it holds*

$$\int_{\mathbb{R}^2} \Big|H\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r)\big) \cdot \Phi(\eta)\Big|\, d\eta < \infty \quad \text{for } \xi \in \mathbb{R}^2,$$

and the functions $(X_k^\lambda \circ T(\varphi)) * \Phi$ *and* $(Y_k^\lambda \circ T(\varphi)) * \Phi$ *are well defined.*

If $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$, *then there is a number* $C_{19}(p, \vartheta, \varphi) > 0$ *such that the ensuing inequality is satisfied for* $\lambda \in \mathbb{C}\backslash\{0\}$ *with* $|\arg\lambda| \le \vartheta$, $j \in \{1, 2, 3\}$, $k \in \{1, 2\}$, $r \in \mathbb{R}$, $\Phi \in L^p(\mathbb{R}^2)$, *and for* $H \in \{\mathcal{X}_j^\lambda, \mathcal{Y}_j^\lambda\}$, $G \in \{X_k^\lambda, Y_K^\lambda\}$:

$$\Big\| \int_{\mathbb{R}^2} H\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r)\big) \cdot \Phi(\eta)\, d\eta \Big\|_p, \quad \big\| (G \circ T(\varphi)) * \Phi \big\|_p$$
$$\le C_{19}(p, \vartheta, \varphi) \cdot \|\Phi\|_p .$$

We recall that for $\varphi = \pi/2$, $T(\varphi)$ is the identity mapping of $\mathbb{R}^2$. Concerning the proof of this lemma, the main difficulty consists in the fact that the kernels $\mathcal{X}_j^\lambda$, X_k^λ contain the singular part $\mathcal{X}_j^\infty$, X_k^∞, respectively. It is due to this property of the kernels $\mathcal{X}_j^\lambda$ and X_k^λ that we shall need the results of Chapter 4.

Proof of Lemma 5.10: Let $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$. Define

$C_{19}(p, \vartheta, \varphi) := 2\cdot\pi \cdot C_{17}(\vartheta) + \max\{ C_{13}(p, \varphi),\ C_{16}(p, \varphi) \}$.

Take $\lambda \in \mathbb{C}\backslash\{0\}$ with $|\arg\lambda| \le \vartheta$, $\Phi \in L^p(\mathbb{R}^2)$, $j \in \{1, 2, 3\}$, $k \in \{1, 2\}$, $r \in \mathbb{R}$. For $\xi, \eta \in \mathbb{R}^2$ with $\xi \ne \eta$, the ensuing inequality may be verified by referring to Lemma 5.8 and 3.4:

$$\big|\mathcal{Y}_j^\lambda\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r)\big)\big|,$$
$$\Big|\chi_{(0,\,|\lambda|^{-1/2})}(|\xi-\eta|) \cdot \mathcal{X}_j^\lambda\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r)\big)$$
$$+ \chi_{(|\lambda|^{-1/2},\,\infty)}(|\xi-\eta|) \cdot (\mathcal{X}_j^\lambda - \mathcal{X}_j^\infty)\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r)\big)\Big|$$
$$\le C_{17}(\vartheta) \cdot \Big(\chi_{(0,\,|\lambda|^{-1/2})}(|\xi-\eta|) \cdot |\lambda| + \chi_{(|\lambda|^{-1/2},\,\infty)}(|\xi-\eta|) \cdot |\lambda|^{-1} \cdot |\xi-\eta|^{-4}\Big);$$

In order to deal with the two-dimensional kernels $Y_k^\lambda \circ T(\varphi)$, $X_j^\lambda \circ T(\varphi)$, we note that $X_k^\lambda(\varrho) = \mathcal{X}_k^\lambda(\varrho, 0)$ and $Y_k^\lambda(\varrho) = \mathcal{Y}_k^\lambda(\varrho, 0)$ for $\varrho \in \mathbb{R}^2$. Thus it follows by once more referring to Lemma 5.8 and 3.4:

$$\big|(Y_k^\lambda \circ T(\varphi))(\xi-\eta)\big|,$$
$$\Big|\chi_{(0,\,|\lambda|^{-1/2})}(|\xi-\eta|) \cdot (X_k^\lambda \circ T(\varphi))(\xi-\eta)$$
$$+ \chi_{(|\lambda|^{-1/2},\,\infty)}(|\xi-\eta|) \cdot \Big((X_k^\lambda - X_k^\infty) \circ T(\varphi)\Big)(\xi-\eta)\Big|$$
$$\le C_{17}(\vartheta) \cdot \Big(\chi_{(0,\,|\lambda|^{-1/2})}(|\xi-\eta|) \cdot |\lambda| + \chi_{(|\lambda|^{-1/2},\,\infty)}(|\xi-\eta|) \cdot |\lambda|^{-1} \cdot |\xi-\eta|^{-4}\Big),$$

where we used the inequality $|T(\varphi)\,(\xi-\eta)| \ge |\xi-\eta|$. But the function

$$\Big(\chi_{(0,\,|\lambda|^{-1/2})}(|id(\mathbb{R}^2)|) \cdot |\lambda|$$
$$+ \chi_{(|\lambda|^{-1/2},\,\infty)}(|id(\mathbb{R}^2)|) \cdot |\lambda|^{-1} \cdot |id(\mathbb{R}^2\backslash\{0\})|^{-4}\Big) * |\Phi|$$

is well defined, and according to Young's inequality (Lemma 4.9), the $L^p(\mathbb{R}^2)$-norm of this mapping is bounded by

$$\int_{\mathbb{R}^2} \Big(\chi_{(0,\,|\lambda|^{-1/2})}(|\xi|) \cdot |\lambda| + \chi_{(|\lambda|^{-1/2},\,\infty)}(|\xi|) \cdot |\lambda|^{-1} \cdot |\xi|^{-4}\Big)\, d\xi \cdot \|\Phi\|_p = 2\cdot\pi \cdot \|\Phi\|_p .$$

Next we note that

$$\mathcal{X}_j^\infty\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r)\big) = -L_j(r, \varphi)(\xi, \eta) \quad \text{for } \xi, \eta \in \mathbb{R}^2,\ \xi \ne \eta .$$

where $L_j(r, \varphi)$ was introduced in Definition 4.1. But we know from Corollary 4.5 that the function $\big(L_j(r, \varphi)\big)_{|\lambda|^{-1/2}} \otimes \Phi$ is well defined, with

$$\big\| \big(L_j(\varphi, r)\big)_{|\lambda|^{-1/2}} \otimes \Phi \big\|_p \le C_{13}(p, \varphi) \cdot \|\Phi\|_p .$$

In addition, Lemma 5.5 states that the function $\big(X_k^\infty \circ T(\varphi)\big)_{|\lambda|^{-1/2}} * \Phi$ is well defined too, and

$$\big\| \big(X_k^\infty \circ T(\varphi)\big)_{|\lambda|^{-1/2}} * \Phi \big\|_p \le C_{16}(p, \varphi) \cdot \|\Phi\|_p .$$

But the preceding estimates imply the lemma, as may be seen by recalling the definition of $C_{19}(p, \vartheta, \varphi)$.

Lemma 5.11. *Let* $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$. *Then there is some number* $C_{20}(p, \vartheta, \varphi) > 0$ *such that it holds for* $\lambda \in \mathbb{C}\backslash\{0\}$ *with* $|\arg\lambda| \le \vartheta$, $j, l \in \{1, 2, 3\}$, $\Phi \in L^p(\mathbb{R}^2)$, $r \in \mathbb{R}$:

$$\Big(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}\Big|\sum_{k=1}^{3} Q_{jkl}^\lambda\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r)\big) \cdot \big(n_k^{(\varphi)} \circ g^{(\varphi)}\big)(\eta) \cdot \Phi(\eta)\Big|\, d\eta\Big)^p d\xi\Big)^{1/p}, \tag{5.23}$$

$$\Big(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}\Big|\sum_{k=1}^{3} Q_{jkl}^\lambda\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r)\big) \cdot \big(n_k^{(\varphi)} \circ g^{(\varphi)}\big)(\xi) \cdot \Phi(\eta)\Big|\, d\eta\Big)^p d\xi\Big)^{1/p}$$
$$\le C_{20}(p, \vartheta, \varphi) \cdot \|\Phi\|_p ,$$

and if in addition $|\lambda| \ge 1$, $R \in (0, \infty)$, *the following estimate is satisfied:*

$$\Big(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R)}\Big|\sum_{k=1}^{3} Q_{jkl}^\lambda\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) \cdot \big(n_k^{(\varphi)} \circ g^{(\varphi)}\big)(\eta) \cdot \Phi(\eta)\Big|\, d\eta\Big)^p d\xi\Big)^{1/p} \tag{5.24}$$
$$\le C_{20}(p, \vartheta, \varphi) \cdot R^{-1} \cdot \|\Phi\|_p .$$

Proof: Put

$$C_{20}(p, \vartheta, \varphi) := \max\Big\{ C_{18}(\vartheta, \varphi) \cdot \big(C_5(p) + C_{10}(p)\big),\quad 4\cdot\pi \cdot C_{18}(\vartheta) \Big\}.$$

Then inequality (5.23) is an immediate consequence of (5.20), Theorem 4.1 and Corollary 4.3. This leaves us to prove (5.24). To this end, take $R \in (0, \infty)$, and assume in addition that $|\lambda| \geq 1$. Using (5.20) and (5.21), we find for $\xi, \eta \in \mathbb{R}^2$ with $\xi \neq \eta$:

$$\Big| \sum_{k=1}^{3} Q^{\lambda}_{jkl}\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) \cdot \big(n_k^{(\varphi)} \circ g^{(\varphi)}\big)(\eta) \cdot \chi_{\mathbb{R}^2\setminus\mathbb{B}_2(0,R)}(\eta) \cdot \Phi(\eta) \Big|$$

$$\leq C_{18}(\vartheta, \varphi) \cdot R^{-1}$$

$$\cdot \Big(\chi_{(0,1)}(|\xi - \eta|) \cdot |\xi - \eta|^{-1} + \chi_{(1,\infty)}(|\xi - \eta|) \cdot |\xi - \eta|^{-3} \Big) \cdot |\Phi(\eta)|.$$

But Young's inequality (Lemma 4.9) implies

$$\Big\| \int_{\mathbb{R}^2} \Big(\chi_{(0,1)}\big(|id(\mathbb{R}^2) - \eta|\big) \cdot |id(\mathbb{R}^2) - \eta|^{-1}$$

$$+ \chi_{(1,\infty)}\big(|id(\mathbb{R}^2) - \eta|\big) \cdot |id(\mathbb{R}^2) - \eta|^{-3} \Big) \cdot |\Phi(\eta)| \, d\eta \Big\|_p$$

$$\leq \int_{\mathbb{R}^2} \Big(\chi_{(0,1)}(|\sigma|) \cdot |\sigma|^{-1} + \chi_{(1,\infty)}(|\sigma|) \cdot |\sigma|^{-3} \Big) \, d\sigma \cdot \|\Phi\|_p.$$

Now inequality (5.24) follows by collecting the preceding estimates.

Lemma 5.12. *For* $\lambda \in \mathbb{C}\setminus(-\infty, 0]$, $\varphi \in (0, \pi/2]$, $j, l \in \{1, 2, 3\}$, $r \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^2\setminus\{0\}$ *with* $\xi \neq \eta$, *abbreviate*

$$f^{(1)}_{jl}(\lambda, \varphi, r) := \sum_{k=1}^{3} \widetilde{\mathcal{D}}^{\lambda}_{jkl}\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r)\big) \cdot \big(n_k^{(\varphi)} \circ g^{(\varphi)}\big)(\eta),$$

$$f^{(2)}_{jl}(\lambda, \varphi, r) := \sum_{k=1}^{3} \widetilde{\mathcal{D}}^{\lambda}_{jkl}\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) + (0,0,r)\big) \cdot \big(n_k^{(\varphi)} \circ g^{(\varphi)}\big)(\xi).$$

For $p \in (1, \infty)$, $\Phi \in L^p(\mathbb{R}^2)$, $\sigma \in \{1, 2\}$, *and for* $\lambda, \varphi, j, l, r$ *as before, the function* $f^{(\sigma)}_{jl}(\lambda, \varphi, r) \otimes \Phi$ *is well defined.*

If $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$, *then there is some number* $C_{21}(p, \vartheta, \varphi) > 0$ *such that*

$$\| f^{(\sigma)}_{jl}(\lambda, \varphi, r) \otimes \Phi \|_p \leq C_{21}(p, \vartheta, \varphi) \cdot \|\Phi\|_p$$

for $r \in \mathbb{R}$, $\sigma \in \{1, 2\}$, $\lambda \in \mathbb{C}\setminus(-\infty, 0]$ *with* $|\arg\lambda| \leq \vartheta$, $j, l \in \{1, 2, 3\}$, $\Phi \in L^p(\mathbb{R}^2)$.

Proof: This lemma combines (5.11), (5.12), Lemma 5.10 and 5.11.

Next we are going to prove certain technical results related to the problem of inverting the operator $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$. These studies will be continued in Chapter 10 - 12.

The result stated in the next lemma was already mentioned in [9, p. 326].

Lemma 5.13. *Let* $\lambda \in \mathbb{C}\setminus(-\infty, 0]$, $j \in \{1, 2\}$. *Then*

$$X^{\lambda}_j,\ Y^{\lambda}_j \in L^p(\mathbb{R}^2) \quad \textit{for } p \in (1, \infty). \tag{5.25}$$

Furthermore, the functions $\underline{X}^{\lambda}_j$, $\underline{Y}^{\lambda}_j$ *introduced in Definition 5.2 satisfy the equations*

$$\overset{\wedge}{X^{\lambda}_j} = \underline{X}^{\lambda}_j, \qquad \overset{\wedge}{Y^{\lambda}_j} = \underline{Y}^{\lambda}_j. \tag{5.26}$$

Finally it holds for $\Phi \in L^2(\mathbb{R}^2)$:

$$(X^{\lambda}_j * \Phi)^{\wedge} = 2\cdot\pi \cdot \underline{X}^{\lambda}_j \cdot \overset{\wedge}{\Phi}, \qquad (Y^{\lambda}_j * \Phi)^{\wedge} = 2\cdot\pi \cdot \underline{Y}^{\lambda}_j \cdot \overset{\wedge}{\Phi}. \tag{5.27}$$

Proof: Since $X^{\lambda}_j(\xi) = \mathcal{X}^{\lambda}_j(\xi, 0)$, $Y^{\lambda}_j(\xi) = \mathcal{Y}^{\lambda}_j(\xi, 0)$ for $\xi \in \mathbb{R}^2\setminus\{0\}$, equation (5.25) follows from (5.14). As for the proof of (5.26), which is by no means trivial, it may be found in [33, p. 214 - 216]. Equation (5.27) may be derived by combining Parseval's theorem ([51, p. 155]) with (5.25) and (5.26).

In the case $\varphi = \pi/2$, equation (5.28) below was used by McCracken [33, p. 216]. Surprisingly, the right-hand side of this equation does not depend on φ.

Lemma 5.14. *Let* $\lambda \in \mathbb{C}\setminus(-\infty, 0]$, $\varphi \in (0, \pi/2]$, $\tau \in \{-1, 1\}$. *Then*

$$\det A^{\lambda,\varphi}_{\tau} = \tau/8 - \tau \cdot 2\cdot\pi^2 \cdot \underline{X}^{\lambda}_1 \cdot \underline{Y}^{\lambda}_1 - \tau \cdot 2\cdot\pi^2 \cdot \underline{X}^{\lambda}_2 \cdot \underline{Y}^{\lambda}_2; \tag{5.28}$$

$$\det A^{\infty,\varphi}_{\tau} = \tau/8. \tag{5.29}$$

Proof: For brevity we set

$a := -2\cdot\pi \cdot \sin\varphi \cdot \cos\varphi \cdot \underline{X}^{\lambda}_1$, $\quad b := -2\cdot\pi \cdot \sin\varphi \cdot \cos\varphi \cdot \underline{Y}^{\lambda}_1$,

$c := -2\cdot\pi \cdot \cos\varphi \cdot \underline{X}^{\lambda}_2$, $\quad d := -2\cdot\pi \cdot \cos\varphi \cdot \underline{Y}^{\lambda}_2$, $\quad \gamma := -\tan\varphi$, $\quad \alpha := \cot\varphi$.

Then we find

$$\det A^{\lambda,\varphi}_{\tau} = \det \begin{pmatrix} \tau/2 + a + b & d & \alpha\cdot b + \gamma\cdot a \\ c & \tau/2 & \gamma\cdot c \\ \alpha\cdot a + \gamma\cdot b & \gamma\cdot d & \tau/2 - a - b \end{pmatrix}.$$

Since $\alpha\cdot\gamma = -1$, it follows

$$\det A^{\lambda,\varphi}_{\tau} = \tau/8 - (\tau/2)\cdot(\gamma^2 + 1)\cdot c\cdot d - (\tau/2)\cdot(\gamma^2 + 2 + \alpha^2)\cdot a\cdot b.$$

On the other hand, we note that

$$\gamma^2+1 = \cos^{-2}(\varphi), \qquad \gamma^2+2+\alpha^2 = \cos^{-2}(\varphi)\cdot\sin^{-2}(\varphi).$$

Collecting our results, we obtain (5.28). Equation (5.29) is obvious.

Lemma 5.15. *There exists a number* $C_{22}>0$ *such that for* $\varphi\in(0,\pi/2]$, $j,k\in\{1,2,3\}$, $m\in\{1,2\}$, $\xi\in\mathbb{R}^2\backslash\{0\}$, $\tau\in\{-1,1\}$, $a\in\mathbb{N}_0^2$ *with* $|a|_*\le 4$, *it holds*

$$|D^a\underline{X}_m^\infty(\xi)|,\quad |D^a(A_\tau^{\infty,\varphi})_{jk}(\xi)|,\quad \left|D^a\Big((A_\tau^{\infty,\varphi})^{-1}\Big)_{jk}(\xi)\right| \quad\le\quad C_{22}\cdot|\xi|^{-|a|_*}.$$

Proof: The lemma is an immediate consequence of Lemma 5.3 and 5.14.

As announced in the remark before Lemma 5.4, the next lemma is the third one – after Lemma 5.4 and 5.8 – containing some estimates which will be essential for our later work.

Lemma 5.16. *Take* $\vartheta\in[0,\pi)$. *Then there is a constant* $C_{23}(\vartheta)>0$ *such that for* $\lambda\in\mathbb{C}\backslash\{0\}$ *with* $|\arg\lambda|\le\vartheta$, $m\in\{1,2\}$, $a\in\mathbb{N}_0^2$ *with* $|a|_*\le 4$, $\kappa\in[0,1]$, *and for* $\xi\in\mathbb{R}^2\backslash\{0\}$, *the following inequalities are fulfilled:*

$$|D^a\underline{X}_m^\lambda(\xi)| \quad\le\quad C_{23}(\vartheta)\cdot|\xi|^{-|a|_*}; \tag{5.30}$$

$$|D^a(\underline{X}_m^\lambda-\underline{X}_m^\infty)(\xi)|,\quad |D^a\underline{Y}_m^\lambda(\xi)| \quad\le\quad C_{23}(\vartheta)\cdot|\xi|^{-|a|_*+\kappa}\cdot|\lambda|^{-\kappa/2}. \tag{5.31}$$

Proof: Let λ, m, a, κ, ξ be given as in the lemma. By distinguishing the cases $\Re(\lambda)\ge 0$ and $\Re(\lambda)<0$, we may show for $\gamma\in[0,\infty)$:

$$|\lambda+\gamma| \quad\ge\quad \sin(\vartheta\vee(\pi/2))\cdot(1/2)\cdot\gamma; \tag{5.32}$$

$$|\lambda+\gamma| \quad\ge\quad \sin(\vartheta\vee(\pi/2))\cdot|\lambda|. \tag{5.33}$$

Moreover, we find for $b\in\mathbb{N}_0^2$, $z\in\mathbb{Z}\backslash\{0\}$ with $z\le 1$:

$$\left|\partial^b/\partial\xi^b\Big((\lambda+|\xi|^2)^{z/2}\Big)\right| \tag{5.34}$$

$$\le\quad \Big(2\cdot|b|_*+|z|\Big)!\cdot(|z|!)^{-1}\cdot|\xi|^{-|b|_*}\cdot|\lambda+|\xi|^2|^{z/2}\cdot 4^{|b|_*}\cdot\sin^{-|b|_*}(\vartheta\vee(\pi/2)).$$

This result may be proved by induction with respect to $|b|_*$. We further mention the inequalities

$$|\gamma+(\lambda+\gamma)^{1/2}| \;\ge\; \gamma, \qquad |\gamma+(\lambda+\gamma)^{1/2}| \;\ge\; \sin^{1/2}(\vartheta\vee(\pi/2))\cdot|\lambda|^{1/2}, \tag{5.35}$$

$L\gamma^2$

with $\gamma\in[0,\infty)$, which follow from (5.33) after a short computation. Finally it holds (compare [33, p. 216, (5.5)]):

$$\underline{Y}_m^\lambda(\xi) \quad=\quad i\cdot\lambda\cdot(4\cdot\pi)^{-1}\cdot\xi_m\cdot(\lambda+|\xi|^2)^{-1/2}\cdot\Big(|\xi|+(\lambda+|\xi|^2)^{1/2}\Big)^{-2}; \tag{5.36}$$

$$\underline{X}_m^\lambda(\xi) \quad=\quad i\cdot(2\cdot\pi)^{-1}\cdot\xi_m\cdot\Big((\lambda+|\xi|^2)^{1/2}+|\xi|\Big)^{-1} \;+\; \underline{X}_m^\infty(\xi). \tag{5.37}$$

Now, by estimating the left-hand side of (5.30) and (5.31) by means of Lemma 5.3 and (5.32) – (5.37), we see there exists a constant $C_{23}(\vartheta)$ having properties as stated in the lemma.

The next lemma is a slight generalization of McCracken's result [33, p. 216/217, Lemma 5.6]. McCracken shows that the left-hand side $|\det A_\tau^{\lambda,\varphi}(\eta)|$ of (5.38) does not take the value 0 if $\varphi=\pi/2$, $|\lambda|=1$. We are going to prove that the term $|\det A_\tau^{\lambda,\varphi}(\eta)|$ is bounded away from zero by a positive constant, uniformly in $\varphi\in(0,\pi/2]$ and $\lambda\in\mathbb{C}$ with $|\arg\lambda|\le\vartheta$. Actually, since the term $\det A_\tau^{\lambda,\varphi}(\eta)$ does not depend on φ (see Lemma 5.14), we only have to write out some arguments which seem to have been regarded as obvious in [33].

Lemma 5.17 *Take* $\vartheta\in[0,\pi)$. *There is a constant* $C_{24}(\vartheta)>0$ *such that*

$$|\det A_\tau^{\lambda,\varphi}(\eta)| \quad\ge\quad C_{24}(\vartheta) \tag{5.38}$$

for $\varphi\in(0,\pi/2]$, $\eta\in\mathbb{R}^2\backslash\{0\}$, $\tau\in\{-1,1\}$, $\lambda\in\mathbb{C}\backslash\{0\}$ *with* $|\arg\lambda|\le\vartheta$. *In particular, the matrix* $A_\tau^{\lambda,\varphi}(\eta)$ *is invertible for* φ, η, τ, λ *as before.*

Proof: For $\lambda\in\mathbb{C}\backslash(-\infty,0]$, $\xi\in\mathbb{R}^2$, $\tau\in\{-1,1\}$, we set

$$F(\tau,\lambda,\xi) := \tau/8 + \tau\cdot\lambda^{-2}\cdot\Big(\lambda\cdot|\xi|^2/2 + |\xi|^4 - (1/8)\cdot\lambda^2\cdot|\xi|\cdot(\lambda+|\xi|^2)^{-1/2}$$
$$- \lambda\cdot|\xi|^3\cdot(\lambda+|\xi|^2)^{-1/2} - |\xi|^5\cdot(\lambda+|\xi|^2)^{-1/2}\Big).$$

Because of (5.33), the function $F(\tau,\lambda,\xi)$ is well defined. It follows from (5.28), after a lengthy calculation (compare [33, p. 216]):

$$\det A_\tau^{\lambda,\varphi}(\xi) \;=\; F(\tau,\lambda,\xi) \qquad \text{for } \xi\in\mathbb{R}^2\backslash\{0\},\ \lambda\in\mathbb{C}\backslash(-\infty,0],\ \tau\in\{-1,1\}. \tag{5.39}$$

Equation (5.39) implies that

$$\det A_\tau^{\lambda,\varphi}(\xi) \quad=\quad \det A_\tau^{\lambda/|\lambda|,\varphi}(|\lambda|^{-1/2}\cdot\xi)$$

for $\xi\in\mathbb{R}^2\backslash\{0\}$, $\lambda\in\mathbb{C}\backslash(-\infty,0]$, $\varphi\in(0,\pi/2]$, $\tau\in\{-1,1\}$. Thus it suffices to construct a constant $C_{24}(\vartheta)$ such that (5.38) holds under the additional assumption $|\lambda|=1$. Therefore, assume that λ, φ, τ are given as in the lemma, and suppose in addition $|\lambda|=1$. By expanding the function $x^{1/2}$ into a power series, it may be deduced from (5.39):

$$\det A_\tau^{\lambda,\varphi}(\xi) = G(\lambda, |\xi|) \qquad \text{for } \xi \in \mathbb{R}^2,$$

with G defined by

$$G(\kappa, r) := \tau/8 - (\tau/8) \cdot \Big(\sum_{n=1}^{\infty} \binom{-1/2}{n} \cdot (\kappa/r^2)^n + \sum_{n=2}^{\infty} \binom{-1/2}{n} \cdot (\kappa/r^2)^{n-1} + \sum_{n=3}^{\infty} \binom{-1/2}{n} \cdot (\kappa/r^2)^{n-2} \Big)$$

for $r \in (1, \infty)$, $\kappa \in \mathbb{C}$ with $|\kappa| = 1$. Hence there is some constant $\mathfrak{C} > 1$ such that $|G(\kappa, r)| \le 1/16$ for $r \in [\mathfrak{C}, \infty)$ and for κ as before. This implies

$$|\det A_\tau^{\lambda,\varphi}(\xi)| \ \ge\ 1/16 \qquad \text{for } \xi \in \mathbb{R}^2 \text{ with } |\xi| \ge \mathfrak{C}. \tag{5.40}$$

As shown by McCracken [33, p. 216/217, Lemma 5.6], the expression $F(\tau, \kappa, \xi)$ is different from zero for any $\xi \in \mathbb{R}^2$, $\kappa \in \mathbb{C}\backslash(-\infty, 0]$ with $|\kappa| = 1$. Furthermore, $F(\tau, \kappa, \xi)$ continuously depends on $\kappa \in \mathbb{C}\backslash(-\infty, 0]$ and $\xi \in \mathbb{R}^2$, as may be seen from (5.33). Thus we conclude from (5.39), for $\xi \in \mathbb{R}^2\backslash\{0\}$:

$$\det A_\tau^{\lambda,\varphi}(\xi) \ge \inf\{ F(\tau, \kappa, \varrho) \ : \ \kappa \in \mathbb{C} \text{ with } |\arg \kappa| \le \vartheta,\ |\kappa| = 1;\ \varrho \in \mathbb{R}^2 \text{ with } |\varrho| \le \mathfrak{C} \} \ > \ 0. \tag{5.41}$$

Existence of a suitable constant $C_{24}(\vartheta)$ follows from (5.40) and (5.41).

Lemma 5.18. *Let* $\vartheta \in [0, \pi)$. *Then there exists some number* $C_{25}(\vartheta) > 0$ *such that for* $\lambda \in \mathbb{C}\backslash\{0\}$ *with* $|\arg\lambda| \le \vartheta$, $j, k \in \{1, 2, 3\}$, $\tau \in \{-1, 1\}$, $\xi \in \mathbb{R}^2\backslash\{0\}$, $a \in \mathbb{N}_0^2$ *with* $|a|_* \le 4$, $\kappa \in [0, 1]$, $\varphi \in (0, \pi/2]$, *the following inequalities hold true:*

$$|D^a(A_\tau^{\lambda,\varphi})_{jk}(\xi)|, \quad \Big|D^a\Big((A_\tau^{\lambda,\varphi})^{-1}\Big)_{jk}(\xi)\Big| \ \le\ C_{25}(\vartheta) \cdot |\xi|^{-|a|_*};$$

$$\Big|D^a\Big((A_\tau^{\lambda,\varphi} - A_\tau^{\infty,\varphi}) \cdot (A_\tau^{\infty,\varphi})^{-1}\Big)_{jk}(\xi)\Big|,$$
$$\Big|D^a\Big(A_\tau^{\infty,\varphi} \cdot \big((A_\tau^{\lambda,\varphi})^{-1} - (A_\tau^{\infty,\varphi})^{-1}\big)\Big)_{jk}(\xi)\Big| \ \le\ C_{25}(\vartheta) \cdot |\xi|^{-|a|_*+\kappa} \cdot |\lambda|^{-\kappa/2}.$$

Proof: The preceding inequalities, some of which were already stated in [9, (3.4), (3.5)] for $\varphi = 0$, are an obvious consequence of Lemma 5.15, 5.16 and 5.17.

At the end of this chapter, we shall present some results which are based on the multiplier theorem [45, p. 96, Theorem 3]. For the convenience of the reader, we repeat this theorem here, in a form which is suitable for our purposes:

Theorem 5.1. *Take* $p \in (1, \infty)$, $E \in (0, \infty)$. *Then there is a constant* $C_{26}(p, E) > 0$ *such that for* $\Phi \in L^p(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, $M \in C^\infty(\mathbb{R}^2\backslash\{0\})$ *with*

$$|D^a M(x)| \le |x|^{-|a|_*} \qquad \text{for } x \in \mathbb{R}^2\backslash\{0\},\ a \in \mathbb{N}_0^2 \text{ with } |a|_* \le 2,$$

the following inequality holds true:

$$\|(M \cdot \hat{\Phi})^\vee\|_p \ \le\ C_{26}(p, E) \cdot \|\Phi\|_p.$$

As a first application of this theorem, we note

Lemma 5.19. *Let* $p \in (1, \infty)$, $\vartheta \in [0, \pi)$. *Then there is a constant* $C_{27}(p, \vartheta) > 0$ *such that*

$$\|(M \cdot \hat{\Phi})^\vee\|_p \ \le\ C_{27}(p, \vartheta) \cdot \|\Phi\|_p$$

for $\lambda \in \mathbb{C}\backslash\{0\}$ *with* $|\arg\lambda| \le \vartheta$, $\varphi \in (0, \pi/2]$, $\Phi \in L^p(\mathbb{R}^2)^3 \cap L^2(\mathbb{R}^2)^3$, $\tau \in \{-1, 1\}$,

$$M \in \Big\{ (1 - \Psi_1) \cdot A_\tau^{\infty,\varphi} \cdot \big((A_\tau^{\lambda,\varphi})^{-1} - (A_\tau^{\infty,\varphi})^{-1}\big),\ (1 - \Psi_1) \cdot (A_\tau^{\lambda,\varphi} - A_\tau^{\infty,\varphi}) \cdot (A_\tau^{\infty,\varphi})^{-1},\ A_\tau^{\infty,\varphi} \cdot (A_\tau^{\lambda,\varphi})^{-1},\ A_\tau^{\lambda,\varphi} \cdot (A_\tau^{\infty,\varphi})^{-1} \Big\}.$$

(The function Ψ_1 *was introduced in Definition 5.3.)*

Proof: The lemma follows from Lemma 5.15, 5.18 and Theorem 5.1.

The next inequality is a slight generalization of a result from [9, p. 332], which we extend to the case $\varphi \ne \pi/2$.

Lemma 5.20. *Let* $p \in (1, \infty)$, $\vartheta \in [0, \pi)$. *Then there is a number* $C_{28}(p, \vartheta) > 0$ *with*

$$\|(\Psi_1 \cdot B \cdot \hat{\Phi})^\vee\|_p \ \le\ C_{28}(p, \vartheta) \cdot |\lambda|^{-1/2} \cdot \|\Phi\|_p$$

for $\lambda \in \mathbb{C}\backslash\{0\}$ *with* $|\arg\lambda| \le \vartheta$, $\tau \in \{-1, 1\}$, $\Phi \in L^p(\mathbb{R}^2)^3 \cap L^2(\mathbb{R}^2)^3$, $\varphi \in (0, \pi/2]$,

$$B \in \Big\{ A_\tau^{\infty,\varphi} \cdot \big((A_\tau^{\lambda,\varphi})^{-1} - (A_\tau^{\infty,\varphi})^{-1}\big),\ (A_\tau^{\lambda,\varphi} - A_\tau^{\infty,\varphi}) \cdot (A_\tau^{\infty,\varphi})^{-1} \Big\}.$$

Proof: Set $C_{28,1}(\vartheta) := 32 \cdot C_{25}(\vartheta) \cdot C_{14}$; $C_{28}(p, \vartheta) := 9 \cdot C_{26}\big(p, C_{28,1}(\vartheta)\big)$.

According to Lemma 5.1, we have $supp(\Psi_1) \subset \mathbb{B}_2(0, 2)$. This fact and Lemma 5.18 imply

$$|D^a(\Psi_1 \cdot B_{jk})(\xi)| \quad \le \quad C_{28,1}(\vartheta)\cdot|\lambda|^{-1/2}\cdot|\xi|^{-|a|_*}$$

for $\xi\in\mathbb{R}^2\setminus\{0\}$, $a\in\mathbb{N}_0^2$ with $|a|_*\le 2$, $j,\,k\in\{1,\,2,\,3\}$, and for $\lambda,\,\varphi,\,B$ as in the lemma. Hence the estimate we are looking for follows from Theorem 5.1.

The ensuing lemma is similar to an inequality in [9, p. 331 below], where the case $\varphi=\pi/2$ was treated.

Lemma 5.21. *Let* $p\in(1,\infty)$, $\vartheta\in[0,\pi)$. *Then there is a constant* $C_{29}(p,\vartheta)>0$ *such that*

$$\Big\|\Big((1-\Psi_1)\cdot B\cdot\overset{\wedge}{\Phi}\Big)^{\vee}\Big\|_p \quad \le \quad C_{29}(p,\vartheta)\cdot|\lambda|^{-1/2}\cdot\sum_{r=1}^{2}\|D_r\Phi\|_p$$

for $\lambda\in\mathbb{C}\setminus\{0\}$ *with* $|\arg\lambda|\le\vartheta$, $\varphi\in(0,\pi/2]$, $\tau\in\{-1,1\}$, $\Phi\in C^1(\mathbb{R}^2)^3\cap L^2(\mathbb{R}^2)^3\cap W^{1,p}(\mathbb{R}^2)^3$, $B\in\Big\{A_\tau^{\infty,\varphi}\cdot\Big(\big(A_\tau^{\lambda,\varphi}\big)^{-1}-\big(A_\tau^{\infty,\varphi}\big)^{-1}\Big),\ \ \big(A_\tau^{\lambda,\varphi}-A_\tau^{\infty,\varphi}\big)\cdot\big(A_\tau^{\infty,\varphi}\big)^{-1}\Big\}$.

Proof: We begin by defining

$$C_{29,1}(\vartheta):=9^3\cdot4^6\cdot C_{25}(\vartheta)\cdot C_{14};\qquad C_{29,2}(p,\vartheta):=C_{26}\big(p,\,C_{29,1}(\vartheta)\big),$$

$$C_{29}(p,\vartheta):=9\cdot C_{29,2}(p,\vartheta).$$

Let $\lambda,\,\varphi,\,\tau,\,\Phi,\,B$ be given as in the lemma, and take $j,\,k\in\{1,\,2,\,3\}$. For $v\in\mathbb{N}$, put $u^{(v)}:=\Psi_v\cdot\Phi$, with Ψ_v from Definition 5.3. Then we have $u^{(v)}\in C_0^1(\mathbb{R}^2)^3$ for $v\in\mathbb{N}$. Using the abbreviation $M:=(1-\Psi_1)\cdot B$, we define a function $F:\mathbb{R}^2\mapsto\mathbb{C}^2$ by setting for $r\in\{1,2\}$, $\xi\in\mathbb{R}^2$:

$$F_r(\xi):=0,\ \text{ if } |\xi|<1/2,\qquad F_r(\xi):=(-i)\cdot|\lambda|^{1/2}\cdot M_{jk}(\xi)\cdot\xi_r\cdot|\xi|^{-2}\ \text{ else.}$$

Since $(1-\Psi_1)\,|\,\mathbb{B}_2(0,1)\ =\ 0$, we have $F_r\in C^\infty(\mathbb{R}^2)$ for $r\in\{1,2\}$. Recalling the fact that $u^{(v)}\in C_0^1(\mathbb{R}^2)^3$, we conclude for $\xi\in\mathbb{R}^2$, $v\in\mathbb{N}$:

$$M_{jk}(\xi)\cdot\big(u_k^{(v)}\big)^{\wedge}(\xi)\quad=\quad|\lambda|^{-1/2}\cdot\sum_{r=1}^{2}F_r(\xi)\cdot\big(D_ru_k^{(v)}\big)^{\wedge}(\xi).\tag{5.42}$$

Applying Lemma 5.18, 5.3 and 5.1, we compute: $|D^aF_r(\xi)|\le C_{29,1}(\vartheta)\cdot|\xi|^{-|a|_*}$ for $\xi\in\mathbb{R}^2\setminus\{0\}$, $\varphi\in(0,\pi/2]$, $r\in\{1,2\}$, $a\in\mathbb{N}_0^2$ with $|a|_*\le2$. Now we may recur to Theorem 5.1, to obtain for $v\in\mathbb{N}$ $r\in\{1,2\}$:

$$\Big\|\Big(F_r\cdot\big(D_ru_k^{(v)}\big)^{\wedge}\Big)^{\vee}\Big\|_p\quad\le\quad C_{29,2}(p,\vartheta)\cdot\|D_ru_k^{(v)}\|_p.$$

This implies by (5.42):

$$\Big\|\Big(M_{jk}\cdot\big(u_k^{(v)}\big)^{\wedge}\Big)^{\vee}\Big\|_p\quad\le\quad C_{29,2}(p,\vartheta)\cdot|\lambda|^{-1/2}\cdot\sum_{r=1}^{2}\|D_ru_k^{(v)}\|_p.$$

On the other hand, we deduce from the properties of the function Ψ_v (see Lemma 5.1): $\|u^{(v)}-\Phi\|_{1,p}\to0$ if $v\to\infty$. Moreover, it follows from Lemma 5.18 and 5.1:

$$|D^aM_{jk}(\xi)|\quad\le\quad C_{29,1}(\vartheta)\cdot|\xi|^{-|a|_*}\qquad\text{for } a\in\mathbb{N}_0^2 \text{ with } |a|_*\le2,\ \xi\in\mathbb{R}^2\setminus\{0\},$$

Thus we may finish our proof by concluding from Theorem 5.1:

$$\Big\|\Big(M_{jk}\cdot\big(\Phi_k-u_k^{(v)}\big)^{\wedge}\Big)^{\vee}\Big\|_p\to0\quad(v\to\infty).$$

Finally, let us point put a result similar to that in [9, (3.8)].

Lemma 5.22. *Let* $\varphi\in(0,\pi/2]$, $\lambda\in\mathbb{C}\setminus(-\infty,0]$, $p\in(1,\infty)$, $\Phi\in L^p(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$, $j,\,k\in\{1,\,2,\,3\}$, $\tau\in\{-1,1\}$,

$M\in\Big\{(1-\Psi_1)\cdot A_\tau^{\infty,\varphi}\cdot\Big(\big(A_\tau^{\lambda,\varphi}\big)^{-1}-\big(A_\tau^{\infty,\varphi}\big)^{-1}\Big),$

$\quad(1-\Psi_1)\cdot\big(A_\tau^{\lambda,\varphi}-A_\tau^{\infty,\varphi}\big)\cdot\big(A_\tau^{\infty,\varphi}\big)^{-1}\Big\}.$

Then it follows that

$$\big(M_{jk}\cdot\overset{\wedge}{\Phi}\big)^{\vee}\quad=\quad L^p(\mathbb{R}^2)-\lim_{v\to\infty}\Big(M_{jk}\cdot\big(\Psi_v\cdot\Phi\big)^{\wedge}\Big)^{\vee}.\tag{5.43}$$

Moreover, it holds for $v,\,t\in\mathbb{N}$, *and for almost every* $\xi\in\mathbb{R}^2$:

$$\Big(M_{jk}\cdot\Psi_t\cdot\big(\Psi_v\cdot\Phi\big)^{\wedge}\Big)^{\vee}(\xi)\tag{5.44}$$

$$=\quad(2\cdot\pi)^{-1}\cdot\int_{\mathbb{R}^2}\Psi_v(\eta)\cdot\Phi(\eta)\cdot\big(M_{jk}\cdot\Psi_t\big)^{\wedge}(\eta-\xi)\ d\eta.$$

Furthermore, let $A\subset\mathbb{R}^2$ *be a measurable set,* $g:A\mapsto\mathbb{R}^2$ *a measurable function, and assume*

$$\Big|\Big(M_{jk}\cdot\Psi_t\cdot\big(\Psi_v\cdot\Phi\big)^{\wedge}\Big)^{\vee}(\xi)\Big|\quad\le\quad|g(\xi)|\qquad\text{for } \xi\in A,\ v,\,t\in\mathbb{N}.\tag{5.45}$$

Then it follows

$$\big\|\big(M_{jk}\cdot\overset{\wedge}{\Phi}\big)^{\vee}\,|\,A\big\|_p\quad\le\quad\|g\|_p.\tag{5.46}$$

Proof: Lebesgue's theorem on dominated convergence yields: $\|(1-\Psi_v)\cdot\Phi\|_p\to0$ for $v\to\infty$. This fact and Lemma 5.19 imply (5.43).

This leaves us to establish (5.44) and (5.46). In order to prove (5.44), we observe that $M_{jk}\cdot\Psi_t$ is a bounded function with compact support ($t\in\mathbb{N}$). Moreover, $\Psi_v\cdot\Phi$ belongs to the space $L^1(\mathbb{R}^2)$ and has compact support too ($v\in\mathbb{N}$). Hence, equation (5.44) is a simple consequence of Fubini's theorem.

Turning to the proof of (5.46), we note that $\Psi_v \cdot \Phi \in L^2(\mathbb{R}^2)$ for $v \in \mathbb{N}$. Since the function M_{jk} is bounded (Lemma 5.18), it follows by another application of Lebesgue's theorem on dominated convergence:

$$\| M_{jk} \cdot (1 - \Psi_t) \cdot (\Psi_v \cdot \Phi)^\wedge \|_2 \to 0 \quad (t \to \infty) \qquad (v \in \mathbb{N}).$$

Now we apply Plancherel's theorem to obtain

$$\left\| \left(M_{jk} \cdot (1 - \Psi_t) \cdot (\Psi_v \cdot \Phi)^\wedge \right)^\vee \right\|_2 \to 0 \quad (t \to \infty) \qquad (v \in \mathbb{N}).$$

By a result from measure theory (see [41, p. 67, Theorem 3.12], for example), it follows that for any $v \in \mathbb{N}$, there is a function $\sigma_v : \mathbb{N} \mapsto \mathbb{N}$ which is strictly monotone increasing and satisfies the following relation for $v \in \mathbb{N}$ and for almost every $\xi \in A$:

$$\left(M_{jk} \cdot (\Psi_v \cdot \Phi)^\wedge \right)^\vee (\xi) \;=\; \lim_{t \to \infty} \left(M_{jk} \cdot \Psi_{\sigma_v(t)} \cdot (\Psi_v \cdot \Phi)^\wedge \right)^\vee (\xi). \tag{5.47}$$

From (5.47) and (5.45), we may infer

$$\left| \left(M_{jk} \cdot (\Psi_v \cdot \Phi)^\wedge \right)^\vee (\xi) \right| \;\leq\; |g(\xi)| \qquad \text{for } v \in \mathbb{N} \text{ and a.e. } \xi \in A.$$

Now (5.46) follows by (5.43) and the preceding inequality.

Chapter 6

Fredholm Properties of Some Layer Potentials

In this chapter, we shall present a device which, in various modifications, turns up frequently when integral operators are considered. In fact, in order to study an operator $H : L^p(B)^\sigma \mapsto L^p(B)^\sigma$, with $p \in (1, \infty)$, $\sigma \in \mathbb{N}$, and B a suitable set, we shall split H into a sum of four mappings, with domain $L^p(B_1)^\sigma$ or $L^p(B_2)^\sigma$, and with range in $L^p(B_1)^\sigma$ or $L^p(B_2)^\sigma$, where B_1, B_2 are sets with $B = B_1 \cup B_2$, $B_1 \cap B_2 = \emptyset$; see Lemma 6.6. Then we shall link the Fredholm properties of H with those of $H_1, \cdots, H_4$. Typically, one of the operators H_1 to H_4 will be a singular integral operator, with the singularity of its kernel arising at the vertex of $\mathbb{K}(\varphi)$, whereas another one of these operators will be singular too, but with a kernel having a singularity at the infinite end of $\mathbb{K}(\varphi)$. The other two operators will be compact.

We shall address another topic in the present chapter, namely, we shall check whether certain operators continuously depend on the vertex angle φ of $\mathbb{K}(\varphi)$, and on the resolvent parameter λ arising in (1.3). These results will be exploited for homotopy arguments; see Lemma 6.17, for example.

Our first aim consists in showing that the integral operators presented in (1.5), (1.8) and (1.9), respectively, are well defined when $\mathbb{K}(\varphi)$ is inserted for B in these definitions. In addition, we are going to introduce some further integral operators which will be needed later on. We begin our discussion by a number of technical lemmas:

Lemma 6.1. *Let* $\varphi \in (0, \pi/2]$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\epsilon \in [0, \infty)$. *Then there are numbers* $\mathfrak{C}_1, \mathfrak{C}_2 > 0$ *such that*

$$\begin{aligned} &\left| \sum_{k=1}^{3} K_k\left(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r) \right) \cdot \left(n_k^{(\varphi,\epsilon)} \circ \gamma^{(\varphi,\epsilon)} \right)(\eta) \right| \\ &\leq \mathfrak{C}_1 \cdot \min\Big\{ \left(|\xi - \eta| + |r| \right)^{-2}, \\ &\qquad (|\xi| + |\eta|)^{-1} \cdot |\xi - \eta|^{-1} \;+\; |r| \cdot \left(|\xi - \eta| + |r| \right)^{-3} \Big\}, \end{aligned} \tag{6.1}$$

$$\Big| \sum_{k=1}^{3} \widetilde{\mathcal{D}}^{\lambda}_{jkl}\big(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)\big) \cdot \big(n_k^{(\varphi,\epsilon)} \circ \gamma^{(\varphi,\epsilon)}\big)(\eta) \Big| \tag{6.2}$$

$$\leq \mathfrak{C}_2 \cdot \min\Big\{ \big(|\xi-\eta| + |r|\big)^{-2}, \; |\xi-\eta|^{-1} + (|\xi|+|\eta|)^{-1} \cdot |\xi-\eta|^{-1} + |r| \cdot \big(|\xi-\eta| + |r|\big)^{-3} \Big\}$$

for $r \in \mathbb{R}$, $j, l \in \{1,2,3\}$, $K \in \big\{ (\overline{\mathcal{D}}_k)_{1 \le k \le 3}, (\mathcal{D}_{jkl})_{1 \le k \le 3} \big\}$, $\xi, \eta \in \mathbb{R}^2 \backslash \{0\}$ *with* $\xi \neq \eta$. *(For our notations in the case* $\epsilon = 0$, *see (3.10).)*

Let φ, λ *be given as before, and take* $\delta \in (0,\infty)$. *Then there exists some number* $\mathfrak{C}_3 > 0$ *such that*

$$\Big| \sum_{k=1}^{3} K_k\big(\gamma^{(\varphi,\delta)}(\xi) - \gamma^{(\varphi,\delta)}(\eta) + (0,0,r)\big) \cdot \big(n_k^{(\varphi,\delta)} \circ \gamma^{(\varphi,\delta)}\big)(\eta) \Big| \tag{6.3}$$

$$\leq \mathfrak{C}_3 \cdot \Big(\min\{ |\xi-\eta|^{-2}, |\xi-\eta|^{-1} \} + |r| \cdot \big(|\xi-\eta| + |r|\big)^{-3} \Big)$$

for $j, l \in \{1,2,3\}$, $K \in \big\{ (\overline{\mathcal{D}}_k)_{1 \le k \le 3}, (\mathcal{D}_{jkl})_{1 \le k \le 3}, (\widetilde{\mathcal{D}}^{\lambda}_{jkl})_{1 \le k \le 3} \big\}$, $\xi, \eta \in \mathbb{R}^2$ *with* $\xi \neq \eta$, $r \in \mathbb{R}$.

Proof: Existence of a constant $\mathfrak{C}_1$ satisfying (6.1) follows from Corollary 3.1, Lemma 3.4 and (5.9). Concerning inequality (6.2), we point out a consequence of (5.3), namely:

$$\big| (D_j \overline{E}^{\lambda}_{kl} + D_k \overline{E}^{\lambda}_{jl})(z) \big| \leq 2 \cdot C_{15}(|\arg\lambda|) \cdot |\lambda|^{1/2} \cdot |z|^{-1} \quad \text{for } z \in \mathbb{R}^3 \backslash \{0\}.$$

Combining this inequality with (5.10), (6.1) and (3.18), we find the left-hand of (6.2) to be bounded by

$$\big(\mathfrak{C}_1 + 6 \cdot C_{15}(|\arg\lambda|)\big) \cdot \Big((|\xi|+|\eta|)^{-1} \cdot |\xi-\eta|^{-1} + |r| \cdot \big(|\xi-\eta| + |r|\big)^{-3} + |\xi-\eta|^{-1} \Big),$$

for ξ, η, r, j, l as in (6.2). On the other hand, from (5.15) and (3.19) we obtain another upper bound of the left-hand side of (6.2), namely

$$C_{17}(|\arg\lambda|) \cdot 16 \cdot \sin^{-2}(\varphi) \cdot \big(|\xi-\eta| + |r|\big)^{-2}.$$

This proves (6.2). Inequality (6.3) may be established by an analogous reasoning, the main difference being that we have to refer to Lemma 3.3 instead of Corollary 3.1.

Lemma 6.2. *Let* $p \in (1,\infty)$, $\varphi \in (0, \pi/2]$, $\epsilon \in [0,\infty)$. *Then there exists* $\mathfrak{C} > 0$ *so that for* $j, l \in \{1,2,3\}$, $r \in \mathbb{R}$, $\Phi \in L^p(\mathbb{R}^2)$, $K \in \big\{ (\overline{\mathcal{D}}_k)_{1 \le k \le 3}, (\mathcal{D}_{jkl})_{1 \le k \le 3} \big\}$, *the following inequality holds true (recall (3.10)):*

$$\Big(\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} \Big| \sum_{k=1}^{3} K_k\big(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)\big) \cdot \big(n_k^{(\varphi,\epsilon)} \circ \gamma^{(\varphi,\epsilon)}\big)(\eta) \cdot \Phi(\eta) \cdot J^{(\varphi,\epsilon)}(\eta) \Big| \, d\eta \Big)^p d\xi \Big)^{1/p},$$

$$\Big(\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} \Big| \sum_{k=1}^{3} K_k\big(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)\big) \cdot \big(n_k^{(\varphi,\epsilon)} \circ \gamma^{(\varphi,\epsilon)}\big)(\xi) \cdot \Phi(\eta) \cdot J^{(\varphi,\epsilon)}(\eta) \Big| \, d\eta \Big)^p d\xi \Big)^{1/p}$$

$$\leq \mathfrak{C} \cdot \|\Phi\|_p.$$

Proof: The lemma is a consequence of Lemma 6.1, Theorem 4.1 and Lemma 4.10.

Lemma 6.3. *Let* $A, B \subset \mathbb{R}^2$ *be measurable and bounded sets,* $\sigma, \sigma' \in \mathbb{N}$ *and* $K : A \times B \mapsto \mathbb{C}^{\sigma \times \sigma'}$ *a measurable function. Furthermore, assume there is some* $\mathfrak{C} > 0$ *with*

$$|K(\xi,\eta)| \leq \mathfrak{C} \cdot |\xi-\eta|^{-1} \quad \textit{for } \xi \in A, \; \eta \in B \textit{ with } \xi \neq \eta.$$

Suppose $R > 0$ *is so large that* $A \cup B \subset \mathbb{B}_2(0,R)$, *and let* $p \in (1,\infty)$. *Then it follows for* $\Phi \in L^p(\mathbb{R}^2)^{\sigma'}$:

$$\Big(\int_A \Big(\int_B |K(\xi,\eta) \cdot \Phi(\eta)| \, d\eta \Big)^p d\xi \Big)^{1/p} \leq \mathfrak{C} \cdot 4 \cdot \pi \cdot R \cdot \|\Phi\|_p. \tag{6.4}$$

In particular, the operator

$$T : L^p(B)^{\sigma'} \mapsto L^p(A)^{\sigma}, \qquad T(\Phi) := K \otimes \Phi \quad \textit{for } \Phi \in L^p(B)^{\sigma'},$$

is well defined and continuous. Furthermore, T *is compact.*

Proof: We have

$$|K(\xi,\eta)| \leq \mathfrak{C} \cdot \chi_{(0,\,2\cdot R)}(|\xi-\eta|) \cdot |\xi-\eta|^{-1} \quad \text{for } \xi \in A, \; \eta \in B.$$

Hence (6.4) follows from Young's inequality (Lemma 4.9). This leaves us to show compactness of T. To this end, we define

$$K_\epsilon(\xi,\eta) := \chi_{(\epsilon,\infty)}(|\xi-\eta|) \cdot K(\xi,\eta) \quad \text{for } \epsilon \in (0,\infty), \; \xi \in A, \; \eta \in B.$$

For any $\epsilon \in (0,\infty)$, the function K_ϵ is bounded, hence

$$\int_A \Big(\int_B |K_\epsilon(\xi,\eta)|^{1/(1-1/p)} \, d\eta \Big)^{p-1} d\xi < \infty.$$

According to [28, p. 271/272; p. 275, Theorem 11.6], this means that the mapping

$$T_\epsilon : L^p(B)^{\sigma'} \mapsto L^p(A)^{\sigma}, \quad T_\epsilon(\Phi)(\xi) := \int_B K_\epsilon(\xi,\eta) \cdot \Phi(\eta) \, d\eta \quad (\xi \in A, \; \Phi \in L^p(B)^{\sigma'}),$$

is a Hille-Tamarkin operator, and thus compact, for any $\epsilon \in (0,\infty)$. On the other hand, it holds for $\epsilon \in (0,\infty)$, $\xi \in A$, $\eta \in B$ with $\xi \neq \eta$:

$$|K(\xi,\eta) - K_\epsilon(\xi,\eta)| \le \mathfrak{C} \cdot \chi_{(0,\epsilon)}(|\xi-\eta|) \cdot |\xi-\eta|^{-1}.$$

Once more referring to Young's inequality (Lemma 4.9), we may conclude for $\Phi \in L^p(B)^{\sigma'}$, $\epsilon \in (0,\infty)$:

$$\|T(\Phi) - T_\epsilon(\Phi)\|_p \le \mathfrak{C}\cdot 2\cdot\pi\cdot\epsilon\cdot\|\Phi\|_p.$$

Thus, for $\epsilon \downarrow 0$, the difference $T - T_\epsilon$ converges to zero with respect to the uniform operator topology. Hence T is compact.

Lemma 6.4. *Let $A, B \subset \mathbb{R}^2$ be measurable sets, with either A or B bounded. Take $\sigma, \sigma' \in \mathbb{N}$, and assume that $K : A\times B \mapsto \mathbb{C}^{\sigma\times\sigma'}$ is a measurable function. Suppose there is some $\mathfrak{C}_1 > 0$ with*

$$|K(\xi,\eta)| \le \mathfrak{C}_1 \cdot \min\{|\xi-\eta|^{-1}, |\xi-\eta|^{-2}\} \quad \text{for } \xi\in A, \eta\in B \text{ with } \xi\neq\eta.$$

Finally, let $p\in(1,\infty)$. Then there is another number $\mathfrak{C}_2 > 0$ with

$$\left(\int_A \left(\int_B |K(\xi,\eta)\cdot\Phi(\eta)|\, d\eta\right)^p d\xi\right)^{1/p} \le \mathfrak{C}_2\cdot\|\Phi\|_p \quad \text{for } \Phi\in L^p(B)^{\sigma'}. \tag{6.5}$$

This means in particular that the operator

$$T : L^p(B)^{\sigma'} \mapsto L^p(A)^\sigma, \qquad T(\Phi) := K\otimes\Phi \quad \text{for } \Phi\in L^p(B)^{\sigma'},$$

is well defined and continuous. In addition, T is compact.

Proof: We consider the case that B is bounded. Otherwise, if the set A is bounded, we may argue in a similar way.

Let $S > 0$ be so large that $B \subset \mathbb{B}_2(0,S)$. According to Lemma 6.3, we have

$$\left(\int_{\mathbb{B}_2(0,2\cdot S)\cap A} \left(\int_B |K(\xi,\eta)\cdot\Phi(\eta)|\, d\eta\right)^p d\xi\right)^{1/p} \le \mathfrak{C}_1\cdot 8\cdot\pi\cdot S\cdot\|\Phi\|_p. \tag{6.6}$$

For $\xi \in A\backslash\mathbb{B}_2(0,2\cdot S)$, $\eta\in B$, we have $|\xi-\eta| \ge |\xi|/2$. Thus it holds for $R\in[2\cdot S,\infty)$, $\Phi\in L^p(B)^{\sigma'}$:

$$\left(\int_{A\backslash\mathbb{B}_2(0,R)} \left(\int_B |K(\xi,\eta)\cdot\Phi(\eta)|\, d\eta\right)^p d\xi\right)^{1/p} \tag{6.7}$$

$$\le 4\cdot\mathfrak{C}_1\cdot\left(\int_{A\backslash\mathbb{B}_2(0,R)} |\xi|^{-2\cdot p}\cdot\left(\int_B |\Phi(\eta)|\, d\eta\right)^p d\xi\right)^{1/p} \le \mathfrak{C}_{2,1}\cdot R^{-2+2/p}\cdot\|\Phi\|_p,$$

with $\mathfrak{C}_{2,1} := 4\cdot\mathfrak{C}_1\cdot((p-1)^{-1}\cdot\pi)^{1/p}\cdot\int_B d\eta$. Setting $R = 2\cdot S$ in inequality (6.7), and

recalling (6.6), we obtain the estimate in (6.5), with an obvious choice of $\mathfrak{C}_2$.

We still have to prove compactness of T. For this purpose, we define for $R\in[2\cdot S,\infty)$:

$$T_R : L^p(B)^{\sigma'} \mapsto L^p(A)^\sigma, \qquad T_R(\Phi) := \chi_{\mathbb{B}_2(0,R)\cap A}\cdot T(\Phi) \quad \text{for } \Phi\in L^p(B)^{\sigma'}.$$

Referring to Lemma 6.3, we see that T_R is compact for any $R\in[2\cdot S,\infty)$. Because of (6.7), we have:

$$\|(T_R - T)(\Phi)\|_p \le \mathfrak{C}_{2,1}\cdot R^{-2+2\cdot p}\cdot\|\Phi\|_p \quad \text{for } R\in[2\cdot S,\infty),\ \Phi\in L^p(B)^{\sigma'}.$$

Hence, for $R\to\infty$, the operator $T_R - T$ converges to 0 with respect to the uniform operator topology. This implies that T is compact.

Lemma 6.5. *Take $p\in(1,\infty)$, $\lambda\in\mathbb{C}\backslash(-\infty,0]$, $\varphi\in(0,\pi/2]$, $\epsilon\in[0,\infty)$, $j,l\in\{1,2,3\}$. For $\xi,\eta\in\mathbb{R}^2\backslash\{0\}$ with $\xi\neq\eta$, $r\in\mathbb{R}$, we write for shortness (recall (3.10)):*

$$K^{(1)}(r)(\xi,\eta)$$

$$:= \sum_{k=1}^{3} \widetilde{\mathcal{D}}^\lambda_{jkl}\big(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)\big)\cdot\big(n_k^{(\varphi,\epsilon)}\circ\gamma^{(\varphi,\epsilon)}\big)(\eta)\cdot J^{(\varphi,\epsilon)}(\eta),$$

$$K^{(2)}(r)(\xi,\eta)$$

$$:= \sum_{k=1}^{3} \widetilde{\mathcal{D}}^\lambda_{jkl}\big(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)\big)\cdot\big(n_k^{(\varphi,\epsilon)}\circ\gamma^{(\varphi,\epsilon)}\big)(\xi)\cdot J^{(\varphi,\epsilon)}(\eta).$$

Then the function $\big(K^{(v)}(r)\big)\otimes\Phi$ is well defined for $v\in\{1,2\}$, $\Phi\in L^p(\mathbb{R}^2)$, $r\in\mathbb{R}$, and there is some $\mathfrak{C}>0$ with

$$\left\|\big(K^{(v)}(r)\big)\otimes\Phi\right\|_p \le \mathfrak{C}\cdot\|\Phi\|_p \quad \text{for } \Phi\in L^p(\mathbb{R}^2),\ r\in\mathbb{R},\ v\in\{1,2\}.$$

Proof: In the case $\epsilon = 0$, we refer to Lemma 5.12. Now assume $\epsilon > 0$. For $\eta\in\mathbb{R}^2\backslash\mathbb{B}_2(0,\epsilon)$, we have $\gamma^{(\varphi,\epsilon)}(\eta) = g^{(\varphi)}(\eta)$, $n^{(\varphi,\epsilon)} = n^{(\varphi)}$; see (3.6), (3.7). Hence, if we define for $v\in\{1,2\}$, $r\in\mathbb{R}$, $\xi,\eta\in\mathbb{R}^2$ with $\xi\neq\eta$:

$$K^{(v,1)}(r)(\xi,\eta) := \chi_{(\epsilon,\infty)}(|\xi|)\cdot\chi_{(\epsilon,\infty)}(|\eta|)\cdot K^{(v)}(r)(\xi,\eta),$$

$$K^{(v,2)}(r)(\xi,\eta) := \chi_{(0,\epsilon)}(|\xi|)\cdot\chi_{(\epsilon,\infty)}(|\eta|)\cdot K^{(v)}(r)(\xi,\eta),$$

$$K^{(v,3)}(r)(\xi,\eta) := \chi_{(\epsilon,\infty)}(|\xi|)\cdot\chi_{(0,\epsilon)}(|\eta|)\cdot K^{(v)}(r)(\xi,\eta),$$

$$K^{(v,4)}(r)(\xi,\eta) := \chi_{(0,\epsilon)}(|\xi|)\cdot\chi_{(0,\epsilon)}(|\eta|)\cdot K^{(v)}(r)(\xi,\eta),$$

then we may conclude from Lemma 5.12: For $\Phi\in L^p(\mathbb{R}^2)$, $v\in\{1,2\}$, $r\in\mathbb{R}$, the function $\big(K^{(v,1)}(r)\big) \otimes \Phi$ is well defined, and there is a number $\mathfrak{C}_1 > 0$ with

$$\Big\| \big(K^{(v,1)}(r)\big) \otimes \Phi \Big\|_p \quad \leq \quad \mathfrak{C}_1 \cdot \|\Phi\|_p \qquad \text{for } \Phi \in L^p(\mathbb{R}^2),\ v \in \{1, 2\},\ r \in \mathbb{R}.$$

According to (6.3), we may find $\mathfrak{C}_2 > 0$ such that for $r \in \mathbb{R}$, $v \in \{1, 2\}$, $\xi, \eta \in \mathbb{R}^2$ with $|\xi| < \epsilon \leq |\eta|$ or $|\xi| \geq \epsilon > |\eta|$, the following inequality holds true:

$$|K^{(v,2)}(r)(\xi, \eta)|, \quad |K^{(v,3)}(r)(\xi, \eta)|$$
$$\leq \mathfrak{C}_2 \cdot \Big(\min\{ |\xi - \eta|^{-2},\ |\xi - \eta|^{-1} \} \ + \ |r| \cdot (|\xi - \eta| + |r|)^{-3} \Big).$$

This implies by Lemma 6.4 and 4.10 that for $r \in \mathbb{R}$, $\Phi \in L^p(\mathbb{R}^2)$, $v \in \{1, 2\}$, the functions $\big(K^{(v,2)}(r)\big) \otimes \Phi$, $\big(K^{(v,3)}(r)\big) \otimes \Phi$ are well defined. In addition, it follows there is some $\mathfrak{C}_3 > 0$ with

$$\Big\| \big(K^{(v,2)}(r)\big) \otimes \Phi \Big\|_p, \quad \Big\| \big(K^{(v,3)}(r)\big) \otimes \Phi \Big\|_p \quad \leq \quad \mathfrak{C}_3 \cdot \|\Phi\|_p$$

for r, Φ, v as before. Due to (6.3), Lemma 4.10 and 6.3, a corresponding result is true for $K^{(v,4)}$. Since $K^{(v)}(r) = \sum_{a=1}^{4} K^{(v,a)}$ for $r \in \mathbb{R}$, $v \in \{1, 2\}$, the lemma is proved.

The operators introduced in the next lemma are in some sense "diagonal": Either the functions belonging to their domain are defined near the vertex of the cone $\mathbb{K}(\varphi)$, and the functions in their range live near the infinite end of $\mathbb{K}(\varphi)$, or the opposite situation is true. These operators turn out to be compact.

Lemma 6.6 *Let* $p \in (1, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\varphi \in (0, \pi/2]$, $j, l \in \{1, 2, 3\}$, $R \in (0, \infty)$, $R_1 \in (R, \infty]$, $R_2 \in [0, R)$, $\epsilon \in [0, \infty)$. *Assume that*

$$K \in \{ (\widetilde{\mathcal{D}}^{\lambda}_{jkl})_{1 \leq k \leq 3},\ (\mathcal{D}_{jkl})_{1 \leq k \leq 3},\ (\overline{\mathcal{D}}_k)_{1 \leq k \leq 3} \}.$$

Define the operators

$$T^{(1)},\ T^{(2)} \ :\ L^p\Big(\mathbb{B}_2(0, R_1) \backslash \mathbb{B}_2(0, R)\Big) \mapsto L^p\Big(\mathbb{B}_2(0, R) \backslash \mathbb{B}_2(0, R_2)\Big),$$
$$T^{(3)},\ T^{(4)} \ :\ L^p\Big(\mathbb{B}_2(0, R) \backslash \mathbb{B}_2(0, R_2)\Big) \mapsto L^p\Big(\mathbb{B}_2(0, R_1) \backslash \mathbb{B}_2(0, R)\Big)$$

in the following way (recall 3.10)):

$$T^{(1)}(\Phi)(\xi) := \int_{\mathbb{B}_2(0,R_1) \backslash \mathbb{B}_2(0,R)} \sum_{k=1}^{3} (n_k^{(\varphi,\epsilon)} \circ \gamma^{(\varphi,\epsilon)})(\eta) \cdot K_k\big(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta)\big) \cdot \Phi(\eta) \cdot J^{(\varphi,\epsilon)}(\eta)\ d\eta,$$

$$T^{(3)}(\Phi)(\xi) := \int_{\mathbb{B}_2(0,R_1) \backslash \mathbb{B}_2(0,R)} \sum_{k=1}^{3} (n_k^{(\varphi,\epsilon)} \circ \gamma^{(\varphi,\epsilon)})(\xi) \cdot K_k\big(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta)\big) \cdot \Phi(\eta) \cdot J^{(\varphi,\epsilon)}(\eta)\ d\eta$$

for $\Phi \in L^p\Big(\mathbb{B}_2(0, R_1) \backslash \mathbb{B}_2(0, R)\Big)$, $\xi \in \mathbb{B}_2(0, R) \backslash \mathbb{B}_2(0, R_2)$;

$$T^{(2)}(\Phi)(\xi) := \int_{\mathbb{B}_2(0,R) \backslash \mathbb{B}_2(0,R_2)} \sum_{k=1}^{3} (n_k^{(\varphi,\epsilon)} \circ \gamma^{(\varphi,\epsilon)})(\eta) \cdot K_k\big(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta)\big) \cdot \Phi(\eta) \cdot J^{(\varphi,\epsilon)}(\eta)\ d\eta,$$

$$T^{(4)}(\Phi)(\xi) := \int_{\mathbb{B}_2(0,R) \backslash \mathbb{B}_2(0,R_2)} \sum_{k=1}^{3} (n_k^{(\varphi,\epsilon)} \circ \gamma^{(\varphi,\epsilon)})(\xi) \cdot K_k\big(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta)\big) \cdot \Phi(\eta) \cdot J^{(\varphi,\epsilon)}(\eta)\ d\eta$$

for $\Phi \in L^p\Big(\mathbb{B}_2(0, R) \backslash \mathbb{B}_2(0, R_2)\Big)$, $\xi \in \mathbb{B}_2(0, R_1) \backslash \mathbb{B}_2(0, R)$.

Then all these operators are compact.

Proof: The lemma follows by a straightforward application of (6.1), (6.2) and Lemma 6.4.

Due to Lemma 6.2, 6.5, and because of (3.1), (3.8), the definitions in (1.5), (1.8) and (1.9) make sense in the case $B = \mathbb{K}(\varphi)$ and $B = \mathbb{L}(\varphi, \epsilon)$. In the following, we shall introduce some further operators, with their definition also based on Lemma 6.2 and 6.5.

Definition 6.1. *Take* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\tau \in \{-1, 1\}$, $R \in (0, \infty)$, $\Omega \in \{ \mathbb{K}(\varphi),\ \mathbb{L}(\varphi, \epsilon) \}$. *Let* $n^{(\Omega)}$ *denote the outward unit normal to* Ω.

Then define the operators $\Gamma^*(\tau, p, \lambda, \Omega)$, $\Lambda^*(\tau, p, \Omega)\ :\ L^p(\partial\Omega)^3 \mapsto L^p(\partial\Omega)^3$ *in the following way: For* $f \in L^p(\partial\Omega)^3$, $x \in \partial\Omega$, *put*

$$\Gamma^*(\tau, p, \lambda, \Omega)(f)(x)$$
$$:= (\tau/2) \cdot f(x) - \Big(\int_{\partial\Omega} \sum_{j,k=1}^{3} \widetilde{\mathcal{D}}^{\lambda}_{jkl}(x - y) \cdot n_k^{(\Omega)}(x) \cdot f_j(y)\ d\Omega(y) \Big)_{1 \leq l \leq 3},$$

$$\Lambda^*(\tau, p, \Omega)(f)(x)$$
$$:= (\tau/2) \cdot f(x) - \Big(\int_{\partial\Omega} \sum_{j,k=1}^{3} \mathcal{D}_{jkl}(x - y) \cdot n_k^{(\Omega)}(x) \cdot f_j(y)\ d\Omega(y) \Big)_{1 \leq l \leq 3}.$$

In addition, we introduce the operator $\Pi^*(\tau, p, \Omega)\ :\ L^p(\partial\Omega) \mapsto L^p(\partial\Omega)$ *by setting for* $h \in L^p(\partial\Omega)$, $x \in \partial\Omega$:

$$\Pi^*(\tau, p, \Omega)(h)(x) := (\tau/2) \cdot h(x) - \int_{\partial\Omega} \sum_{k=1}^{3} \overline{\mathcal{D}}_k(x - y) \cdot n_k^{(\Omega)}(x) \cdot h(y)\ d\Omega(y).$$

Furthermore, we shall consider the operator

$$\Gamma^{(inf)}(\tau, p, \lambda, \varphi, R)\ :\ L^p\Big(g^{(\varphi)}(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R))\Big)^3 \mapsto L^p\Big(g^{(\varphi)}(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R))\Big)^3,$$

with

$$\Gamma^{(inf)}(\tau, p, \lambda, \varphi, R)(f)(x) \quad := \quad (\tau/2) \cdot f(x)$$
$$+ \Big(\int_{g^{(\varphi)}(\mathbb{R}^2\backslash\mathbb{B}_2(0,R))} \sum_{j,k=1}^{3} \widetilde{\mathcal{D}}^{\lambda}_{jkl}(x-y) \cdot n_k^{(\varphi)}(y) \cdot f_j(y) \; d\,\mathbb{K}(\varphi)(y) \Big)_{1 \le l \le 3}$$

for $f \in L^p\big(g^{(\varphi)}(\mathbb{R}^2\backslash\mathbb{B}_2(0,R))\big)^3$, $x \in g^{(\varphi)}(\mathbb{R}^2\backslash\mathbb{B}_2(0,R))$. *The superscription "inf" indicates that the preceding operator acts on functions defined in a neighbourhood of the infinite end of the cone* $\mathbb{K}(\varphi)$. *On the other hand, we introduce the operators*

$$\Gamma^{(ver)}(\tau, p, \lambda, \varphi, R), \quad \Lambda^{(ver)}(\tau, p, \varphi, R), \quad \Lambda^{*\,(ver)}(\tau, p, \varphi, R) \; :$$
$$L^p\big(g^{(\varphi)}(\mathbb{B}_2(0,R))\big)^3 \mapsto L^p\big(g^{(\varphi)}(\mathbb{B}_2(0,R))\big)^3$$

by setting

$$\Gamma^{(ver)}(\tau, p, \lambda, \varphi, R)(f)(x) \quad := \quad (\tau/2) \cdot f(x)$$
$$+ \Big(\int_{g^{(\varphi)}(\mathbb{B}_2(0,R))} \sum_{j,k=1}^{3} \widetilde{\mathcal{D}}^{\lambda}_{jkl}(x-y) \cdot n_k^{(\varphi)}(y) \cdot f_j(y) \; d\,\mathbb{K}(\varphi)(y) \Big)_{1 \le l \le 3},$$

$$\Lambda^{(ver)}(\tau, p, \varphi, R)(f)(x) \quad := \quad (\tau/2) \cdot f(x)$$
$$+ \Big(\int_{g^{(\varphi)}(\mathbb{B}_2(0,R))} \sum_{j,k=1}^{3} \mathcal{D}_{jkl}(x-y) \cdot n_k^{(\varphi)}(y) \cdot f_j(y) \; d\,\mathbb{K}(\varphi)(y) \Big)_{1 \le l \le 3},$$

$$\Lambda^{*\,(ver)}(\tau, p, \varphi, R)(f)(x) \quad := \quad (\tau/2) \cdot f(x)$$
$$- \Big(\int_{g^{(\varphi)}(\mathbb{B}_2(0,R))} \sum_{j,k=1}^{3} \mathcal{D}_{jkl}(x-y) \cdot n_k^{(\varphi)}(x) \cdot f_j(y) \; d\,\mathbb{K}(\varphi)(y) \Big)_{1 \le l \le 3}.$$

for $f \in L^p\big(g^{(\varphi)}(\mathbb{B}_2(0,R))\big)^3$, $x \in g^{(\varphi)}(\mathbb{B}_2(0,R))$. *Note that the domain of these operators consists of functions living near the vertex of* $\mathbb{K}(\varphi)$.

Observe that $\Gamma^*(\tau, p, \lambda, \mathbb{K}(\varphi))$, $\Lambda^*(\tau, p, \mathbb{K}(\varphi))$, $\Pi^*(\tau, p, \mathbb{K}(\varphi))$ are the adjoint operators to $\Gamma(\tau, q, \lambda, \mathbb{K}(\varphi))$, $\Lambda(\tau, q, \mathbb{K}(\varphi))$ and $\Pi(\tau, p, \mathbb{K}(\varphi))$, respectively, with $q := (1-1/p)^{-1}$. We further note that $\Pi^*(\tau, p, \mathbb{K}(\varphi))$ is related to solutions of Laplace's equation (1.15) under Neumann boundary conditions; see [28, p. 211-217], for example. Moreover, the operators $\Gamma^*(\tau, p, \lambda, \mathbb{K}(\varphi))$ and $\Lambda^*(\tau, p, \mathbb{K}(\varphi))$ may be used in order to construct solutions of (1.12) and (1.18), respectively, satisfying a "slip condition" on the boundary; see Theorem 9.1 and Corollary 9.2.

In the next definition, we introduce some trivial extension operators:

Definition 6.2. *Take* $R \in [0, \infty)$, $S \in [1, \infty)$, $\sigma \in \mathbb{N}$.

For any function $\Phi : \mathbb{R}^2\backslash\mathbb{B}_2(0, R{\cdot}S) \mapsto \mathbb{C}^\sigma$, *let* $\mathcal{F}(R, S)(\Phi)$ *denote the zero extension of* Φ *to* $\mathbb{R}^2\backslash\mathbb{B}_2(0,R)$.

If Φ *is an arbitrary function from* $\mathbb{R}^2\backslash\mathbb{B}_2(0,R)$ *into* $\mathbb{C}^\sigma$, *then* $\widetilde{\mathcal{F}}(R)(\Phi)$ *is to denote the trivial extension of* Φ *to* $\mathbb{R}^2$.

Now suppose $R \in (0, \infty]$ *and* $S \in [1, \infty)$, $\sigma \in \mathbb{N}$. *Then, for any function* Φ : $\mathbb{B}_2(0,R) \mapsto \mathbb{C}^\sigma$, *let* $\widetilde{G}(R)(\Phi)$ *denote the zero extension of* Φ *to* $\mathbb{R}^2$, *and* $G(R, S)(\Phi)$ *the zero extension of* Φ *to* $\mathbb{B}_2(0, R{\cdot}S)$.

We finally introduce a number of mappings which arise when certain operators on $\partial\mathbb{K}(\varphi)$ are written in local coordinates:

Definition 6.3. *Let* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $R \in [0, \infty)$, $S \in [1, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\tau \in \{-1, 1\}$, $\epsilon \in (0, \infty)$.

Define the operator $F^*(\tau, p, \varphi, R, S) : L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S)) \mapsto L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0,R))$ *by*

$$F^*(\tau, p, \varphi, R, S)(\Phi)$$
$$:= \Big(-\Pi^*(-\tau, p, \mathbb{K}(\varphi)) \big(\widetilde{\mathcal{F}}(R\cdot S)(\Phi) \circ (g^{(\varphi)})^{-1} \big) \Big) \circ g^{(\varphi)} \Big|\, \mathbb{R}^2\backslash\mathbb{B}_2(0,R)$$

for $\Phi \in L^p\big(\mathbb{R}^2\backslash\mathbb{B}_2(0, R{\cdot}S)\big)$. *This means that* $F^*(\tau, p, \varphi, R, S)$ *is obtained by transforming the operator* $-\Pi^*(-\tau, p, \mathbb{K}(\varphi))$ *into local coordinates and then restricting its domain to* $L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R{\cdot}S))$, *and its range to* $L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0,R))$. *It follows for* Φ *as before, and for* $\xi \in \mathbb{R}^2\backslash\mathbb{B}_2(0,R)$:

$$F^*(\tau, p, \varphi, R, S)(\Phi)(\xi) \;=\; (\tau/2) \cdot \mathcal{F}(R, S)(\Phi)(\xi) \tag{6.8}$$
$$- \int_{\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S)} (4\cdot\pi)^{-1} \cdot \big((n^{(\varphi)} \circ g^{(\varphi)})(\xi) \cdot (g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \big)$$
$$\cdot \big| g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) \big|^{-3} \cdot \Phi(\eta) \cdot \sin^{-1}(\varphi) \; d\eta.$$

We further observe, for $\Phi \in L^p(\mathbb{R}^2)$:

$$F^*(\tau, p, \varphi, 0, 1)(\Phi) \;=\; \Big(-\Pi^*(-\tau, p, \mathbb{K}(\varphi)) \big(\Phi \circ (g^{(\varphi)})^{-1} \big) \Big) \circ g^{(\varphi)}. \tag{6.9}$$

Moreover, we introduce the operator

$$H^*(\tau, p, \varphi, R, S) : L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S))^3 \mapsto L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0,R))^3$$

by setting

$$H^*(\tau, p, \varphi, R, S)(\Phi) \quad := \quad (\tau/2) \cdot \mathcal{F}(R, S)(\Phi)$$
$$+ \; \sin^{-1}(\varphi) \cdot \Big(L_j(0, \varphi) \otimes_p \big((n^{(\varphi)} \circ g^{(\varphi)}) \cdot \Phi \big) \Big)_{1 \le j \le 3} \Big|\, \mathbb{R}^2\backslash\mathbb{B}_2(0,R)$$

for $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R{\cdot}S))^3$; *see Corollary 4.2. Inserting the definition of the function* $L_j(0, \varphi)$, *and writing out the definition of the operation "⊗", we obtain for* Φ *as before,* *and for* $\xi \in \mathbb{R}^2 \backslash \mathbb{B}_2(0, R)$:

$$H^*(\tau, p, \varphi, R, S)(\Phi)(\xi) \quad = \quad (\tau/2) \cdot \mathcal{F}(R, S)(\Phi)(\xi) \tag{6.10}$$
$$- \Bigg(L^p(\mathbb{R}^2) - \lim_{\epsilon \downarrow 0} \Big(\int_{\mathbb{R}^2 \backslash \mathbb{B}_2(0, R \cdot S)} \chi_{(\epsilon, \infty)}(|id(\mathbb{R}^2) - \eta|) \cdot (4 \cdot \pi)^{-1} \cdot (id(\mathbb{R}^2) - \eta)_l$$
$$\cdot \big| id(\mathbb{R}^2) - \eta \big|^{-3} \cdot \big((n^{(\varphi)} \circ g^{(\varphi)})(\eta) \cdot \Phi(\eta) \big) \cdot \sin^{-1}(\varphi) \; d\eta \Big)_{1 \le l \le 3} \Bigg)(\xi) \, .$$

Next we transform the operator $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ *into local coordinates, and then restrict its domain and range in the same way as we did when defining* $F^*(\tau, p, \varphi, R, S)$. *This leads to a mapping*

$$J(\tau, p, \lambda, \varphi, R, S) \, : \, L^p\big(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R \cdot S)\big)^3 \mapsto L^p\big(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R)\big)^3$$

with

$$J(\tau, p, \lambda, \varphi, R, S)(\Phi)$$
$$:= \Big(\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi)) \big(\tilde{\mathcal{F}}(R \cdot S)(\Phi) \circ (g^{(\varphi)})^{-1} \big) \Big) \circ g^{(\varphi)} \Big| \mathbb{R}^2 \backslash \mathbb{B}_2(0, R)$$

for $\Phi \in L^p\big(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R \cdot S)\big)^3$. *It follows for* Φ *as above, and for* $\xi \in \mathbb{R}^2 \backslash \mathbb{B}_2(0, R)$:

$$J(\tau, p, \varphi, R, S)(\Phi)(\xi) \quad = \quad (\tau/2) \cdot \mathcal{F}(R, S)(\Phi)(\xi) \tag{6.11}$$
$$+ \Big(\int_{\mathbb{R}^2 \backslash \mathbb{B}_2(0, R \cdot S)} \sum_{j,k=1}^{3} \tilde{\mathcal{D}}^{\lambda}_{jkl}(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \cdot (n_k^{(\varphi)} \circ g^{(\varphi)})(\eta)$$
$$\cdot \Phi_j(\eta) \cdot \sin^{-1}(\varphi) \; d\eta \Big)_{1 \le l \le 3} .$$

Let us write out the preceding definitions for some special values of R *and* S. *In fact, it holds for* $\Phi \in L^p\big(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R)\big)^3$, $\Psi \in L^p(\mathbb{R}^2)^3$:

$$J(\tau, p, \varphi, R, 1)(\Phi) \tag{6.12}$$
$$= \Big(\Gamma^{(infty)}(\tau, p, \lambda, \varphi, R) \big(\Phi \circ (g^{(\varphi)})^{-1} \big| g^{(\varphi)}(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R)) \big) \Big) \circ g^{(\varphi)} \Big| \mathbb{R}^2 \backslash \mathbb{B}_2(0, R)$$

and

$$J(\tau, p, \lambda, \varphi, 0, 1)(\Psi) \quad = \quad \Big(\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi)) \big(\Psi \circ (g^{(\varphi)})^{-1} \big) \Big) \circ g^{(\varphi)} . \tag{6.13}$$

Next, we introduce some notations concerning operators related to $\mathbb{L}(\varphi, \epsilon)$. *Recall that we assumed* $\epsilon > 0$. *By transforming* $-\Pi^*(-\tau, p, \mathbb{L}(\varphi, \epsilon))$ *into local coordinates, we are led to an operator* $G^*(\tau, p, \varphi, \epsilon) : L^p(\mathbb{R}^2) \mapsto L^p(\mathbb{R}^2)$ *with*

$$G^*(\tau, p, \varphi, \epsilon)(\Phi) := - \Big(\Pi^*(-\tau, p, \mathbb{L}(\varphi, \epsilon)) \big(\Phi \circ (\gamma^{(\varphi, \epsilon)})^{-1} \big) \Big) \circ \gamma^{(\varphi, \epsilon)}$$

for $\Phi \in L^p(\mathbb{R}^2)$, *that is,*

$$G^*(\tau, p, \varphi, \epsilon)(\Phi)(\xi) \tag{6.14}$$
$$= \quad (\tau/2) \cdot \Phi(\xi) \; - \; \int_{\mathbb{R}^2} (4 \cdot \pi)^{-1} \cdot \Big((n^{(\varphi)} \circ g^{(\varphi)})(\xi) \cdot (\gamma^{(\varphi, \epsilon)}(\xi) - \gamma^{(\varphi, \epsilon)}(\eta)) \Big)$$
$$\cdot \big| \gamma^{(\varphi, \epsilon)}(\xi) - \gamma^{(\varphi, \epsilon)}(\eta) \big|^{-3} \cdot \Phi(\eta) \cdot J^{(\varphi, \epsilon)}(\eta) \; d\eta$$

for $\Phi \in L^p(\mathbb{R}^2)$, $\xi \in \mathbb{R}^2$. *Writing* $-\Pi(-\tau, p, \mathbb{L}(\varphi, \epsilon))$ *in local coordinates, we obtain an operator* $G(\tau, p, \varphi, \epsilon) : L^p(\mathbb{R}^2) \mapsto L^p(\mathbb{R}^2)$ *defined by*

$$G(\tau, p, \varphi, \epsilon) := - \Big(\Pi(-\tau, p, \mathbb{L}(\varphi, \epsilon)) \big(\Phi \circ (\gamma^{(\varphi, \epsilon)})^{-1} \big) \Big) \circ \gamma^{(\varphi, \epsilon)}$$

for $\Phi \in L^p(\mathbb{R}^2)$. *Later on we shall need an explicit representation of* $G(\tau, p, \varphi, \epsilon)$, *analogous to (6.14). Therefore, we note for* $\Phi \in L^p(\mathbb{R}^2)$, $\xi \in \mathbb{R}^2$:

$$G(\tau, p, \varphi, \epsilon)(\Phi)(\xi) \tag{6.15}$$
$$= \quad (\tau/2) \cdot \Phi(\xi) \; + \; \int_{\mathbb{R}^2} (4 \cdot \pi)^{-1} \cdot \Big((n^{(\varphi)} \circ g^{(\varphi)})(\eta) \cdot (\gamma^{(\varphi, \epsilon)}(\xi) - \gamma^{(\varphi, \epsilon)}(\eta)) \Big)$$
$$\cdot \big| \gamma^{(\varphi, \epsilon)}(\xi) - \gamma^{(\varphi, \epsilon)}(\eta) \big|^{-3} \cdot \Phi(\eta) \cdot J^{(\varphi, \epsilon)}(\eta) \; d\eta .$$

Now, turning to $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$, *we define a mapping* $M(\tau, p, \lambda, \varphi, \epsilon) : L^p(\mathbb{R}^2)^3 \mapsto L^p(\mathbb{R}^2)^3$ *by*

$$M(\tau, p, \lambda, \varphi, \epsilon)(\Phi) := \Big(\Gamma(\tau, p, \lambda, \mathbb{L}(\varphi, \epsilon)) \big(\Phi \circ (\gamma^{(\varphi, \epsilon)})^{-1} \big) \Big) \circ \gamma^{(\varphi, \epsilon)}$$

for $\Phi \in L^p(\mathbb{R}^2)^3$. *This implies for* $\xi \in \mathbb{R}^2$ *and for* Φ *as before:*

$$M(\tau, p, \lambda, \varphi, \epsilon)(\Phi)(\xi) \tag{6.16}$$
$$= \quad (\tau/2) \cdot (\Phi)(\xi) \; + \; \Big(\int_{\mathbb{R}^2} \sum_{j,k=1}^{3} \tilde{\mathcal{D}}^{\lambda}_{jkl}(\gamma^{(\varphi, \epsilon)}(\xi) - \gamma^{(\varphi, \epsilon)}(\eta)) \cdot (n_k^{(\varphi, \epsilon)} \circ \gamma^{(\varphi, \epsilon)})(\eta)$$
$$\cdot \Phi_j(\eta) \cdot J^{(\varphi, \epsilon)}(\eta) \; d\eta \Big)_{1 \le l \le 3} .$$

Finally, we shall need the operator $M^*(\tau, p, \lambda, \varphi, \epsilon) : L^p(\mathbb{R}^2)^3 \mapsto L^p(\mathbb{R}^2)^3$ *defined by*

$$M^*(\tau, p, \lambda, \varphi, \epsilon)(\Phi) := \Big(\Gamma^*(\tau, p, \lambda, \mathbb{L}(\varphi, \epsilon)) \big(\Phi \circ (\gamma^{(\varphi, \epsilon)})^{-1} \big) \Big) \circ \gamma^{(\varphi, \epsilon)} ,$$

with $\Phi \in L^p(\mathbb{R}^2)^3$. *Correspondig to (6.16), it holds for* $\Phi \in L^p(\mathbb{R}^2)^3$, $\xi \in \mathbb{R}^2$:

$$M^*(\tau, p, \lambda, \varphi, \epsilon)(\Phi)(\xi) \tag{6.17}$$
$$= \quad (\tau/2) \cdot \Phi(\xi) \; - \; \Big(\int_{\mathbb{R}^2} \sum_{j,k=1}^{3} \tilde{\mathcal{D}}^{\lambda}_{jkl}(\gamma^{(\varphi, \epsilon)}(\xi) - \gamma^{(\varphi, \epsilon)}(\eta)) \cdot (n_k^{(\varphi, \epsilon)} \circ \gamma^{(\varphi, \epsilon)})(\xi)$$
$$\cdot \Phi_j(\eta) \cdot J^{(\varphi, \epsilon)}(\eta) \; d\eta \Big)_{1 \le l \le 3} .$$

Next we consider operators acting on functions which are defined near the vertex of $\mathbb{K}(\varphi)$. *To this end, take* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $R \in (0, \infty]$, $S \in [1, \infty)$, $\tau \in \{-1, 1\}$, $\lambda \in \mathbb{C} \backslash (-\infty, 0]$. *Then we introduce the operators*

$A(\tau, p, \varphi, R, S)\,,\quad A^*(\tau, p, \varphi, R, S)\; :\; L^p(\mathbb{B}_2(0, R))^3 \mapsto L^p(\mathbb{B}_2(0, R\cdot S))^3$

by setting for for $\Phi \in L^p(\mathbb{B}_2(0, R))^3$:

$$A(\tau, p, \varphi, R, S)(\Phi) := \Big(\Lambda(\tau, p, \mathbb{K}(\varphi))\big(\widetilde{G}(R)(\Phi)\circ (g^{(\varphi)})^{-1}\big)\Big) \circ g^{(\varphi)}\Big|\mathbb{B}_2(0, R\cdot S),$$

$$A^*(\tau, p, \varphi, R, S)(\Phi) := \Big(\Lambda^*(\tau, p, \mathbb{K}(\varphi))\big(\widetilde{G}(R)(\Phi)\circ (g^{(\varphi)})^{-1}\big)\Big) \circ g^{(\varphi)}\Big|\mathbb{B}_2(0, R\cdot S).$$

Recalling the definition of $\Lambda(\tau, p, \mathbb{K}(\varphi))$, *we obtain for* $\Phi \in L^p(\mathbb{B}_2(0, R))^3$, $\xi \in \mathbb{B}_2(0, R\cdot S)$:

$$\begin{aligned} A(\tau, p, \varphi, R, S)(\Phi)(\xi) \;&=\; (\tau/2)\cdot G(R, S)(\Phi)(\xi) \\ &+ \Big(\int_{\mathbb{B}_2(0,R)} \sum_{j,k=1}^{3} \mathcal{D}_{jkl}(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta))\cdot (n_k^{(\varphi)}\circ g^{(\varphi)})(\eta) \\ &\qquad \cdot \Phi_j(\eta)\cdot \sin^{-1}(\varphi)\; d\eta\Big)_{1\le l\le 3}, \end{aligned} \tag{6.18}$$

For later references, we also state the corresponding formula for $A^*(\tau, p, \varphi, R, S)$:

$$\begin{aligned} A^*(\tau, p, \varphi, R, S)(\Phi)(\xi) \;&=\; (\tau/2)\cdot G(R, S)(\Phi)(\xi) \\ &- \Big(\int_{\mathbb{B}_2(0,R)} \sum_{j,k=1}^{3} \mathcal{D}_{jkl}(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta))\cdot (n_k^{(\varphi)}\circ g^{(\varphi)})(\xi) \\ &\qquad \cdot \Phi_j(\eta)\cdot \sin^{-1}(\varphi)\; d\eta\Big)_{1\le l\le 3}. \end{aligned} \tag{6.19}$$

Let us take note of some special cases. Thus, for $R = \infty$, $S = 1$, *it holds*

$$A(\tau, p, \varphi, \infty, 1)(\Phi) \;=\; \Big(\Lambda(\tau, p, \mathbb{B}(\varphi))\big(\Phi\circ (g^{(\varphi)})^{-1}\big)\Big)\circ g^{(\varphi)} \tag{6.20}$$

for $\Phi \in L^p(\mathbb{R}^2)^3$, *and if* $S = 1$, *we find*

$$\begin{aligned} &A(\tau, p, \varphi, R, 1)(\Phi) \\ &= \Big(\Lambda^{(ver)}(\tau, p, \varphi, R)\big(\Phi\circ (g^{(\varphi)})^{-1}\big|g^{(\varphi)}(\mathbb{B}_2(0, R))\big)\Big)\circ g^{(\varphi)}\Big|\mathbb{B}_2(0, R) \end{aligned} \tag{6.21}$$

for $\Phi \in L^p(\mathbb{B}_2(0, R))^3$, *with corresponding formulae for* $A^*(\tau, p, \varphi, \infty, 1)$ *and* $A^*(\tau, p, \varphi, R, 1)$.

Lastly we introduce the mapping $L(\tau, p, \lambda, \varphi, R)\; :\; L^p(\mathbb{B}_2(0, R))^3 \mapsto L^p(\mathbb{B}_2(0, R))^3$ *by*

$$\begin{aligned} &L(\tau, p, \lambda, \varphi, R)(\Phi) \\ &:= \Big(\Gamma^{(ver)}(\tau, p, \lambda, \varphi, R)\big(\Phi\circ (g^{(\varphi)})^{-1}\big|g^{(\varphi)}(\mathbb{B}_2(0, R))\big)\Big)\circ g^{(\varphi)}\Big|\mathbb{B}_2(0, R) \end{aligned}$$

for $\Phi \in L^p(\mathbb{B}_2(0, R))^3$. *Thus* $L(\tau, p, \lambda, \varphi, R)$ *is obtained by transforming the operator* $\Gamma^{(ver)}(\tau, p, \lambda, \varphi, R)$ *into local coordinates. Hence, for* $\Phi \in L^p(\mathbb{B}_2(0, R))^3$, $\xi \in \mathbb{B}_2(0, R)$, *it holds*

$$\begin{aligned} &L(\tau, p, \lambda, \varphi, R)(\Phi)(\xi) \\ &= (\tau/2)\cdot\Phi(\xi) + \Big(\int_{\mathbb{B}_2(0,R)} \sum_{j,k=1}^{3} \widetilde{\mathcal{D}}^{\lambda}_{jkl}(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta))\cdot (n_k^{(\varphi)}\circ g^{(\varphi)})(\eta) \\ &\qquad \cdot \Phi_j(\eta)\cdot \sin^{-1}(\varphi)\; d\eta\Big)_{1\le l\le 3}. \end{aligned} \tag{6.22}$$

Now we are going to present some – rather obvious – properties of the operators introduced above. First we observe that all of the preceding operators are bounded:

Lemma 6.7. *Let* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $\tau \in \{-1, 1\}$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $R \in [0, \infty)$, $T \in (0, \infty]$, $S \in [1, \infty)$, $\epsilon \in (0, \infty)$. *Then the operators*

$F^*(\tau, p, \varphi, R, S)$, $H^*(\tau, p, \varphi, R, S)$, $J(\tau, p, \lambda, \varphi, R, S)$, $G^*(\tau, p, \varphi, \epsilon)$, $G(\tau, p, \varphi, \epsilon)$,

$M^*(\tau, p, \lambda, \varphi, \epsilon)$, $M(\tau, p, \lambda, \varphi, \epsilon)$, $A(\tau, p, \varphi, T, S)$, $A^*(\tau, p, \varphi, T, S)$

are linear and continuous.

Proof: We refer to Lemma 6.2, 6.5, and to Corollary 4.2.

Lemma 6.8. *Take* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $R \in (0, \infty)$, $S \in [1, \infty)$, $\tau \in \{-1, 1\}$, $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S))^3$. *Then*

$$(n^{(\varphi)}\circ g^{(\varphi)})\cdot H^*(\tau, p, \varphi, R, S)(\Phi) \;=\; F^*(\tau, p, \varphi, R, S)\big((n^{(\varphi)}\circ g^{(\varphi)})\cdot\Phi\big).$$

Proof: The proof is based on Lebesgue's theorem on dominated convergence, which may be applied in a suitable way by referring to (6.8), (6.10), Lemma 3.1, Satz 4.1 and Corollary 4.2. We omit the details.

Next we study the Fredholm properties of the operators defined before. From now on, the notion of a "F_+-operator" will arise frequently. We refer to our definition in Chapter 2.

The following lemma contains a basic result which was already announced at the beginning of this chapter, and which concerns Fredholm properties of a sum of certain operators.

Lemma 6.9. *Let* A *be a non-empty set,* $\mathcal{B}$ *a* σ*-algebra in* A, *and* μ *a (positive) measure on* $\mathcal{B}$. *Moreover, take* $p \in (1, \infty)$ *and* $\sigma \in \mathbb{N}$.

For any subset G *of* A, *and for any function* $f : G \mapsto \mathbb{C}^\sigma$, *let* $F_G(f)$ *denote the trivial extension of* f *to* A. *Furthermore, if* $G \subset A$ *is* $\mathcal{B}$*-measurable, then* $L^p(G, \mu)$ *is to denote the space of all* $\mathcal{B}$*-measurable functions* $f : G \mapsto \mathbb{C}$ *with* $\int_G |f|^p\, d\mu < \infty$.

Assume that $A_1\ A_2$ *are* $\mathcal{B}$*-measurable subsets of* A *with* $A_1 \cap A_2 = \emptyset$, $A_1 \cup A_2 = A$. *Let* $K : L^p(A, \mu)^\sigma \mapsto L^p(A, \mu)^\sigma$ *be a linear operator which is bounded with respect to the usual norm on* $L^p(A, \mu)^\sigma$. *Then we define the operators*

$$K^{(1)} : L^p(A_1, \mu)^\sigma \mapsto L^p(A_1, \mu)^\sigma, \qquad K^{(2)} : L^p(A_2, \mu)^\sigma \mapsto L^p(A_2, \mu)^\sigma,$$

$$K^{(3)} : L^p(A_1, \mu)^\sigma \mapsto L^p(A_2, \mu)^\sigma, \qquad K^{(4)} : L^p(A_2, \mu)^\sigma \mapsto L^p(A_1, \mu)^\sigma$$

by

$$K^{(j)}(\Phi) := K(F_{A_j}(\Phi)) \,|\, A_j \quad \text{for } j \in \{1, 2\},\ \Phi \in L^p(A_j, \mu)^\sigma;$$

$$K^{(3)}(\Phi) := K(F_{A_1}(\Phi)) \,|\, A_2 \quad \text{for } \Phi \in L^p(A_1, \mu)^\sigma;$$

$$K^{(4)}(\Phi) := K(F_{A_2}(\Phi)) \,|\, A_1 \quad \text{for } \Phi \in L^p(A_2, \mu)^\sigma.$$

Suppose that $K^{(3)}$ *and* $K^{(4)}$ *are compact.*

Then K *is Fredholm (or* F_+*) if and only if* $K^{(1)}$ *and* $K^{(2)}$ *are Fredholm (or* F_+*) too. In case* K, $K^{(1)}$ *and* $K^{(2)}$ *are* F_+*-operators, it follows*

$$index(K) = index(K^{(1)}) + index(K^{(2)}).$$

Proof: The operators

$$\widetilde{K}^{(3)} : L^p(A, \mu)^\sigma \mapsto L^p(A, \mu)^\sigma, \quad \widetilde{K}^{(3)}(\Phi) := F_{A_2}\big(K^{(3)}(\Phi \,|\, A_1)\big) \quad \text{for } \Phi \in L^p(A, \mu)^\sigma,$$

$$\widetilde{K}^{(4)} : L^p(A, \mu)^\sigma \mapsto L^p(A, \mu)^\sigma, \quad \widetilde{K}^{(4)}(\Phi) := F_{A_1}\big(K^{(4)}(\Phi \,|\, A_2)\big) \quad \text{for } \Phi \in L^p(A, \mu)^\sigma$$

are compact since $K^{(3)}$ and $K^{(4)}$ have the same property. Hence it follows with [34, p. 25, Theorem 3.8]: K Fredholm (or F_+) if and only if $K - \widetilde{K}^{(3)} - \widetilde{K}^{(4)}$ Fredholm (or F_+). If one of these assumptions is true, then the same reference yields:

$$index(K) = index(K - \widetilde{K}^{(3)} - \widetilde{K}^{(4)}).$$

On the other hand, we have for $\Phi \in L^p(A, \mu)^\sigma$:

$$(K - \widetilde{K}^{(3)} - \widetilde{K}^{(4)})(\Phi) = F_{A_1}\big(K^{(1)}(\Phi \,|\, A_1)\big) + F_{A_2}\big(K^{(2)}(\Phi \,|\, A_2)\big).$$

Defining the operator $\widetilde{K} : L^p(A_1, \mu)^\sigma \times L^p(A_2, \mu)^\sigma \mapsto L^p(A_1, \mu)^\sigma \times L^p(A_2, \mu)^\sigma$ by

$$\widetilde{K}(\Phi_1, \Phi_2) := \big(K^{(1)}(\Phi_1),\ K^{(2)}(\Phi_2)\big) \quad \text{for } \Phi_1 \in L^p(A_1, \mu)^\sigma,\ \Phi_2 \in L^p(A_2, \mu)^\sigma,$$

we may infer that $K - \widetilde{K}^{(3)} - \widetilde{K}^{(4)}$ is Fredholm (or F_+) if and only if $\widetilde{K}$ is Fredholm (or F_+) as well. In case both $K - \widetilde{K}^{(3)} - \widetilde{K}^{(4)}$ and $\widetilde{K}$ are F_+-operators, they have the same index.

Next observe that $\widetilde{K}$ is Fredholm (or F_+) if and only if $K^{(1)}$ and $K^{(2)}$ have the same property. Furthermore, if $\widetilde{K}$, $K^{(1)}$ and $K^{(2)}$ are F_+-operators, it follows at once that

$$index(\widetilde{K}) = index(K^{(1)}) + index(K^{(2)}).$$

Collecting our results, we obtain the lemma.

Lemma 6.10. *Let* A, $\mathcal{B}$, μ, p, σ, A_1, A_2 *be given as in the preceding lemma. Consider a linear operator* $K : L^p(A_1, \mu)^\sigma \mapsto L^p(A, \mu)^\sigma$ *which is bounded with respect to the usual norm of* $L^p(A_1, \mu)^\sigma$ *and* $L^p(A, \mu)^\sigma$, *respectively. For* $j \in \{1, 2\}$, *we define the operator* $M^{(j)} : L^p(A_1, \mu)^\sigma \mapsto L^p(A_j, \mu)^\sigma$ *by*

$$M^{(j)}(\Phi) := K(\Phi) \,|\, A_j \quad \text{for } \Phi \in L^p(A_1, \mu)^\sigma.$$

Suppose in addition that $M^{(2)}$ *is compact. Then* K *is an* F_+*-operator if and only if* $M^{(1)}$ *has the same property.*

Proof: For any subst G of A and for any function $f : G \mapsto \mathbb{C}^\sigma$, let $F_G(f)$ denote the trivial extension of f to A. Define the operators $\widetilde{M}^{(1)}$, $\widetilde{M}^{(2)} : L^p(A_1, \mu)^\sigma \mapsto L^p(A, \mu)^\sigma$ by

$$\widetilde{M}^{(j)}(\Phi) := F_{A_j}\big(M^{(j)}(\Phi)\big) = F_{A_j}\big(K(\Phi) \,|\, A_j\big) \quad \text{for } \Phi \in L^p(A_1, \mu)^\sigma,\ j \in \{1, 2\}.$$

Then $\widetilde{M}^{(2)}$ is compact, and it holds: $K = \widetilde{M}^{(1)} + \widetilde{M}^{(2)}$. Hence it follows by [34, p. 25, Theorem 3.8]): K is F_+ if and only if $\widetilde{M}^{(1)}$ is F_+ too. On the other hand, observe that $kern(M^{(1)}) = kern(\widetilde{M}^{(1)})$, and $im(M^{(1)})$ is closed if and only if the set $im(\widetilde{M}^{(1)})$ has the same property. Hence we may conclude that $M^{(1)}$ is F_+ if and only if $\widetilde{M}^{(1)}$ is also a F_+-operator. Gathering up our information, we obtain the lemma.

Corollary 6.1. *Let* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $R \in (0, \infty)$, $S \in [1, \infty)$, $\tau \in \{-1, 1\}$. *Assume that* $(L^{(1)}, L^{(2)})$ *coincides with one of the following pairs of operators:*

$$\big(J(\tau, p, \lambda, \varphi, R, S),\ J(\tau, p, \lambda, \varphi, R{\cdot}S, 1)\big), \quad \big(A(\tau, p, \varphi, R, S),\ A(\tau, p, \varphi, R, 1)\big),$$

$$\big(A^*(\tau, p, \varphi, R, S),\ A^*(\tau, p, \varphi, R, 1)\big), \quad \big(F^*(\tau, p, \varphi, R, S),\ F^*(\tau, p, \varphi, R{\cdot}S, 1)\big).$$

Then $L^{(1)}$ *is* F_+ *if and only if* $L^{(2)}$ *has the same property.*

Proof: Define the operators K, $M^{(1)}$, $M^{(2)}$ in the following way:

$$K := J(\tau, p, \lambda, \varphi, R, S); \qquad M^{(1)} := J(\tau, p, \lambda, \varphi, R{\cdot}S, 1);$$

$$M^{(2)} : L^p(\mathbb{R}^3\backslash \mathbb{B}_2(0, R{\cdot}S))^3 \mapsto L^p(\mathbb{B}_2(0, R{\cdot}S) \backslash \mathbb{B}_2(0, R))^3,$$

$$M^{(2)}(\Phi) := J(\tau, p, \lambda, \varphi, R, S)(\Phi) \big|\, \mathbb{B}_2(0, R{\cdot}S) \backslash \mathbb{B}_2(0, R)$$

for $\Phi \in L^p(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R\cdot S))^3$.

It follows by (6.11) and Lemma 6.6 that the operator $M^{(2)}$ is compact. Now Lemma 6.10 yields the corollary in the case $L^{(1)} = J(\tau, p, \lambda, \varphi, R, S)$, $L^{(2)} = J(\tau, p, \lambda, \varphi, R\cdot S, 1)$. The other three cases may be treated in an analogous way, the main difference being that the reference to (6.11) must be replaced by another one to (6.8), (6.18) and (6.19), respectively.

Corollary 6.2. *Let* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $R_1, R_2 \in (0, \infty)$, $\tau \in \{-1, 1\}$. *Choose* $(L^{(1)}, L^{(2)})$ *among the the following pairs of operators:*

$$\Big(J(\tau, p, \lambda, \varphi, R_1, 1),\ J(\tau, p, \lambda, \varphi, R_2, 1)\Big), \Big(F^*(\tau, p, \varphi, R_1, 1),\ F^*(\tau, p, \varphi, R_2, 1)\Big),$$

$$\Big(A(\tau, p, \varphi, R_1, 1),\ A(\tau, p, \varphi, R_2, 1)\Big), \quad \Big(A^*(\tau, p, \varphi, R_1, 1),\ A^*(\tau, p, \varphi, R_2, 1)\Big).$$

Then $L^{(1)}$ *is Fredholm (or* F_+*) if and only if* $L^{(2)}$ *is Fredholm (or* F_+*) too. In case* $L^{(1)}$ *and* $L^{(2)}$ *are* F_+*-operators, they have the same index.*

Proof: The corollary is a consequence of Lemma 6.9, combined with the integral representations given by (6.8), (6.11), (6.18), (6.19) for the operators mentioned in the lemma. As an example, consider the case $L^{(1)} = A^*(\tau, p, \varphi, R_1, 1)$, $L^{(2)} = A^*(\tau, p, \varphi, R_2, 1)$. Without restriction of generality, we may assume $R_2 > R_1$. Let us apply Lemma 6.9 with

$$\sigma := 3, \quad A := \mathbb{B}_2(0, R_2), \quad A_1 := \mathbb{B}_2(0, R_1), \quad K := A^*(\tau, p, \varphi, R_2, 1).$$

For $\mathcal{B}$, we take the Lebesgue-measurable subsets of A, and for μ, the Lebesgue measure. Let $K^{(1)}, \cdots, K^{(4)}$ be defined as in the lemma just mentioned. Note that K is a linear bounded operator from $L^p(A)^\sigma$ into $L^p(A)^\sigma$ (Lemma 6.7). According to Lemma 6.6, $K^{(3)}$ and $K^{(4)}$ are compact. The operator $K^{(2)} - (\tau/2)\cdot id(A_2)$ is compact as well; see (6.1), (6.2) and Lemma 6.3. Hence, by [34, p. 24, Theorem 3.4], the operator $K^{(2)}$ is Fredholm with $index(K^{(2)}) = 0$. Finally we point our that $K^{(1)} = A^*(\tau, p, \varphi, R_1, 1)$. Thus Lemma 6.9 yields the corollary for the case considered presently. The other cases may be treated in an analougous way.

Corollary 6.3. *Take* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $R \in (0, \infty)$, $\tau \in \{-1, 1\}$. *Let* $(L^{(1)}, L^{(2)})$ *be one of the following pairs of operators:*

$$\Big(F^*(\tau, p, \varphi, 0, 1),\ F^*(\tau, p, \varphi, R, 1)\Big), \quad \Big(A(\tau, p, \varphi, \infty, 1),\ A(\tau, p, \varphi, R, 1)\Big),$$

$$\Big(A^*(\tau, p, \varphi, \infty, 1),\ A^*(\tau, p, \varphi, R, 1)\Big).$$

Assume that $L^{(1)}$ *is a Fredholm operator. Then* $L^{(2)}$ *is Fredholm too.*

Proof: This result may be established by a reasoning similar to the one used in the proof of Corollary 6.2. We omit the details.

Corollary 6.4. *Let* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $R \in (0, \infty)$, $\tau \in \{-1, 1\}$. *In addition, let* $(L^{(1)}, L^{(2)})$ *coincide with either*

$$\Big(F^*(\tau, p, \varphi, R, 1),\ G^*(\tau, p, \varphi, R)\Big) \quad or \quad \Big(J(\tau, p, \lambda, \varphi, R, 1),\ M(\tau, p, \lambda, \varphi, R)\Big).$$

Assume $L^{(1)}$ *to be Fredholm. Then* $L^{(2)}$ *is also Fredholm, and it holds:* $index(L^{(1)}) = index(L^{(2)})$.

Proof: Consider the case $L^{(1)} = F^*(\tau, p, \varphi, R, 1)$, $L^{(2)} = G^*(\tau, p, \varphi, R)$. Let us apply Lemma 6.9 with

$$\sigma := 1, \quad A := \mathbb{R}^2, \quad A_1 := \mathbb{B}_2(0, R), \quad K := G^*(\tau, p, \varphi, R).$$

Moreover, we take $\mathcal{B}$ as the σ-algebra of Lebesgue-measurable subsets of $\mathbb{R}^2$, and μ as the Lebesgue measure on $\mathbb{R}^2$. Under these assumptions, define $K^{(1)}, \cdots, K^{(4)}$ as in Lemma 6.9. We observe that K is a linear and bounded mapping from the space $L^p(A)^\sigma$ into itself (Lemma 6.7). Equation (6.14) and Lemma 6.6 yield that $K^{(3)}$ and $K^{(4)}$ are compact. The operator $K^{(1)} - (\tau/2)\cdot id(A_1)$ is compact too, as may be seen from (6.14), (6.3) and Lemma 6.3. Hence, by [34, p. 24, Theorem 3.4], $K^{(1)}$ is Fredholm with index zero. On the other hand, due to (3.6), (6.8) and (6.14), we have $K^{(2)} = F^*(\tau, p, \varphi, R, 1)$. It follows by the assumptions in the corollary that $K^{(2)}$ is Fredholm. Now Lemma 6.9 yields the corollary for the case considered presently.

If instead $L^{(1)} = J(\tau, p, \lambda, \varphi, R, 1)$, $L^{(2)} = M(\tau, p, \lambda, \varphi, R)$, then we may argue in an analogous way, with the references (6.14), (6.8) replaced by (6.16), (6.11), respectively.

Corollary 6.5. *Let* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $R \in (0, \infty)$, $\tau \in \{-1, 1\}$. *Then* $J(\tau, p, \lambda, \varphi, 0, 1)$ *is Fredholm if and only if* $J(\tau, p, \lambda, \varphi, R, 1)$ *and* $L(\tau, p, \lambda, \varphi, R)$ *have the same property. If these conditions are true, then*

$$index\Big(J(\tau, p, \lambda, \varphi, 0, 1)\Big) = index\Big(L(\tau, p, \lambda, \varphi, R)\Big) + index\Big(J(\tau, p, \lambda, \varphi, R, 1)\Big).$$

Proof: We apply Lemma 6.9 with

$$\sigma := 3, \quad A := \mathbb{R}^2, \quad A_1 := \mathbb{B}_2(0, R), \quad K := J(\tau, p, \varphi, 0, 1).$$

We further assume that $\mathcal{B}$ is the Lebesgue-σ-algebra in $\mathbb{R}^2$, and μ the corresponding

Lebegue measure. Let $K^{(1)}, \cdots, K^{(4)}$ be defined as in Lemma 6.9. Then $K^{(3)}$ and $K^{(4)}$ are compact operators, as follows from (6.11) and Lemma 6.6. In addition, we conclude from (6.11), (6.22):

$$K^{(1)} = L(\tau, p, \lambda, \varphi, R), \qquad K^{(2)} = J(\tau, p, \lambda, \varphi, R, 1).$$

Hence the corollary follows from Lemma 6.9.

Lemma 6.11. *Suppose X, X', Y, Y' are Banach spaces, $K : X \mapsto Y$ is a linear bounded operator, and $F : X' \mapsto X$, $G : Y \mapsto Y'$ are linear topological operators. Then K is Fredholm (or F_+) if and only if $G \circ K \circ F$ is also Fredholm (or F_+). In case these operators are F_+, they have the same index.*

The proof of this result is simple, and we omit it.

Lemma 6.12. *Let $p \in (1, \infty)$, $A, B \subset \mathbb{R}^2$ measurable, $\sigma \in \mathbb{N}$, $\varphi \in (0, \pi/2]$, $\epsilon \in (0, \infty)$, $h \in \{ g^{(\varphi)}, \gamma^{(\varphi, \epsilon)} \}$. In addition, suppose that*

$$K : L^p(A)^\sigma \mapsto L^p(B)^\sigma, \qquad M : L^p(h(A))^\sigma \mapsto L^p(h(B))^\sigma$$

are linear and bounded operators with

$$\Big(M\big(\Phi \circ h^{-1} \big| h(A) \big) \Big) \circ h \,|\, B \quad = \quad K(\Phi) \qquad \text{for } \Phi \in L^p(A)^\sigma .$$

Then K is Fredholm (or F_+) if and only if M is Fredholm (or F_+). When any of these conditions is true, it follows that $index(K) = index(M)$.

Proof: Apply (3.1), (3.8), as well as Lemma 6.11 with

$$G : L^p(B)^\sigma \mapsto L^p(h(B))^\sigma, \qquad G(\Psi) := \Psi \circ h^{-1} \,|\, h(B) \quad \text{for } \Psi \in L^p(B)^\sigma;$$
$$F : L^p(h(A))^\sigma \mapsto L^p(A)^\sigma, \qquad F(\Psi) := \Psi \circ h \,|\, A \quad \text{for } \Psi \in L^p(h(A))^\sigma .$$

Corollary 6.6. *Take $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $R \in (0, \infty)$, $\tau \in \{-1, 1\}$. Choose $(L^{(1)}, L^{(2)})$ among the following pairs of operators:*

$$\Big(\Pi^*(-\tau, p, \mathbb{K}(\varphi)),\ F^*(\tau, p, \varphi, 0, 1)\Big), \quad \Big(\Gamma^{(inf)}(\tau, p, \lambda, \varphi, R),\ J(\tau, p, \lambda, \varphi, R, 1)\Big),$$

$$\Big(\Gamma(\tau, p, \lambda, \mathbb{L}(\varphi, R)),\ M(\tau, p, \lambda, \varphi, R)\Big), \quad \Big(\Gamma^{(ver)}(\tau, p, \lambda, \varphi, R),\ L(\tau, p, \lambda, \varphi, R)\Big),$$

$$\Big(\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi)),\ J(\tau, p, \lambda, \varphi, 0, 1)\Big), \quad \Big(\Lambda^{*\,(ver)}(\tau, p, \varphi, R),\ A^*(\tau, p, \varphi, R, 1)\Big),$$

$$\Big(\Lambda^{(ver)}(\tau, p, \varphi, R),\ A(\tau, p, \varphi, R, 1)\Big), \quad \Big(\Lambda(\tau, p, \mathbb{K}(\varphi)),\ A(\tau, p, \varphi, \infty, 1)\Big),$$

$$\Big(\Lambda^*(\tau, p, \mathbb{K}(\varphi)),\ A^*(\tau, p, \varphi, \infty, 1)\Big).$$

Then $L^{(1)}$ is Fredholm (or F_+) if and only if $L^{(2)}$ has the same property. In case $L^{(1)}$ and $L^{(2)}$ are F_+-operators, it follows: $index(L^{(1)}) = index(L^{(2)})$.

Proof: Use Lemma 6.12 and (6.9), (6.12), (6.13), (6.20), (6.21).

Lemma 6.13. *Let $p \in (1, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $R \in [0, \infty)$, $\tau \in \{-1, 1\}$. Then the operator $J(\tau, p, \lambda, \pi/2, R, 1)$ is Fredholm with index 0.*

Proof: We note for $\eta \in \mathbb{R}^2$: $g^{(\pi/2)}(\eta) = (\eta, 0)$, $(n^{(\pi/2)} \circ g^{(\pi/2)})(\eta) = (0, 0, -1)$. Hence we have for $\xi, \eta \in \mathbb{R}^2$:

$$(n^{(\pi/2)} \circ g^{(\pi/2)})(\eta) \cdot (g^{(\pi/2)}(\xi) - g^{(\pi/2)}(\eta)) \quad = \quad 0 .$$

By (5.11), it holds for $\xi, \eta \in \mathbb{R}^2$ with $\xi \neq \eta$, $l \in \{1, 2, 3\}$:

$$\sum_{j,k=1}^{3} \widetilde{\mathcal{D}}^{\lambda}_{jkl}(g^{(\pi/2)}(\xi) - g^{(\pi/2)}(\eta)) \cdot (n_k^{(\pi/2)} \circ g^{(\pi/2)})(\eta) \cdot \Phi_j(\eta) \tag{6.23}$$
$$= \sum_{j=1}^{3} - P^{\lambda}_{j3l}(\xi - \eta, 0) \cdot \Phi_j(\eta).$$

Combining this equation with (6.11), Lemma 6.7 and [33, p. 219, Theorem 5.8], we may conclude the operator $J(\tau, p, \lambda, \pi/2, 0, 1)$ is continuous, one-to-one and onto. In particular, it is Fredholm with index 0. This proves the lemma in the case $R = 0$. Now assume $R > 0$. Then Corollary 6.5 yields that $J(\tau, p, \lambda, \pi/2, R, 1)$ and $L(\tau, p, \lambda, \pi/2, R)$ are Fredholm with

$$index(J(\tau, p, \lambda, \pi/2, R, 1)) + index(L(\tau, p, \lambda, \pi/2, R) \quad = \quad 0. \tag{6.24}$$

But from (5.12) and (5.14), it follows

$$|P^{\lambda}_{jkl}(\eta, 0)| \leq C_{17}(|\arg \lambda|) \cdot |\lambda| \qquad \text{for } \eta \in \mathbb{R}^2\backslash\{0\}.$$

This implies by (6.23), (6.22) and Lemma 6.3: The operator

$$L(\tau, p, \lambda, \pi/2, R) - (\tau/2) \cdot id(\mathbb{B}_2(0, R))$$

is compact. Thus, referring to [34, p. 24, Theorem 3.4], we obtain that $L(\tau, p, \lambda, \pi/2, R)$ has index 0. Now the lemma follows by (6.24).

Next we turn to the question whether our operators depend continuously on certain parameters. We begin by an abstract result:

Theorem 6.1. *Let X, Y be Banach spaces, and assume that $A : [0, 1] \mapsto \mathcal{LB}(X, Y)$ is a continuous mapping. Moreover, suppose that $A(t)$ is a F_+-operator for any $t \in [0, 1]$. Then it holds: $index(A(0)) = index(A(1))$.*

Proof: We refer to the proof of [34, p. 25, Theorem 3.11]. There, reference [34, p. 24, Theorem 3.5], which states that the index of a Fredholm operator is invariant with respect to small perturbations, should be replaced by a reference to [29, p. 235, Theorem 5.17]. The latter theorem concerns stability of the index of semi-Fredholm operators.

Now we shall check whether the assumptions of the preceding theorem are satisfied by some concrete operators:

Lemma 6.14. *Let $p \in (1, \infty)$, $\varphi_0 \in (0, \pi/2]$. Then there is a constant $C_{30}(p, \varphi_0) > 0$ such that for $\varphi, \varphi' \in [\varphi_0, \pi/2]$, $\tau \in \{-1, 1\}$, $\Psi \in L^p(\mathbb{R}^2)$, $\Phi \in L^p(\mathbb{R}^2)^3$, the ensuing four expressions*

$$\sum_{l=1}^{3}\Big(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}\Big|\sum_{j,k=1}^{3}\mathcal{D}_{jkl}(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta))(\eta)\cdot(n_k^{(\varphi)}\circ g^{(\varphi)})(\eta)\cdot\Phi_j(\eta)\cdot\sin^{-1}(\varphi)$$
$$-\sum_{j,k=1}^{3}\mathcal{D}_{jkl}(g^{(\varphi')}(\xi)-g^{(\varphi')}(\eta))\cdot(n_k^{(\varphi')}\circ g^{(\varphi')})(\eta)\cdot\Phi_j(\eta)\cdot\sin^{-1}(\varphi')\Big|\,d\eta\Big)^p\Big)^{1/p}, \tag{6.25}$$

$$\|A(\tau, p, \varphi, \infty, 1)(\Phi) - A(\tau, p, \varphi', \infty, 1)(\Phi)\|_p, \tag{6.26}$$

$$\|A^*(\tau, p, \varphi, \infty, 1)(\Phi) - A^*(\tau, p, \varphi', \infty, 1)(\Phi)\|_p, \tag{6.27}$$

$$\|F^*(\tau, p, \varphi, 0, 1)(\Psi) - F^*(\tau, p, \varphi', 0, 1)(\Psi)\|_p \tag{6.28}$$

are bounded by the term $C_{30}(p, \varphi_0)\cdot|\varphi-\varphi'|\cdot\|\Phi\|_p$ and $C_{30}(p, \varphi_0)\cdot|\varphi-\varphi'|\cdot\|\Psi\|_p$, respectively.

Proof: Put

$C_{30,1}(p, \varphi_0) := (27/\pi)\cdot\sin^{-2}(\varphi_0)\cdot\cos\varphi_0\cdot C_5(p)$;

$C_{30,2}(\varphi_0) := 21\cdot(2\cdot\pi)^{-1}\cdot\sin^{-2}(\varphi_0)$;

$C_{30,3}(p, \varphi_0) := C_{30,2}(\varphi_0)\cdot 36\cdot\sin^{-1}(\varphi_0)\cdot\cos\varphi_0\cdot C_5(p)$;

$C_{30,4}(p, \varphi_0) := \sin^{-1}(\varphi)\cdot(27/\pi)\cdot C_5(p)$; $\quad C_{30}(p, \varphi_0) := \sum_{j\in\{1,3,4\}} C_{30,j}(p, \varphi_0)$.

Now let φ, φ', τ, Ψ, Φ be given as in the lemma. It is sufficient to estimate the expression in (6.25). Due to (6.18), we know that the term in (6.26) is smaller than the one appearing in (6.26). As for the expression in (6.27), it may be dealt with by a duality argument; see (6.19). Finally, the term in (6.28) may be evaluated by replacing Φ with $((1/3)\cdot\delta_{jl}\cdot\Psi)_{1\le j\le 3}$, in (6.27) ($l \in \{1, 2, 3\}$), taking the sum in l, using (5.9), (6.9), (6.19), and then applying the estimate in (6.27).

In order to evaluate the term in (6.25), let us introduce some abbreviations. For $z \in \mathbb{R}^3\backslash\{0\}$, $\tilde\varphi \in \{\varphi, \varphi'\}$, $\xi, \eta \in \mathbb{R}^2$, we write

$K(z) := -3\cdot(4\cdot\pi)^{-1}\cdot(z_j\cdot z_l\cdot|z|^{-5})_{1\le l,j\le 3}$,

$h(\tilde\varphi, \xi, \eta) := (n^{(\tilde\varphi)}\circ g^{(\tilde\varphi)})(\eta)\cdot(g^{(\tilde\varphi)}(\xi) - g^{(\tilde\varphi)}(\eta))$.

It follows for $j, l \in \{1, 2, 3\}$, $z \in \mathbb{R}^3\backslash\{0\}$:

$$|D_3K_{jl}(z)| \le 21\cdot(4\cdot\pi)^{-1}\cdot|z|^{-4}. \tag{6.29}$$

Due to (5.9), the expression in (6.25) is bounded by $\sum_{m=1}^{3} R_m$, where

$$R_1 := |\sin^{-1}(\varphi) - \sin^{-1}(\varphi')|\cdot\Big(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}\Big|h(\varphi, \xi, \eta)\cdot K(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta))\cdot\Phi(\eta)\Big|\,d\eta\Big)^p d\xi\Big)^{1/p};$$

$$R_2 := \sin^{-1}(\varphi')\cdot\Big(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}|h(\varphi, \xi, \eta)|\cdot\Big|\Big(K(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) - K(g^{(\varphi')}(\xi) - g^{(\varphi')}(\eta))\Big)\cdot\Phi(\eta)\Big|\,d\eta\Big)^p d\xi\Big)^{1/p};$$

$$R_3 := \sin^{-1}(\varphi')\cdot\Big(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}|h(\varphi, \xi, \eta) - h(\varphi', \xi, \eta)|\cdot|K(g^{(\varphi')}(\xi) - g^{(\varphi')}(\eta))\cdot\Phi(\eta)|\,d\eta\Big)^p d\xi\Big)^{1/p}.$$

Since

$$|\sin^{-1}(\alpha) - \sin^{-1}(\alpha')| \le \sin^{-2}(\alpha\wedge\alpha')\cdot|\alpha-\alpha'| \quad \text{for } \alpha, \alpha' \in (0, \pi/2], \tag{6.30}$$

we conclude from (3.12) and Theorem 4.1:

$$R_1 \le C_{30,1}(p, \varphi_0)\cdot|\varphi-\varphi'|\cdot\|\Phi\|_p. \tag{6.31}$$

We further observe, for $\xi, \eta \in \mathbb{R}^2$ with $\xi \ne \eta$, $j, l \in \{1, 2, 3\}$:

$$\Big|K_{jl}(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) - K_{jl}(g^{(\varphi')}(\xi) - g^{(\varphi')}(\eta))\Big|$$
$$= \Big|(\cot\varphi - \cot\varphi')\cdot(|\xi|-|\eta|)\cdot\int_0^1 D_3K_{jl}\Big((\xi-\eta), \Big(\cot\varphi' + \vartheta\cdot(\cot\varphi - \cot\varphi')\Big)\cdot(|\xi|-|\eta|)\Big)\,d\vartheta\Big|$$

$$\leq \; 21 \cdot (4 \cdot \pi)^{-1} \cdot \left| \cot\varphi - \cot\varphi' \right| \cdot |\xi - \eta|^{-3} \quad \leq \quad C_{30,2}(\vartheta_0) \cdot |\varphi - \varphi'| \cdot |\xi - \eta|^{-3},$$

where the first inequality follows from (6.29). Applying (3.12) and Theorem 4.1 once more, we get

$$R_2 \quad \leq \quad C_{30,3}(p, \varphi_0) \cdot |\varphi - \varphi'| \cdot \|\Phi\|_p . \tag{6.32}$$

Finally we conclude from (3.13) and Theorem 4.1:

$$R_3 \quad \leq \quad C_{30,4}(p, \varphi_0) \cdot |\varphi - \varphi'| \cdot \|\Phi\|_p . \tag{6.33}$$

Now, combining (6.31) - (6.33) with the definition of $C_{30}(p, \varphi_0)$, it follows that the expression in (6.25) is bounded by $C_{30}(p, \varphi_0) \cdot |\varphi - \varphi'| \cdot \|\Phi\|_p$.

Lemma 6.15. *Take* $p \in (1, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\varphi_0 \in (0, \pi/2]$. *Then there is a number* $C_{31}(p, \varphi_0, \lambda) > 0$ *such that it holds for* $\varphi, \varphi' \in [\varphi_0\ \pi/2]$, $\tau \in \{-1, 1\}$, $\Phi \in L^p(\mathbb{R}^2)^3$:

$$\| J(\tau, p, \lambda, \varphi, 0, 1)(\Phi) - J(\tau, p, \lambda, \varphi', 0, 1)(\Phi) \|_p \quad \leq \quad C_{31}(p, \varphi_0, \lambda) \cdot |\varphi - \varphi'| \cdot \|\Phi\|_p .$$

Proof: We begin by defining the constant $C_{31}(p, \varphi_0, \lambda)$. To this end, we set

$$C_{31,1}(p, \varphi_0) := 9 \cdot C_9(p, \varphi_0) \cdot \left(\sin^{-2}(\varphi_0) \; + \; 2 \cdot \sin^{-1}(\varphi_0) \right);$$

$$C_{31,2}(\varphi_0, \lambda) := 108 \cdot \sin^{-2}(\varphi_0) \cdot C_{15}(|\arg\lambda|) \cdot (|\lambda| \vee |\lambda|^{-1});$$

$$C_{31}(p, \varphi_0, \lambda) := C_{31}(p, \varphi_0) \; + \; C_{31,1}(p, \varphi_0,) \; + \; 8 \cdot \pi \cdot \sin^{-1}(\varphi_0) \cdot C_{31,2}(\varphi_0, \lambda).$$

Let τ, φ, φ' be given as in the lemma. For $\widetilde{\varphi} \in \{\varphi, \varphi'\}$, $\xi, \eta \in \mathbb{R}^2\backslash\{0\}$ with $\xi \neq \eta$, we set

$$K(\widetilde{\varphi}, \xi, \eta) := \left(\sin^{-1}(\widetilde{\varphi}) \cdot \sum_{k=1}^{3} \widetilde{\mathcal{D}}^{\lambda}_{jkl}(g^{(\widetilde{\varphi})}(\xi) - g^{(\widetilde{\varphi})}(\eta)) \cdot (n_k^{(\widetilde{\varphi})} \circ g^{(\widetilde{\varphi})})(\eta) \right)_{1 \leq l, j \leq 3}.$$

Moreover, we define $K_1(\widetilde{\varphi}, \xi, \eta), \cdots, K_4(\widetilde{\varphi}, \xi, \eta)$ in the same way as $K(\widetilde{\varphi}, \xi, \eta)$, but with the function $\widetilde{\mathcal{D}}^{\lambda}_{jkl}$ replaced by $\mathcal{D}_{jkl}$, $D_j\overline{E}^{\lambda}_{kl} + D_k\overline{E}^{\lambda}_{jl}$, $D_j\widetilde{E}^{\lambda}_{kl} + D_k\widetilde{E}^{\lambda}_{jl}$, $-\delta_{jk} \cdot E_{4l}$, respectively $(j, k, l \in \{1, 2, 3\})$.

By the definition of $\widetilde{\mathcal{D}}^{\lambda}_{jkl}$ and $\mathcal{D}_{jkl}$ in (1.4), (1.7), respectively, and because of (5.1), it holds:

$$K \; = \; K_1 + K_2 \; = \; K_3 + K_4 .$$

Thus we have chosen two different ways of writing K as a sum of two summands. It follows by (6.11):

$$\| J(\tau, p, \lambda, \varphi, 0, 1)(\Phi) - J(\tau, p, \lambda, \varphi', 0, 1)(\Phi) \|_p \quad \leq \quad \sum_{m=1}^{4} R_m , \tag{6.34}$$

with

$$R_m := \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \chi_{(0,1)}(|\xi - \eta|) \cdot \left| K_m(\varphi, \xi, \eta) \cdot \Phi(\eta) - K_m(\varphi', \xi, \eta) \cdot \Phi(\eta) \right| d\eta \right)^p d\xi \right)^{1/p}$$

for $m \in \{1, 2\}$, and

$$R_m := \left(\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} \chi_{(1,\infty)}(|\xi - \eta|) \cdot \left(K_m(\varphi, \xi, \eta) \cdot \Phi(\eta) - K_m(\varphi', \xi, \eta) \cdot \Phi(\eta) \right) d\eta \right|^p d\xi \right)^{1/p}$$

for $m \in \{3, 4\}$. Now we apply Lemma 6.14 to obtain

$$R_1 \quad \leq \quad C_{30}(p, \varphi_0) \cdot |\varphi - \varphi'| \cdot \|\Phi\|_p . \tag{6.35}$$

Next we write R_4 in the following way:

$$R_4 \quad = \quad \left\| \left(\sin^{-1}(\varphi) \cdot (L_l(0, \varphi))_1 \otimes \left((n^{(\varphi)} \circ g^{(\varphi)}) \cdot \Phi \right) - \sin^{-1}(\varphi') \cdot (L_l(0, \varphi'))_1 \otimes \left((n^{(\varphi')} \circ g^{(\varphi')}) \cdot \Phi \right) \right)_{1 \leq l \leq 3} \right\|_p ,$$

with $L_l(0, \varphi)$, $L_l(0, \varphi')$ introduced in Definition 4.1. Recalling Corollary 4.2 and (3.2), we may infer from the preceding equation:

$$R_4 \quad \leq \quad C_{31,1}(p, \varphi_0) \cdot |\varphi - \varphi'| \cdot \|\Phi\|_p . \tag{6.36}$$

In order to evaluate R_2 and R_3, we define for $\xi, \eta \in \mathbb{R}^2$ with $\xi \neq \eta$:

$$K_{2,1}(\xi, \eta) := \sum_{k=1}^{3} \left(\sin^{-1}(\varphi) \cdot (n_k^{(\varphi)} \circ g^{(\varphi)})(\eta) - \sin^{-1}(\varphi') \cdot (n_k^{(\varphi')} \circ g^{(\varphi')})(\eta) \right) \cdot \left((D_j\overline{E}^{\lambda}_{kl} + D_k\overline{E}^{\lambda}_{jl})(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \right)_{1 \leq l, j \leq 3},$$

$$K_{2,2}(\xi, \eta) := \sum_{k=1}^{3} \sin^{-1}(\varphi') \cdot (n_k^{(\varphi')} \circ g^{(\varphi')})(\eta) \cdot \left((D_j\overline{E}^{\lambda}_{kl} + D_k\overline{E}^{\lambda}_{jl})(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) - (D_j\overline{E}^{\lambda}_{kl} + D_k\overline{E}^{\lambda}_{jl})(g^{(\varphi')}(\xi) - g^{(\varphi')}(\eta)) \right)_{1 \leq l, j \leq 3}.$$

Let $K_{3,1}$, $K_{3,2}$ be defined in the same way as $K_{2,1}$, $K_{2,2}$, but with the function $D_j\overline{E}^{\lambda}_{kl} + D_k\overline{E}^{\lambda}_{jl}$ replaced by $D_j\widetilde{E}^{\lambda}_{kl} + D_k\widetilde{E}^{\lambda}_{jl}$, for $j, k, l \in \{1, 2, 3\}$. Then it holds

$$K_m(\varphi, \xi, \eta) \; - \; K_m(\varphi', \xi, \eta) \quad = \quad K_{m,1}(\xi, \eta) \; + \; K_{m,2}(\xi, \eta) \tag{6.37}$$

for $m \in \{1, 2\}$, $\xi, \eta \in \mathbb{R}^2$ with $\xi \neq \eta$. We are going to use this equation in order to evaluate $R_2 + R_3$. To this end, we first note a consequence of (3.2) and Lemma 5.4. In fact, it holds for $\xi, \eta \in \mathbb{R}^2\backslash\{0\}$ with $\xi \neq \eta$:

$$|K_{2,1}(\xi, \eta) \cdot \Phi(\eta)| \leq C_{31,2}(\varphi_0, \lambda) \cdot |\varphi - \varphi'| \cdot |\Phi(\eta)|, \tag{6.38}$$

$$|K_{3,1}(\xi, \eta) \cdot \Phi(\eta)| \leq C_{31,2}(\varphi_0, \lambda) \cdot |\varphi - \varphi'| \cdot |\xi - \eta|^{-4} \cdot |\Phi(\eta)|. \tag{6.39}$$

In order to deal with $K_{2,2}$ and $K_{3,2}$, we take note of the ensuing estimate, which is valid for $j, k, l \in \{1, 2, 3\}$, $\xi, \eta \in \mathbb{R}^2\backslash\{0\}$ with $\xi \neq \eta$:

$$\begin{aligned}
&\big| D_j\overline{E}^{\lambda}_{kl}\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) - D_j\overline{E}^{\lambda}_{kl}\big(g^{(\varphi')}(\xi) - g^{(\varphi')}(\eta)\big)\big| \\
&= \Big| (\cot\varphi - \cot\varphi') \cdot (|\xi| - |\eta|) \\
&\qquad \cdot \int_0^1 D_3D_j\overline{E}^{\lambda}_{kl}\Big(\xi - \eta,\ \big(\cot\varphi' + \vartheta \cdot (\cot\varphi - \cot\varphi')\big) \cdot (|\xi| - |\eta|)\Big)\, d\vartheta \Big| \\
&\leq C_{15}(|\arg\lambda|) \cdot |\lambda| \cdot |\xi - \eta|^{-1} \cdot |\cot\varphi - \cot\varphi'| \cdot \big||\xi| - |\eta|\big| \\
&\leq C_{15}(|\arg\lambda|) \cdot |\lambda| \cdot 2 \cdot \sin^{-2}(\varphi_0) \cdot |\varphi - \varphi'|,
\end{aligned}$$

with the first inequality implied by Lemma 5.4. An analogous argument yields, for j, k, l, ξ, η as before:

$$\begin{aligned}
&\big| D_j\widetilde{E}^{\lambda}_{kl}\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) - D_j\widetilde{E}^{\lambda}_{kl}\big(g^{(\varphi')}(\xi) - g^{(\varphi')}(\eta)\big)\big| \\
&\leq C_{15}(|\arg\lambda|) \cdot |\lambda|^{-1} \cdot 2 \cdot \sin^{-2}(\varphi_0) \cdot |\varphi - \varphi'| \cdot |\xi - \eta|^{-4}.
\end{aligned}$$

Now it follows for $\xi, \eta \in \mathbb{R}^2\backslash\{0\}$ with $\xi \neq \eta$:

$$|K_{2,2}(\xi, \eta) \cdot \Phi(\eta)| \leq C_{31,2}(\varphi_0, \lambda) \cdot \sin^{-1}(\varphi_0) \cdot |\varphi - \varphi'| \cdot |\Phi(\eta)|, \tag{6.40}$$

$$|K_{3,2}(\xi, \eta) \cdot \Phi(\eta)| \leq C_{31,2}(\varphi_0, \lambda) \cdot \sin^{-1}(\varphi_0) \cdot |\varphi - \varphi'| \cdot |\xi - \eta|^{-4} \cdot |\Phi(\eta)|. \tag{6.41}$$

Recalling (6.37), we may conclude from (6.38) - (6.41):

$$\begin{aligned}
R_2 + R_3 \ &\leq\ 4 \cdot \sin^{-1}(\varphi_0) \cdot C_{31,2}(\varphi_0, \lambda) \cdot |\varphi - \varphi'| \\
&\cdot \Big(\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} \big(\chi_{(0,1)}(|\xi - \eta|) + \chi_{(1,\infty)}(|\xi - \eta|) \cdot |\xi - \eta|^{-4} \big) \\
&\qquad \cdot |\Phi(\eta)|\, d\eta \Big)^p d\xi \Big)^{1/p}.
\end{aligned}$$

But the right-hand side of the preceding inequality may be estimated by Lemma 4.9 (Young's inequality). It follows

$$R_2 + R_3 \leq 8 \cdot \pi \cdot \sin^{-1}(\varphi_0) \cdot C_{31,2}(\varphi_0, \lambda) \cdot |\varphi - \varphi'| \cdot \|\Phi\|_p. \tag{6.42}$$

Now the lemma is implied by (6.34), (6.35), (6.36) and (6.42).

Lemma 6.16. *Let* $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$, $\tau \in \{-1, 1\}$, $R \in (0, \infty)$, $\Phi \in L^p(\mathbb{R}^2)^3$, $\gamma, \gamma' \in [-\vartheta, \vartheta]$. *Then*

$$\begin{aligned}
&\| J(\tau, p, R \cdot e^{i\cdot\gamma}, \varphi, 0, 1)(\Phi) - J(\tau, p, R \cdot e^{i\cdot\gamma'}, \varphi, 0, 1)(\Phi) \|_p \\
&\leq 27 \cdot \pi \cdot C_{15}(\vartheta) \cdot \sin^{-1}(\varphi) \cdot (R + R^{-1}) \cdot |\gamma - \gamma'| \cdot \|\Phi\|_p.
\end{aligned} \tag{6.43}$$

Proof: For $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\xi, \eta \in \mathbb{R}^2$ with $\xi \neq \eta$, we put

$$\begin{aligned}
K^{(1)}(\lambda, \xi, \eta) &:= \Big(\chi_{(0,1)}(|\xi - \eta|) \cdot \sin^{-1}(\varphi) \\
&\quad \cdot \sum_{k=1}^{3} \big(D_j\overline{E}^{\lambda}_{kl} + D_k\overline{E}^{\lambda}_{jl} \big)\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) \cdot \big(n_k^{(\varphi)} \circ g^{(\varphi)}\big)(\eta) \Big)_{1 \leq l, j \leq 3}; \\
K^{(2)}(\lambda, \xi, \eta) &:= \Big(\chi_{(1,\infty)}(|\xi - \eta|) \cdot \sin^{-1}(\varphi) \\
&\quad \cdot \sum_{k=1}^{3} \big(D_j\widetilde{E}^{\lambda}_{kl} + D_k\widetilde{E}^{\lambda}_{jl} \big)\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) \cdot \big(n_k^{(\varphi)} \circ g^{(\varphi)}\big)(\eta) \Big)_{1 \leq l, j \leq 3}.
\end{aligned}$$

From the definition of $\widetilde{\mathcal{D}}^{\lambda}_{jkl}$ in (1.4), and from (5.1), (6.11), we obtain for $\lambda, \lambda' \in \mathbb{C}\backslash(-\infty, 0]$:

$$\begin{aligned}
&\| J(\tau, p, \lambda, \varphi, 0, 1)(\Phi) - J(\tau, p, \lambda', \varphi, 0, 1)(\Phi) \|_p \\
&= \Big(\int_{\mathbb{R}^2} \Big| \Big(\sum_{v=1}^{2} \sum_{j=1}^{3} \int_{\mathbb{R}^2} \big(K^{(v)}_{jl}(\lambda, \xi, \eta) - K^{(v)}_{jl}(\lambda', \xi, \eta) \big) \\
&\qquad \cdot \Phi_j(\eta)\, d\eta \Big)_{1 \leq l \leq 3} \Big|^p d\xi \Big)^{1/p}.
\end{aligned}$$

By Lemma 5.4 and (3.18), it follows that the left-hand side in (6.43) is bounded by

$$\begin{aligned}
&C_{15}(\vartheta) \cdot |\gamma - \gamma'| \cdot 27 \cdot \sin^{-1}(\varphi) \cdot \Big(\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} \big(R \cdot \chi_{(0,1)}(|\xi - \eta|) \\
&\qquad + R^{-1} \cdot \chi_{(1,\infty)}(|\xi - \eta|) \cdot |\xi - \eta|^{-4} \big) \cdot |\Phi(\eta)|\, d\eta \Big)^p d\xi \Big)^{1/p}.
\end{aligned}$$

Applying Young's inequality (Lemma 4.9), we see that the preceding expression may be estimated against the right-hand side of (6.43).

Lemma 6.17. *Let* $p \in (1, \infty)$, $\varphi_0 \in (0, \pi/2]$, $\tau \in \{-1, 1\}$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $R \in [0, \infty)$, $S \in (0, \infty]$. *According to Lemma 6.7, the following mappings are well defined:*

$$\Psi_1 : [\varphi_0, \pi/2] \mapsto \mathcal{LB}\big(L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R)),\ L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R)) \big),$$

$$\Psi_1(\varphi) := F^*(\tau, p, \varphi, R, 1) \qquad \textit{for } \varphi \in [\varphi_0, \pi/2];$$

$\Psi_2 : [\varphi_0, \pi/2] \mapsto \mathcal{LB}\Big(L^p(\mathbb{B}_2(0,S))^3,\ L^p(\mathbb{B}_2(0,S))^3\Big),$
$\Psi_2(\varphi) := A(\tau, p, \varphi, S, 1) \quad$ *for* $\varphi \in [\varphi_0, \pi/2]$;

$\Psi_3 : [\varphi_0, \pi/2] \mapsto \mathcal{LB}\Big(L^p(\mathbb{B}_2(0,S))^3,\ L^p(\mathbb{B}_2(0,S))^3\Big),$
$\Psi_3(\varphi) := A^*(\tau, p, \varphi, S, 1) \quad$ *for* $\varphi \in [\varphi_0, \pi/2]$;

$\Psi_4 : [\varphi_0, \pi/2] \mapsto \mathcal{LB}\Big(L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0,R))^3,\ L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0,R))^3\Big),$
$\Psi_4(\varphi) := J(\tau, p, \lambda, \varphi, R, 1) \quad$ *for* $\varphi \in [\varphi_0, \pi/2]$.

Let $j \in \{1, 2, 3, 4\}$, *and suppose that for any* $\varphi \in [\varphi_0, \pi/2]$, *the operator* $\Psi_j(\varphi)$ *has property* F_+. *Then it follows* $index(\Psi_j(\varphi)) = 0$ *for* $\varphi \in [\varphi_0, \pi/2]$.

Proof: Since

$$F(\tau, p, \pi/2, R, 1) = (\tau/2) \cdot id\Big(L^p\big(\mathbb{R}^2\backslash\mathbb{B}_2(0,R)\big)\Big),$$
$$A(\tau, p, \pi/2, S, 1) = A^*(\tau, p, \pi/2, S, 1) = (\tau/2) \cdot id\Big(L^p\big(\mathbb{B}_2(0,S)\big)^3\Big),$$

the index of the three preceding operators has value zero. Furthermore, we know from Lemma 6.13 that the operator $J(\tau, p, \pi/2, R, 1)$ has index zero too. On the other hand, it is clear by Lemma 6.14 and 6.15 that Ψ_j, for $j \in \{1, 2, 3, 4\}$, is a continuous mapping. Hence Lemma 6.17 follows from Theorem 6.1.

Chapter 7

Fourier Analysis on Locally Compact Abelian Groups

In this chapter, we are going to present some results from Fourier analysis on locally compact abelian groups. We shall apply these results in the next chapter in order to investigate the behaviour of the operators $\Lambda(\tau, p, \mathbb{K}(\varphi))$ and $\Pi(\tau, p, \mathbb{K}(\varphi))$.

Consider a non-empty set G, which is equipped with a group operation $\oplus$ ("addition"), and with a topology $\mathcal{T}$. Assume the group $(G, \oplus)$ to be abelian, and the topological space $(G, \mathcal{T})$ to be Hausdorff and locally compact. Suppose in addition that the triple $(G, \oplus, \mathcal{T})$ forms a locally compact abelian group ([42, p. 253]). If $x \in G$, the symbol x^{-1} stands for the inverse of x with respect to the operation $\oplus$. Let $\mathcal{B}$ denote the Borel-σ-algebra generated by $\mathcal{T}$ on G (see [3, p. 205]). Then there exists a non-negative regular measure m on $\mathcal{B}$ with the properties to follow:

$$m(V) > 0 \quad \text{for } V \in \mathcal{T} \text{ with } V \neq \emptyset, \qquad m(x \oplus E) = m(E) \quad \text{for } E \in \mathcal{B}. \tag{7.1}$$

(Concerning the concept of a "regular measure", we refer to [3, p. 206].) If m' is another non-negative regular measure on $\mathcal{B}$ satisfying the properties listed in (7.1), then there is a number $c \in (0, \infty)$ with $m = c \cdot m'$.

A proof of the preceding results may be found in [3, p. 303 - 310, Section 9.2]. A regular measure with properties as in (7.1) is called a "Haar measure" on G. From now on, such a Haar measure m on G is assumed to be fixed.

If $f : G \mapsto \mathbb{C}$ is a $\mathcal{B}$-measurable function, integrable with respect to m, then the corresponding integral is written as

$$\int_G f(x)\ dm(x).$$

We point out the following equations, which hold for $\mathcal{B}$-measurable and m-integrable functions $f : G \mapsto \mathbb{C}$:

$$\int_G f(x \oplus y)\ dm(x) \quad = \quad \int_G f(x)\ dm(x) \quad \text{for } y \in G, \tag{7.2}$$

$$\int_G f(x^{-1})\ dm(x) \quad = \quad \int_G f(x)\ dm(x).$$

For a proof, first consider the case $f = \chi_A$, for a Borel set $A \in \mathcal{B}$. Then the first equation in (7.2) is valid due to (7.1), and the second one holds according to [42, p. 3, 1.1.4]. As for the general case, we may reduce it to the preceding one by choosing a sequence of simple functions (finite linear combinations of characteristic functions of Borel sets) converging to f in a suitable sense; see [3, p. 51; p. 54, Proposition 2.1.7; p. 64/65; p. 70, Theorem 2.4.1].

If $p \in [1, \infty)$, let $L^p(G, m)$ denote the set of all $\mathcal{B}$-measurable mappings f from G into $\mathbb{C}$ with

$$\int_G |f(x)|^p\ dm(x) \quad < \quad \infty.$$

For $p \in [1, \infty)$, $f \in L^p(G, m)$, we put

$$\|f\|_{p,G} := \Big(\int_G |f(x)|^p\ dm(x)\Big)^{1/p}.$$

As usual, we shall identify functions from G into $\mathbb{C}$ if they only differ on a set with m-measure zero. Hence the mapping $\|\ \|_{p,G}$ may be considered as a norm on $L^p(G, m)$.

According to [3, p. 318/319, Proposition 9.4.1], we have for $f, g \in L^1(G, m)$:

$$\int_G |\,f(x \oplus y^{-1}) \cdot g(y)\,|\ dm(y) \quad < \quad \infty \qquad \text{for } m\text{-almost every } x \in G. \tag{7.3}$$

Thus, for $f, g \in L^1(G, m)$, the convolution

$$f \triangle g : G \mapsto \mathbb{C}, \qquad (f \triangle g)(x) := \int_G f(x \oplus y^{-1}) \cdot g(y)\ dm(y) \qquad \text{for } m\text{-a.e. } x \in G$$

is well defined. Furthermore, we have $f \triangle g \in L^1(G, m)$, and it holds

$$\|f \triangle g\|_{1,G} \quad \le \quad \|f\|_{1,G} \cdot \|g\|_{1,G} \qquad \text{for } f, g \in L^1(G, m); \tag{7.4}$$

see [3, p. 318/319, Proposition 9.4.1]. A mapping $\gamma : G \mapsto \mathbb{C}$ with

$$\gamma(x \oplus y) \quad = \quad \gamma(x) \cdot \gamma(y), \qquad |\gamma(x)| = 1 \qquad \text{for } x, y \in G,$$

is called a "character" of G. By Γ, we denote the set of all <u>continuous</u> characters of G. For $\gamma, \gamma' \in \Gamma$, the mapping $\gamma \square \gamma' : G \mapsto \mathbb{C}$ is to be defined by the usual multiplication of complex-valued functions. The set Γ equipped with the operation $\square$ is a group, the so-called "dual group" of G. If $f \in L^1(G, m)$, we call the function

$$\widehat{f} : \Gamma \mapsto \mathbb{C}, \qquad \widehat{f}(\gamma) := \int_G f(x) \cdot \gamma(x^{-1})\ dm(x) \qquad \text{for } \gamma \in \Gamma,$$

the "Fourier transform" of f (with respect to the Haar measure m). Note that

$$\{\, f \in L^1(G, m) \ :\ \widehat{f}(\gamma) = 0 \ \text{ for } \gamma \in \Gamma \,\} \quad = \quad \{\text{zero function in } L^1(G, m)\,\}; \tag{7.5}$$

see [42, p. 16, 1.3.4; p. 29, 1.7.3 (b)]. Define

$$A(\Gamma) := \{\, \widehat{f} \ :\ f \in L^1(G, m) \,\}.$$

According to [42, p. 9, Theorem 1.2.4 (a)], $A(\Gamma)$ is a subalgebra of the space of continuous functions from Γ into $\mathbb{C}$. We equip Γ with the weak topology induced by the functions $f \in A(\Gamma)$; see [42, p. 8/9, 1.2.3; p. 249, A10], [50, p. 55, Definition 8.9]. In this way, Γ becomes a locally compact Hausdorff space ([42, p. 10, 1.2.5; p. 262/263, D4]).

For $f, g \in L^1(G, m)$, the Fourier transform of $f \triangle g$ is given by $\widehat{f} \cdot \widehat{g}$ ([42, p. 9, Theorem 1.2.4 (b)]).

If $B \in L^1(G, m)^{3\times 3}$, $f \in L^1(G, m)^3$, we define the Fourier transform $\widehat{B}$, $\widehat{f}$ of B, f, respectively, and the convolution $B \triangle f$ in the following way:

$$\widehat{B} := (\widehat{B}_{jk})_{1 \le j,k \le 3}, \qquad \widehat{f} := (\widehat{f}_l)_{1 \le l \le 3}, \qquad B \triangle f := \Big(\sum_{k=1}^{3} B_{jk} \triangle f_k\Big)_{1 \le j \le 3}.$$

Lemma 7.1. *Take $\gamma_0 \in \Gamma$. Then there is some $A \in \mathcal{B}$ with $m(A) < \infty$ and $\widehat{\chi_A}(\gamma_0) \ne 0$.*

Proof: Assume $\widehat{\chi_A}(\gamma_0) = 0$ for all $A \in \mathcal{B}$ with $m(A) < \infty$. Put

$$\mathcal{E} := \Big\{ \sum_{j=1}^{n} \alpha_j \cdot \chi_{A_j} \ :\ n \in \mathbb{N},\ \alpha_1, \cdots, \alpha_n \in \mathbb{C},\ A_1, \cdots, A_n \in \mathcal{B} \ \text{ with } m(A_i) < \infty \text{ for } i \in \{1, \cdots, n\} \Big\}.$$

Then it follows $\widehat{f}(\gamma_0) = 0$ for any $f \in \mathcal{E}$.

Now take $f \in L^1(G, m)$. According to [3, p .51; p. 52, Proposition 2.1.7; p. 64/65; p. 70, Theorem 2.4.1], there is a sequence (f_n) in $\mathcal{E}$ with $\|f - f_n\|_{1,G} \to 0$ for $n \to \infty$. But for $n \in \mathbb{N}$, we have

$$|\,\widehat{f}(\gamma_0) - \widehat{f}_n(\gamma_0)\,| \quad \le \quad \int_G |f(x) - f_n(x)| \cdot |\gamma_0(x^{-1})|\ dm(x) \quad = \quad \|f - f_n\|_{1,G},$$

so that $f(\gamma_0) = 0$. Since f is an arbitrary function from $L^1(G, m)$, we have arrived at a contradiction to [42, p. 49, Theorem 2.6.2].

Theorem 7.1. *Let $B \in L^1(G, m)^{3\times 3}$, $p \in (1, \infty)$. Then the operator $K : L^p(G, m)^3 \mapsto L^p(G, m)^3$, with*

$$K(f)\,(x) := f(x) + \int_G B(x \oplus y^{-1}) \cdot f(y)\ dm(y) \quad \text{for } f \in L^p(G, m)^3,\ x \in G, \tag{7.6}$$

is well defined and continuous.

The convolution on the right-hand side of (7.6) will be abbreviated in the same way as in the case $f \in L^1(G, m)$ *(see (7.3)), that is, by* $(B\triangle f)(x)$.

If there is some $\epsilon > 0$ *with* $|\det(E_3 + \widehat{B}(\gamma))| \geq \epsilon$ *for* $\gamma \in \Gamma$, *then* K *is bijective.*

Conversely, if K *is bijective, then it holds* $\det(E_3 + \widehat{B}(\gamma)) \neq 0$ *for* $\gamma \in \Gamma$.

Proof: Take $h \in L^1(G, m)$. Applying Hölder's inequality in an analogous way as in the proof of Lemma 4.9, and recalling (7.2), we find for $f \in L^p(G, m)$:

$$\left(\int_G \left(\int_G |h(x \oplus y^{-1})| \cdot |f(y)| \, dm(y)\right)^p dm(x)\right)^{1/p} \leq \|h\|_{1,G} \cdot \|f\|_{p,G}. \tag{7.7}$$

This means in particular (see [3, p. 159; p. 68, Corollary 2.3.12; p. 64/65]):

$$\int_G |h(x \oplus y^{-1})| \cdot |f(y)| \, dm(y) < \infty \quad \text{for } m\text{-almost every } x \in G,$$

so that the integral on the right-hand side of (7.6) exists for m-almost every $x \in G$. In addition, we may conclude from (7.7) that the integral on the right-hand side of (7.6) defines a function belonging to $L^p(G, m)^3$. This means the operator K is well defined. It further follows from (7.7) that K is continuous.

Now suppose there is some $\epsilon > 0$ with

$$|\det(E_3 + \widehat{B}(\gamma))| \geq \epsilon \quad \text{for } \gamma \in \Gamma. \tag{7.8}$$

Without loss of generality we may assume $\epsilon < 1$. Since $A(\Gamma)$ is an algebra, as mentioned above, we have $\det(E_3 + \widehat{B}(\gamma)) - 1 \in A(\Gamma)$. So there is some $f \in L^1(G, m)$ with

$$\det(E_3 + \widehat{B}(\gamma)) = 1 + \widehat{f}(\gamma) \quad \text{for } \gamma \in \Gamma. \tag{7.9}$$

Put

$$E := \{z \in \mathbb{C} : |z+1| > \epsilon/2\} = \mathbb{R}^2 \backslash \overline{\mathbb{B}_2((-1,0), \epsilon/2)}.$$

Then E is a domain in $\mathbb{R}^2$, with $0 \in E$. Define the function $F : E \mapsto \mathbb{C}$ by $F(z) := z \cdot (1+z)^{-1}$ for $z \in E$. This function is real-analytic, with $F(0) = 0$. Due to (7.8), the function f introduced in (7.9) satisfies the relation

$$im(\widehat{f}) \subset \mathbb{R}^2 \backslash \mathbb{B}_2((-1,0), \epsilon).$$

Recurring to a theorem by Wiener and Levy (see [42, p. 133, Theorem 6.2.4]), we may conclude that $F(\widehat{f}) \in A(\Gamma)$, that is, there is some $f_0 \in L^1(G, m)$ with $\widehat{f_0} = F(\widehat{f})$. Recalling the definition of F, we obtain $\widehat{f_0} = \widehat{f} \cdot (1 + \widehat{f})^{-1}$. Now define the operator

$$\widetilde{K}^{(1)} : L^1(G, m)^3 \cap L^p(G, m)^3 \mapsto L^1(G, m)^3 \cap L^p(G, m)^3$$

by

$$\widetilde{K}^{(1)}(g) := g - f_0 \triangle g \quad \text{for } g \in L^1(G, m)^3 \cap L^p(G, m)^3.$$

Because of (7.3), (7.4) and (7.7), this definition makes sense. For g as before, we have

$$\left(\widetilde{K}^{(1)}(g)\right)^\wedge = \widehat{g} - \widehat{f_0} \cdot \widehat{g} = (1+\widehat{f})^{-1} \cdot \widehat{g}. \tag{7.10}$$

If $j \in \{1, 2, 3\}$, we put $\sigma(j) := \max\{1, 2, 3\} \backslash \{j\}$, $\varrho(j) := \min\{1, 2, 3\} \backslash \{j\}$. Using these notations, we introduce the operator

$$\widetilde{K}^{(2)} : L^1(G, m)^3 \cap L^p(G, m)^3 \mapsto L^1(G, m)^3 \cap L^p(G, m)^3$$

by setting for $i \in \{1, 2, 3\}$, $g \in L^1(G, m)^3 \cap L^p(G, m)^3$:

$$\begin{aligned} &\widetilde{K}_i^{(2)}(g) \\ &:= \sum_{j=1}^{3} (-1)^{i+j} \cdot \Big(\delta_{\sigma(j),\sigma(i)} \cdot \delta_{\varrho(j),\varrho(i)} \cdot \widetilde{K}_j^{(1)}(g) + \delta_{\sigma(j),\sigma(i)} \cdot B_{\varrho(j),\varrho(i)} \triangle \widetilde{K}_j^{(1)}(g) \\ &\quad + \delta_{\varrho(j),\varrho(i)} \cdot B_{\sigma(j),\sigma(i)} \triangle \widetilde{K}_j^{(1)}(g) + B_{\sigma(j),\sigma(i)} \triangle B_{\varrho(j),\varrho(i)} \triangle \widetilde{K}_j^{(1)}(g) \\ &\quad - \delta_{\sigma(j),\varrho(i)} \cdot \delta_{\varrho(j),\sigma(i)} \cdot \widetilde{K}_j^{(1)}(g) - \delta_{\sigma(j),\varrho(i)} \cdot B_{\varrho(j),\sigma(i)} \triangle \widetilde{K}_j^{(1)}(g) \\ &\quad - \delta_{\varrho(j),\sigma(i)} \cdot B_{\sigma(j),\varrho(i)} \triangle \widetilde{K}_j^{(1)}(g) - B_{\sigma(j),\varrho(i)} \triangle B_{\varrho(j),\sigma(i)} \triangle \widetilde{K}_j^{(1)}(g)\Big). \end{aligned}$$

Referring to (7.3), (7.4) and (7.7) once more, we see that $\widetilde{K}^{(2)}$ is well defined.

For any matrix $A \in \mathbb{C}^{3\times3}$ and for $i, j \in \{1, 2, 3\}$, let $A^{(i,j)} \in \mathbb{C}^{2\times2}$ denote the matrix obtained from A by deleting the i-th row and the j-th column. Then it holds for $g \in L^1(G, m)^3 \cap L^p(G, m)^3$, $\gamma \in \Gamma$:

$$\begin{aligned} &\left(\widetilde{K}^{(2)}(g)\right)^\wedge(\gamma) \\ &= \left((-1)^{i+j} \cdot \det\left[\left(E_3 + \widehat{B}(\gamma)\right)^{(j,i)}\right]\right)_{1\leq i,j\leq 3} \cdot \left(\widetilde{K}^{(1)}(g)\right)^\wedge(\gamma). \end{aligned} \tag{7.11}$$

Now it follows, with g, γ as in (7.11):

$$\begin{aligned} \left(K(\widetilde{K}^{(2)}(g))\right)^\wedge(\gamma) &= \left(\widetilde{K}^{(2)}(g)\right)^\wedge(\gamma) + \widehat{B}(\gamma) \cdot \left(\widetilde{K}^{(2)}(g)\right)^\wedge(\gamma) \\ &= (E_3 + \widehat{B}(\gamma)) \cdot \left(\widetilde{K}^{(2)}(g)\right)^\wedge(\gamma) = \det(E_3 + \widehat{B}(\gamma)) \cdot \left(\widetilde{K}^{(1)}(g)\right)^\wedge(\gamma). \end{aligned} \tag{7.12}$$

The last equation is a consequence of (7.11). From (7.10), (7.9) and (7.12) we infer, for g, γ as in (7.11):

$$\left(K(\widetilde{K}^{(2)}(g))\right)^\wedge(\gamma) = \widehat{g}(\gamma).$$

Because of (7.5), this implies

$$K(\widetilde{K}^{(2)}(g)) = g \quad \text{for } g \in L^1(G, m)^3 \cap L^p(G, m)^3. \tag{7.13}$$

It may be shown in a similar way that

$$\widetilde{K}^{(2)}(K(g)) = g \quad \text{for } g \in L^1(G,m)^3 \cap L^p(G,m)^3. \tag{7.14}$$

Now put

$$\mathfrak{C}_1 := 6 \cdot \max\Big\{ 1 + 2 \cdot \|B_{jk}\|_{1,G} + \|B_{jk}\|_{1,G}^2 \;:\; j,k \in \{1,2,3\} \Big\};$$

$$\mathfrak{C}_2 := \mathfrak{C}_1 \cdot 9 \cdot (1 + \|f_0\|_{1,G}).$$

Then we infer from (7.7), if $g \in L^1(G,m)^3 \cap L^p(G,m)^3$:

$$\|\widetilde{K}^{(2)}(g)\|_{p,G} \;\leq\; \mathfrak{C}_1 \cdot \sum_{j=1}^{3} \|\widetilde{K}_j^{(1)}(g)\|_{p,G} \;\leq\; \mathfrak{C}_2 \cdot \|g\|_{p,G}.$$

Moreover, the operator $\widetilde{K}^{(2)}$ is linear. Since the set $L^1(G,m)^3 \cap L^p(G,m)^3$ is dense in $L^p(G,m)^3$ (see [3, p. 108, Proposition 3.4.2]), it follows that $\widetilde{K}^{(2)}$ has a unique extension to a linear bounded operator $\widetilde{K} : L^p(G,m)^3 \mapsto L^p(G,m)^3$. Using the fact that K and $\widetilde{K}$ are continuous, and recalling (7.13) and (7.14), we conclude

$$(K \circ \widetilde{K})(g) = g, \qquad (\widetilde{K} \circ K)(g) = g \quad \text{for } g \in L^p(G,m)^3.$$

But these equations imply that K is bijective.

Conversely, let us assume that the latter statement is true. Then, since K is continuous, we may apply the open mapping theorem to obtain some $\mathfrak{C}_3 > 0$ with

$$\|g\|_{p,G} \;\leq\; \mathfrak{C}_3 \cdot \big\| g + B \triangle g \big\|_{p,G} \quad \text{for } g \in L^p(G,m)^3. \tag{7.15}$$

Now take $\gamma_0 \in \Gamma$, and assume $\det(E_3 + \widehat{B}(\gamma_0)) = 0$. Then there is a vector $\alpha \in \mathbb{C}^3 \backslash \{0\}$ with

$$(E_3 + \widehat{B}(\gamma_0)) \cdot \alpha = 0. \tag{7.16}$$

According to [42, p. 51, Theorem 2.6.5], there is an open neighbourhood U of γ_0 in Γ, and a function $H_{jl} \in L^1(G,m)$ for $j,l \in \{1,2,3\}$ such that

$$\|H_{jl}\|_{1,G} \;\leq\; (18 \cdot \mathfrak{C}_3)^{-1}, \tag{7.17}$$

$$\delta_{jl} + \widehat{B}_{jl}(\gamma) - \widehat{H}_{jl}(\gamma) = \delta_{jl} + \widehat{B}_{jl}(\gamma_0) \quad \text{for } \gamma \in U. \tag{7.18}$$

Put $H := (H_{jl})_{1 \leq j,l \leq 3}$. Let V be a compact neighbourhood of γ_0 with $V \subset U$. Referring to [42, p. 49, Theorem 2.6.2], we may choose a function $f_1 \in L^1(G,m)$ with

$$\widehat{f}_1(\gamma) = 1 \quad \text{for } \gamma \in V, \qquad \widehat{f}_1(\gamma) = 0 \quad \text{for } \gamma \in \Gamma \backslash U.$$

Next, applying Lemma 7.1, we fix some $A \in \mathcal{B}$ with $m(A) < \infty$ and $\widehat{\chi_A}(\gamma_0) \neq 0$. Put $f_2 := \chi_A \triangle f_1$. Due to (7.4) and (7.7), this function f_2 belongs to $L^1(G,m) \cap L^p(G,m)$. In addition, we observe that

$$\widehat{f}_2(\gamma_0) = \widehat{\chi_A}(\gamma_0) \cdot \widehat{f}_1(\gamma_0) \neq 0, \qquad \widehat{f}_2(\gamma) = 0 \quad \text{for } \gamma \in \Gamma \backslash U.$$

Set $g_0 := (\alpha_1 \cdot f_2, \alpha_2 \cdot f_2, \alpha_3 \cdot f_2)$, with α from (7.16). It follows that $g_0 \in L^1(G,m)^3 \cap L^p(G,m)^3$, and

$$\big(g_0 + B \triangle g_0 - H \triangle g_0\big)^{\wedge}(\gamma) = \big(E_3 + \widehat{B}(\gamma) - \widehat{H}(\gamma)\big) \cdot \widehat{g}_0(\gamma) \quad \text{for } \gamma \in \Gamma.$$

But $\widehat{g}_0(\gamma)$ vanishes for $\gamma \in \Gamma \backslash U$. Furthermore, if γ is any element of Γ, then it holds $\widehat{g}_0(\gamma) = \widehat{f}_2(\gamma) \cdot \alpha$. Recalling (7.16) and (7.18), we thus arrive at the equation

$$\big(E_3 + \widehat{B}(\gamma) - \widehat{H}(\gamma)\big) \cdot \widehat{g}_0(\gamma) = 0 \quad \text{for } \gamma \in \Gamma,$$

hence,

$$\big(E_3 + B \triangle g_0 - H \triangle g_0\big)^{\wedge}(\gamma) = 0 \quad \text{for } \gamma \in \Gamma.$$

This result and (7.5) imply that $E_3 + B \triangle g_0 - H \triangle g_0$ is the zero function. Now we obtain

$$\|g_0\|_{p,G} \;\leq\; \mathfrak{C}_3 \cdot \|H \triangle g_0\|_{p,G} \;\leq\; \mathfrak{C}_3 \cdot \sum_{j,k=1}^{3} \| H_{jk} \triangle (g_0)_k \|_{p,G} \tag{7.19}$$

$$\leq\; \mathfrak{C}_3 \cdot \sum_{j,k=1}^{3} \|H_{jk}\|_{1,G} \cdot \|(g_0)_k\|_{p,G} \;\leq\; (1/2) \cdot \|g_0\|_{p,G},$$

where the first inequality follows from (7.15), (7.18), the third one from (7.7), and the fourth one from (7.17). Thus we have shown $\|g_0\|_{p,G} = 0$. This implies $g_0 = 0$ m-almost everywhere, so that $\|g_0\|_{1,G} = 0$. On the other hand, we know that $\widehat{g}_0(\gamma_0) \neq 0$, hence $\|g_0\|_{1,G} > 0$ due to (7.5). Thus we have arrived at a contradiction, and it follows $\det(E_3 + \widehat{B}(\gamma_0)) = 0$. This finishes our proof.

Now let us consider some concrete locally compact abelian groups.

Of course, the set $\mathbb{R}$ of real numbers with the usual addition and with the euclidean topology forms a group of this type. As a Haar measure on this group, we may take the Lebesgue measure on $\mathbb{R}$, as follow at once from [3, p. 26, Proposition 1.4.1; p. 30, Proposition 1.4.5]. The corresponding dual group consists precisely of all functions $\exp(i \cdot \alpha \cdot id(\mathbb{R}))$, with $\alpha \in \mathbb{R}$ (see [42, p. 12]).

Definition 7.1. *Consider the set* $\mathbb{T} := \{ e^{i \cdot \varphi} \;:\; \varphi \in \mathbb{R} \}$, *equipped with multiplication in* $\mathbb{C}$ *as group operation, and with the topology induced by the euclidean topology of* $\mathbb{R}^2$. *Let* $\mathcal{B}_{\mathbb{T}}$ *denote the corresponging Borel-σ-algebra on* $\mathbb{T}$, *that is,* $\mathcal{B}_{\mathbb{T}}$ *coincides with the restriction to* $\mathbb{T}$ *of the usual Borel-σ-algebra on* $\mathbb{R}^2$; *see [3, p. 205, Lemma 7.2.2]. In particular, every* $\mathcal{B}_{\mathbb{T}}$*-measurable set is Lebesgue-measurable too. This of course implies a corresponding result with respect to integration. Moreover, we set*

$$m_{\mathbb{T}}(A) := \int_0^{2\cdot\pi} \chi_A(e^{i\cdot\theta})\, d\theta \quad (A \in \mathcal{B}_{\mathbb{T}}); \qquad \Gamma_{\mathbb{T}} := \Big\{ \exp\big(i \cdot n \cdot (\arg \circ\, id(\mathbb{T}))\big) \;:\; n \in \mathbb{Z} \Big\}.$$

We further consider the set $\mathbb{R}_+ := (0, \infty)$, *equipped with the usual multiplication as group operation, and with the topology inherited from the euclidean space* $\mathbb{R}$. *Let* $\mathcal{B}_+$ *denote the corresponding Borel-σ-algebra. Thus* $\mathcal{B}_+$ *is the restriction to* $(0, \infty)$ *of the usual Borel-σ-algebra on* $\mathbb{R}$, *so that any* $\mathcal{B}_+$*-measurable set is Lebesgue measurable as well, with a corresonding statement holding true with respect to integration. Finally, we put*

$$m_+(A) := \int_0^\infty \chi_A(r) \cdot r^{-1}\, dr \quad \text{for } A \in \mathcal{B}_+; \qquad \Gamma_+ := \left\{ \left(id((0,\infty)) \right)^{i\cdot\alpha} : \alpha \in \mathbb{R} \right\}.$$

According to [3, p. 304], $\mathbb{T}$ is a locally compact abelian group, and the mapping $m_\mathbb{T}$ defined above is a Haar measure on $\mathbb{T}$. Moreover, $\Gamma_\mathbb{T}$ is the dual group of $\mathbb{T}$, as explained in [42, p. 13]. As for $\mathbb{R}_+$, m_+, Γ_+, we note the following lemma, which is well known, although we cannot cite a reference for its proof:

Lemma 7.2. *The space* $\mathbb{R}_+$ *from Definition 7.1 is a locally compact abelian group, the mapping* m_+ *is a Haar measure on* $\mathbb{R}_+$, *and* Γ_+ *is the dual group of* $\mathbb{R}_+$.

Proof: It is clear that $\mathbb{R}_+$ is a locally compact abelian group. For $A \in \mathcal{B}_+$, $t \in (0, \infty)$, it holds

$$\int_0^\infty \chi_{t\cdot A}(r) \cdot r^{-1}\, dr = \int_0^\infty \chi_A(t^{-1} \cdot r) \cdot r^{-1}\, dr = \int_0^\infty \chi_A(r) \cdot r^{-1}\, dr.$$

For any open set $U \subset \mathbb{R}_+$, we have $m_+(U) > 0$. The measure m_+ is regular, as follows from [3, p. 206, Proposition 7.2.3]. For any $\alpha \in \mathbb{R}$, the function $(id(\mathbb{R}_+))^{i\cdot\alpha}$ mapping $\mathbb{R}_+$ into $\mathbb{C}$ is continuous, with

$$(r \cdot s)^{i\cdot\alpha} = r^{i\cdot\alpha} \cdot s^{i\cdot\alpha}, \qquad |r^{i\cdot\alpha}| = 1, \qquad \text{for } r, s \in (0, \infty).$$

This implies $(id(\mathbb{R}_+))^{i\cdot\alpha} \in \Gamma_+$ for $\alpha \in \mathbb{R}$. Conversely, take $\gamma \in \Gamma_+$. We define a function $F : \mathbb{R} \mapsto \mathbb{C}$ by $F(t) := \gamma(e^t)$ for $t \in \mathbb{R}$. Then

$$F(s+t) = \gamma(e^{s+t}) = \gamma(e^s) \cdot \gamma(e^t) = F(s) \cdot F(t), \qquad \text{for } s, t \in \mathbb{R}.$$

This means that F belongs to the dual group of the locally compact abelian group $\mathbb{R}$, where $\mathbb{R}$ is understood to be equipped with the usual addition and topology. As mentioned above, this implies there is some $\alpha \in \mathbb{R}$ such that $F(t) = e^{i\cdot\alpha\cdot t}$ for $t \in \mathbb{R}$. It follows

$$\gamma(r) = \gamma(e^{\ln r}) = e^{i\cdot\alpha\cdot\ln r} = r^{i\cdot\alpha} \qquad \text{for } r \in (0, \infty).$$

This proves the last statement of the lemma.

For $j \in \{1, 2\}$, let G_j be a locally compact abelian group with topology $\mathcal{T}_j$, Haar

measure m_j, and with dual group Γ_j. We assume that for $j \in \{1, 2\}$, the topology $\mathcal{T}_j$ has a countable base. Let $\mathcal{B}_j$ denote the Borel-σ-algebra on G_j generated by $\mathcal{T}_j$ $(j \in \{1, 2\})$.

On the cartesian product $G_1 \times G_2$, we introduce the group operation which acts by addition in each coordinate ("complete direct sum"; see [42, p. 254/255, B7]). Moreover, we equip $G_1 \times G_2$ with the product topology $\mathcal{T}_1 \odot \mathcal{T}_2$ of $\mathcal{T}_1$ and $\mathcal{T}_2$. In this way, $G_1 \times G_2$ becomes a locally compact abelian group ([42, p. 254/255, B7].

Denote by $\mathcal{B}(G_1, G_2)$ the Borel-σ-algebra generated by $\mathcal{T}_1 \odot \mathcal{T}_2$ on $G_1 \times G_2$. Since $\mathcal{T}_j$ has a countable base $(j \in \{1, 2\})$, the product $\mathcal{B}_1 \odot \mathcal{B}_2$ of the σ-algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ coincides with $\mathcal{B}(G_1, G_2)$ (see [3, p. 154 - 158, Section 5.1]. Hence the product measure $m_1 \odot m_2$ on $\mathcal{B}_1 \odot \mathcal{B}_2$ ([3, p. 154 - 158, Section 5.1]) is a Haar measure with respect to the locally compact abelian group $G_1 \times G_2$ introduced before. In particular, the measure $m_1 \odot m_2$ is regular; see [3, p. 242, Proposition 7.6.2]. The dual group of $\Gamma_1 \odot \Gamma_2$ of $G_1 \times G_2$ consists precisely of the mappings $\gamma : G_1 \times G_2 \mapsto \mathbb{C}$ having the property that there are $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$ with

$$\gamma((x_1, x_2)) = \gamma_1(x_1) \cdot \gamma_2(x_2) \qquad \text{for } (x_1, x_2) \in G_1 \times G_2;$$

see [42, p. 37/38, 2.2.2]. Hence it follows for $f \in L^1(G_1 \times G_2, m_1 \odot m_2)$ that the Fourier transform $\hat{f}$ of f with respect to $m_1 \odot m_2$ is given by

$$\hat{f}((\gamma_1, \gamma_2)) = \int_{G_1} \int_{G_2} f((x_1, x_2)) \cdot \gamma_1((x_1)^{-1}) \cdot \gamma_2((x_2)^{-1})\, dm_2(x_2)\, dm_1(x_1)$$

for $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$. This implies in the case $G_1 = \mathbb{R}_+$, $G_2 = \mathbb{T}$ (recall Lemma 7.2):

Corollary 7.1. *The product measure* $m_{\mathbb{R}_+} \odot m_\mathbb{T}$ *is a Haar measure on the locally compact abelian group* $\mathbb{R}_+ \times \mathbb{T}$. *For* $f \in L^1(\mathbb{R}_+ \times \mathbb{T}, m_{\mathbb{R}_+} \odot m_\mathbb{T})$, *it holds*

$$\int_{\mathbb{R}_+ \times \mathbb{T}} f\, d(m_{\mathbb{R}_+} \odot m_\mathbb{T}) = \int_0^{2\cdot\pi} \int_0^\infty f(r, e^{i\cdot\theta}) \cdot r^{-1}\, dr\, d\theta.$$

The dual group $\Gamma_+ \odot \Gamma_\mathbb{T}$ *of* $\mathbb{R}_+ \times \mathbb{T}$ *consists precisely of the mappings* $\gamma_{\alpha,n} : \mathbb{R}_+ \times \mathbb{T} \mapsto \mathbb{C}$ *with*

$$\gamma_{\alpha,n}(r, z) := r^{i\cdot\alpha} \cdot e^{i\cdot n\cdot \arg z} \qquad \text{for } r \in (0, \infty),\ z \in \mathbb{T},$$

where $\alpha \in \mathbb{R}$, $n \in \mathbb{Z}$. *If* $f \in L^1(\mathbb{R}_+ \times \mathbb{T}, m_{\mathbb{R}_+} \odot m_\mathbb{T})$, *then the Fourier transform* $\hat{f}$ *of* f *with respect to* $m_{\mathbb{R}_+} \odot m_\mathbb{T}$ *satisfies the equation*

$$\hat{f}(\gamma_{\alpha,n}) = \int_0^{2\cdot\pi} \int_0^\infty f(r, e^{i\cdot\theta}) \cdot r^{i\cdot\alpha} \cdot e^{i\cdot n\cdot\theta} \cdot r^{-1}\, dr\, d\theta \qquad \text{for } \alpha \in \mathbb{R},\ n \in \mathbb{Z}.$$

Moreover, if $\in L^1(\mathbb{R}_+ \times \mathbb{T}, m_{\mathbb{R}_+} \odot m_\mathbb{T})$, $p \in [1, \infty)$, $g \in L^p(\mathbb{R}_+ \times \mathbb{T}, m_{\mathbb{R}_+} \odot m_\mathbb{T})$, *then it holds for* $r \in (0, \infty)$, $\theta \in [0, 2\cdot\pi]$:

$$(f \Delta g)(r, e^{i\cdot\theta}) = \int_0^{2\cdot\pi} \int_0^\infty f(r/s, e^{i\cdot(\theta-\sigma)}) \cdot g(s, \sigma) \cdot s^{-1}\, ds\, d\sigma.$$

Chapter 8

The Operators $\Lambda(\tau, p, \mathbb{K}(\varphi))$ and $\Pi(\tau, p, \mathbb{K}(\varphi))$

In this chapter, we want to find out which are the values $p \in (1, \infty)$ having the property that the operators $\Pi(\tau, p, \mathbb{K}(\varphi))$ and $\Lambda(\tau, p, \mathbb{K}(\varphi))$ are invertible. To this end, we shall apply the results from Chapter 7 related to Fourier analysis on locally compact abelian groups. However, only a partial result will be obtained: For certain values of $\tau \in \{-1, 1\}$, $\varphi \in (0, \pi/2]$ and $p \in (1, 2)$, these operators are not invertible (Theorem 8.2). On the other hand, $\Pi(\tau, p, \mathbb{K}(\varphi))$ will turn out to be topological for $p \in [2, \infty)$ (Corollary 8.3). This leaves open the question how the operator $\Lambda(\tau, p, \mathbb{K}(\varphi))$ behaves if $p \in [2, \infty)$. We shall deal with this question in Chapter 13 by recurring to an additional tool (L^2-theory for the Stokes system on bounded Lipschitz domains).

Definition 8.1. *Let $\mathbb{R}_+$, $\mathbb{T}$ be the locally compact abelian groups introduced in Definition 7.1, and m_+, $m_{\mathbb{T}}$ the Haar measures defined there. For $p \in [1, \infty)$, $\sigma \in \mathbb{N}$, we set* $L^p_* := L^p(\mathbb{R}_+ \times \mathbb{T}, \; m_+ \odot m_{\mathbb{T}})$.

According to Corollary 7.1, this means that L^p_* contains those and only those measurable functions $f : \mathbb{R}_+ \times \mathbb{T} \mapsto \mathbb{C}$ having the property that

$$\int_0^{2\cdot\pi} \int_0^{\infty} |f(r, e^{i\cdot\theta})|^p \cdot r^{-1} \, dr \, d\theta \quad < \quad \infty. \tag{8.1}$$

Let $\varphi \in (0, \pi/2]$. We put for $r \in (0, \infty)$, $\theta \in [0, 2\cdot\pi]$:

$$g_0^{(\varphi)}(r, \theta) := r^2 + 1 - 2\cdot r \cdot (\cos^2(\varphi) + \cos\theta \cdot \sin^2(\varphi))$$
$$= (r-1)^2 + 2\cdot r \cdot (1 - \cos\theta) \cdot \sin^2(\varphi).$$

Note that $g_0^{(\varphi)}(r, \theta) \neq 0$ if and only if $(r, \theta) \in \mathcal{P}$, with

$$\mathcal{P} := \{ (r, \theta) \in (0, \infty) \times [0, 2\cdot\pi] \;\; : \;\; r \neq 1 \;\; or \;\; \theta \notin \{0, 2\cdot\pi\} \}.$$

We further define the mappings $R^{(\varphi)} : \mathbb{C} \times \mathcal{P} \mapsto \mathbb{C}$, $S^{(\varphi)} : \mathbb{C} \times \mathcal{P} \mapsto \mathbb{C}^{3\times 3}$ *by*

$$R^{(\varphi)}(z, r, \theta) := (2\cdot\pi)^{-1} \cdot \cos\varphi \cdot \sin^2(\varphi) \cdot (1-\cos\theta) \cdot r^z \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-3/2},$$

$$S_{jk}^{(\varphi)}(z, r, \theta) := 3\cdot(2\cdot\pi)^{-1} \cdot \cos\varphi \cdot \sin^2(\varphi) \cdot (1-\cos\theta) \cdot r^z \cdot \widetilde{S}_j^{(a)}(\varphi, r, \theta) \cdot \widetilde{S}_k^{(b)}(\varphi, r, \theta) \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-5/2}$$

for $j, k \in \{1, 2, 3\}$, $z \in \mathbb{C}$, $(r, \theta) \in \mathcal{P}$, *where*

$$\widetilde{S}_1^{(a)}(\varphi, r, \theta) := (r - \cos\theta) \cdot \sin\varphi, \qquad \widetilde{S}_1^{(b)}(\varphi, r, \theta) := (r\cdot\cos\theta - 1) \cdot \sin\varphi,$$

$$\widetilde{S}_2^{(a)}(\varphi, r, \theta) := -\sin\theta \cdot \sin\varphi, \qquad \widetilde{S}_2^{(b)}(\varphi, r, \theta) := -r \cdot \sin\theta \cdot \sin\varphi,$$

$$\widetilde{S}_3^{(a)}(\varphi, r, \theta) := \widetilde{S}_3^{(b)}(\varphi, r, \theta) := (r-1) \cdot \cos\varphi, \qquad \text{for } r \in (0, \infty),\ \theta \in [0, 2\cdot\pi].$$

When the operator $\Pi(\tau, p, \mathbb{K}(\varphi))$ *is transformed into polar coordinates, it turns into an integral operator with kernel* $R^{(\varphi)}$. *Similarly, after transforming* $\Lambda(\tau, p, \mathbb{K}(\varphi))$ *into polar coordinates and performing some additional computations, we obtain an integral operator with matrix-valued kernel* $S^{(\varphi)}$; *see the proof of Lemma 8.3. When studying these transformed operators, it will be convenient to use the following functions, defined for* $z \in \mathbb{C}$, $(r, \theta) \in \mathcal{P}$:

$$a_1^{(\varphi)}(z, r, \theta) := \gamma_\varphi \cdot \sin^2(\varphi) \cdot (1-\cos\theta)^2 \cdot r^z \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-5/2};$$

$$a_2^{(\varphi)}(z, r, \theta) := \gamma_\varphi \cdot \sin^2(\varphi) \cdot \cos\theta \cdot (1-\cos\theta)^2 \cdot r^z \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-5/2},$$

$$a_3^{(\varphi)}(z, r, \theta) := \gamma_\varphi \cdot \sin^2(\varphi) \cdot \sin\theta \cdot (1-\cos\theta)^2 \cdot r^z \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-5/2},$$

$$a_4^{(\varphi)}(z, r, \theta) := \gamma_\varphi \cdot \sin^2(\varphi) \cdot \sin\theta \cdot (1-\cos\theta) \cdot r^z \cdot (r-1) \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-5/2},$$

$$a_5^{(\varphi)}(z, r, \theta) := \gamma_\varphi \cdot \sin^2(\varphi) \cdot \cos\theta \cdot \sin\theta \cdot (1-\cos\theta) \cdot r^z \cdot (r-1) \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-5/2},$$

$$a_6^{(\varphi)}(z, r, \theta) := \gamma_\varphi \cdot (1-\cos\theta) \cdot r^z \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-3/2},$$

$$a_7^{(\varphi)}(z, r, \theta) := \gamma_\varphi \cdot \cos\theta \cdot (1-\cos\theta) \cdot r^z \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-3/2},$$

where we used the abbreviation $\gamma_\varphi := 3\cdot(2\cdot\pi)^{-1} \cdot \cos\varphi \cdot \sin^2(\varphi)$.

The next lemma shows how the functions $R^{(\varphi)}$, $S^{(\varphi)}$ and $a_1^{(\varphi)}, \cdots, a_7^{(\varphi)}$ are related.

Lemma 8.1. *Let* $\varphi \in (0, \pi/2]$. *Then it holds for* $z \in \mathbb{C}$, $(r, \theta) \in \mathcal{P}$:

$$R^{(\varphi)}(z, r, \theta) = (1/3) \cdot a_6^{(\varphi)}(z, r, \theta),$$

$$S_{11}^{(\varphi)}(z, r, \theta) = -a_1^{(\varphi)}(z+1, r, \theta) + \big(1 - 2\cdot\sin^2(\varphi)\big) \cdot a_2^{(\varphi)}(z+1, r, \theta) + \sin^2(\varphi) \cdot a_7^{(\varphi)}(z, r, \theta),$$

$$S_{12}^{(\varphi)}(z, r, \theta) = -a_4^{(\varphi)}(z+1, r, \theta) - a_3^{(\varphi)}(z+1, r, \theta),$$

$$S_{13}^{(\varphi)}(z, r, \theta) = \cos\varphi \cdot \sin\varphi \cdot a_6^{(\varphi)}(z, r, \theta) - (2\cdot\cos\varphi \cdot \sin\varphi - \cot\varphi) \cdot a_1^{(\varphi)}(z+1, r, \theta) - \cot\varphi \cdot a_1^{(\varphi)}(z, r, \theta)$$

$$S_{21}^{(\varphi)}(z, r, \theta) = -a_5^{(\varphi)}(z, r, \theta) + a_3^{(\varphi)}(z, r, \theta),$$

$$S_{22}^{(\varphi)}(z, r, \theta) = a_1^{(\varphi)}(z+1, r, \theta) + a_2^{(\varphi)}(z+1, r, \theta),$$

$$S_{23}^{(\varphi)}(z, r, \theta) = -\cot\varphi \cdot a_4^{(\varphi)}(z, r, \theta),$$

$$S_{31}^{(\varphi)}(z, r, \theta) = \cos\varphi \cdot \sin\varphi \cdot a_7^{(\varphi)}(z, r, \theta) - 2\cdot\cos\varphi \cdot \sin\varphi \cdot a_2^{(\varphi)}(z+1, r, \theta) - \cot\varphi \cdot a_1^{(\varphi)}(z+1, r, \theta) + \cot\varphi \cdot a_1^{(\varphi)}(z, r, \theta),$$

$$S_{32}^{(\varphi)}(z, r, \theta) = -\cot\varphi \cdot a_4^{(\varphi)}(z+1, r, \theta),$$

$$S_{33}^{(\varphi)}(z, r, \theta) = \cos^2(\varphi) \cdot a_6^{(\varphi)}(z, r, \theta) - 2\cdot\cos^2(\varphi) \cdot a_1^{(\varphi)}(z+1, r, \theta).$$

These equations may be proved by some simple computations, which we do not write out here.

In the following lemma, we are going to check existence of certain integrals needed later on.

Lemma 8.2. *Let* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$. *Then it holds for* $\zeta \in [0, 4)$:

$$\int_0^{2\cdot\pi} \int_0^\infty (1-\cos\theta)^{15/8} \cdot r^\zeta \cdot (1 + |\ln r|) \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-5/2}\, dr\, d\theta < \infty,$$

and for $\zeta \in [0, 3)$:

$$\int_0^{2\cdot\pi} \int_0^\infty (1-\cos\theta)^{11/8} \cdot |r-1| \cdot r^\zeta \cdot (1 + |\ln r|) \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-5/2}\, dr\, d\theta < \infty,$$

and finally for $\zeta \in [0, 2)$:

$$\int_0^{2\cdot\pi} \int_0^\infty (1-\cos\theta)^{7/8} \cdot r^\zeta \cdot (1 + |\ln r|) \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-3/2}\, dr\, d\theta < \infty.$$

Proof: We shall split the domain of integration $(0, \infty)$ into the parts $(0, 1/2)$, $(1/2, 2)$ and $(2, \infty)$, and then evaluate the three integrals arising in this way. Our calculations are based on the following inequalities, which hold for $r \in (0, \infty)$, $\theta \in [0, 2\cdot\pi]$:

$$g_0^{(\varphi)}(r, \theta) \geq (r-1)^2, \tag{8.2}$$

$$g_0^{(\varphi)}(r, \theta) \geq 2\cdot(1-\cos\theta) \cdot r \cdot \sin^2(\varphi). \tag{8.3}$$

Take $\zeta_1 \in [0, 4)$, $\zeta_2 \in [0, 3)$, $\zeta_3 \in [0, 2)$. For brevity, we set for $(r, \theta) \in \mathcal{P}$:

$$A_1(r, \theta) := (1-\cos\theta)^{15/8} \cdot r^{\zeta_1} \cdot (1 + |\ln r|) \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-5/2},$$

$$A_2(r,\theta) := (1-\cos\theta)^{11/8} \cdot |r-1| \cdot r^{\zeta_2} \cdot (1+|\ln r|) \cdot \big(g_0^{(\varphi)}(r,\theta)\big)^{-5/2},$$

$$A_3(r,\theta) := (1-\cos\theta)^{7/8} \cdot r^{\zeta_3} \cdot (1+|\ln r|) \cdot \big(g_0^{(\varphi)}(r,\theta)\big)^{-3/2}.$$

In order to evaluate the integral of A_v over $(0, 2\cdot\pi)\times(0, 1/2)$, we use inequality (8.2), which implies $g_0^{(\varphi)}(r,\theta) \geq 1/4$ for $r \in (0, 1/2)$, $\theta \in [0, 2\cdot\pi]$. Hence it follows for $v \in \{1, 2, 3\}$:

$$\int_0^{2\cdot\pi}\int_0^{1/2} A_v(r,\theta)\ dr\ d\theta \quad\leq\quad 4^4\cdot\pi\cdot\int_0^{1/2}(1-\ln r)\ dr \quad<\quad \infty. \tag{8.4}$$

Now we turn to integrating A_v over $(0, 2\cdot\pi)\times(2,\infty)$, starting with the inequality $g_0^{(\varphi)}(r,\theta) \geq r^2/4$ $(r \in (2,\infty),\ \theta\in[0, 2\cdot\pi])$, which is also a consequence of (8.2), and which implies that

$$r^{\zeta_1}\cdot\big(g_0^{(\varphi)}(r,\theta)\big)^{-5/2} \quad\leq\quad 32\cdot r^{\zeta_1-5}, \qquad r^{\zeta_2}\cdot|r-1|\cdot\big(g_0^{(\varphi)}(r,\theta)\big)^{-5/2} \quad\leq\quad 16\cdot r^{\zeta_2-4},$$

$$r^{\zeta_3}\cdot\big(g_0^{(\varphi)}(r,\theta)\big)^{-3/2} \quad\leq\quad 8\cdot r^{\zeta_3-3}.$$

Since $\epsilon := \min\{5-\zeta_1,\ 4-\zeta_2,\ 3-\zeta_3\} > 1$, we conclude for $v\in\{1,2,3\}$:

$$\int_0^{2\cdot\pi}\int_2^{\infty} A_v(r,\theta)\ dr\ d\theta \quad\leq\quad 2^8\cdot\pi\cdot\int_2^{\infty}(1+\ln r)\cdot r^{-\epsilon}\ dr \quad<\quad\infty. \tag{8.5}$$

Finally we are going to integrate $A_v(r,\theta)$ with respect to $\varphi\in(0, 2\cdot\pi)$ and $r\in(1/2, 2)$. Due to (8.2) and (8.3), we obtain for $(r,\theta)\in\mathcal{P}$ with $r\in(1/2, 2]$:

$$\begin{aligned}
&r^{\zeta_1}\cdot(1-\cos\theta)^{15/8}\cdot\big(g_0^{(\varphi)}(r,\theta)\big)^{-5/2}\\
&\quad\leq\ \big(2\cdot\sin^2(\varphi)\big)^{-9/4}\cdot|r-1|^{-1/2}\cdot(1-\cos\theta)^{-3/8}\cdot r^{\zeta_1-9/4}\\
&\quad\leq\ \sin^{-9/2}(\varphi)\cdot|r-1|^{-1/2}\cdot(1-\cos\theta)^{-3/8};
\end{aligned}$$

$$\begin{aligned}
&r^{\zeta_2}\cdot(1-\cos\theta)^{11/8}\cdot|r-1|\cdot\big(g_0^{(\varphi)}(r,\theta)\big)^{-5/2} \quad\leq\quad r^{\zeta_2}\cdot(1-\cos\theta)^{11/8}\cdot\big(g_0^{(\varphi)}(r,\theta)\big)^{-2}\\
&\quad\leq\ \big(2\cdot\sin^2(\varphi)\big)^{-7/4}\cdot|r-1|^{-1/2}\cdot(1-\cos\theta)^{-3/8}\cdot r^{\zeta_2-7/4}\\
&\quad\leq\ \sin^{-7/2}(\varphi)\cdot|r-1|^{-1/2}\cdot(1-\cos\theta)^{-3/8};
\end{aligned}$$

$$\begin{aligned}
&r^{\zeta_3}\cdot(1-\cos\theta)^{7/8}\cdot\big(g_0^{(\varphi)}(r,\theta)\big)^{-3/2}\\
&\quad\leq\ \big(2\cdot\sin^2(\varphi)\big)^{-5/4}\cdot|r-1|^{-1/2}\cdot(1-\cos\theta)^{-3/8}\cdot r^{\zeta_3-5/4}\\
&\quad\leq\ \sin^{-5/2}(\varphi)\cdot|r-1|^{-1/2}\cdot(1-\cos\theta)^{-3/8}.
\end{aligned}$$

Note that it holds for $\theta\in[0, 2\cdot\pi]$:

$$1-\cos\theta \quad=\quad 1-\cos^2(\theta/2)+\sin^2(\theta/2) \quad=\quad 2\cdot\sin^2(\theta/2).$$

Thus, applying the substitution rule according to the equation $\sin(\theta/2) = 2\cdot t\cdot(1+t^2)^{-1}$, we obtain for $v\in\{1,\cdots,7\}$:

$$\int_0^{2\cdot\pi}\int_{1/2}^{2} A_v(r,\theta)\ dr\ d\theta \quad<\quad\infty. \tag{8.6}$$

The lemma now follows from (8.4), (8.5) and (8.6).

Corollary 8.1. *Let* $\varphi\in(0,\pi/2]$, $j\in\{1,\cdots,7\}$, $k,l\in\{1,2,3\}$, $z\in\mathbb{C}$ *with* $1\leq\Re(z)<3$. *Define the functions* G, H_{kl} : $\mathbb{R}_+\times\mathbb{T}\mapsto\mathbb{C}$ *by*

$$G(r, e^{i\cdot\theta}) := R^{(\varphi)}(z,r,\theta), \qquad H_{kl}(r, e^{i\cdot\theta}) := S_{kl}^{(\varphi)}(z,r,\theta) \qquad \textit{for } (r,\theta)\in\mathcal{P}.$$

Then G *and* H_{kl} *belong to* L^1_*.

Proof: Referring to Lemma 8.1 and 8.2, it is easy to check that the functions G and H_{kl} satisfy the condition in (8.1), which is necessary and sufficient for any function to belong to L^1_*.

Due to Corollary 8.1, Theorem 7.1, Corollary 7.1 and Lemma 8.2, we are able to introduce the following notations:

Definition 8.2. *Let* $p\in(1,\infty)$, $\varphi\in(0,\pi/2]$, $\tau\in\{-1,1\}$. *Then we define the operators* $\mathcal{R}(\tau,p,\varphi): L^p_*\mapsto L^p_*$, $\mathcal{S}(\tau,p,\varphi): (L^p_*)^3\mapsto(L^p_*)^3$ *by*

$$\begin{aligned}
\mathcal{R}(\tau,p,\varphi)(\Phi)(r, e^{i\cdot\theta}) \quad:=&\quad \tau\cdot\Phi(r, e^{i\cdot\theta})\\
&+\int_0^{2\cdot\pi}\int_0^{\infty} R^{(\varphi)}(1+2/p,\ r/s,\ e^{i\cdot(\theta-\sigma)})\cdot\Phi(s, e^{i\cdot\sigma})\cdot s^{-1}\ ds\ d\sigma,
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}(\tau,p,\varphi)(\Psi)(r, e^{i\cdot\theta}) \quad:=&\quad \tau\cdot\Psi(r, e^{i\cdot\theta})\\
&+\Big(\int_0^{2\cdot\pi}\int_0^{\infty}\sum_{j=1}^{3} S_{ji}^{(\varphi)}(1+2/p,\ r/s,\ e^{i\cdot(\theta-\sigma)})\cdot\Psi_j(s, e^{i\cdot\sigma})\cdot s^{-1}\ ds\ d\sigma\Big)_{1\leq i\leq 3}
\end{aligned}$$

where $\Phi\in L^p_*$, $\Psi\in(L^p_*)^3$, $r\in(0,\infty)$, $\theta\in[0, 2\cdot\pi]$.

Furthermore, we set for $z\in[0,2)\times\mathbb{R}$, $n\in\mathbb{Z}$:

$$Y^{(\varphi)}(z,n) := \int_0^{2\cdot\pi}\int_0^{\infty} R^{(\varphi)}(z,r,\theta)\cdot e^{i\cdot n\cdot\theta}\ dr\ d\theta,$$

$$Z^{(\varphi)}(z,n) := \int_0^{2\cdot\pi}\int_0^{\infty} S^{(\varphi)}(z,r,\theta)\cdot e^{i\cdot n\cdot\theta}\ dr\ d\theta.$$

Put $\zeta_j := 4$ *for* $j\in\{1,2,3\}$, $\zeta_j := 3$ *for* $j\in\{4,5\}$, *and* $\zeta_j := 2$ *if* $j\in\{6,7\}$.

Then we define for $j \in \{1, \cdots, 7\}$, $z \in [0, \zeta_j) \times \mathbb{R}$, $n \in \mathbb{Z}$:

$$A_j^{(\varphi)}(z, n) := \int_0^{2\cdot\pi} \int_0^{\infty} a_j^{(\varphi)}(z, r, \theta) \cdot e^{i\cdot n\cdot\theta} \, dr \, d\theta . \tag{8.7}$$

As will be seen in the proof of the next lemma, the operator $\mathcal{R}(\tau, p, \varphi)$ arises by transforming $\Pi(\tau, p, \mathbb{K}(\varphi))$ into polar coordinates. A comparison with Corollary 7.1 shows that $\mathcal{R}(\tau, p, \varphi)$ may be considered as a convolution operator with respect to the group $\mathbb{R}_+ \times \mathbf{T}$. Unfortunately, when $\Lambda(\tau, p, \mathbb{K}(\varphi))$ is transformed into polar coordinates, no such convolution may be identified. However, by further computations involving addition formulas of trigonometry, an operator of suitable form – namely $\mathcal{S}(\tau, p, \varphi)$ – may be obtained.

Lemma 8.3. *Let* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $\tau \in \{-1, 1\}$. *Then the operator* $\Pi(\tau, p, \mathbb{K}(\varphi))$ *is topological if and only if the mapping* $\mathcal{R}(\tau, p, \varphi)$ *is bijective. An analogous relation holds true with respect to* $\Lambda(\tau, p, \mathbb{K}(\varphi))$ *and* $\mathcal{S}(\tau, p, \varphi)$.

Proof: We shall show that $\Lambda(\tau, p, \mathbb{K}(\varphi))$ is topological if and only if $\mathcal{S}(\tau, p, \varphi)$ is bijective. As for the analogous relation between $\Pi(\tau, p, \mathbb{K}(\varphi))$ and $\mathcal{R}(\tau, p, \varphi)$ stated in the lemma, it may be shown by similar arguments. However, the calculations arising in latter case are much shorter than in the former one, due to the above mentioned fact that $\Pi(\tau, p, \mathbb{K}(\varphi))$, upon transformation into polar coordinates, turns out to be a convolution operator with respect to the group $\mathbb{R}_+ \times \mathbf{T}$.

We recall the parametric representation of $h^{(\varphi)}$ of $\partial\mathbb{K}(\varphi)$ introduced in Chapter 3.

Let $f \in L^p(\partial\mathbb{K}(\varphi))^3$ be given. Due to (3.3) - (3.5) and (5.9), we get for $r \in (0, \infty)$, $\theta \in [0, 2\cdot\pi]$:

$$\begin{aligned}\Big(\Lambda(\tau, p, \mathbb{K}(\varphi))(f) \circ h^{(\varphi)}\Big)(r, \theta) &= (\tau/2) \cdot (f \circ h^{(\varphi)})(r, \theta) \\ &\quad + \Big(\int_{\partial\mathbb{K}(\varphi)} \sum_{j,k=1}^{3} \mathcal{D}_{jkl}\big(h^{(\varphi)}(r, \theta) - y\big) \cdot n_k^{(\varphi)}(y) \cdot f_j(y) \, d\,\partial\mathbb{K}(\varphi)(y)\Big)_{1\le l\le 3} \\ &= (\tau/2) \cdot (f \circ h^{(\varphi)})(r, \theta) \\ &\quad + \int_0^{2\cdot\pi} \int_0^{\infty} 3 \cdot (4\cdot\pi)^{-1} \cdot \cos\varphi \cdot \sin^2(\varphi) \cdot r \cdot s \cdot (1 - \cos(\theta - \sigma)) \\ &\qquad \cdot \big(\tilde{g}(r, s, \theta - \sigma)\big)^{-5/2} \cdot Q(r, s, \theta, \sigma) \cdot (f \circ h^{(\varphi)})(s, \theta) \, ds \, d\theta ,\end{aligned} \tag{8.8}$$

where the functions $\tilde{g} : (0, \infty)^2 \times [0, 2\cdot\pi] \mapsto \mathbb{R}$, $Q : (0, \infty)^2 \times [0, 2\cdot\pi]^2 \mapsto \mathbb{R}^{3\times 3}$ are defined by

$$\tilde{g}(r, s, \varrho) := r^2 + s^2 - 2\cdot r\cdot s \cdot (\cos^2(\varphi) + \cos\varrho \cdot \sin^2(\varphi)) \quad \text{for } r, s \in (0, \infty),\ \varrho \in [0, 2\cdot\pi];$$

$$Q(r, s, \theta, \sigma) := \Big(q_j(r, s, \theta, \sigma) \cdot q_l(r, s, \theta, \sigma)\Big)_{1\le j,l\le 3},$$

with

$$\begin{aligned} q_1(r, s, \theta, \sigma) &:= r \cdot \cos\theta \cdot \sin\varphi - s \cdot \cos\sigma \cdot \sin\varphi , \\ q_2(r, s, \theta, \sigma) &:= r \cdot \sin\theta \cdot \sin\varphi - s \cdot \sin\sigma \cdot \sin\varphi , \\ q_3(r, s, \theta, \sigma) &:= (r - s) \cdot \cos\varphi , \quad \text{for } r, s \in (0, \infty),\ \theta, \sigma \in [0, 2\cdot\pi]. \end{aligned}$$

For $\theta \in [0, 2\cdot\pi]$, we put $B(\theta) := \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. In addition, we introduce the functions $p_j^{(a)}, p_j^{(b)} : (0, \infty)^2 \times [0, 2\cdot\pi] \mapsto \mathbb{R}$ $(j \in \{1, 2, 3\})$ by setting

$$\begin{aligned} p_1^{(a)}(r, s, \delta) &:= (r - s \cdot \cos\delta) \cdot \sin\varphi , & p_1^{(b)}(r, s, \delta) &:= (r \cdot \cos\delta - s) \cdot \sin\varphi , \\ p_2^{(a)}(r, s, \delta) &:= -s \cdot \sin\delta \cdot \sin\varphi , & p_2^{(b)}(r, s, \delta) &:= -r \cdot \sin\delta \cdot \sin\varphi , \end{aligned}$$

$$p_3^{(a)}(r, s, \delta) := p_3^{(b)}(r, s, \delta) := (r - s) \cdot \cos\varphi , \quad \text{for } r, s \in (0, \infty),\ \delta \in [0, 2\cdot\pi].$$

We further define for r, s, δ as before:

$$P(r, s, \delta) := \Big(p_j^{(a)}(r, s, \delta) \cdot p_l^{(b)}(r, s, \delta)\Big)_{1\le j,l\le 3}.$$

Then it holds for $r, s \in (0, \infty)$, $\theta, \sigma \in [0, 2\cdot\pi]$:

$$Q(r, s, \theta, \sigma) = B(\theta) \cdot P(r, s, \theta - \sigma) \cdot B(\sigma). \tag{8.9}$$

This equation involves a lengthy calculation based on addition formulas for trigonometric functions. Combining (8.8) and (8.9), we obtain for $r \in (0, \infty)$, $\theta \in [0, 2\cdot\pi]$:

$$\begin{aligned} r^{2/p} \cdot B(\theta) \cdot \Big(\Lambda(\tau, p, \mathbb{K}(\varphi))(f) \circ h^{(\varphi)}\Big)(r, \theta) \\ = (\tau/2) \cdot r^{2/p} \cdot B(\theta) \cdot (f \circ h^{(\varphi)})(r, \theta) \\ + 3 \cdot (4\cdot\pi)^{-1} \cdot \cos\varphi \cdot \sin^2(\varphi) \cdot r^{1+2/p} \\ \cdot \int_0^{2\cdot\pi} \int_0^{\infty} s^{2-2/p} \cdot (1 - \cos(\theta - \sigma)) \cdot P(r, s, \theta - \sigma) \cdot (\tilde{g}(r, s, \theta - \sigma))^{-5/2} \\ \cdot s^{2/p} \cdot B(\sigma) \cdot (f \circ h^{(\varphi)})(s, \sigma) \cdot s^{-1} \, ds \, d\theta \\ = (\tau/2) \cdot r^{2/p} \cdot B(\theta) \cdot (f \circ h^{(\varphi)})(r, \theta) \\ + (1/2) \cdot \int_0^{2\cdot\pi} \int_0^{\infty} S^{(\varphi)}(1 + 2/p,\ r/s,\ \theta - \sigma) \cdot s^{2/p} \cdot B(\sigma) \cdot (f \circ h^{(\varphi)})(s, \sigma) \\ \cdot s^{-1} \, ds \, d\sigma , \end{aligned} \tag{8.10}$$

where f was chosen arbitrarily in $L^p(\partial\mathbb{K}(\varphi))^3$. For any $f \in L^p(\partial\mathbb{K}(\varphi))^3$, we define the function $F(f) : \mathbb{R}_+ \times \mathbf{T} \mapsto \mathbb{C}^3$ by

$$F(f)\,(r\,,\,e^{i\cdot\theta}) \;:=\; r^{2/p}\cdot B(\theta)\cdot(f\circ h^{(\varphi)})(r,\theta) \qquad (r\in(0,\infty)\,,\;\theta\in[0\,,\,2\cdot\pi])\,. \tag{8.11}$$

Futhermore, we put $\widetilde{h} := \left(h^{(\varphi)}\,\big|\,(0,\infty)\times(0\,,\,2\cdot\pi)\right)^{-1}$. Finally, for $\Phi\in(L^p_*)^3$, we introduce the mapping $G(\Phi):\partial\mathbb{K}(\varphi)\mapsto\mathbb{C}^3$ by

$$G(\Phi)\,(x) := \left(\widetilde{h}_1(x)\right)^{-2/p}\cdot B(\widetilde{h}_2(x))\cdot\Phi(\widetilde{h}_1(x)\,,\,e^{i\cdot\widetilde{h}_2(x)}) \tag{8.12}$$

for $x\in\partial\mathbb{K}(\varphi)\backslash\{0\}$ with $x\notin\{r\cdot(\sin\varphi\,,\,0\,,\,\cos\varphi)\;:\;r\in(0,\infty)\}$. Using (3.3), we find

$$F(f)\in(L^p_*)^3 \quad\text{for } f\in L^p(\partial\mathbb{K}(\varphi))^3, \qquad G(\Phi)\in L^p(\partial\mathbb{K}(\varphi))^3 \quad\text{for } \Phi\in(L^p_*)^3,$$

so that the definitions in (8.11) and (8.12) introduce an operator $F: L^p(\partial\mathbb{K}(\varphi))^3\mapsto(L^p_*)^3$ and $G:(L^p_*)^3\mapsto L^p(\partial\mathbb{K}(\varphi))^3$, respectively. Now, recalling Definition 8.2, we deduce from (8.10), for $f\in L^p(\partial\mathbb{K}(\varphi))^3$, $r\in(0,\infty)$, $\theta\in[0\,,\,2\cdot\pi]$:

$$\begin{aligned} &F\Big(\Lambda(\tau,p,\mathbb{K}(\varphi))(f)\Big)\,(r\,,\,e^{i\cdot\theta}) \qquad (8.13)\\ &= (\tau/2)\cdot F(f)\,(r\,,\,e^{i\cdot\theta}) \;+\; (1/2)\cdot\int_0^{2\cdot\pi}\!\!\int_0^{\infty} S^{(\varphi)}(1+2/p\,,\,r/s\,,\,\theta-\sigma)\\ &\qquad\qquad\cdot F(f)\,(s\,,\,e^{i\cdot\sigma})\cdot s^{-1}\;ds\;d\sigma\\ &= (1/2)\cdot\mathcal{S}(\tau,p,\varphi)(F(f))(r\,,\,e^{i\cdot\theta})\,. \end{aligned}$$

On the other hand, it is easily checked that $G(F(f))$ equals f, and $F(G(\Phi))$ coincides with Φ, if $f\in L^p(\partial\mathbb{K}(\varphi))^3$ and $\Phi\in(L^p_*)^3$. This means the operator F is bijective. Now equation (8.13) yields that $\Lambda(\tau,p,\mathbb{K}(\varphi))$ is bijective if and only if $\mathcal{S}(\tau,p,\varphi)$ has the same property. But according to Lemma 6.7, the operator $\Lambda(\tau,p,\mathbb{K}(\varphi))$ is continuous. Hence, by the open mapping theorem, the properties "bijective" and "topological" are equivalent for this operator. Thus we have shown that $\Lambda(\tau,p,\mathbb{K}(\varphi))$ is topological if and only if $\mathcal{S}(\tau,p,\varphi)$ is bijective.

Now, referring to the results presented in Chapter 7, we are able to study the behaviour of $\Pi(\tau,p,\mathbb{K}(\varphi))$ and $\Lambda(\tau,p,\mathbb{K}(\varphi))$ by means of Fourier analysis. To this end, the Fourier transform related to the locally compact abelian group $\mathbb{R}_+\times\mathbb{T}$ will be applied to the operators $\mathcal{R}(\tau,p,\varphi)$ and $\mathcal{S}(\tau,p,\varphi)$. Then, recurring to Theorem 7.1 and Lemma 8.3, we shall obtain a symbol for each of these operators. These symbols are given by the functions $\tau+Y^{(\varphi)}(2/p+i\cdot\xi,n)$ and $det(\tau+Z^{(\varphi)}(2/p+i\cdot\xi,n))$, respectively, with the variables $\xi\in\mathbb{R}$ and $n\in\mathbb{Z}$. By looking for zeros of these symbols, we may obtain information concerning the behaviour of the corresponding operator. These indications will be made precise in the following corollary and its proof:

Corollary 8.2. *Let* $p\in(1,\infty)$, $\varphi\in(0,\pi/2]$, $\tau\in\{-1,1\}$. *Then it holds:*

If there exists some $\epsilon>0$ *with*

$$|\tau\;+\;Y^{(\varphi)}(2/p+i\cdot\xi\,,\,n)|\;\geq\;\epsilon \qquad \text{for } \xi\in\mathbb{R}\,,\;n\in\mathbb{Z}\,,$$

then $\Pi(\tau,p,\mathbb{K}(\varphi))$ *is topological.*

If there is $\epsilon>0$ *with*

$$\left|\det\!\left(\tau\cdot E_3\;+\;Z^{(\varphi)}(2/p+i\cdot\xi\,,\,n)\right)\right|\;\geq\;\epsilon \qquad \text{for } \xi\in\mathbb{R}\,,\;n\in\mathbb{Z}\,,$$

then $\Lambda(\tau,p,\mathbb{K}(\varphi))$ *is topological.*

Conversely, if $\Pi(\tau,p,\mathbb{K}(\varphi))$ *is topological, it follows*

$$\tau\;+\;Y^{(\varphi)}(2/p+i\cdot\xi\,,\,n)\;\neq\;0 \qquad \text{for } \xi\in\mathbb{R}\,,\;n\in\mathbb{Z}\,,$$

and if $\Lambda(\tau,p,\mathbb{K}(\varphi))$ *is topological, then*

$$\det\!\left(\tau\cdot E_3\;+\;Z^{(\varphi)}(2/p+i\cdot\xi\,,\,n)\right)\;\neq\;0 \qquad \text{for } \xi\in\mathbb{R}\,,\;n\in\mathbb{Z}\,.$$

Proof: Define the mappings $F:\mathbb{R}_+\times\mathbb{T}\mapsto\mathbb{C}$ and $H:\mathbb{R}_+\times\mathbb{T}\mapsto\mathbb{C}^{3\times3}$ by

$$F(r\,,\,e^{i\cdot\theta}) := R^{(\varphi)}(1+2/p\,,\,r,\theta)\,, \qquad H(r\,,\,e^{i\cdot\theta}) := S^{(\varphi)}(1+2/p\,,\,r,\theta)$$

for $(r,\theta)\in\mathcal{P}$. Let Δ denote convolution with respect to $m_+\odot m_{\mathbb{T}}$ on $\mathbb{R}_+\times\mathbb{T}$; see Theorem 7.1 and Corollary 7.1. Then we have by Definition 8.2:

$$\mathcal{R}(\tau,p,\varphi)(\Phi) \;=\; \tau\cdot\Phi\;+\;F\,\Delta\,\Phi \qquad \text{for } \Phi\in L^p_*\,, \tag{8.14}$$

$$\mathcal{S}(\tau,p,\varphi)(\Phi) \;=\; \tau\cdot\Phi\;+\;H\,\Delta\,\Phi \qquad \text{for } \Phi\in(L^p_*)^3\,. \tag{8.15}$$

Denote by $\widetilde{R}$ and $\widetilde{S}$ the Fourier transform of F and H, respectively, related to the Haar measure $m_+\odot m_{\mathbb{T}}$. Let γ be an element of the dual group $\Gamma_+\odot\Gamma_{\mathbb{T}}$ of $\mathbb{R}_+\times\mathbb{T}$. According to Corollary 7.1, there is some $\xi\in\mathbb{R}$, $n\in\mathbb{Z}$ with

$$\gamma(r,z) \;=\; r^{i\cdot\xi}\cdot e^{i\cdot n\cdot\arg z} \qquad \text{for } r\in(0,\infty)\,,\;z\in\mathbb{T}\,.$$

Referring to Definition 8.2 and Corollary 7.1, we find

$$\widetilde{R}(\gamma) \;=\; Y^{(\varphi)}(2/p+i\cdot\xi\,,\,n)\,, \qquad \widetilde{S}(\gamma) \;=\; Z^{(\varphi)}(2/p+i\cdot\xi\,,\,n)\,.$$

Due to these equations, and because of (8.14), (8.15), the assumptions of Theorem 7.1 are fulfilled if $G=\mathbb{R}_+\times\mathbb{T}$, $m=m_+\odot m_{\mathbb{T}}$. Now the corollary follows by that theorem and Lemma 8.3.

Lemma 8.4. *Let* $\theta\in(0\,,\,2\cdot\pi)$, $\varphi\in(0,\pi/2]$. *Then the following integrals* $I_1,\cdots,I_6$ *are equal to* 1 :

$$I_1 := \int_0^\infty (1-\cos\theta) \cdot \sin^2(\varphi) \cdot \left(g_0^{(\varphi)}(r,\theta)\right)^{-3/2} \, dr,$$

$$I_2 := \int_0^\infty (1-\cos\theta) \cdot \sin^2(\varphi) \cdot r \cdot \left(g_0^{(\varphi)}(r,\theta)\right)^{-3/2} \, dr,$$

$$I_3 := \int_0^\infty 3 \cdot (1-\cos\theta)^2 \cdot \sin^4(\varphi) \cdot r \cdot \left(g_0^{(\varphi)}(r,\theta)\right)^{-5/2} \, dr,$$

$$I_4 := \int_0^\infty 3 \cdot (1-\cos\theta)^2 \cdot \sin^4(\varphi) \cdot r^2 \cdot \left(g_0^{(\varphi)}(r,\theta)\right)^{-5/2} \, dr,$$

$$I_5 := \int_0^\infty 3 \cdot (1-\cos\theta) \cdot \sin^2(\varphi) \cdot (r-1)^2 \cdot \left(g_0^{(\varphi)}(r,\theta)\right)^{-5/2} \, dr,$$

$$I_6 := \int_0^\infty 3 \cdot (1-\cos\theta) \cdot \sin^2(\varphi) \cdot (r-1)^2 \cdot r \cdot \left(g_0^{(\varphi)}(r,\theta)\right)^{-5/2} \, dr.$$

Furthermore, it holds

$$I_7 := \int_0^\infty 3 \cdot (1-\cos\theta)^2 \cdot \sin^4(\varphi) \cdot r^3 \cdot \left(g_0^{(\varphi)}(r,\theta)\right)^{-5/2} \, dr$$

$$= 2 - \cos^2(\varphi) - \cos\theta \cdot \sin^2(\varphi) = 1 + (1-\cos\theta) \cdot \sin^2(\varphi).$$

Proof: In a first step, we compute I_1, I_3, and the integral

$$I_8 := \int_0^\infty 3 \cdot (1-\cos\theta)^2 \cdot \sin^4(\varphi) \cdot \left(g_0^{(\varphi)}(r,\theta)\right)^{-5/2} \, dr.$$

To this end, following Rathsfeld [39], we choose an angle $\alpha \in (0, \pi/2]$ satisfying the relation $\cos\alpha = \cos^2(\varphi) + \sin^2(\varphi) \cdot \cos\theta$, and then apply the substitution rule with $r = s \cdot \sin\alpha + \cos\alpha$, to get

$$I_1 = 1, \quad I_3 = 1, \quad I_8 = 2 - \cos^2(\varphi) - \cos\theta \cdot \sin^2(\varphi).$$

By the change of variable $r = 1/s$ in I_1, I_3 and I_8, we obtain the other integrals listed in the lemma, either directly, or as a linear combination of integrals already computed.

Lemma 8.5. *Let B be a Banach space, $A : B \mapsto B$ a linear bounded operator, $A^* : B \mapsto B$ the adjoint of A. Then A is topological if and only if A^* has the same property.*

Proof: According to [29, p. 154/155, Section III.3], the operator A^* is linear and bounded too. Assume A to be topological. Then $im(A) = B$ and $kern(A) = \{0\}$. Hence it follows from [29, p. 234, Theorem 5.13] ("closed range theorem") that A^* is bijective. Since A^* is bounded, we may apply the open mappping theorem, which

yields that A^* is topological. An analogous argument yields the opposite direction of the equivalence claimed in the lemma.

Now we are able to prove that the symbol $\tau + Y^{(\varphi)}(2/p + i \cdot \xi, n)$ of $\Pi(\tau, p, \mathbb{K}(\varphi))$ has no zeros for $p \in [2, \infty)$. Consequently, the latter operator is topological for these values of p, where φ is supposed to be an arbitrary element from $(0, \pi/2]$. This is a generalization of a result in [20, p. 102]. As for the symbol $det(\tau \cdot E_3 + Z^{(\varphi)}(2/p + i \cdot \xi))$ of $\Lambda(\tau, p, \mathbb{K}(\varphi))$, we could not prove directly that it has no zeros for $p \in [2, \infty)$. Instead, we shall establish by a different reasoning that $\Lambda(\tau, p, \mathbb{K}(\varphi))$ is topological if $2 \le p$ (Theorem 13.1), so that by Corollary 8.2, the symbol of this operator cannot vanish for these values of p. Actually, we are not interested in the latter result for its own sake. It is the former fact – invertibility of the operator $\Lambda(\tau, p, \mathbb{K}(\varphi))$ – which we want to establish.

Corollary 8.3. *Let $p \in [2, \infty)$, $\varphi \in (0, \pi/2]$, $\tau \in \{-1, 1\}$. Then the operators $\Pi(\tau, p, \mathbb{K}(\varphi))$ and $\Pi^*(\tau, (1-1/p)^{-1}, \mathbb{K}(\varphi))$ are topological.*

Proof: Recalling Definition 8.2, we find for $\xi \in \mathbb{R}$, $n \in \mathbb{N}$:

$$\left| Y^{(\varphi)}(2/p + i \cdot \xi, n) \right|$$

$$\le (2 \cdot \pi)^{-1} \cdot \cos\varphi \cdot \sin^2(\varphi) \cdot \int_0^{2\cdot\pi} \int_0^\infty (1-\cos\theta) \cdot r^{2/p} \cdot \left(g_0^{(\varphi)}(r,\theta)\right)^{-3/2} \, dr \, d\theta$$

$$\le (2 \cdot \pi)^{-1} \cdot \cos\varphi$$

$$\cdot \int_0^{2\cdot\pi} \Big((2/p) \cdot \int_0^\infty \sin^2(\varphi) \cdot (1-\cos\theta) \cdot r \cdot \left(g_0^{(\varphi)}(r,\theta)\right)^{-3/2} \, dr$$

$$+ (1-2/p) \cdot \int_0^\infty \sin^2(\varphi) \cdot (1-\cos\theta) \cdot \left(g_0^{(\varphi)}(r,\theta)\right)^{-3/2} \, dr \Big) \, d\theta$$

$$= \cos\varphi.$$

The last equation follows from Lemma 8.4, whereas the second inequality is implied by the assumption $p \ge 2$ allowing us to apply the estimate

$$a \cdot b \le \gamma^{-1} \cdot a^\gamma + (1 - 1/\gamma) \cdot b^{1/(1-1/\gamma)} \quad \text{for } a, b \in [0, \infty), \ \gamma \in (0, 1),$$

which arises in any proof of Hölder's inequality. Thus we have for $\xi \in \mathbb{R}$, $n \in \mathbb{N}$:

$$\left| \tau + Y^{(\varphi)}(2/p + i \cdot \xi, n) \right| \ge 1 - \cos\varphi.$$

Since $\varphi \in (0, \pi/2]$, the term $1 - \cos\varphi$ is positive. Now we may conclude from Corollary 8.2 that the operator $\Pi(\tau, p, \mathbb{K}(\varphi))$ is topological.

The mapping $\Pi^*(\tau, (1-1/p)^{-1}, \mathbb{K}(\varphi))$ is the adjoint of $\Pi^*(\tau, p, \mathbb{K}(\varphi))$. According to Lemma 6.7, these operators are bounded. Now Lemma 8.5 yields that $\Pi^*(\tau, (1 -$

$1/p)^{-1}, \ \mathbb{K}(\varphi))$ is topological.

In the rest of this chapter, we shall show that for certain values of φ and τ, there is a discreet subset of $(1, 2)$ such that $\Pi(\tau, p, \mathbb{K}(\varphi))$ and $\Lambda(\tau, p, \mathbb{K}(\varphi))$ are not invertible for p from this subset. Our proof, inspired by [20, p. 102], is based on the fact that the functions $Y^{(\varphi)}(\cdot, n)$ and $Z^{(\varphi)}(\cdot, n)$ are holomorphic on $(0, 2)\times\mathbb{R}$. The desired result may then be deduced by the identity theorem for holomorphic functions. However, the technical details are somewhat involved. In particular, we shall have to show that the functions $\tau + Y^{(\varphi)}(z, n)$ and $det(\tau\cdot E_3 + Z^{(\varphi)}(z, n))$ decay for $|\Im(z)| \to \infty$ as well as for $n \to \infty$. This will lead to a sharpened version of Corollary 8.2 (see Corollary 8.5), which in turn will enable us to obtain our results on non-invertibility by considering the functions $\tau + Y^{(\varphi)}(\cdot, n)$ and $det(\tau \cdot E_3 + Z^{(\varphi)}(\cdot, n))$ for $n = 0$.

We begin by a lemma which will be needed in order to expand $A^{(\varphi)}_{(v)}$ into a series.

Lemma 8.6. *Take $\varphi \in (0, \pi/2]$, $\theta \in (0, 2\cdot\pi)$, $\xi \in \mathbb{R}$. Then it holds for $\zeta \in [0, 4)$:*

$$\int_0^\infty r^{\zeta+i\cdot\xi}\cdot\left(g_0^{(\varphi)}(r,\theta)\right)^{-5/2} dr = \Gamma(\zeta+i\cdot\xi+1)\cdot\Gamma(4-\zeta-i\cdot\xi)\cdot(1/12)\cdot\sum_{k=0}^{\infty}\Big(\prod_{j=1}^{k}(\zeta+i\cdot\xi+j)\Big)\cdot\Big(\prod_{j=1}^{k}(3-\zeta-i\cdot\xi+j)\Big)\cdot\Big(k!\cdot(k+2)!\Big)^{-1}\cdot\Big(1/2 + \cos^2(\varphi)/2 + \cos\theta\cdot\sin^2(\varphi)/2\Big)^k.$$

If $\zeta \in [0, 2)$, the ensuing equation is satisfied:

$$\int_0^\infty r^{\zeta+i\cdot\xi}\cdot\left(g_0^{(\varphi)}(r,\theta)\right)^{-3/2} dr = \Gamma(\zeta+i\cdot\xi+1)\cdot\Gamma(2-\zeta-i\cdot\xi)\cdot(1/2)\cdot\sum_{k=0}^{\infty}\Big(\prod_{j=1}^{k}(\zeta+i\cdot\xi+j)\Big)\cdot\Big(\prod_{j=1}^{k}(1-\zeta-i\cdot\xi+j)\Big)\cdot\Big(k!\cdot(k+1)!\Big)^{-1}\cdot\Big(1/2 + \cos^2(\varphi)/2 + \cos\theta\cdot\sin^2(\varphi)/2\Big)^k.$$

As should be expected, these results may be looked up in integral tables. We refer to [19, p. 310, 6.2 (22)] and [17, p. 423, 10.13], for example. However, it turned out that the formulas given there are false. The reader may check this by himself, setting $s = 2$, $v = 5/2$ or $s = 1$, $v = 3/2$ in [17, p. 423, 10.13], and then comparing the result with Lemma 8.4. To clarify the matter, we shall carry out the proof of the preceding lemma in full detail, following the indications in [18, p. 160].

Proof of Lemma 8.6: Let $\sigma \in (-1, 1)$, $v \in \mathbb{R}$ with $v - 1/2 \in \mathbb{N}$, and $z \in \mathbb{C}$ with $-1 < \Re(z) < 2\cdot v - 1$.

For $t \in (0, \infty)$, we have $2\cdot t\cdot(1+t)^{-2} \leq 1/2$. Since $|\sigma - 1| < 2$, it follows for $t \in (0, \infty)$:

$$|2\cdot t\cdot(\sigma-1)\cdot(1+t)^{-2}| \leq (1-\sigma)/2 < 1. \tag{8.16}$$

Hence we may use the expansion of the function x^{-v} into a power series, to obtain for $t \in (0, \infty)$:

$$\Big(1 + 2\cdot t\cdot(1+t)^{-2}\cdot(\sigma-1)\Big)^{-v} = \sum_{k=0}^{\infty}\binom{-v}{k}\cdot\Big(2\cdot t\cdot(1+t)^{-2}\cdot(\sigma-1)\Big)^k, \tag{8.17}$$

where the series on the right-hand side converges absolutely. In addition, it holds due to (8.16):

$$\sum_{k=0}^{\infty}\left|t^z\cdot(1+t)^{-2\cdot v}\cdot\binom{-v}{k}\cdot\Big(2\cdot t\cdot(1+t)^{-2}\cdot(\sigma-1)\Big)^k\right| \leq t^{\Re(z)}\cdot(1+t)^{-2\cdot v}\cdot\sum_{k=0}^{\infty}\left|\binom{-v}{k}\right|\cdot((1-\sigma)/2)^k. \tag{8.18}$$

Since $-1 < \Re(z) < 2\cdot v - 1$, we have

$$\int_0^\infty t^{\Re(z)}\cdot(1+t)^{-2\cdot v}\, dt < \infty.$$

Thus, combining (8.17) and (8.18) with Lebesgue's theorem on dominated convergence, we infer that

$$\int_0^\infty t^z\cdot(1+t)^{-2\cdot v}\cdot\Big(1 + 2\cdot t\cdot(1+t)^{-2}\cdot(\sigma-1)\Big)^{-v} dt = \sum_{k=0}^{\infty}\binom{-v}{k}\cdot(\sigma-1)^k\cdot\int_0^\infty t^z\cdot(1+t)^{-2\cdot v}\cdot\Big(2\cdot t\cdot(1+t)^{-2}\Big)^k dt. \tag{8.19}$$

Now we find

$$\begin{aligned}\int_0^\infty t^z\cdot(1 + t^2 + 2\cdot\sigma\cdot t)^{-v}\, dt &= \sum_{k=0}^{\infty}\binom{-v}{k}\cdot 2^k\cdot(\sigma-1)^k\cdot\int_0^\infty t^{z+k}\cdot(1+t)^{-2\cdot v-2\cdot k}\, dt\\ &= \sum_{k=0}^{\infty}\binom{-v}{k}\cdot 2^k\cdot(\sigma-1)^k\cdot\int_1^\infty (t-1)^{z+k}\cdot t^{-2\cdot v-2\cdot k}\, dt\\ &= \sum_{k=0}^{\infty}\binom{-v}{k}\cdot 2^k\cdot(\sigma-1)^k\cdot\int_0^1 (1-s)^{z+k}\cdot s^{-z+k+2\cdot v-2}\, ds\\ &= \sum_{k=0}^{\infty}\binom{-v}{k}\cdot 2^k\cdot(\sigma-1)^k\cdot B(z+k+1,\ k+2\cdot v-z-1),\end{aligned}$$

where the first equation is a consequence of (8.19). The last one is valid by the definition of the Beta function; see [38, p. 37, (1.11)]; note that $\Re(z) > -1$, $\Re(2\cdot v - z - 1) > 0$. Referring to a well known relation between Beta and Gamma function (see [38, p. 37]), we may deduce from the preceding relation:

$$\int_0^\infty t^z \cdot (1 + t^2 + 2\cdot\sigma\cdot t)^{-v}\, dt \tag{8.20}$$

$$= \sum_{k=0}^\infty \binom{-v}{k} \cdot 2^k \cdot (\sigma - 1)^k \cdot \Gamma(z+k+1) \cdot \Gamma(k+2\cdot v - z - 1) / \Gamma(2\cdot k + 2\cdot v)$$

$$= \sum_{k=0}^\infty \Big(\prod_{j=0}^{k-1}(v+j)\Big) \cdot (k!)^{-1} \cdot 2^k \cdot (1-\sigma)^k \cdot \Gamma(z+k+1) \cdot \Gamma(k+2\cdot v - z - 1) / (2\cdot k + 2\cdot v - 1)\,!\,.$$

Concerning the last equation, we remark that $v - 1/2 \in \mathbb{N}$. This implies in addition that $2\cdot k + 2\cdot v - 1$ is an even positive integer if $k \in \mathbb{N}_0$. Hence it follows for such k:

$$(2\cdot k + 2\cdot v - 1)\,! = (k + v - 1/2)\,! \cdot \Big(\prod_{j=1}^{v-1/2}(j - 1/2)\Big) \cdot \Big(\prod_{j=0}^{k-1}(j+v)\Big) \cdot 2^{2\cdot k + 2\cdot v - 1}.$$

Inserting this result into (8.20), and expanding the terms $\Gamma(z+k+1)$ and $\Gamma(k+2\cdot v-1)$ into products, according to the fundamental recurrence formula for the Gamma function (see [38, p. 32, (1.03)]), we arrive at the equation

$$\int_0^\infty t^z \cdot (1 + t^2 + 2\cdot\sigma\cdot t)^{-v}\, dt$$

$$= 2^{-2\cdot v + 1} \cdot \Big(\prod_{j=1}^{v-1/2}(j - 1/2)\Big)^{-1} \cdot \Gamma(z+1) \cdot \Gamma(2\cdot v - z - 1) \cdot \sum_{k=0}^\infty ((1-\sigma)/2)^k \cdot \Big(\prod_{j=1}^k (z+j)\Big) \cdot \Big(\prod_{j=1}^k (2\cdot v - z - 2 + j)\Big) \cdot \Big(k\,! \cdot (k + v - 1/2)\,!\Big)^{-1}.$$

Setting $\sigma = -\cos^2(\varphi) - \cos\theta \cdot \sin^2(\varphi)$, as well as $v = 5/2$ and $v = 3/2$, respectively, we obtain the two formulas stated in the lemma.

Now we are going to derive some inequalities which, together with the preceding result, will enable us to estimate $Y^{(\varphi)}(z, n)$ and $Z^{(\varphi)}(z, n)$ for large values of $\Im(z)$ and n.

Lemma 8.7. *Let $k \in \mathbb{N}$, $\xi \in \mathbb{R}$. Then it holds in the case $k \geq |\xi|$:*

$$\prod_{j=k}^\infty (j^2/(j^2+\xi^2)) \leq \exp(-(5/6)\cdot\xi^2/k), \tag{8.21}$$

and if $k \leq |\xi|$:

$$\prod_{j=k}^\infty (j^2/(j^2+\xi^2)) \leq \exp((-\pi/2 + \ln 2)\cdot|\xi|). \tag{8.22}$$

Proof: Without loss of generality, we may assume $\xi \neq 0$. Take $j \in \mathbb{N}$ with $j \geq k$. Then

$$\ln(x^2/(x^2+\xi^2)) \geq \ln(j^2/(j^2+\xi^2)) \quad \text{for } x \in [j, \infty),$$

hence,

$$\ln\Big(\prod_{j=k}^\infty (j^2/(j^2+\xi^2))\Big) = \sum_{j=k}^\infty \ln(j^2/(j^2+\xi^2)) \leq \int_k^\infty \ln(x^2/(x^2+\xi^2))\, dx \tag{8.23}$$

$$= \Big[2\cdot x\cdot\ln x - x\cdot\ln(x^2+\xi^2) - 2\cdot\xi\cdot\arctan(x/\xi)\Big]_{x=k}^{x=\infty}$$

$$= -\xi\cdot\pi + k\cdot\ln(1 + \xi^2/k^2) + 2\cdot\xi\cdot\arctan(k/\xi).$$

Concerning the last equation, we point out that

$$2\cdot x\cdot\ln x - x\cdot\ln(x^2+\xi^2) = x\cdot\ln(x^2/(x^2+\xi^2)) \longrightarrow 0 \quad (x\to\infty),$$

as may be deduced from the relation

$$x\cdot\ln(x^2/(x^2+\xi^2)) = x\cdot\xi^2\cdot(x^2+\xi^2)^{-1}\cdot\sum_{k=1}^\infty (1/k)\cdot(\xi^2/(\xi^2+x^2))^{k-1},$$

which is valid for $x \in (0, \infty)$. If $\xi, x \in (0, \infty)$, we set

$$f(\xi, x) := -\xi\cdot\pi + x\cdot\ln(1 + \xi^2/x^2) + 2\cdot\xi\cdot\arctan(x/\xi).$$

Therefore, $\partial/\partial x\, f(\xi, x) = \ln(1+\xi^2/x^2) > 0$ for $x, \xi \in (0, \infty)$, so that

$$f(\xi, x) \leq f(\xi, \xi) = (-\pi/2 + \ln 2)\cdot\xi, \quad \text{with } \xi \in (0,\infty),\ x \in (0, \xi].$$

Now (8.22) follows from the preceding inequality and (8.23).

Next, recurring to the series expansions of the logarithm and arc tangent, we find for $\xi \in (0, \infty)$, $x \in (\xi, \infty)$:

$$f(x, \xi) = x\cdot\sum_{v=0}^\infty (v+1)^{-1}\cdot(v+2)^{-1}\cdot(-\xi^2/x^2)^{v+1},$$

so that

$$f(x, \xi) \leq -\xi^2/x + (1/6)\cdot\xi^4/x^3 \leq -(5/6)\cdot\xi^2/x.$$

Combining this estimate with (8.23) yields (8.21).

Now we are able to show that the functions $Y^{(\varphi)}(z, n)$ and $Z^{(\varphi)}(z, n)$ decay for $\Im(z)$ tending to infinity.

Lemma 8.8. *Let* $\varphi \in (0, \pi/2]$, $v \in \{1, \cdots, 7\}$. *In addition, take* $\gamma_v \in [0, 4)$ *in the case* $v \in \{1, 2, 3\}$, $\gamma_v \in [0, 3)$ *if* $v \in \{4, 5\}$, *and* $\gamma_v \in [0, 2)$ *in the case* $v \in \{6, 7\}$. *It follows* $A_v^{(\varphi)}(\zeta + i \cdot \xi,\ n) \to 0$ *for* $|\xi| \to \infty$, *uniformly in* $\zeta \in [0, \gamma_v]$, $n \in \mathbb{Z}$.

Due to Lemma 8.1 and Definition 8.2, this means in particular, for $\gamma \in [0, 2)$:

$$Y^{(\varphi)}(\zeta + i \cdot \xi,\ n) \to 0, \qquad \det\Big(\tau \cdot E_3 \ + \ Z^{(\varphi)}(\zeta + i \cdot \xi,\ n)\Big) \to \tau$$

for $|\xi| \to \infty$, *uniformly in* $n \in \mathbb{Z}$, $\zeta \in [0, \gamma]$.

Proof: If $\theta \in (0,\ 2 \cdot \pi)$, $r \in (0, \infty)$, $\xi \in \mathbb{R}$, $\zeta \in [0, 4)$, $\tilde{\zeta} \in [0, 2)$, then we set

$$\sigma(\theta) := (1/2) \ + \ \cos^2(\varphi)/2 \ + \ \cos\theta \cdot \sin^2(\varphi)/2,$$

$$B(\zeta, \xi, \theta) := \int_0^\infty r^{\zeta + i \cdot \xi} \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-5/2} \ dr,$$

$$\tilde{B}(\tilde{\zeta}, \xi, \theta) := \int_0^\infty r^{\tilde{\zeta} + i \cdot \xi} \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-3/2} \ dr,$$

$$f_1(\theta) := \gamma_\varphi \cdot \sin^2(\varphi) \cdot (1 - \cos\theta)^2, \qquad f_2(\theta) := \gamma_\varphi \cdot \sin^2(\varphi) \cdot \cos\theta \cdot (1 - \cos\theta)^2,$$

$$f_3(\theta) := \gamma_\varphi \cdot \sin^2(\varphi) \cdot \sin\theta \cdot (1 - \cos\theta)^2, \qquad f_4(\theta) := \gamma_\varphi \cdot \sin^2(\varphi) \cdot \sin\theta \cdot (1 - \cos\theta),$$

$$f_5(\theta) := \gamma_\varphi \cdot \sin^2(\varphi) \cdot \cos\theta \cdot \sin\theta \cdot (1 - \cos\theta), \qquad f_6(\theta) := \gamma_\varphi \cdot (1 - \cos\theta),$$

$$f_7(\theta) := \gamma_\varphi \cdot \cos\theta \cdot (1 - \cos\theta).$$

Recalling Definition 8.1 and (8.4), we find in the case $v \in \{1, 2, 3\}$:

$$A_v^{(\varphi)}(\zeta + i \cdot \xi,\ n) \ = \ \int_0^{2 \cdot \pi} f_v(\theta) \cdot B(\zeta, \xi, \theta) \cdot e^{i \cdot n \cdot \theta} \ d\theta \tag{8.24}$$

for $\zeta \in [0, 4)$, $\xi \in \mathbb{R}$, $n \in \mathbb{Z}$. In the case $v \in \{4, 5\}$, we have

$$A_v^{(\varphi)}(\zeta + i \cdot \xi,\ n) \tag{8.25}$$

$$= \int_0^{2 \cdot \pi} f_v(\theta) \cdot \Big(B(\zeta + 1, \xi, \theta) - B(\zeta, \xi, \theta)\Big) \cdot e^{i \cdot n \cdot \theta} \ d\theta$$

for $\zeta \in [0, 3)$, $\xi \in \mathbb{R}$, $n \in \mathbb{Z}$. If $v \in \{6, 7\}$, it holds:

$$A_v^{(\varphi)}(\zeta + i \cdot \xi,\ n) \ = \ \int_0^{2 \cdot \pi} f_v(\theta) \cdot \tilde{B}(\zeta, \xi, \theta) \cdot e^{i \cdot n \cdot \theta} \ d\theta$$

for $\zeta \in [0, 2)$, $\xi \in \mathbb{R}$, $n \in \mathbb{Z}$. Let us estimate the kernel functions of the preceding integrals. It follows from Lemma 8.6, for $\zeta \in [0, 4)$, $\xi \in \mathbb{R}$, $\theta \in (0,\ 2 \cdot \pi)$:

$$|B(\zeta, \xi, \theta)| \ \le \ (1/12) \cdot \big|\Gamma(\zeta + i \cdot \xi + 1) \cdot \Gamma(4 - \zeta - i \cdot \xi)\big| \tag{8.26}$$

$$\cdot \sum_{s=0}^{\infty} \prod_{j=1}^{s} \big|(-\zeta - i \cdot \xi + 3 + j) \cdot (\zeta + i \cdot \xi + j)\big| \cdot \Big(s\,! \cdot (s+2)\,!\Big)^{-1} \cdot \big(\sigma(\theta)\big)^s.$$

In order to evaluate the right-hand side of (8.26), we take note of the ensuing estimate, which holds for $s \in \mathbb{N}$, $\zeta \in [0, 4)$, $\xi \in \mathbb{R}$:

$$\prod_{j=1}^{s} \big|(-\zeta - i \cdot \xi + 3 + j) \cdot (\zeta + i \cdot \xi + j)\big|$$

$$= \prod_{j=1}^{s} \Big((-\zeta + 3 + j)^2 \ + \ \xi^2\Big)^{1/2} \cdot \Big((\zeta + j)^2 \ + \ \xi^2\Big)^{1/2} \ \le \ \prod_{j=1}^{s+4} (j^2 + \xi^2)$$

$$= \ \big((s+4)\,!\big)^2 \cdot \prod_{j=1}^{s+4} \big(1 \ + \ \xi^2/j^2\big).$$

According to [27, p. 484, (89.5.16)], we have

$$\prod_{j=1}^{\infty} \big(1 \ + \ \xi^2/j^2\big) \ = \ \big(2 \cdot \pi \cdot |\xi|\big)^{-1} \cdot \big(e^{\pi \cdot |\xi|} - e^{-\pi \cdot |\xi|}\big) \qquad (\xi \in \mathbb{R}).$$

Thus we get for $s \in \mathbb{N}$, $\zeta \in [0, 4)$, $\xi \in \mathbb{R}$:

$$\prod_{j=1}^{s} \big|(-\zeta - i \cdot \xi + 3 + j) \cdot (\zeta + i \cdot \xi + j)\big| \tag{8.27}$$

$$\le \ \big((s+4)\,!\big)^2 \cdot \big(2 \cdot \pi \cdot |\xi|\big)^{-1} \cdot \big(e^{\pi \cdot |\xi|} - e^{-\pi \cdot |\xi|}\big) \cdot \Big(\prod_{j=s+5}^{\infty} j^2/(j^2 + \xi^2)\Big).$$

Note that in the case $s = 0$, this estimate is trivial. As another preparatory result, we mention the ensuing equation, which is valid for $\zeta \in [0, 4)$, $\xi \in \mathbb{R}$:

$$\Gamma(\zeta + i \cdot \xi + 1) \cdot \Gamma(4 - \zeta - i \cdot \xi) \tag{8.28}$$

$$= \Big(\prod_{j=0}^{3} (j - \zeta - i \cdot \xi)\Big) \cdot \Gamma(\zeta + i \cdot \xi + 1) \cdot \Gamma(-\zeta - i \cdot \xi)$$

$$= \Big(\prod_{j=0}^{3} (j - \zeta - i \cdot \xi)\Big) \cdot \pi \cdot \sin^{-1}\big((-\zeta - i \cdot \xi) \cdot \pi\big)$$

$$= \Big(\prod_{j=0}^{3} (j - \zeta - i \cdot \xi)\Big) \cdot 2 \cdot \pi \cdot i \cdot \Big(e^{\pi \cdot (-i \cdot \zeta + \xi)} - e^{\pi \cdot (i \cdot \zeta - \xi)}\Big)^{-1}.$$

The first of these equations follows by applying the fundamental recurrence formula for the Gamma function ([38, p. 32, (1.03)], the second one is implied by the reflection formula for the Gamma function ([38, p. 35, (1.07)]. As may be shown by some elementary calculations, there is some constant $\mathfrak{C}_1 > 0$ such that it holds for $\zeta \in [0, 4)$, $\xi \in \mathbb{R}$ with $|\xi| \ge 1$:

$$\Big|\prod_{j=0}^{3}(j-\zeta-i\cdot\xi)\Big|\cdot 2\cdot\pi\cdot\Big|e^{\pi\cdot(-i\cdot\zeta+\xi)}-e^{\pi\cdot(i\cdot\zeta-\xi)}\Big|^{-1}\cdot(2\cdot\pi\cdot|\xi|)^{-1}\cdot\big(e^{\pi\cdot|\xi|}-e^{-\pi\cdot|\xi|}\big) \tag{8.29}$$
$$\leq\ \mathfrak{C}_1\cdot|\xi|^3.$$

Gathering up the results in (8.26) - (8.29), we may construct a constant $\mathfrak{C}_2>0$ such that the following inequality is fulfilled if $\zeta\in[0,4)$, $\theta\in(0,2\cdot\pi)$, $\xi\in\mathbb{R}$ with $|\xi|\geq 1$:

$$|B(\zeta,\xi,\theta)| \tag{8.30}$$
$$\leq\ \mathfrak{C}_2\cdot|\xi|^3\cdot\sum_{s=0}^{\infty}\big((s+4)!\big)^2\cdot\Big(\prod_{j=s+5}^{\infty}j^2/(j^2+\xi^2)\Big)\cdot(\sigma(\theta))^s\cdot\big(s!\cdot(s+2)!\big)^{-1}$$
$$\leq\ \mathfrak{C}_2\cdot|\xi|^3\cdot\sum_{s=0}^{\infty}\Big(\prod_{j=1}^{4}(s+j)\Big)\cdot\Big(\prod_{j=3}^{4}(s+j)\Big)\cdot\Big(\prod_{j=s+5}^{\infty}j^2/(j^2+\xi^2)\Big)\cdot(\sigma(\theta))^s.$$

Now we choose $\mathfrak{C}_3>0$ so large that it holds for $x\in[0,\infty)$:

$$x^2\cdot\exp(-(5/6)\cdot x)\ \leq\ \mathfrak{C}_3,\qquad x^4\cdot\exp((-\pi/2+\ln 2)\cdot x)\ \leq\ \mathfrak{C}_3.$$

Then, by Lemma 8.7, we get for $s\in\mathbb{N}_0$, $\xi\in\mathbb{R}$, in the case $s+5\geq|\xi|$:

$$\big(\xi^4/(s+5)^2\big)\cdot\Big(\prod_{j=s+5}^{\infty}j^2/(j^2+\xi^2)\Big)$$
$$\leq\ \big(\xi^4/(s+5)^2\big)\cdot\exp(-(5/6)\cdot\xi^2/(s+5))\ \leq\ \mathfrak{C}_3,$$

and in the case $s+5\leq|\xi|$:

$$\xi^4\cdot\Big(\prod_{j=s+5}^{\infty}j^2/(j^2+\xi^2)\Big)\ \leq\ \xi^4\cdot\exp((-\pi/2+\ln 2)\cdot|\xi|)\ \leq\ \mathfrak{C}_3.$$

Now we apply (8.30), for $\zeta\in[0,4)$, $\xi\in\mathbb{R}$ with $|\xi|\geq 1$, $\theta\in(0,2\cdot\pi)$. This leads to the inequality

$$|B(\zeta,\xi,\theta)|$$
$$\leq\ \mathfrak{C}_2\cdot\mathfrak{C}_3\cdot|\xi|^{-1}\cdot\sum_{s=0}^{\infty}\Big(\prod_{j=1}^{4}(s+j)\Big)\cdot\Big(\prod_{j=3}^{4}(s+j)\Big)\cdot(s+5)^2\cdot(\sigma(\theta))^s.$$

Abbreviating $\mathfrak{C}_4:=\mathfrak{C}_2\cdot\mathfrak{C}_3$, it follows for ζ,ξ,θ as before:

$$|B(\zeta,\xi,\theta)|\ \leq\ \mathfrak{C}_4\cdot|\xi|^{-1}\cdot\sum_{s=0}^{\infty}(s+5)^8\cdot(\sigma(\theta))^s. \tag{8.31}$$

Now fix $\epsilon\in(0,\infty)$, and take $\zeta\in[0,\gamma_v]$, $\xi\in\mathbb{R}$, $\theta\in(0,2\cdot\pi)$, $n\in\mathbb{Z}$. Then we find in the case $v\in\{1,2,3\}$:

$$|f_v(\theta)\cdot B(\zeta,\xi,\theta)\cdot e^{i\cdot n\cdot\theta}| \tag{8.32}$$
$$\leq\ \gamma_\varphi\cdot\int_0^\infty(1-\cos\theta)^2\cdot r^\zeta\cdot\big(g_0^{(\varphi)}(r,\theta)\big)^{-5/2}\,dr$$
$$\leq\ \gamma_\varphi\cdot\int_0^\infty(1-\cos\theta)^2\cdot(1+r^{\gamma_v})\cdot\big(g_0^{(\varphi)}(r,\theta)\big)^{-5/2}\,dr,$$

and in the case $v\in\{4,5\}$:

$$\Big|f_v(\theta)\cdot\big(B(\zeta+1,\xi,\theta)-B(\zeta,\xi,\theta)\big)\cdot e^{i\cdot n\cdot\theta}\Big| \tag{8.33}$$
$$\leq\ 2\cdot\gamma_\varphi\cdot\int_0^\infty(1-\cos\theta)^{3/2}\cdot(1+r^{\gamma_v})\cdot|r-1|\cdot\big(g_0^{(\varphi)}(r,\theta)\big)^{-5/2}\,dr.$$

Due to Lemma 8.2, we may choose $\kappa\in(0,\pi)$ in such a way that it holds in the case $v\in\{1,2,3\}$:

$$\gamma_\varphi\cdot\int_{(0,\kappa)\cup(2\cdot\pi-\kappa,\,2\cdot\pi)}\int_0^\infty(1-\cos\theta)^2\cdot(1+r^{\gamma_v})\cdot\big(g_0^{(\varphi)}(r,\theta)\big)^{-5/2}\,dr\,d\theta \tag{8.34}$$
$$\leq\ \epsilon/2,$$

and in the case $v\in\{4,5\}$:

$$2\cdot\gamma_\varphi\cdot\int_{(0,\kappa)\cup(2\cdot\pi-\kappa,\,2\cdot\pi)}\int_0^\infty(1-\cos\theta)^{3/2}\cdot(1+r^{\gamma_v})\cdot|r-1|\cdot\big(g_0^{(\varphi)}(r,\theta)\big)^{-5/2}\,dr\,d\theta \tag{8.35}$$
$$\leq\ \epsilon/2.$$

Combining (8.32) - (8.35), it follows for $\zeta\in[0,\gamma_v]$, $\xi\in\mathbb{R}$, $n\in\mathbb{Z}$, if $v\in\{1,2,3\}$:

$$\int_{(0,\kappa)\cup(2\cdot\pi-\kappa,\,2\cdot\pi)}|f_v(\theta)\cdot B(\zeta,\xi,\theta)\cdot e^{i\cdot n\cdot\theta}|\,d\theta\ \leq\ \epsilon/2, \tag{8.36}$$

and if $v\in\{4,5\}$:

$$\int_{(0,\kappa)\cup(2\cdot\pi-\kappa,\,2\cdot\pi)}\Big|f_v(\theta)\cdot\big(B(\zeta+1,\xi,\theta)-B(\zeta,\xi,\theta)\big)\cdot e^{i\cdot n\cdot\theta}\Big|\,d\theta\ \leq\ \epsilon/2. \tag{8.37}$$

There exists some $\delta\in(0,1)$ such that $\sigma(\theta)\in[-\delta,\delta]$ for $\theta\in[\kappa,2\cdot\pi-\kappa]$. Thus inequality (8.31) yields for $\zeta\in[0,4)$, $\xi\in\mathbb{R}$ with $|\xi|\geq 1$, $\theta\in[\kappa,2\cdot\pi-\kappa]$:

$$|B(\zeta,\xi,\theta)|\ \leq\ \mathfrak{C}_4\cdot|\xi|^{-1}\cdot\sum_{s=0}^{\infty}(s+5)^8\cdot\delta^s,$$

where $\sum_{s=0}^{\infty}(s+5)^8\cdot\delta^s<\infty$ since $\delta<1$. Hence we may choose $r_0\geq 1$ such that it holds for $\zeta\in[0,4)$, $n\in\mathbb{Z}$, $\xi\in\mathbb{R}$ with $|\xi|\geq r_0$:

$$\int_\kappa^{2\cdot\pi-\kappa}|f_v(\theta)\cdot B(\zeta,\xi,\theta)\cdot e^{i\cdot n\cdot\theta}|\,d\theta\ \leq\ \epsilon/2 \tag{8.38}$$

in the case $v\in\{1,2,3\}$, and

$$\int_\kappa^{2\cdot\pi-\kappa}\Big|f_v(\theta)\cdot\big(B(\zeta+1,\xi,\theta)-B(\zeta,\xi,\theta)\big)\cdot e^{i\cdot n\cdot\theta}\Big|\,d\theta\ \leq\ \epsilon/2 \tag{8.39}$$

in the case $v \in \{4, 5\}$. From (8.24), (8.25) and (8.36) - (8.39), we obtain in the case $v \in \{1, \cdots, 5\}$, for $\zeta \in [0, \gamma_v]$, $\xi \in \mathbb{R}$ with $|\xi| \geq r_0$, $n \in \mathbb{Z}$:

$$|A_v^{(\varphi)}(\zeta + i \cdot \xi, n)| \leq \epsilon.$$

Thus we have proved the lemma if $v \in \{1, \cdots, 5\}$. In the case $v \in \{6, 7\}$, which is somewhat easier to treat, we may use similar arguments.

In the next lemma, we collect some technical results which will be used frequently in the following.

Lemma 8.9. *Put* $\zeta_j := 4$ *for* $j \in \{1, 2, 3\}$, $\zeta_j := 3$ *for* $j \in \{4, 5\}$, *and* $\zeta_j := 2$ *for* $j \in \{6, 7\}$. *Take* $\varphi \in (0, \pi/2]$. *Then the following equations hold true:*

$$A_j^{(\varphi)}(z, n) = \int_0^{2\cdot\pi} \int_0^\infty \cos(n \cdot \theta) \cdot a_j^{(\varphi)}(z, r, \theta)\, dr\, d\theta \tag{8.40}$$

for $j \in \{1, 2, 6, 7\}$, $z \in [0, \zeta_j) \times \mathbb{R}$, $n \in \mathbb{Z}$;

$$A_j^{(\varphi)}(z, n) = i \cdot \int_0^{2\cdot\pi} \int_0^\infty \sin(n \cdot \theta) \cdot a_j^{(\varphi)}(z, r, \theta)\, dr\, d\theta \tag{8.41}$$

for $j \in \{3, 4, 5\}$, $z \in [0, \zeta_j) \times \mathbb{R}$, $n \in \mathbb{Z}$;

$$A_j^{(\varphi)}(z, 0) = 0, \qquad \Re\big(A_j^{(\varphi)}(\gamma, n)\big) = 0 \tag{8.42}$$

for $j \in \{3, 4, 5\}$, $z \in [0, \zeta_j) \times \mathbb{R}$, $\gamma \in [0, \zeta_j)$, $n \in \mathbb{Z}$.

$$\Im\big(A_j^{(\varphi)}(\gamma, 0)\big) = 0 \quad \textit{if } j \in \{1, \cdots, 7\},\ \gamma \in [0, \zeta_j); \tag{8.43}$$

$$A_j^{(\varphi)}(1, n) = 0 \quad \textit{if } j \in \{4, 5\},\ n \in \mathbb{Z}, \textit{ or if } j \in \{1, 6\},\ n \in \mathbb{Z}\backslash\{0\}, \tag{8.44}$$

$$A_j^{(\varphi)}(1, 0) = 0 \quad \textit{for } j \in \{2, 7\}, \qquad A_1^{(\varphi)}(1, 0) = (1/3) \cdot A_6^{(\varphi)}(1, 0) = \cos\varphi,$$

$$A_2^{(\varphi)}(1, n) = (1/3) \cdot A_7^{(\varphi)}(1, n)$$
$$= \cos\varphi \cdot (2\cdot\pi)^{-1} \cdot \int_0^{2\cdot\pi} \cos(n\cdot\theta) \cdot \cos\theta\, d\theta \quad \textit{for } n \in \mathbb{Z}\backslash\{0\},$$

$$A_3^{(\varphi)}(1, n) = i \cdot \cos\varphi \cdot (2\cdot\pi)^{-1} \cdot \int_0^{2\cdot\pi} \sin(n\cdot\theta) \cdot \sin\theta\, d\theta \quad \textit{for } n \in \mathbb{Z}\backslash\{0\};$$

$$\Re\big(A_j^{(\varphi)}(2 + i\cdot\xi, 0)\big) = \Re\big(A_j^{(\varphi)}(1 + i\cdot\xi, 0)\big), \tag{8.45}$$

$$\Im\big(A_j^{(\varphi)}(2 + i\cdot\xi, 0)\big) = -\Im\big(A_j^{(\varphi)}(1 + i\cdot\xi, 0)\big),$$

$$\Re\big(A_j^{(\varphi)}(2, n)\big) = \Re\big(A_j^{(\varphi)}(1, n)\big), \qquad A_3^{(\varphi)}(2, n) = A_3^{(\varphi)}(1, n),$$

$$A_4^{(\varphi)}(2, n) = i \cdot \cos\varphi \cdot \sin^2(\varphi) \cdot (2\cdot\pi)^{-1} \cdot \int_0^{2\cdot\pi} \sin(n\cdot\theta) \cdot \sin\theta\, d\theta$$

for $j \in \{1, 2, 3\}$, $n \in \mathbb{Z}$, $\xi \in \mathbb{R}$;

$$A_1^{(\varphi)}(3, 0) = \cos\varphi \cdot (1 + \sin^2(\varphi)), \quad A_2^{(\varphi)}(3, 0) = (-1/2) \cdot \cos\varphi \cdot \sin^2(\varphi); \tag{8.46}$$

$$A_6^{(\varphi)}(\sigma, 0) \to \infty \quad (\sigma \uparrow 2); \tag{8.47}$$

$$A_7^{(\varphi)}(z, 0) \tag{8.48}$$
$$= (-1/2) \cdot A_6^{(\varphi)}(z, 0) + (3/2) \cdot A_1^{(\varphi)}(z+1, 0) + (3/2) \cdot A_2^{(\varphi)}(z+1, 0)$$

for $z \in [0, 2) \times \mathbb{R}$.

Proof: The equations in (8.40) - (8.46) follow by referring to Definition 8.1, (8.7), Lemma 8.4, and the substitution rule, applied with $r = 1/s$. As for equation (8.48), it may be derived by performing an integration by parts with respect to θ in the integral defining $A_7^{(\varphi)}$; see (8.7). The proof of (8.47) is based on the observation that

$$g_0^{(\varphi)}(r, \theta) \geq (r-1)^2 \geq (1/4) \cdot r^2 \quad \text{for } \theta \in [0, 2\cdot\pi],\ r \in [4, \infty).$$

Thus it follows by the definition of $A_6^{(\varphi)}$ in (8.7), for $\sigma \in [0, 2)$:

$$A_6^{(\varphi)}(\sigma, 0) \geq \gamma_\varphi \cdot \int_0^{2\cdot\pi} \int_4^\infty (1 - \cos\theta) \cdot r^\sigma \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-3/2}\, dr\, d\theta$$

$$\geq 2\cdot\pi \cdot \gamma_\varphi \cdot 8 \cdot \int_4^\infty r^{\sigma-3}\, dr \geq \pi \cdot \gamma_\varphi \cdot (2-\sigma)^{-1}.$$

This implies (8.47).

Next we study how the function $A_v^{(\varphi)}(z, n)$ decays for n tending to ∞, in contrast to Lemma 8.8, where decay for $|\Im(z)| \to \infty$ was considered.

Lemma 8.10 *Take* $\varphi \in (0, \pi/2]$, $v \in \{1, \cdots, 7\}$. *Let* $\gamma_v \in [0, 4)$ *in the case* $v \in \{1, 2, 3\}$, $\gamma_v \in [0, 3)$ *in the case* $v \in \{4, 5\}$, *and* $\gamma_v \in [0, 2)$ *if* $j \in \{6, 7\}$. *It follows* $A_v^{(\varphi)}(z, n) \to 0$ *for* $n \to \pm\infty$, *uniformly in* $z \in [0, \gamma_v] \times \mathbb{R}$.

Proof: The main device consists in performing an integration by parts with respect to θ in the definition of $A_v^{(\varphi)}$. Before entering into details, let us introduce some abbreviations. Put

$$\varrho_v := 3 \cdot (2\cdot\pi)^{-1} \cdot \sin^4(\varphi) \cdot \cos\varphi \qquad \text{in the case } v \in \{1, \cdots, 5\};$$

$$\varrho_v := 3 \cdot (2\cdot\pi)^{-1} \cdot \sin^2(\varphi) \cdot \cos\varphi \qquad \text{in the case } v \in \{6, 7\}.$$

If $r \in (0, \infty)\backslash\{1\}$, $\theta \in [0, 2\cdot\pi]$, we set

$$f_1(r, \theta) := 2 \cdot \sin\theta \cdot (1-\cos\theta) - (1-\cos\theta)^2 \cdot 5 \cdot r \cdot \sin^2(\varphi) \cdot \sin\theta \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-1},$$

$$f_2(r, \theta) := 2 \cdot \cos\theta \cdot \sin\theta \cdot (1-\cos\theta) - \sin\theta \cdot (1-\cos\theta)^2$$
$$- \cos\theta \cdot (1-\cos\theta)^2 \cdot 5 \cdot r \cdot \sin^2(\varphi) \cdot \sin\theta \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-1},$$

$$f_3(r, \theta) := 2 \cdot \sin^2(\theta) \cdot (1-\cos\theta) + \cos\theta \cdot (1-\cos\theta)^2$$
$$- \sin^2(\theta) \cdot (1-\cos\theta)^2 \cdot 5 \cdot r \cdot \sin^2(\varphi) \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-1},$$

$$f_4(r, \theta) := \sin^2(\theta) + \cos\theta \cdot (1-\cos\theta)$$
$$- \sin^2(\theta) \cdot (1-\cos\theta) \cdot 5 \cdot r \cdot \sin^2(\varphi) \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-1},$$

$$f_5(r, \theta) := \cos\theta \cdot \sin^2(\theta) - \sin^2(\theta) \cdot (1-\cos\theta) + \cos^2(\theta) \cdot (1-\cos\theta)$$
$$- \cos\theta \cdot \sin^2(\theta) \cdot (1-\cos\theta) \cdot 5 \cdot r \cdot \sin^2(\varphi) \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-1},$$

$$f_6(r, \theta) := \sin\theta - (1-\cos\theta) \cdot 3 \cdot r \cdot \sin^2(\varphi) \cdot \sin\theta \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-1},$$

$$f_7(r, \theta) := \cos\theta \cdot \sin\theta - \sin\theta \cdot (1-\cos\theta)$$
$$- \cos\theta \cdot (1-\cos\theta) \cdot 3 \cdot r \cdot \sin^2(\varphi) \cdot \sin\theta \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-1}.$$

For $r \in (0, \infty)\backslash\{1\}$, $z \in [0, \gamma_v] \times \mathbb{R}$, $\theta \in [0, 2\cdot\pi]$, we define

$$h_v(z, r, \theta) := \varrho_v \cdot r^z \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-5/2} \qquad \text{in the case } v \in \{1, 2, 3\},$$

$$h_v(z, r, \theta) := \varrho_v \cdot (r^{z+1} - r^z) \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-5/2} \qquad \text{in the case } v \in \{4, 5\}, \text{and}$$

$$h_v(z, r, \theta) := \varrho_v \cdot r^z \cdot \big(g_0^{(\varphi)}(r, \theta)\big)^{-3/2} \qquad \text{if } v \in \{6, 7\}.$$

Note that $g_0^{(\varphi)}(r, \theta) \neq 0$ for any $r \in (0, \infty)\backslash\{1\}$ and $\theta \in [0, 2\cdot\pi]$.

If $r \in (0, \infty)\backslash\{1\}$, $\theta \in [0, 2\cdot\pi]$, $z \in [0, \gamma_v] \times \mathbb{R}$, $n \in \mathbb{N}$, then it holds in the case $v \in \{1, 2, 6, 7\}$:

$$\partial/\partial\theta \Big(a_v^{(\varphi)}(z, r, \theta) \cdot n^{-1} \cdot \sin(n\cdot\theta) \Big)$$
$$= \cos(n\cdot\theta) \cdot a_v^{(\varphi)}(z, r, \theta) + n^{-1} \cdot \sin(n\cdot\theta) \cdot f_v(r, \theta) \cdot h_v(z, r, \theta),$$

and in the case $v \in \{3, 4, 5\}$:

$$\partial/\partial\theta \Big(a_v^{(\varphi)}(z, r, \theta) \cdot n^{-1} \cdot \cos(n\cdot\theta) \Big)$$
$$= -\sin(n\cdot\theta) \cdot a_v^{(\varphi)}(z, r, \theta) + n^{-1} \cdot \cos(n\cdot\theta) \cdot f_v(r, \theta) \cdot h_v(z, r, \theta).$$

For $r \in (0, \infty)\backslash\{1\}$, $z \in [0, \gamma_v] \times \mathbb{R}$, $n \in \mathbb{N}$, we obtain in the case $v \in \{1, 2, 6, 7\}$:

$$\int_0^{2\cdot\pi} \cos(n\cdot\theta) \cdot a_v^{(\varphi)}(z, r, \theta)\, d\theta \tag{8.49}$$
$$= (-1/n) \cdot \int_0^{2\cdot\pi} \sin(n\cdot\theta) \cdot f_v(r, \theta) \cdot h_v(z, r, \theta)\, d\theta,$$

and if $v \in \{3, 4, 5\}$:

$$\int_0^{2\cdot\pi} \sin(n\cdot\theta) \cdot a_v^{(\varphi)}(z, r, \theta)\, d\theta \tag{8.50}$$
$$= (1/n) \cdot \int_0^{2\cdot\pi} \cos(n\cdot\theta) \cdot f_v(r, \theta) \cdot h_v(z, r, \theta)\, d\theta.$$

If $v \in \{4, 5\}$, we introduce an additional term on the right-hand side of (8.50), to obtain:

$$\int_0^{2\cdot\pi} \sin(n\cdot\theta) \cdot a_v^{(\varphi)}(z, r, \theta)\, d\theta \tag{8.51}$$
$$= (1/n) \cdot \int_0^{2\cdot\pi} \Big(\cos(n\cdot\theta) - \cos^n(\theta) \Big) \cdot f_v(r, \theta) \cdot h_v(z, r, \theta)\, d\theta$$
$$+ \int_0^{2\cdot\pi} \cos^{n-1}(\theta) \cdot \sin\theta \cdot a_v^{(\varphi)}(z, r, \theta)\, d\theta$$

for $r \in (0, \infty)\backslash\{1\}$, $z \in [0, \gamma_v] \times \mathbb{R}$, $n \in \mathbb{N}$. It may be shown by induction that

$$|\sin(n\cdot\theta)| \leq n \cdot |\sin\theta|,$$

$$|\cos(n\cdot\theta) - \cos^n(\theta)| \leq n \cdot |\sin\theta| \qquad \text{for } n \in \mathbb{N},\ \theta \in [0, 2\cdot\pi].$$

This implies for θ, n as before:

$$|\sin(n\cdot\theta)| \leq |\sin(n\cdot\theta)|^{3/4} \leq n^{3/4} \cdot |\sin\theta|^{3/4} \leq 2 \cdot n^{3/4} \cdot (1-\cos\theta)^{3/8},$$

$$|\cos(n\cdot\theta) - \cos^n(\theta)| \leq 4 \cdot n^{3/4} \cdot (1-\cos\theta)^{3/8}.$$

We further observe that

$$|r^z| = r^{\Re(z)} \leq 1 + r^{\gamma_v} \qquad \text{for } r \in (0, \infty),\ z \in [0, \gamma_v] \times \mathbb{R}. \tag{8.52}$$

Thus, gathering up our information, we see there is a constant $\mathfrak{C} > 0$ such that we have for $r \in (0, \infty)\backslash\{1\}$, $n \in \mathbb{N}$, $\theta \in [0, 2\cdot\pi]$, $z \in [0, \gamma_v] \times \mathbb{R}$:

$$\Big|\sin(n\cdot\theta) \cdot f_v(r, \theta) \cdot h_v(z, r, \theta)\Big| \leq \mathfrak{C} \cdot n^{3/4} \cdot G_v(r, \theta) \tag{8.53}$$

in the case $v \in \{1, 2, 6, 7\}$, and

$$\left|\cos(n\cdot\theta)\cdot f_v(r,\theta)\cdot h_v(z,r,\theta)\right| \;\le\; \mathfrak{C}\cdot G_v(r,\theta) \tag{8.54}$$

if $v=3$, as well as

$$\left|\left(\cos(n\cdot\theta) - \cos^n(\theta)\right)\cdot f_v(r,\theta)\cdot h_v(z,r,\theta)\right| \;\le\; \mathfrak{C}\cdot n^{3/4}\cdot G_v(r,\theta) \tag{8.55}$$

in the case $v\in\{4,5\}$, where the function G_v is defined in the following way:

$$G_v(r,\theta) := (1-\cos\theta)^{15/8}\cdot(1+r^{\gamma_v})\cdot\left(g_0^{(\varphi)}(r,\theta)\right)^{-5/2} \quad \text{for } v\in\{1,2,3\},$$

$$G_v(r,\theta) := (1-\cos\theta)^{11/8}\cdot|r-1|\cdot(1+r^{\gamma_v})\cdot\left(g_0^{(\varphi)}(r,\theta)\right)^{-5/2} \quad \text{if } v\in\{4,5\},$$

$$G_v(r,\theta) := (1-\cos\theta)^{7/8}\cdot(1+r^{\gamma_v})\cdot\left(g_0^{(\varphi)}(r,\theta)\right)^{-3/2} \quad \text{in the case } v\in\{6,7\},$$

with $r\in(0,\infty)\backslash\{1\}$, $\theta\in[0,2\cdot\pi]$. If $v\in\{4,5\}$, then we further obtain by referring to (8.52), for $n\in\mathbb{N}$, $\theta\in[0,2\cdot\pi]$, $r\in(0,\infty)\backslash\{1\}$, $z\in[0,\gamma_v]\times\mathbb{R}$:

$$\left|\cos^{n-1}(\theta)\cdot\sin\theta\cdot a_v^{(\varphi)}(z,r,\theta)\right| \;\le\; |\cos\theta|^{n-1}\cdot\tilde{h}_v(r,\theta), \tag{8.56}$$

where $\tilde{h}$ is defined by

$$\tilde{h}_v(r,\theta) := 2\cdot\varrho_v\cdot(1-\cos\theta)^{3/2}\cdot|r-1|\cdot(1+r^{\gamma_v})\cdot\left(g_0^{(\varphi)}(r,\theta)\right)^{-5/2},$$

with the constant ϱ_v introduced at the beginning of the proof. In the case $v\in\{1,2,3,6,7\}$, we set $\tilde{h}(r,\theta):=0$ for $r\in(0,\infty)$, $\theta\in[0,2\cdot\pi]$.

Now, combining (8.40), (8.41), (8.49), (8.51) and (8.53) - (8.56), we obtain for $n\in\mathbb{N}$, $z\in[0,\gamma_v]\times\mathbb{R}$:

$$|A_v^{(\varphi)}(z,n)| \tag{8.57}$$
$$\le\; \mathfrak{C}\cdot n^{-1/4}\cdot\int_0^{2\cdot\pi}\int_0^{\infty} G_v(r,\theta)\,dr\,d\eta \;+\; \int_0^{2\cdot\pi}\int_0^{\infty}|\cos\theta|^{n-1}\cdot\tilde{h}_v(r,\theta)\,dr\,d\theta.$$

But according to Lemma 8.2, the functions G_v and $\tilde{h}_v$ are integrable on $(0,\infty)\times(0,2\cdot\pi)$. Hence the lemma follows by (8.57) and Lebesgue's theorem on dominated convergence.

Lemma 8.11. *Take $\varphi\in(0,\pi/2]$, $n\in\mathbb{Z}$, $j\in\{1,\cdots,7\}$. Set $\zeta_j:=4$ in the case $j\in\{1,2,3\}$, $\zeta_j:=3$ in the case $j\in\{4,5\}$, and $\zeta_j:=2$ if $j\in\{6,7\}$. Then the function $A_j^{(\varphi)}(\cdot,n)\,|\,(0,\zeta_j)\times\mathbb{R}$ is holomorphic, and the function $A_j^{(\varphi)}(\cdot,n)$, which maps $[0,\zeta_j)\times\mathbb{R}$ into $\mathbb{C}$, is continuous.*

Proof: Fix $\epsilon\in(0,\infty)$. For $r\in(0,\infty)$, $\theta\in[0,2\cdot\pi]$, the function

$$f_{r,\theta}:(0,\zeta_j-\epsilon)\times\mathbb{R}\mapsto\mathbb{C},\qquad f_{r,\theta}(z) := a_j^{(\varphi)}(z,r,\theta)\qquad (z\in(0,\zeta_j-\epsilon)\times\mathbb{R}),$$

is holomorphic, as may be seen by the definition of $a_v^{(\varphi)}$ in Defintion 8.1. In addition, by referring to Definition 8.1 once more, we find for $z\in(0,\zeta_j-\epsilon)\times\mathbb{R}$, $r\in(0,\infty)$, $\theta\in[0,2\cdot\pi]$ that

$$|\,d/dz\,(f_{r,\theta}(z))\,| \;=\; |\ln r\cdot a_j^{(\varphi)}(z,r,\theta)| \;\le\; h(r,\theta),$$

where the function h is defined by

$$h(r,\theta) := |\ln r|\cdot\left(|a_j^{(\varphi)}(0,r,\theta)| + |a_j^{(\varphi)}(\zeta_j-\epsilon,r,\theta)|\right),$$

for r,θ as before. From Lemma 8.2 we know that the function h is integrable on $(0,\infty)\times(0,2\cdot\pi)$. Hence, referring to the definition of $A_j^{(\varphi)}$ in (8.7), and to Lebesgue's theorem on dominated convergence, we may conclude that the function $A_j^{(\varphi)}\,|\,(0,\zeta_j-\epsilon)\times\mathbb{R}$ is holomorphic. Since ϵ was arbitrary from $(0,\infty)$, we have shown that $A_j^{(\varphi)}\,|\,(0,\zeta_j)\times\mathbb{R}$ is holomorphic too. Continuity of $A_j^{(\varphi)}$ may be proved by recurring once more to Lemma 8.2 and Lebesgue's theorem on dominated convergence.

Corollary 8.4. *Let $\varphi\in(0,\pi/2]$, $\tau\in\{-1,1\}$, $n\in\mathbb{Z}$. Then the functions*

$$\tau + Y^{(\varphi)}(\cdot,n)\,\big|\,(0,2)\times\mathbb{R} \qquad\text{and}\qquad \det\left(\tau\cdot E_3 + Z^{(\varphi)}(\cdot,n)\right)\big|\,(0,2)\times\mathbb{R}$$

are holomorphic. Furthermore, $\tau+Y^{(\varphi)}(\cdot,n)$ and $\tau\cdot E_3+Z^{(\varphi)}(\cdot,n)$ are continuous functions on $[0,2)\times\mathbb{R}$.

This corollary readily follows from Definition 8.2, Lemma 8.1 and 8.11.

Now we are able to prove a sharpened version of Corollary 8.2:

Corollary 8.5. *Let $p\in(1,\infty)$, $\varphi\in(0,\pi/2]$, $\tau\in\{-1,1\}$. Then it holds:*

$\tau + Y^{(\varphi)}(2/p+i\cdot\xi,n)\neq 0$ for all $\xi\in\mathbb{R}$, $n\in\mathbb{Z}$ if and only if $\Pi(\tau,p,\mathbb{K}(\varphi))$ is topological;

$\det\left(\tau\cdot E_3 + Z^{(\varphi)}(2/p+i\cdot\xi,n)\right)\neq 0$ for all $\xi\in\mathbb{R}$, $n\in\mathbb{Z}$ if and only if $\Lambda(\tau,p,\mathbb{K}(\varphi))$ is topological.

Proof: Due to Corollary 8.2, we only have to show:

If $\tau + Y^{(\varphi)}(2/p+i\cdot\xi,n)\neq 0$ for any $\xi\in\mathbb{R}$, $n\in\mathbb{Z}$, then there is some $\epsilon>0$ with $\left|\tau + Y^{(\varphi)}(2/p+i\cdot\xi,n)\right|\ge\epsilon$ for $\xi\in\mathbb{R}$, $n\in\mathbb{Z}$;

if $\det\left(\tau\cdot E_3 + Z^{(\varphi)}(2/p+i\cdot\xi,n)\right)\neq 0$ for any $\xi\in\mathbb{R}$, $n\in\mathbb{Z}$, then there exists some $\epsilon>0$ with

$$\left|\det\left(\tau\cdot E_3 + Z^{(\varphi)}(2/p+i\cdot\xi,\, n)\right)\right| \;\geq\; \epsilon \qquad \text{for } \xi\in\mathbb{R},\ n\in\mathbb{Z}. \tag{8.58}$$

We proof the second implication; the first one follows in an analogous way. According to Lemma 8.10, there is $n_0 \in \mathbb{N}$ so that it holds for $\xi\in\mathbb{R}$, $n\in\mathbb{Z}$ with $|n| > n_0$:

$$\left|\det\left(\tau\cdot E_3 + Z^{(\varphi)}(2/p+i\cdot\xi,\, n)\right)\right| \;\geq\; 1/2. \tag{8.59}$$

Recalling Lemma 8.8, we may choose $r_0 > 0$ such that inequality (8.59) is valid for $\xi\in\mathbb{R}$ with $|\xi| \geq r_0$, and for $n\in\mathbb{Z}$ with $|n|\leq n_0$. On the other hand, the function $A_n : [-r_0, r_0] \mapsto [0,\infty)$, with

$$A_n(\xi) := \left|\det\left(\tau\cdot E_3 + Z^{(\varphi)}(2/p+i\cdot\xi,\, n)\right)\right| \qquad (\xi\in[-r_0, r_0]),$$

is continuous $(n\in\mathbb{N})$; see Corollary 8.4. Now suppose

$$\det\left(\tau\cdot E_3 + Z^{(\varphi)}(2/p+i\cdot\xi,\, n)\right) \;\neq\; 0 \qquad \text{for } \xi\in\mathbb{R},\ n\in\mathbb{Z}.$$

Then, for $n\in\mathbb{Z}$ with $|n|\leq n_0$, there is some $\epsilon_n > 0$ such that $A_n(x) \geq \epsilon_n$ for $\xi\in[-r_0, r_0]$. Recalling (8.59), we now see that inequality (8.58) is satisfied if we set $\epsilon := \min\{1/2, \epsilon_{-n_0}, \cdots, \epsilon_0, \cdots, \epsilon_{n_0}\}$.

Lemma 8.12. *Let* $\varphi\in(0,\pi/2]$, $\tau\in\{-1,1\}$, $n\in\mathbb{Z}$. *Then*

$$\det\left(\tau\cdot E_3 + Z^{(\varphi)}(1,n)\right) \;\neq\; 0 \qquad \textit{for } n\in\mathbb{Z}; \tag{8.60}$$

$$\tau + Y^{(\varphi)}(1,n) \;=\; \tau \quad \textit{for } n\in\mathbb{Z}\backslash\{0\}; \qquad \tau + Y^{(\varphi)}(1,0) \;=\; \tau + \cos\varphi; \tag{8.61}$$

$$\det\left(\tau\cdot E_3 + Z^{(\varphi)}(1,0)\right) \;=\; (1-\cos^2(\varphi))\cdot(\tau+\cos^3(\varphi)). \tag{8.62}$$

Proof: Combining (8.42), (8.44), (8.45), Lemma 8.1 and Definition 8.2, we obtain

$$\tau\cdot E_3 + Z^{(\varphi)}(1,0) \;=\; \begin{pmatrix} \tau-\cos\varphi & 0 & \cos^2(\varphi)\cdot\sin\varphi \\ 0 & \tau+\cos\varphi & 0 \\ 0 & 0 & \tau+\cos^3(\varphi) \end{pmatrix}.$$

This implies (8.62), as well as (8.60) in the case $n = 0$. Now take $n\in\mathbb{Z}\backslash\{0\}$. For brevity we write $B := \tau\cdot E_3 + Z^{(\varphi)}(1,n)$. Then, applying (8.42) - (8.46) and Lemma 8.1, we compute:

$$B_{11} \;=\; \tau \;+\; \cos\varphi\cdot(1+\sin^2(\varphi))\cdot(2\cdot\pi)^{-1}\cdot\int_0^{2\cdot\pi}\cos\theta\cdot\cos(n\cdot\theta)\;d\theta,$$

$$B_{22} \;=\; \tau \;+\; \cos\varphi\cdot(2\cdot\pi)^{-1}\cdot\int_0^{2\cdot\pi}\cos\varphi\cdot\cos(n\cdot\theta)\;d\theta, \qquad B_{33} \;=\; \tau,$$

$$B_{21} \;=\; i\cdot\cos\varphi\cdot(2\cdot\pi)^{-1}\cdot\int_0^{2\cdot\pi}\sin\theta\cdot\sin(n\cdot\theta)\;d\theta,$$

$$B_{12} \;=\; -i\cdot\cos\varphi\cdot(1+\sin^2(\varphi))\cdot(2\cdot\pi)^{-1}\cdot\int_0^{2\cdot\pi}\sin\theta\cdot\sin(n\cdot\theta)\;d\theta,$$

$$B_{jl} \;=\; 0 \qquad \text{for } (j,l)\in\{(1,3),\,(2,3)\}.$$

Observe that in the case $n\in\mathbb{Z}\backslash\{1,-1\}$, it holds

$$\int_0^{2\cdot\pi}\cos\theta\cdot\cos(n\cdot\theta)\;d\theta \;=\; \int_0^{2\cdot\pi}\sin\theta\cdot\sin(n\cdot\theta)\;d\theta \;=\; 0.$$

Thus it follows by expanding B according to the last column:

$$\det\left(\tau\cdot E_3 + Z^{(\varphi)}(1,n)\right) \;=\; \tau^3.$$

Now take $n\in\{-1,1\}$. Then we find

$$\int_0^{2\cdot\pi}\cos\theta\cdot\cos(n\cdot\theta)\;d\theta \;=\; \int_0^{2\cdot\pi}\sin\theta\cdot\sin(n\cdot\theta)\;d\theta \;=\; \pi,$$

so that

$$\det\left(\tau\cdot E_3 + Z^{(\varphi)}(1,n)\right) \;=\; \tau\cdot\left(1 \;+\; \tau\cdot\cos\varphi\cdot\left(1 \;+\; (1/2)\cdot\sin^2(\varphi)\right)\right).$$

If $\sigma\in[0,\pi/2]$, we set $f(\sigma) := \cos\sigma\cdot\left(1 + (1/2)\cdot\sin^2(\sigma)\right)$. Using the fact that

$$f'(\sigma) \;=\; \sin\sigma\cdot\left(-1 \;-\; (1/2)\cdot\sin^2(\sigma) \;+\; \cos^2(\sigma)\right) \;<\; 0$$

for $\sigma\in(0,\pi/2]$, we obtain

$$f(\sigma) \;<\; f(0) \;=\; 1, \qquad f(\sigma) \;\geq\; f(\pi/2) \;=\; 0 \qquad (\sigma\in(0,\pi/2]). \tag{8.63}$$

It follows

$$\left|\det\left(\tau\cdot E_3 + Z^{(\varphi)}(1,n)\right)\right| \;>\; 0 \qquad \text{for } \varphi\in(0,\pi/2].$$

Collecting our results, we arrive at inequality (8.60). The equations in (8.61) are an immediate consequence of (8.44).

Now we are in a position to prove the main results of this chapter:

Theorem 8.1. *Let* $\varphi\in(0,\pi/2]$, $\tau\in\{-1,1\}$. *Then the following sets are countable:*

$P_1 := \{p\in(1,\infty) \;:\; \Pi(\tau, p, \mathbb{K}(\varphi))$ *is not topological*$.\}$,

$P_2 := \{p\in(1,\infty) \;:\; \Lambda(\tau, p, \mathbb{K}(\varphi))$ *is not topological*$.\}$,

$P_3 := \{p\in(1,\infty) \;:\; \Pi^*(\tau, p, \mathbb{K}(\varphi))$ *is not topological*$.\}$,

$P_4 := \{p\in(1,\infty) \;:\; \Lambda^*(\tau, p, \mathbb{K}(\varphi))$ *is not topological*$.\}$.

Proof: We shall show that P_2 is countable. As for P_1, we may proceed in an analogous way, with some details turning out to be somewhat simpler, due to the fact that $Y^{(\varphi)}$, contrary to $Z^{(\varphi)}$, is a scalar function.

But once the sets P_1 and P_2 are known to be countable, we may recur to the fact that the operator $\Pi^*(\tau, p, \mathbb{K}(\varphi))$ is adjoint to $\Pi(\tau, (1-1/p)^{-1}, \mathbb{K}(\varphi))$, and $\Lambda^*(\tau, p, \mathbb{K}(\varphi))$ to $\Lambda(\tau, (1-1/p)^{-1}, \mathbb{K}(\varphi))$. On the other hand, these operators are continuous, as mentioned in Lemma 6.7. Hence we deduce from Lemma 8.5 that P_3 and P_4 are countable sets.

In order to prove that P_2 is countable, we first note the ensuing equation, which is a consequence of Corollary 8.5:

$$P_2 = \Big\{ p \in (1, \infty) \; : \; \text{There is some } \xi \in \mathbb{R} \text{ and some } n \in \mathbb{Z} \text{ with } \det\Big(\tau \cdot E_3 + Z^{(\varphi)}(2/p + i \cdot \xi, n)\Big) = 0. \Big\}. \tag{8.64}$$

For $n \in \mathbb{N}$, we set

$$Q_n := \Big\{ p \in (1, \infty) \; : \; \text{There is some } \xi \in \mathbb{R} \text{ with } \det\Big(\tau \cdot E_3 + Z^{(\varphi)}(2/p + i \cdot \xi, n)\Big) = 0. \Big\}.$$

Because of (8.64), we know that

$$P_2 = \bigcup_{n \in \mathbb{N}} Q_n. \tag{8.65}$$

If $n \in \mathbb{Z}$, $m \in \mathbb{N}$, define

$$Q_{n,m} := \Big\{ p \in [1 + 1/m, m] \; : \; \text{There is } \xi \in [-m, m] \text{ with } \det\Big(\tau \cdot E_3 + Z^{(\varphi)}(2/p + i \cdot \xi, n)\Big) = 0. \Big\}.$$

Then we conclude for $n \in \mathbb{Z}$:

$$Q_n = \bigcup_{m \in \mathbb{N}} Q_{n,m}. \tag{8.66}$$

We are going to prove that $Q_{n,m}$ is finite for $n \in \mathbb{Z}$, $m \in \mathbb{N}$. Then equations (8.65) and (8.66) yield that P_2 is countable. In order to establish our claim, we take $n \in \mathbb{Z}$, $m \in \mathbb{N}$ and assume that $Q_{n,m}$ is an infinite set. Then we may choose a sequence (p_v) in $Q_{n,m}$ so that $p_v \neq p_\mu$ for $\mu, v \in \mathbb{N}$ with $\mu \neq v$. For any $v \in \mathbb{N}$, there is some $\xi_v \in [-m, m]$ with

$$\det\Big(\tau \cdot E_3 + Z^{(\varphi)}(2/p_v + i \cdot \xi_v, n)\Big) = 0.$$

Define the function $G : (0, 2) \times \mathbb{R} \mapsto \mathbb{C}$ by setting

$$G(\zeta + i \cdot \xi) := \det\Big(\tau \cdot E_3 + Z^{(\varphi)}(\zeta + i \cdot \xi, n)\Big) \quad \text{for } \zeta \in (0, 2), \; \xi \in \mathbb{R}.$$

Putting $\gamma_v := 2/p_v + i \cdot \xi_v$, we see that $\gamma_v \in [2/m, 2 \cdot m/(m+1)] \times [-m, m]$, and $G(\gamma_v) = 0$ for $v \in \mathbb{N}$. Furthermore, any two terms of the sequence (γ_v) are different. Hence there is a mapping $\sigma : \mathbb{N} \mapsto \mathbb{N}$ strictly monotone increasing, as well as an element $\gamma_0 \in [2/m, 2 \cdot m/(m+1)] \times [-m, m]$, such that $\gamma_{\sigma(v)} \to \gamma_0$ $(v \to \infty)$ and $\gamma_{\sigma(v)} \neq \gamma_0$ $(v \in \mathbb{N})$. Moreover, we have $G(\gamma_{\sigma(v)}) = 0$ for $v \in \mathbb{N}$. Since $G : (0, 2) \times \mathbb{R} \mapsto \mathbb{C}$ is holomorphic (Corollary 8.4), it follows by the identity theorem for holomorphic functions (see [41, p. 209, Theorem 10.18]):

$$G(z) = 0 \quad \text{for } z \in (0, 2) \times \mathbb{R}. \tag{8.67}$$

On the other hand, it holds by Lemma 8.12:

$$\det\Big(\tau \cdot E_3 + Z^{(\varphi)}(1, k)\Big) \neq 0 \quad \text{for } k \in \mathbb{Z}$$

– a contradiction to (8.67). Thus our assumption that $Q_{n,m}$ is an infinite set turned out to be false. This completes our proof.

We still have to show that the sets appearing in Theorem 8.1 are not empty. In fact, we shall establish are more precise result:

Theorem 8.2. *If $\tau \in \{-1, 1\}$, $\varphi \in (0, \pi/2]$ we define*

$$Q_1(\tau, \varphi) := \{ p \in (1, 2) \; : \; \Pi(\tau, p, \mathbb{K}(\varphi)) \text{ is not topological.} \},$$

$$Q_1^*(\tau, \varphi) := \{ p \in (2, \infty) \; : \; \Pi^*(\tau, p, \mathbb{K}(\varphi)) \text{ is not topological.} \},$$

$$Q_2(\tau, \varphi) := \{ p \in (1, 2) \; : \; \Lambda(\tau, p, \mathbb{K}(\varphi)) \text{ is not topological.} \},$$

$$Q_2^*(\tau, \varphi) := \{ p \in (2, \infty) \; : \; \Lambda^*(\tau, p, \mathbb{K}(\varphi)) \text{ is not topological.} \}.$$

Then, for any $\varphi \in (0, \pi/2)$, the sets $Q_1(-1, \varphi)$ and $Q_1^*(-1, \varphi)$ are not empty.

In addition, for $\tau \in \{-1, 1\}$, there is some angle $\varphi \in (0, \pi/2)$ such that $Q_2(\tau, \varphi) \neq \emptyset$ and $Q_2^*(\tau, \varphi) \neq \emptyset$.

Proof: According to Corollary 8.5, it holds for $\tau \in \{-1, 1\}$, $\varphi \in (0, \pi/2]$:

$$Q_1(\tau, \varphi) = \Big\{ p \in (1, 2) \; : \; \tau + Y^{(\varphi)}(2/p + i \cdot \xi, n) = 0 \text{ for some } \xi \in \mathbb{R} \text{ and some } n \in \mathbb{Z} \Big\}, \tag{8.68}$$

$$Q_2(\tau, \varphi) = \Big\{ p \in (1, 2) \; : \; \det\Big(\tau \cdot E_3 + Z^{(\varphi)}(2/p + i \cdot \xi, n)\Big) = 0 \text{ for some } \xi \in \mathbb{R} \text{ and some } n \in \mathbb{Z} \Big\}. \tag{8.69}$$

Now take $\varphi \in (0, \pi/2)$. Recalling Definition 8.2 and 8.1, we find for $p \in (1, 2]$:

$$-1 + Y^{(\varphi)}(2/p, 0) = -1 + (1/3) \cdot A_8^{(\varphi)}(2/p, 0). \tag{8.70}$$

On the other hand, we know from (8.44) and (8.47):

$A_6^{(\varphi)}(1, 0) = \cos\varphi, \quad A_6^{(\varphi)}(\sigma, 0) \to \infty \quad \text{for } \sigma \uparrow 2.$

Hence there is some $p \in (1, 2)$ with $1 - (1/3) \cdot A_6^{(\varphi)}(2/p, 0) = 0$. Due to (8.70) and (8.68), this implies $p \in Q_1(-1, \varphi)$.

For $p \in (1, \infty)$, the operator $\Pi^*(-1, (1-1/p)^{-1}, \mathbb{K}(\varphi))$ is adjoint to $\Pi(-1, p, \mathbb{K}(\varphi))$. Furthermore, these operators are continuous (Lemma 6.7). Hence, referring to Lemma 8.5, we conclude that $Q_1^*(-1, \varphi) \neq \emptyset$.

Turning to $Q_2(\tau, \varphi)$, we first introduce some abbreviations by defining for $\tau \in \{-1, 1\}$, $\varphi \in (0, \pi/2]$, $\gamma \in [0, 3)$:

$$B_1(\tau, \varphi, \gamma) := \tau - A_1^{(\varphi)}(\gamma+1, 0) + (1 - 2\cdot\sin^2(\varphi)) \cdot A_2^{(\varphi)}(\gamma+1, 0),$$

$$B_2(\tau, \varphi, \gamma) := \tau + A_1^{(\varphi)}(\gamma+1, 0) + A_2^{(\varphi)}(\gamma+1, 0),$$

$$B_3(\tau, \varphi, \gamma) := \tau - 2\cdot\cos^2(\varphi) \cdot A_1^{(\varphi)}(\gamma+1, 0),$$

$$B_4(\tau, \varphi, \gamma) := -(2\cdot\cos\varphi\cdot\sin\varphi - \cot\varphi) \cdot A_1^{(\varphi)}(\gamma+1, 0) - \cot\varphi \cdot A_1^{(\varphi)}(\gamma, 0),$$

$$B_5(\tau, \varphi, \gamma) := -2\cdot\cos\varphi\cdot\sin\varphi \cdot A_2^{(\varphi)}(\gamma+1, 0) - \cot\varphi \cdot A_1^{(\varphi)}(\gamma+1, 0) + \cot\varphi \cdot A_1^{(\varphi)}(\gamma, 0).$$

Recalling Lemma 8.1, Definition 8.2, 8.1, as well as (8.42), we obtain for $\tau \in \{-1, 1\}$, $\varphi \in (0, \pi/2]$, $\gamma \in [0, 2)$:

$$\det\Big(\tau \cdot E_3 + Z^{(\varphi)}(\gamma, 0)\Big) = B_2(\tau, \varphi, \gamma) \tag{8.71}$$
$$\cdot\bigg(\Big(B_1(\tau, \varphi, \gamma) + \sin^2(\varphi) \cdot A_7^{(\varphi)}(\gamma, 0)\Big) \cdot \Big(B_3(\tau, \varphi, \gamma) + \cos^2(\varphi) \cdot A_6^{(\varphi)}(\gamma, 0)\Big)$$
$$- \Big(B_4(\tau, \varphi, \gamma) + \cos\varphi \cdot \sin\varphi \cdot A_6^{(\varphi)}(\gamma, 0)\Big) \cdot \Big(B_5(\tau, \varphi, \gamma) + \cos\varphi \cdot \sin\varphi \cdot A_7^{(\varphi)}(\gamma, 0)\Big)\bigg).$$

For $\tau \in \{-1, 1\}$, $\varphi \in (0, \pi/2]$, $\gamma \in [0, 4)$, we define

$$B_6(\tau, \varphi, \gamma) := B_1(\tau, \varphi, \gamma) \cdot B_3(\tau, \varphi, \gamma) - B_4(\tau, \varphi, \gamma) \cdot B_5(\tau, \varphi, \gamma),$$

$$B_7(\tau, \varphi, \gamma) := \cos^2(\varphi) \cdot B_1(\tau, \varphi, \gamma) - \cos\varphi \cdot \sin\varphi \cdot B_5(\tau, \varphi, \gamma),$$

$$B_8(\tau, \varphi, \gamma) := \sin^2(\varphi) \cdot B_3(\tau, \varphi, \gamma) - \cos\varphi \cdot \sin\varphi \cdot B_4(\tau, \varphi, \gamma).$$

Then it follows from (8.71), for τ, φ, γ as given there:

$$\det\Big(\tau \cdot E_3 + Z^{(\varphi)}(\gamma, 0)\Big) = B_2(\tau, \varphi, \gamma) \tag{8.72}$$
$$\cdot\Big(B_6(\tau, \varphi, \gamma) + B_7(\tau, \varphi, \gamma) \cdot A_6^{(\varphi)}(\gamma, 0) + B_8(\tau, \varphi, \gamma) \cdot A_7^{(\varphi)}(\gamma, 0)\Big).$$

Next, we set for $\tau \in \{-1, 1\}$, $\varphi \in (0, \pi/2]$, $\gamma \in [0, 3)$:

$$B_9(\tau, \varphi, \gamma) := B_6(\tau, \varphi, \gamma) + B_8(\tau, \varphi, \gamma) \cdot (3/2) \cdot \Big(A_1^{(\varphi)}(\gamma+1, 0) + A_2^{(\varphi)}(\gamma+1, 0)\Big),$$

$$B_{10}(\tau, \varphi, \gamma) := B_7(\tau, \varphi, \gamma) - (1/2) \cdot B_8(\tau, \varphi, \gamma).$$

Referring to (8.72) and (8.48), we find for τ, φ, γ as in (8.71):

$$\det\Big(\tau \cdot E_3 + Z^{(\varphi)}(\gamma, 0)\Big) = B_2(\tau, \varphi, \gamma) \cdot \Big(B_9(\tau, \varphi, \gamma) + B_{10}(\tau, \varphi, \gamma) \cdot A_6^{(\varphi)}(\gamma, 0)\Big). \tag{8.73}$$

On the other hand, Lemma 8.11 yields

$$B_j(\tau, \varphi, \gamma) \to B_j(\tau, \varphi, 2) \quad (\gamma \uparrow 2) \qquad \text{for } j \in \{2, 9, 10\}. \tag{8.74}$$

This reference to Lemma 8.11 is justified, since only the functions $A_1^{(\varphi)}(\gamma, 0)$, $A_1^{(\varphi)}(\gamma+1, 0)$, $A_2^{(\varphi)}(\gamma, 0)$ and $A_2^{(\varphi)}(\gamma+1, 0)$ enter into the definition of $B_j(\tau, \varphi, \gamma)$, for $j \in \{1, \cdots, 10\}$. Next we point out a consequence of (8.46), namely

$$B_2(\tau, \varphi, 2) = \tau + \cos\varphi \cdot \Big(1 + (1/2) \cdot \sin^2(\varphi)\Big),$$

so that it follows from (8.63):

$$\operatorname{sign}(B_2(\tau, \varphi, 2)) = \operatorname{sign}(\tau) \qquad (\tau \in \{1, 1\},\ \varphi \in (0, \pi/2]). \tag{8.75}$$

In addition, we conclude from (8.44) - (8.46):

$$B_{10}(\tau, \varphi, 2) = -(\tau/2) \cdot \sin^2(\varphi) + \tau \cdot \cos^2(\varphi) - \cos^3(\varphi) \tag{8.76}$$

for $\tau \in \{-1, 1\}$, $\varphi \in (0, \pi/2]$. But there is some $\varphi_1 \in (0, \pi/2)$ with

$$(1/2) \cdot \sin^2(\varphi_1) - \cos^2(\varphi_1) - \cos^3(\varphi_1) < 0.$$

This means by (8.76): $B_{10}(-1, \varphi_1, 2) < 0$. On the other hand, we know from (8.75) that $B_2(-1, \varphi_1, 2) < 0$. Now it follows from (8.73), (8.74) and (8.47):

$$\det\Big(-E_3 + Z^{(\varphi_1)}(\gamma, 0)\Big) \to \infty \quad \text{for } \gamma \uparrow 2.$$

We further recall equation (8.62), which yields

$$\det\Big(-E_3 + Z^{(\varphi_1)}(1, 0)\Big) < 0.$$

Thus there is some $\gamma \in (1, 2)$ with

$$\det\Big(-E_3 + Z^{(\varphi_1)}(\gamma, 0)\Big) = 0.$$

Referring to (8.69), we may conclude $Q_2(-1, \varphi_1) \neq \emptyset$. Next we point out there is some $\varphi_2 \in (0, \pi/2)$ with

$$-(1/2)\cdot\sin^2(\varphi_2) \;+\; \cos^2(\varphi_2) \;-\; \cos^3(\varphi_2) \;<\; 0\,.$$

This means by (8.76): $B_{10}(1, \varphi_2, 2) < 0$. According to (8.75), we have $B_2(1, \varphi_2, 2) > 0$. Now we recall (8.73), (8.74) and (8.47) to obtain

$$\det\Big(E_3 \;+\; Z^{(\varphi_2)}(\gamma, 0)\Big) \;\to\; -\infty \quad \text{for } \gamma \uparrow 2\,.$$

On the other hand, we know from (8.62):

$$\det\Big(E_3 \;+\; Z^{(\varphi_2)}(1, 0)\Big) \;>\; 0\,.$$

Hence there is some $\gamma \in (1, 2)$ with

$$\det\Big(E_3 \;+\; Z^{(\varphi_2)}(\gamma, 0)\Big) \;=\; 0\,,$$

so that it follows by (8.69): $Q_2(1, \varphi_2) \neq \emptyset$.

The operator $\Lambda^*(\tau, (1-1/p)^{-1}, \mathbb{K}(\varphi))$ is continuous (Lemma 6.7) and adjoint to $\Lambda(\tau, p, \mathbb{K}(\varphi))$, for any $\tau \in \{-1, 1\}$, $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$. Now Lemma 8.5 yields that $Q_2^*(-1, \varphi_1) \neq \emptyset$ and $Q_2^*(1, \varphi_2) \neq \emptyset$.

We close this chapter by a lemma needed in Chapter 13. It states that in order to prove invertibility of $\Lambda(\tau, p, \mathbb{K}(\varphi))$ for $p \in (2, \infty)\backslash\{4\}$, it is sufficient to consider either the range $p \in (2, 4)$ or $p \in (4, \infty)$.

Lemma 8.13. *Let* $\varphi \in (0, \pi/2]$, $\tau \in \{-1, 1\}$, $p \in (2, \infty)$. *Assume the operator* $\Lambda(\tau, p, \mathbb{K}(\varphi))$ *to be topological. Thus* $\Lambda(\tau, 2\cdot p/(p-2), \mathbb{K}(\varphi))$ *has the same property.*

Proof: According to Corollary 8.5, we have

$$\det\Big(\tau\cdot E_3 \;+\; Z^{(\varphi)}(2/p + i\cdot\xi,\, n)\Big) \;\neq\; 0 \qquad \text{for all } \xi \in \mathbb{R},\; n \in \mathbb{Z}\,.$$

Applying the substitution rule with $r = 1/s$ in the definition of $Z^{(\varphi)}$ (see Definition 8.2), we deduce that

$$\det\Big(\tau\cdot E_3 \;+\; Z^{(\varphi)}(1 - 2/p - i\cdot\xi,\, n)\Big) \;\neq\; 0 \qquad \text{for } \xi \in \mathbb{R},\; n \in \mathbb{Z}\,.$$

Hence, by referring to Corollary 8.5 again, we see that that $\Lambda(\tau, 2\cdot p/(p-2), \mathbb{K}(\varphi))$ is topological.

Chapter 9

Some Uniqueness Results

It is the aim of this chapter to show that the operators $\Gamma^*(\tau, p, \lambda, \mathbb{K}(\varphi))$, $\Gamma^*(\tau, p, \lambda, \mathbb{L}(\varphi, \epsilon))$ and $\Pi^*(\tau, p, \mathbb{L}(\varphi, \epsilon))$ $(\epsilon > 0)$ are one-to-one for certain values of p (Corollary 9.3, Theorem 9.4). These results are based on a suitable application of the Divergence theorem (see (9.43), (9.52)). However, it will take some effort to work out the details of this approach, due to the fact that the surface $\partial\mathbb{L}(\varphi, \epsilon)$ is not bounded.

Our approach yields somewhat stronger results than will be needed later in this book. For example, there will be no occasion to use the fact that $\Gamma^*(\tau, p, \lambda, \mathbb{L}(\varphi, \epsilon))$ is one-to-one not only for $p = 2$, but also for $p \in (1, 2)$, if $\epsilon > 0$. However, almost no additional work was necessary in order to attain the present level of generality. So there is no reason to omit any of the results presented in this chapter.

In a first step, we shall establish a "jump relation" for the potential functions we are going to introduce now. Note that the following definitions make sense due to Lemma 6.2 and 6.5.

Definition 9.1. *Take* $p \in (1, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\varphi \in (0, \pi/2]$, $\epsilon \in [0, \infty)$, $r \in \mathbb{R}$. *Then we define the operators*

$$R^*(p, \mathbb{L}(\varphi, \epsilon), r),\;\; \widetilde{R}^*(p, \lambda, \mathbb{L}(\varphi, \epsilon), r) \;:\; L^p(\partial\mathbb{L}(\varphi, \epsilon))^3 \mapsto L^p(\partial\mathbb{L}(\varphi, \epsilon))^3$$

by setting for $f \in L^p(\partial\mathbb{L}(\varphi, \epsilon))^3$, $x \in \partial\mathbb{L}(\varphi, \epsilon)$:

$$R^*(p, \mathbb{L}(\varphi, \epsilon), r)(f)(x) := \Big(-\int_{\partial\mathbb{L}(\varphi,\epsilon)} \sum_{l,k=1}^{3} \mathcal{D}_{jkl}(x - y + (0,0,r)) \cdot n_k^{(\varphi,\epsilon)}(x)\cdot f_l(y)\; d\mathbb{L}(\varphi, \epsilon)(y)\Big)_{1\le j\le 3},$$

$$\widetilde{R}^*(p, \lambda, \mathbb{L}(\varphi, \epsilon), r)(f)(x) := \Big(-\int_{\partial\mathbb{L}(\varphi,\epsilon)} \sum_{l,k=1}^{3} \widetilde{\mathcal{D}}_{jkl}^{\lambda}(x - y + (0,0,r)) \cdot n_k^{(\varphi,\epsilon)}(x)\cdot f_l(y)\; d\mathbb{L}(\varphi, \epsilon)(y)\Big)_{1\le j\le 3}.$$

The following theorem is called a "jump relation" because the L^p-limit function appearing in this theorem comes out different when r tends to zero from either the positive or negative side of the real line. We shall reduce this result to a corresponding theorem pertaining to bounded domains with smooth boundary.

Theorem 9.1 (jump relation). *Let* $p \in (1,\infty)$, $\varphi \in (0,\pi/2]$, $\lambda \in \mathbb{C}\backslash(-\infty,0]$, $\epsilon \in [0,\infty)$, $\tau \in \{-1,1\}$, $f \in L^p\big(\partial\mathbb{L}(\varphi,\epsilon)\big)^3$. *It follows:*

$$\int_{\partial\mathbb{L}(\varphi,\epsilon)} \Big| \Lambda^*(\tau,p,\mathbb{L}(\varphi,\epsilon))(f)(x) - R^*(p,\mathbb{L}(\varphi,\epsilon),-\tau\cdot r)(f)(x)\Big|^p \, d\mathbb{L}(\varphi,\epsilon)(x)$$

$$\longrightarrow 0 \quad \textit{for } r\downarrow 0;$$

$$\int_{\partial\mathbb{L}(\varphi,\epsilon)} \Big| \Gamma^*(\tau,p,\lambda,\mathbb{L}(\varphi,\epsilon))(f)(x) - \widetilde{R}^*(p,\lambda,\mathbb{L}(\varphi,\epsilon),-\tau\cdot r)(f)(x)\Big|^p \, d\mathbb{L}(\varphi,\epsilon)(x)$$

$$\longrightarrow 0 \quad \textit{for } r\downarrow 0.$$

Proof: Fix $j,l \in \{1,2,3\}$. For brevity, we set

$$S^{(1)}_{jkl} := \mathcal{D}_{jkl}, \quad S^{(2)}_{jkl} := \widetilde{\mathcal{D}}^{\lambda}_{jkl} \quad \text{for } k\in\{1,2,3\}; \qquad \Psi := f_l \circ \gamma^{(\varphi,\epsilon)};$$

$$K^{(v)}(r)(\xi,\eta) := -\sum_{k=1}^{3} S^{(v)}_{jkl}\big(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)\big) \cdot \big(n_k^{(\varphi,\epsilon)}\circ\gamma^{(\varphi,\epsilon)}\big)(\xi)\cdot J^{(\varphi,\epsilon)}(\eta)$$

for $r\in\mathbb{R}$, $v\in\{1,2\}$, $\xi,\eta\in\mathbb{R}^2$ with $\xi\neq\eta$. According to Lemma 6.2, 6.5, the function $K^{(v)}(r)\otimes\gamma$ is well defined if $r\in\mathbb{R}$, $\gamma\in L^p(\mathbb{R}^2)$, $v\in\{1,2\}$. We are going to show that for $v\in\{1,2\}$, the following relation holds true:

$$\big\| (\tau\cdot\delta_{jl}/2)\cdot\Psi + K^{(v)}(0)\otimes\Psi - K^{(v)}(-\tau\cdot r)\otimes\Psi \big\|_p \longrightarrow 0 \quad (r\downarrow 0). \tag{9.1}$$

Recalling (3.1) and (3.8), we see the lemma is implied by the preceding convergence result.

In order to establish (9.1), we first note that by Lemma 6.2, 6.5, there is some $\mathfrak{C}_1 > 0$ such that we have for $\gamma\in L^p(\mathbb{R}^2)$, $r\in\mathbb{R}\backslash\{0\}$, $v\in\{1,2\}$:

$$\big\| (\tau\cdot\delta_{jl}/2)\cdot\gamma + K^{(v)}(0)\otimes\gamma \big\|_p, \ \ \big\| K^{(v)}(r)\otimes\gamma \big\|_p \ \le\ \mathfrak{C}_1\cdot\|\gamma\|_p. \tag{9.2}$$

Now let $\kappa\in(0,\infty)$ be given. Choose $\Phi\in C^0(\mathbb{R}^2)$ with $supp(\Phi)$ compact, and with $\|\Phi-\Psi\|_p \le \kappa/(4\cdot\mathfrak{C}_1)$. In the case $\epsilon = 0$, we assume in addition that $supp(\Phi)\subset\mathbb{R}^2\backslash\{0\}$. From (9.2), we conclude for $r\in\mathbb{R}\backslash\{0\}$, $v\in\{1,2\}$:

$$\big\| (\tau\cdot\delta_{jl}/2)\cdot(\Psi-\Phi) + K^{(v)}(0)\otimes(\Psi-\Phi) \big\|_p, \ \ \big\| K^{(v)}(r)\otimes(\Psi-\Phi) \big\|_p \tag{9.3}$$

$$\le \kappa/4.$$

Choose $\delta\in(0,\infty)$ in such a way that in the case $\epsilon=0$, we have $supp(\Phi)\subset \mathbb{B}_2(0,1/\delta)\backslash\mathbb{B}_2(0,\delta)$, whereas in the case $\epsilon>0$, we require $\delta<1/\epsilon$, $supp(\Phi)\subset\mathbb{B}_2(0,1/\delta)$.

Next we construct a bounded domain Ω with C^∞-boundary by setting

$$A_0 := \mathbb{L}(\varphi,\delta/3) \cap \overline{\mathbb{B}_3\big(0,\,3\cdot(\delta\cdot\sin\varphi)^{-1}\big)} \quad \text{in the case } \epsilon=0;$$

$$A_0 := \mathbb{L}(\varphi,\epsilon) \cap \overline{\mathbb{B}_3\big(0,\,3\cdot(\delta\cdot\sin\varphi)^{-1}\big)} \quad \text{if } \epsilon>0,$$

and then defining

$$\Omega := A_0 \cup \{\,9\cdot(\delta\cdot\sin\varphi)^{-2}\cdot|x|^{-2}\cdot x \ : \ x\in A_0\,\}.$$

This means that Ω is the union of A_0 with the set obtained by ~~inverting~~ A_0 ~~with respect to~~ the spherical surface $\partial\mathbb{B}_3\big(0,\,3\cdot(\delta\cdot\sin\varphi)^{-1}\big)$. Let m denote the outward unit normal to Ω. For brevity, we set for $\sigma\in[1,3]$:

$$A(\sigma) := \mathbb{B}_2(0,\sigma/\delta)\backslash\mathbb{B}_2(0,\delta/\sigma) \quad \text{in the case } \epsilon=0,$$

$$A(\sigma) := \mathbb{B}_2(0,\sigma/\delta) \quad \text{if } \epsilon>0.$$

Then $\gamma^{(\varphi,\epsilon)}\,|\,A(3)$ is a parametric representation of

$$\gamma^{(\varphi,\epsilon)}(A(3))\cap\partial\Omega \;=\; \gamma^{(\varphi,\epsilon)}(A(3))\cap\partial\mathbb{L}(\varphi,\epsilon),$$

and for ~~a member~~ x of this set, we have $n^{(\varphi,\epsilon)}(x) = m(x)$. Moreover, there is $\delta_0>0$ such that

$$x+(0,0,r)\in\Omega, \quad x+(0,0,-r)\in\mathbb{R}^3\backslash\overline{\Omega} \qquad \text{for } x\in\gamma^{(\varphi,\epsilon)}(A(2)),\ r\in(0,\delta_0).$$

Define the mapping $h:\partial\Omega\mapsto\mathbb{C}$ by

$$h(x) := \big(\Phi\circ(\gamma^{(\varphi,\epsilon)})^{-1}\big)(x) \qquad \text{for } x\in\gamma^{(\varphi,\epsilon)}(A(2)),$$

$$h(x) := 0 \qquad \text{for } x\in\partial\Omega\backslash\gamma^{(\varphi,\epsilon)}(A(2)).$$

Then we have $h\in C^0(\partial\Omega)$ and $supp(h)\subset\gamma^{(\varphi,\epsilon)}(A(1))$. Now, in the case $v=1$, we may recur to the jump relation for double-layer potentials on bounded smooth domains, as given in [30, p. 56-58], [5, p. 51-56], or [13, p. 136, Satz 4.1], with a detailed proof in [13, p. 113-129, p. 137-139]. It follows

$$\int_{\gamma^{(\varphi,\epsilon)}(A(1))} \sum_{k=1}^{3} S^{(v)}_{jkl}\big(x-y+(0,0,-\tau\cdot r)\big)\cdot m_k(y)\cdot h(y)\, d\Omega(y) \tag{9.4}$$

$$\longrightarrow \ (-\tau\cdot\delta_{jl}/2)\cdot h(x) + \int_{\gamma^{(\varphi,\epsilon)}(A(1))} \sum_{k=1}^{3} S^{(v)}_{jkl}(x-y)\cdot m_k(y)\cdot h(y)\, d\Omega(y) \qquad \text{for } r\downarrow 0,$$

uniformly in $x\in\gamma^{(\varphi,\epsilon)}(A(2))$, $v\in\{1,2\}$. If $v=2$, we note that

$$\widetilde{\mathcal{D}}^{\lambda}_{jkl} - \mathcal{D}_{jkl} = D_j\overline{E}^{\lambda}_{jk} + D_k\overline{E}^{\lambda}_{kl} \qquad (j,\, k,\, l \in \{1,\, 2,\, 3\}),$$

with the right-hand side being bounded; see (5.3). Therefore, in the case $v = 2$, the relation in (9.4) is easily reduced to the case $v = 1$; compare the arguments in [13, p. 141-143]. Next we observe there is some $\mathfrak{C}_2 > 0$ with

$$|m(x) - m(y)| = \left| n^{(\varphi,\epsilon)}(x) - n^{(\varphi,\epsilon)}(y) \right| \le \mathfrak{C}_2 \cdot |x-y|$$

for $x,\, y \in \gamma^{(\varphi,\epsilon)}(A(2))$. This is clear in the case $\epsilon > 0$ since $n^{(\varphi,\epsilon)} \circ \gamma^{(\varphi,\epsilon)} \in C^\infty(\mathbb{R}^2)^3$. If $\epsilon = 0$, we point out that

$$n^{(\varphi,\epsilon)} \circ \left(\gamma^{(\varphi,\epsilon)} \,|\, \mathbb{R}^2\backslash\{0\}\right) = n^{(\varphi)} \circ \left(g^{(\varphi)} \,|\, \mathbb{R}^2\backslash\{0\}\right) \in C^\infty\left(\mathbb{R}^2\backslash\{0\}\right)^3,$$

$$A(2) \subset \mathbb{R}^2\backslash\mathbb{B}_2(0,\, \delta/2).$$

Thus, recalling (5.9), (5.15) and Lemma 3.4, we see there is a constant $\mathfrak{C}_3 > 0$ with

$$\left| \sum_{k=1}^{3} S^{(v)}_{jkl}(x - y + (0,0,r)) \cdot (m_k(x) - m_k(y)) \cdot h(y) \right| \le \mathfrak{C}_3 \cdot |\xi - \eta|^{-1} \cdot |h(y)|$$

for $x,\, y \in \gamma^{(\varphi,\epsilon)}(A(2))$, $r \in \mathbb{R}$, $v \in \{1,\, 2\}$. It follows (compare [13, p. 141-143]):

$$\int_{\gamma^{(\varphi,\epsilon)}(A(1))} \sum_{k=1}^{3} S^{(v)}_{jkl}(x - y + (0,\, 0,\, -\tau\cdot r)) \cdot (m_k(x) - m_k(y)) \cdot h(y)\; d\Omega(y)$$

$$\longrightarrow \int_{\gamma^{(\varphi,\epsilon)}(A(1))} \sum_{k=1}^{3} S^{(v)}_{jkl}(x - y) \cdot (m_k(x) - m_k(y)) \cdot h(y)\; d\Omega(y) \qquad \text{for } r \downarrow 0,$$

uniformly in $x \in \gamma^{(\varphi,\epsilon)}(A(2))$, $v \in \{1,\, 2\}$. This result and (9.4) imply:

$$-\int_{\gamma^{(\varphi,\epsilon)}(A(1))} \sum_{k=1}^{3} S^{(v)}_{jkl}(x - y + (0,\, 0,\, -\tau\cdot r)) \cdot m_k(x) \cdot h(y)\; d\Omega(y)$$

$$\longrightarrow (\tau\cdot\delta_{jl}/2)\cdot h(x) - \int_{\gamma^{(\varphi,\epsilon)}(A(1))} \sum_{k=1}^{3} S^{(v)}_{jkl}(x - y) \cdot m_k(x) \cdot h(y)\; d\Omega(y) \qquad \text{for } r \downarrow 0,$$

uniformly in $x \in \gamma^{(\varphi,\epsilon)}(A(2))$, $v \in \{1,\, 2\}$. This means

$$(\tau\cdot\delta_{jl}/2)\cdot\Phi(\xi) + (K^{(v)}(0) \otimes \Phi)(\xi) - (K^{(v)}(-\tau\cdot r) \otimes \Phi)(\xi) \longrightarrow 0$$

for $r \downarrow 0$, uniformly in $\xi \in A(2)$. In particular, there is a number $r_1 \in (0, \infty)$ with

$$\int_{A(2)} \left| (\tau\cdot\delta_{jl}/2)\cdot\Phi(\xi) + (K^{(v)}(0) \otimes \Phi)(\xi) - (K^{(v)}(-\tau\cdot r) \otimes \Phi)(\xi) \right|^p d\xi \tag{9.5}$$

$$\le (\kappa/4)^p \qquad \text{for } r \in (0,\, r_1),\; v \in \{1,\, 2\}.$$

Take $\eta \in supp(\Phi)$. Then we have $|\xi - \eta| \ge |\xi|/2$ for $\xi \in \mathbb{R}^2\backslash\mathbb{B}_2(0,\, 2/\delta)$. If $\epsilon = 0$, it holds in addition:

$$|\xi - \eta| \ge (1/2)\cdot|\eta| \ge (1/2)\cdot\delta \qquad \text{for } \xi \in \mathbb{B}_2(0,\, \delta/2).$$

According to Lemma 6.1, there exists some number $\mathfrak{C}_4 > 0$ so that

$$\left| K^{(v)}(r)\,(\xi,\, \eta) \right| \le \mathfrak{C}_4 \cdot |\xi - \eta|^{-2} \quad \text{for } \xi,\, \eta \in \mathbb{R}^2\backslash\{0\} \text{ with } \xi \ne \eta,\; r \in \mathbb{R},\; v \in \{1,\, 2\}.$$

It follows for $\eta \in supp(\Phi)$, $\xi \in \mathbb{R}^2\backslash A(2)$, $r \in \mathbb{R}$, $v \in \{1,\, 2\}$:

$$\left| K^{(v)}(r)\,(\xi,\, \eta) \right| \le 4\cdot\mathfrak{C}_4\cdot|\xi|^{-2} \quad \text{if } \xi \in \mathbb{R}^2\backslash\mathbb{B}_2(0,\, 2/\delta); \tag{9.6}$$

$$\left| K^{(v)}(r)\,(\xi,\, \eta) \right| \le 4\cdot\mathfrak{C}_4\cdot\delta^{-2} \quad \text{in the case } \epsilon = 0,\; \xi \in \mathbb{B}_2(0,\, \delta/2).$$

In a first step, these estimates imply by Lebesgue's theorem on dominated convergence:

$$\int_{\mathbb{R}^2} K^{(v)}(r)\,(\xi,\, \eta)\cdot\Phi(\eta)\; d\eta \longrightarrow \int_{\mathbb{R}^2} K^{(v)}(0)\,(\xi,\, \eta)\cdot\Phi(\eta)\; d\eta \qquad (r \to 0),$$

with $\xi \in \mathbb{R}^2\backslash A(2)$, $v \in \{1,\, 2\}$. It further follows from (9.6), for $\xi \in \mathbb{R}^2\backslash\mathbb{B}_2(0,\, 2/\delta)$, $r \in \mathbb{R}$, $v \in \{1,\, 2\}$:

$$\left| \int_{\mathbb{R}^2} K^{(v)}(r)\,(\xi,\, \eta)\cdot\Phi(\eta)\; d\eta \right|^p \le \mathfrak{C}_5 \cdot |\xi|^{-2\cdot p},$$

where we have set $\mathfrak{C}_5 := \left(|\Phi|_0 \cdot 4\cdot\mathfrak{C}_4 \cdot \int_{supp(\Phi)} d\eta \right)^p$. If $\epsilon = 0$, we get in addition:

$$\left| \int_{\mathbb{R}^2} K^{(v)}(r)\,(\xi,\, \eta)\cdot\Phi(\eta)\; d\eta \right|^p \le \mathfrak{C}_5 \cdot \delta^{-2\cdot p}$$

for $\xi \in \mathbb{B}_2(0,\, \delta/2)$, $r \in \mathbb{R}$, $v \in \{1,\, 2\}$. But the function $L : \mathbb{R}^2\backslash A(2) \mapsto [0, \infty)$, with $L(\xi)$ defined by

$$L(\xi) := \mathfrak{C}_5 \cdot |\xi|^{-2\cdot p} \quad \text{if } \xi \in \mathbb{R}^2\backslash\mathbb{B}_2(0,\, 2/\delta),$$

$$L(\xi) := \mathfrak{C}_5 \cdot \delta^{-2\cdot p} \quad \text{in the case } \epsilon = 0,\; \xi \in \mathbb{B}_2(0,\, \delta/2),$$

is integrable. Thus, applying Lebesgue's theorem once more, we may find some number $r_2 \in (0, \infty)$ such that it holds for $v \in \{1,\, 2\}$, $r \in (-r_2,\, r_2)$:

$$\int_{\mathbb{R}^2\backslash A(2)} \left| (K^{(v)}(0) \otimes \Phi)(\xi) - (K^{(v)}(r) \otimes \Phi)(\xi) \right|^p d\xi \le (\kappa/4)^p. \tag{9.7}$$

Since $\Phi(\xi) = 0$ for $\xi \in \mathbb{R}^2\backslash A(2)$, the integrand in (9.7) may be replaced by

$$\left| (\tau\cdot\delta_{jl}/2)\cdot\Phi(\xi) + (K^{(v)}(0) \otimes \Phi)(\xi) - (K^{(v)}(r) \otimes \Phi)(\xi) \right|^p.$$

Now we combine (9.3), (9.5) and (9.7) to obtain for $v \in \{1,\, 2\}$, $r \in (0,\, (r_1 \wedge r_2))$:

$$\left\| (\tau\cdot\delta_{jl}/2)\cdot\Psi + K^{(v)}(0) \otimes \Psi - K^{(v)}(-\tau\cdot r) \otimes \Psi \right\|_p \le \kappa.$$

Since κ is arbitrary from $(0, \infty)$, the proof of (9.1) is completed.

Theorem 9.2 (Hardy-Littlewood-Sobolev inequality in $\mathbb{R}^2$). *Let* $p \in (1, \infty)$, $\alpha \in$

$(0, 2/p)$. *Then there is a constant* $C_{32}(p, \alpha) > 0$ *such that*

$$\Big(\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} |\xi - \eta|^{-2+\alpha} \cdot |\Phi(\eta)| \; d\eta \Big)^{1/(1/p - \alpha/2)} \; d\xi \Big)^{1/p - \alpha/2} \;\le\; C_{32}(p, \alpha) \cdot \|\Phi\|_p .$$

For a proof, we refer to [45, p. 119].

Lemma 9.1. *Let* $\varphi \in (0, \pi/2]$, $\epsilon \in [0, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $j, k \in \{1, 2, 3\}$, $p \in (1, 2)$, $q \in (1, \infty)$. *In the case* $q < 2$, *define* $t := (1/q - 1/2)^{-1}$. *If* $q \ge 2$, *take* $t \in (q, \infty)$.

Then there are constants $\mathfrak{C}_1, \mathfrak{C}_2 > 0$ *with the properties to follow:*

$$\Big(\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} | S(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)) \cdot \Phi(\eta) | \cdot J^{(\varphi,\epsilon)}(\eta) \; d\eta \Big)^{1/(1/p - 1/2)} \; d\xi \Big)^{1/p - 1/2} \le \mathfrak{C}_1 \cdot \|\Phi\|_p \tag{9.8}$$

for $\Phi \in L^p(\mathbb{R}^2)$, $r \in \mathbb{R}$, $S \in \{\overline{E}, E_{jk}\}$;

$$\Big(\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} | \widetilde{E}^{\lambda}_{jk}(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)) \cdot \Phi(\eta) | \cdot J^{(\varphi,\epsilon)}(\eta) \; d\eta \Big)^{t} \; d\xi \Big)^{1/t} \le \mathfrak{C}_2 \cdot \|\Phi\|_q \tag{9.9}$$

for $\Phi \in L^q(\mathbb{R}^2)$, $r \in \mathbb{R}$.

Proof: Due to Lemma 3.4, we have for $r \in \mathbb{R}$, $S \in \{\overline{E}, E_{jk}\}$, $\xi, \eta \in \mathbb{R}^2$ with $\xi \ne \eta$:

$$| S(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)) | \;\le\; (\pi \cdot \sin\varphi)^{-1} \cdot (|\xi - \eta| + |r|)^{-1}. \tag{9.10}$$

Thus inequality (9.8) follows from Theorem 9.2. According to Lemma 5.4 and 3.4, it holds for $\xi, \eta \in \mathbb{R}^2$ with $\xi \ne \eta$, and for $r \in \mathbb{R}$, $\alpha \in [0, 1]$:

$$| \widetilde{E}^{\lambda}_{jk}(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)) | \le (4/\sin\varphi)^{2-\alpha} \cdot C_{15}(|\arg\lambda|) \cdot |\lambda|^{(-1+\alpha)/2} \cdot (|\xi - \eta| + |r|)^{-2+\alpha}. \tag{9.11}$$

Now, by referring to (9.11) with $\alpha = 1$ if $q < 2$, and with $\alpha = 2/q - 2/t$ else, we see that inequality (9.9) may also be deduced from Theorem 9.2.

Lemma 9.2. *Let* $\varphi \in (0, \pi/2]$, $\epsilon \in [0, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $j, k \in \{1, 2, 3\}$, $p \in (1, 2)$, $\Phi \in L^p(\mathbb{R}^2)$, $q \in (1, \infty)$, $\gamma \in L^q(\mathbb{R}^2)$. *Then*

$$\int_{\mathbb{R}^2} | S(x - \gamma^{(\varphi,\epsilon)}(\eta)) \cdot \Phi(\eta) | \cdot J^{(\varphi,\epsilon)}(\eta) \; d\eta \;<\; \infty$$

for $S \in \{\overline{E}, E_{jk}\}$, $x \in \mathbb{R}^3\backslash \partial\mathbb{L}(\varphi, \epsilon)$, *and for almost every* $x \in \partial\mathbb{L}(\varphi, \epsilon)$;

$$\int_{\mathbb{R}^2} | \widetilde{E}^{\lambda}_{jk}(x - \gamma^{(\varphi,\epsilon)}(\eta)) \cdot \gamma(\eta) | \cdot J^{(\varphi,\epsilon)}(\eta) \; d\eta \;<\; \infty$$

for $x \in \mathbb{R}^3\backslash \partial\mathbb{L}(\varphi, \epsilon)$, *and for almost every* $x \in \partial\mathbb{L}(\varphi, \epsilon)$;

$$\int_{\mathbb{R}^2} | E_{4k}(x - \gamma^{(\varphi,\epsilon)}(\eta)) \cdot \gamma(\eta) | \cdot J^{(\varphi,\epsilon)}(\eta) \; d\eta \;<\; \infty \quad \textit{for } x \in \mathbb{R}^3\backslash \partial\mathbb{L}(\varphi, \epsilon).$$

Proof: Take $x \in \mathbb{R}^3\backslash \partial\mathbb{L}(\varphi, \epsilon)$. According to Lemma 3.5, there exists some $\xi \in \mathbb{R}^2$, $r \in \mathbb{R}\backslash\{0\}$ with $x = \gamma^{(\varphi,\epsilon)}(\xi) + (0,0,r)$. By (9.10), (9.11), we conclude there is $\mathfrak{C} > 0$ such that for $\eta \in \mathbb{R}^2$, $S \in \{\overline{E}, E_{jk}\}$, we have

$$| S(x - \gamma^{(\varphi,\epsilon)}(\eta)) | \;\le\; \mathfrak{C} \cdot (|\xi - \eta| + |r|)^{-1},$$

$$| \widetilde{E}^{\lambda}_{jk}(x - \gamma^{(\varphi,\epsilon)}(\eta)) |, \quad | E_{4k}(x - \gamma^{(\varphi,\epsilon)}(\eta)) | \;\le\; \mathfrak{C} \cdot (|\xi - \eta| + |r|)^{-2}.$$

Now, by Hölder's inequality, it readily follows that the integrals appearing in the lemma are finite for $x \in \mathbb{R}^3\backslash \partial\mathbb{L}(\varphi, \epsilon)$. Moreover, by Lemma 9.1, the first two integrals exist for almost every $x \in \partial\mathbb{L}(\varphi, \epsilon)$.

Due to Lemma 9.2, we are able to introduce some so-called "single-layer potentials":

Definition 9.2. *Let* $\varphi \in (0, \pi/2]$, $\epsilon \in [0, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $f \in \bigcup\{ L^p(\partial\mathbb{L}(\varphi, \epsilon))^3 : p \in (1, 2) \}$, $h \in \bigcup\{ L^p(\partial\mathbb{L}(\varphi, \epsilon))^3 : p \in (1, \infty) \}$.

Then the functions

$$\overline{V}(\partial\mathbb{L}(\varphi, \epsilon))(f_1) : \mathbb{R}^3 \mapsto \mathbb{C}, \qquad V(\partial\mathbb{L}(\varphi, \epsilon))(f) : \mathbb{R}^3 \mapsto \mathbb{C}^3,$$

$$\widetilde{V}^{\lambda}(\partial\mathbb{L}(\varphi, \epsilon))(h) : \mathbb{R}^3 \mapsto \mathbb{C}^3, \qquad Q(\partial\mathbb{L}(\varphi, \epsilon))(h) : \mathbb{R}^3\backslash \partial\mathbb{L}(\varphi, \epsilon) \mapsto \mathbb{C}$$

are to be defined by

$$\overline{V}(\partial\mathbb{L}(\varphi, \epsilon))(f_1)(x) := \int_{\partial\mathbb{L}(\varphi,\epsilon)} \overline{E}(x - y) \cdot f_1(y) \; d\mathbb{L}(\varphi, \epsilon)(y),$$

$$V(\partial\mathbb{L}(\varphi, \epsilon))(f)(x) := \Big(\int_{\partial\mathbb{L}(\varphi,\epsilon)} \sum_{k=1}^{3} E_{jk}(x - y) \cdot f_k(y) \; d\mathbb{L}(\varphi, \epsilon)(y) \Big)_{1 \le j \le 3},$$

$$\widetilde{V}^{\lambda}(\partial\mathbb{L}(\varphi, \epsilon))(h)(x) := \Big(\int_{\partial\mathbb{L}(\varphi,\epsilon)} \sum_{k=1}^{3} \widetilde{E}^{\lambda}_{jk}(x - y) \cdot h_k(y) \; d\mathbb{L}(\varphi, \epsilon)(y) \Big)_{1 \le j \le 3}$$

for $x \in \mathbb{R}^3\backslash\partial\mathbb{L}(\varphi, \epsilon)$, *and for almost every* $x \in \partial\mathbb{L}(\varphi, \epsilon)$;

$$Q(\,\partial\mathbb{L}(\varphi,\epsilon)\,)(h)\,(x) := \int_{\partial\mathbb{L}(\varphi,\epsilon)} \sum_{k=1}^{3} E_{4k}(x-y)\cdot h_k(y)\ d\,\mathbb{L}(\varphi,\epsilon)\,(y)$$

for $x \in \mathbb{R}^3\backslash\,\partial\mathbb{L}(\varphi,\epsilon)$.

Lemma 9.3. *Let* $\varphi \in (0,\pi/2]$, $\epsilon\in[0,\infty)$, $\lambda\in\mathbb{C}\backslash(-\infty,0]$, $j,k\in\{1,2,3\}$, $S\in\{\overline{E},\,E_{jk},\,\widetilde{E}^{\lambda}_{jk},\,E_{4k}\}$, $a\in\mathbb{N}_0^3$ *with* $1\le|a|_*$. *Furthermore, take* $K\subset\mathbb{R}^3\backslash\,\partial\mathbb{L}(\varphi,\epsilon)$ *with* $\overline{K}\subset\mathbb{R}^3\backslash\,\partial\mathbb{L}(\varphi,\epsilon)$.

Then there are constants $\mathfrak{C},\,\delta>0$ *with*

$$|\,D^aS(x-\gamma^{(\varphi,\epsilon)}(\eta))\,|\cdot J^{(\varphi,\epsilon)}(\eta)\quad\le\quad \mathfrak{C}\cdot(|\eta|\vee\delta)^{-2}\qquad \text{for } x\in K,\ \eta\in\mathbb{R}^2.$$

Proof: By Lemma 5.3, 5.4, we may find some number $\mathfrak{C}_1>0$ such that

$$|\,D^aS(x-y)\,|\ \le\ \mathfrak{C}_1\cdot|x-y|^{-1-|a|_*},\qquad |\,D^bE_{4k}(x-y)\,|\ \le\ \mathfrak{C}_1\cdot|x-y|^{-2-|b|_*}\qquad (9.12)$$

for $S\in\{\overline{E},\,E_{jk}\}$, $x,y\in\mathbb{R}^3$ mit $x\neq y$. Furthermore, using Lemma 5.3 and the properties of the exponential function, we may construct a constant $\mathfrak{C}_2>0$ with

$$|D^a\widetilde{E}^{\lambda}_{jk}(x-y)|\quad\le\quad \mathfrak{C}_2\cdot|x-y|^{-1-|a|_*}\qquad \text{for } x,\,y\in\mathbb{R}^3 \text{ with } x\neq y.\qquad (9.13)$$

Finally, by referring to Lemma 3.5, we see there is a number $\delta>0$ such that $|r|\ge\delta$ for $(\xi,r)\in(T^{(\varphi,\epsilon)})^{-1}(K)$. Hence, using (9.12), (9.13) and Lemma 3.4, we are able to construct a constant $\mathfrak{C}_3>0$ with

$$|\,D^aS(T^{(\varphi,\epsilon)}(\xi,r)\ -\ \gamma^{(\varphi,\epsilon)}(\eta))\,|\ \le\ \mathfrak{C}_3\cdot\delta^{-2}\qquad \text{for } (\xi,r)\in(T^{(\varphi,\epsilon)})^{-1}(K),\ \eta\in\mathbb{R}^2.$$

Now the lemma follows by the boundedness of K.

Corollary 9.1. *Let* $\varphi\in(0,\pi/2]$, $\epsilon\in[0,\infty)$, $\lambda\in\mathbb{C}\backslash(-\infty,0]$, *and assume that*

$$f\in\bigcup\{\,L^p(\,\partial\mathbb{L}(\varphi,\epsilon))^3\ :\ p\in(1,2)\,\},\qquad h\in\bigcup\{\,L^p(\,\partial\mathbb{L}(\varphi,\epsilon))^3\ :\ p\in(1,\infty)\,\}.$$

Then, for $l\in\{1,2,3\}$, *the functions*

$$\overline{V}(\,\partial\mathbb{L}(\varphi,\epsilon))(f_1)\,\big|\,\mathbb{R}^3\backslash\,\partial\mathbb{L}(\varphi,\epsilon),\quad V_l(\,\partial\mathbb{L}(\varphi,\epsilon))(f)\,\big|\,\mathbb{R}^3\backslash\,\partial\mathbb{L}(\varphi,\epsilon),$$

$$\widetilde{V}_l^{\lambda}(\,\partial\mathbb{L}(\varphi,\epsilon))(h)\,\big|\,\mathbb{R}^3\backslash\,\partial\mathbb{L}(\varphi,\epsilon),\quad Q(\,\partial\mathbb{L}(\varphi,\epsilon))(h)$$

all belong to $C^\infty(\mathbb{R}^3\backslash\,\partial\mathbb{L}(\varphi,\epsilon))$. *Moreover, let* $x\in\mathbb{R}^3\backslash\,\partial\mathbb{L}(\varphi,\epsilon)$, $l\in\{1,2,3\}$, $a\in\mathbb{N}_0^3$. *Then*

$$\partial^a/\partial x^a\big(\,\overline{V}(\,\partial\mathbb{L}(\varphi,\epsilon))(f_1)\,(x)\big)\ =\ \int_{\partial\mathbb{L}(\varphi,\epsilon)} D^a\overline{E}(x-y)\cdot f_1(y)\ d\mathbb{L}(\varphi,\epsilon)\,(y),$$

$$\partial^a/\partial x^a\big(\,V_l(\,\partial\mathbb{L}(\varphi,\epsilon))(f)\,(x)\big)\ =\ \int_{\partial\mathbb{L}(\varphi,\epsilon)}\sum_{k=1}^{3} D^aE_{lk}(x-y)\cdot f_k(y)\ d\,\mathbb{L}(\varphi,\epsilon)\,(y),$$

$$\partial^a/\partial x^a\big(\,\widetilde{V}_l^{\lambda}(\,\partial\mathbb{L}(\varphi,\epsilon))(h)\,(x)\big)\ =\ \int_{\partial\mathbb{L}(\varphi,\epsilon)}\sum_{k=1}^{3} D^a\widetilde{E}^{\lambda}_{lk}(x-y)\cdot h_k(y)\ d\,\mathbb{L}(\varphi,\epsilon)\,(y),$$

$$\partial^a/\partial x^a\big(\,Q(\,\partial\mathbb{L}(\varphi,\epsilon))(h)\,(x)\big)\ =\ \int_{\partial\mathbb{L}(\varphi,\epsilon)}\sum_{k=1}^{3} D^aE_{4k}(x-y)\cdot h_k(y)\ d\,\mathbb{L}(\varphi,\epsilon)\,(y).$$

Proof: The corollary is proved by induction with respect to $|a|_*$. Use Lebesgue's theorem on dominated convergence, which may be applied due to Lemma 9.3.

Corollary 9.2. *Let* $\varphi,\,\epsilon,\,\lambda,\,f,h$ *be given as in the preceding corollary. Then the function* $\overline{V}(\,\partial\mathbb{L}(\varphi,\epsilon))(f_1)\,|\,\partial\mathbb{L}(\varphi,\epsilon)$ *solves Laplace's equation (1.15). For brevity, put*

$$(G^{(1)},\,H^{(1)}) := \Big(V(\,\partial\mathbb{L}(\varphi,\epsilon))(f)\,\Big|\,\mathbb{R}^3\backslash\partial\mathbb{L}(\varphi,\epsilon),\ \ Q(\,\partial\mathbb{L}(\varphi,\epsilon))(f)\Big)$$

$$(G^{(2)},\,H^{(2)}) := \Big(\widetilde{V}^{\lambda}(\,\partial\mathbb{L}(\varphi,\epsilon))(h)\,\Big|\,\mathbb{R}^3\backslash\partial\mathbb{L}(\varphi,\epsilon),\ \ Q(\,\partial\mathbb{L}(\varphi,\epsilon))(h)\Big)$$

Then the pair of functions $(G^{(1)},H^{(1)})$ *satisfies the Stokes system (1.18), whereas* $(G^{(2)},H^{(2)})$ *is a solution of the resolvent problem (1.12).*

Let $r\in\mathbb{R}\backslash\{0\}$, *and put*

$$M^{(1)}(r) := R^*(p,\,\mathbb{L}(\varphi,\epsilon),\,r)(f),\quad M^{(2)}(r) := \widetilde{R}^*(p,\,\lambda,\,\mathbb{L}(\varphi,\epsilon),\,r)(h).$$

Then it holds for almost every $x\in\partial\mathbb{L}(\varphi,\epsilon)$, *and for* $v\in\{1,2\}$:

$$M^{(v)}(r)(x) \qquad (9.14)$$

$$=\ \Big(-\sum_{k=1}^{3}\big(D_jG_k^{(v)}+D_kG_j^{(v)}-\delta_{jk}H^{(v)}\big)(x+(0,0,r))\cdot n_k^{(\varphi,\epsilon)}(x)\Big)_{1\le j\le 3}$$

When the right-hand side of (9.14) ~~is restricted to $\mathbb{L}(\varphi,\epsilon)$, the boundary value of this restriction~~ coincides with the function $\Lambda^*(-1,p,\mathbb{L}(\varphi,\epsilon))(f)$ if $v=1$, and with $\Gamma^*(-1,p,\lambda,\mathbb{L}(\varphi,\epsilon))(h)$ in the case $v=2$. This follows from ~~(9.14) and~~ Theorem 9.1. Corresponding results are valid if the right-hand side of (9.14) ~~is restricted to $\mathbb{R}^3\backslash\overline{\mathbb{L}(\varphi,\epsilon)}$~~. Prescribing a boundary value ~~for~~ the right-hand side of (9.14) is called a "slip condition".

Proof of Corollary 9.2: All the results stated in this corollary, with the exception of (9.14), are a consequence of Corollary 9.1 and the properties of the fundamental solutions introduced in (1.2), (1.6). As for equation (9.14), it follows by referring to Corollary 9.1 again, and by recalling the definitions presented in (1.4), (1.7), Definition 9.1 and 9.2.

Lemma 9.4. *Take* $\varphi \in (0, \pi/2]$, $\epsilon \in [0, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $g \in C^0(\partial\mathbb{L}(\varphi, \epsilon))$ *with* $supp(g)$ *compact,* $p \in (1, 2)$, $f \in L^p(\partial\mathbb{L}(\varphi, \epsilon))^3$, $q \in (1, \infty)$, $h \in L^q(\partial\mathbb{L}(\varphi, \epsilon))^3$. *Then it holds for* $t \in (1, (1/p - 1/2)^{-1}]$:

$$\int_{\partial\mathbb{L}(\varphi,\epsilon)} \Big| V(\partial\mathbb{L}(\varphi, \epsilon))(f)\,(x + (0,0,r)) - V(\partial\mathbb{L}(\varphi, \epsilon))(f)\,(x)\Big|^t \cdot |g(x)|\; d\mathbb{L}(\varphi, \epsilon)\,(x) \longrightarrow 0 \quad (r \to 0). \tag{9.15}$$

In addition, the following relations are valid for $t \in (1, \infty)$ *in the case* $q \geq 2$, *and for* $t \in (1, (1/q - 1/2)^{-1}]$ *if* $q < 2$:

$$\int_{\partial\mathbb{L}(\varphi,\epsilon)} \Big| \widetilde{V}^\lambda(\partial\mathbb{L}(\varphi, \epsilon))(h)\,(x + (0,0,r)) - \widetilde{V}^\lambda(\partial\mathbb{L}(\varphi, \epsilon))(h)\,(x)\Big|^t \cdot |g(x)|\; d\mathbb{L}(\varphi, \epsilon)\,(x) \longrightarrow 0 \quad (r \to 0). \tag{9.16}$$

Finally, there exists a set $\mathcal{G} \subset \partial\mathbb{L}(\varphi, \epsilon)$ *of measure zero such that it holds for* $x \in \partial\mathbb{L}(\varphi, \epsilon) \setminus \mathcal{G}$:

$$V(\partial\mathbb{L}(\varphi, \epsilon))(f)\,(x + (0,0,r)) \;\to\; V(\partial\mathbb{L}(\varphi, \epsilon))(f)\,(x) \quad (r \to 0), \tag{9.17}$$

$$\widetilde{V}^\lambda(\partial\mathbb{L}(\varphi, \epsilon))(h)\,(x + (0,0,r)) \;\to\; \widetilde{V}^\lambda(\partial\mathbb{L}(\varphi, \epsilon))(h)\,(x) \quad (r \to 0). \tag{9.18}$$

Proof: Set $p_1 := p$, $p_2 := q$, $t_1 := (1/p - 1/2)^{-1}$. If $q \geq 2$, we take $t_2 \in (q, \infty)$, whereas in the case $q < 2$, we set $t_2 := (1/q - 1/2)^{-1}$. We further define $\alpha_1 := 1$, as well as $\alpha_2 := 1$ if $q < 2$, and $\alpha_2 := 2/q - 2/t_2$ if $q \geq 2$. Then it follows for $v \in \{1, 2\}$:

$$\alpha_v \in (0, 1] \cap (0, 2/p_v), \quad 1/t_v = 1/p_v - \alpha_v/2. \tag{9.19}$$

In addition, we define for $\xi, \eta \in \mathbb{R}^2\backslash\{0\}$ with $\xi \neq \eta$, $r \in \mathbb{R}$:

$$K^{(1)}(r)\,(\xi, \eta) := \Big(E_{jk}(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)) \cdot J^{(\varphi,\epsilon)}(\eta) \Big)_{1 \leq j,k \leq 3},$$

$$K^{(2)}(r)\,(\xi, \eta) := \Big(\widetilde{E}^\lambda_{jk}(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)) \cdot J^{(\varphi,\epsilon)}(\eta) \Big)_{1 \leq j,k \leq 3},$$

$$e^{(1)} := f \circ \gamma^{(\varphi,\epsilon)}, \quad e^{(2)} := h \circ \gamma^{(\varphi,\epsilon)}.$$

Note that $e^{(v)} \in L^{p_v}(\mathbb{R}^2)^3$ for $v \in \{1, 2\}$. Let us show there is a set $\mathcal{N} \subset \mathbb{R}^2$ of measure zero such that

$$\int_{\mathbb{R}^2} \Big| \Big(K^{(v)}(r)\,(\xi, \eta) - K^{(v)}(0)\,(\xi, \eta) \Big) \cdot e^{(v)}(\eta) \Big|\; d\eta \;\longrightarrow\; 0 \quad (r \to 0), \tag{9.20}$$

for $\xi \in \mathbb{R}^2\backslash\mathcal{N}$, $v \in \{1, 2\}$. This relation yields (9.17) and (9.18), as may be seen by (3.1), (3.8). Furthermore, we shall establish the following result:

$$\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} \Big| \Big(K^{(v)}(r)\,(\xi, \eta) - K^{(v)}(0)\,(\xi, \eta) \Big) \cdot e^{(v)}(\eta) \Big|\; d\eta \Big)^{t_v} \cdot \big| (g \circ \gamma^{(\varphi,\epsilon)})(\eta) \big|\; d\eta \longrightarrow 0 \quad (r \to 0) \quad \text{for } v \in \{1, 2\}. \tag{9.21}$$

Because of (3.1), (3.8), this proves the relation in (9.15) for $t = (1/p - 1/2)^{-1}$. In addition, if $q < 2$, the convergence result in (9.21) implies (9.16) for $t = (1/q - 1/2)^{-1}$. In the case $q \geq 2$, the statement in (9.21) yields (9.16) with $t = t_2$. But in the latter case, t_2 was chosen arbitrarily in (q, ∞), so that if (9.21) is proved, the relation in (9.16) is established for any $t \in (q, \infty)$. Since $supp(g)$ is compact, we may then conclude that (9.15) is true for $t \in [1, (1/p - 1/2)^{-1}]$, and (9.16) is valid for $t \in [1, (1/p - 1/2)^{-1}]$ in the case $q < 2$, and for $t \in [1, \infty)$ if $q \geq 2$.

This leaves us to show (9.20) and (9.21). For this purpose, take $v \in \{1, 2\}$. Then we observe there is $\mathfrak{C}_1 > 0$ so that

$$|K^{(v)}(r)\,(\xi, \eta)| \;\leq\; \mathfrak{C}_1 \cdot |\xi - \eta|^{-2+\alpha_v} \quad \text{for } \xi, \eta \in \mathbb{R}^2 \text{ with } \xi \neq \eta, \; r \in \mathbb{R}; \tag{9.22}$$

see (9.10), (9.11). Because of (9.19) and Theorem 9.2, we have

$$\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} |\xi - \eta|^{-2+\alpha_v} \cdot |e^{(v)}(\eta)|\; d\eta \Big)^{t_v}\; d\xi \;<\; \infty. \tag{9.23}$$

Thus there is a set $\mathcal{N} \subset \mathbb{R}^2$ of measure zero such that for any $\xi \in \mathbb{R}^2\backslash\mathcal{N}$, the function

$$F_\xi : \mathbb{R}^2 \mapsto [0, \infty), \quad F_\xi(\eta) := |\xi - \eta|^{-2+\alpha_v} \cdot |e^{(v)}(\eta)| \quad \text{for } \eta \in \mathbb{R}^2,$$

is integrable. Applying (9.22) and Lebesgue's theorem on dominated convergence, we obtain (9.20). Finally the convergence result in (9.21) follows by again referring to Lebesgue's theorem, which may be used due to (9.20), (9.22), (9.23).

Lemma 9.5. *Let* $\varphi \in (0, \pi/2]$, $\epsilon \in [0, \infty)$, $\tau \in \{-1, 1\}$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $g \in C^0(\partial\mathbb{L}(\varphi, \epsilon))$ *with* $supp(g)$ *compact,* $p \in [4/3, 2)$, $f \in L^p(\mathbb{R}^2)^3$, $q \in [4/3, \infty)$, $h \in L^q(\mathbb{R}^2)^3$. *Then*

$$\int_{\partial\mathbb{L}(\varphi,\epsilon)} \bigg(\Big(V(\partial\mathbb{L}(\varphi, \epsilon))(f)\,(x + \tau \cdot (0,0,r)) \Big)^{cc} \cdot R^*(p, \mathbb{L}(\varphi, \epsilon), \tau \cdot r)(f)\,(x) - \Big(V(\partial\mathbb{L}(\varphi, \epsilon))(f)\,(x) \Big)^{cc} \cdot \Lambda^*(-\tau, p, \mathbb{L}(\varphi, \epsilon))(f)\,(x) \bigg) \cdot g(x)\; d\mathbb{L}(\varphi, \epsilon)\,(x) \longrightarrow 0 \quad (r \downarrow 0), \tag{9.24}$$

$$\int_{\partial\mathbb{L}(\varphi,\epsilon)} \bigg(\Big(\widetilde{V}^\lambda(\partial\mathbb{L}(\varphi, \epsilon))(h)\,(x + \tau \cdot (0,0,r)) \Big)^{cc} \cdot \widetilde{R}^*(q, \lambda, \mathbb{L}(\varphi, \epsilon), \tau \cdot r)(h)\,(x) \tag{9.25}$$

$$- \Big(\widetilde{V}^{\lambda}(\partial\mathbb{L}(\varphi,\epsilon))(h)(x) \Big)^{cc} \cdot \Gamma^*(-\tau,\, q,\, \lambda,\, \mathbb{L}(\varphi,\epsilon))(h)(x) \Big) \cdot g(x)\; d\,\mathbb{L}(\varphi,\epsilon)(x)$$

$$\longrightarrow 0 \qquad (r \downarrow 0).$$

Proof: Since $p \in [4/3,\, 2)$ and $q \geq 4/3$, we have

$$(1-1/q)^{-1} \leq (1/q-1/2)^{-1} \;\text{ if } q < 2, \quad (1-1/p)^{-1} \leq (1/p-1/2)^{-1}. \tag{9.26}$$

Put $p_1 := p$, $p_2 := q$, $F^{(1)} := V(\partial\mathbb{L}(\varphi,\epsilon))(f)$, $F^{(2)} := \widetilde{V}^{\lambda}(\partial\mathbb{L}(\varphi,\epsilon))(h)$. Then Lemma 9.4 implies for $v \in \{1,\, 2\}$:

$$\int_{\partial\mathbb{L}(\varphi,\epsilon)} \big| F^{(v)}(x + (0,0,r)) - F^{(v)}(x) \big|^{1/(1-1/p_v)} \cdot |g(x)|\; d\,\mathbb{L}(\varphi,\epsilon)(x) \longrightarrow 0$$

for $r \to 0$. It is known from Theorem 9.1:

$$\| R^*(p,\, \mathbb{L}(\varphi,\epsilon),\, \tau\cdot r)(f) - \Lambda^*(-\tau,\, p,\, \mathbb{L}(\varphi,\epsilon))(f) \|_p \longrightarrow 0 \qquad (r\downarrow 0),$$

$$\| \widetilde{R}^*(q,\, \lambda,\, \mathbb{L}(\varphi,\epsilon),\, \tau\cdot r)(h) - \Gamma^*(-\tau,\, q,\, \lambda,\, \mathbb{L}(\varphi,\epsilon))(h) \|_q \longrightarrow 0 \qquad (r\downarrow 0).$$

If $v = 1$, or if $v = 2$ and $q < 2$, it follows from (9.26), for $r \in \mathbb{R}$:

$$\Big(\int_{\partial\mathbb{L}(\varphi,\epsilon)} | F^{(v)}(x + (0,0,r)) |^{1/(1-1/p_v)} \cdot |g(x)|\; d\,\mathbb{L}(\varphi,\epsilon)(x) \Big)^{1-1/p_v}$$
$$\leq \Big(\int_{\partial\mathbb{L}(\varphi,\epsilon)\cap supp(g)} d\,\mathbb{L}(\varphi,\epsilon) \Big)^{3/2-2/p_v}$$
$$\cdot \Big(\int_{\partial\mathbb{L}(\varphi,\epsilon)} | F^{(v)}(x + (0,0,r)) |^{1/(1/p_v-1/2)}\; d\,\mathbb{L}(\varphi,\epsilon)(x) \Big)^{1/p_v-1/2}.$$

Furthermore, in the case $v = 2$, $q \geq 2$, we choose $t \in (q,\, \infty)$ and then obtain for $r \in \mathbb{R}$:

$$\Big(\int_{\partial\mathbb{L}(\varphi,\epsilon)} | F^{(2)}(x + (0,0,r)) |^{1/(1-1/q)} \cdot |g(x)|\; d\,\mathbb{L}(\varphi,\epsilon)(x) \Big)^{1-1/q}$$
$$\leq \Big(\int_{\partial\mathbb{L}(\varphi,\epsilon)\cap supp(g)} d\,\mathbb{L}(\varphi,\epsilon) \Big)^{1-1/q-1/t}$$
$$\cdot \Big(\int_{\partial\mathbb{L}(\varphi,\epsilon)} | F^{(2)}(x + (0,0,r)) |^{t}\; d\,\mathbb{L}(\varphi,\epsilon)(x) \Big)^{1/t}.$$

Now Lemma 9.1 implies there is $\mathfrak{C}_1 > 0$ such that it holds for $v \in \{1,\, 2\}$, $r \in \mathbb{R}$:

$$\Big(\int_{\partial\mathbb{L}(\varphi,\epsilon)} | F^{(v)}(x + (0,0,r)) |^{1/(1-1/p_v)} \cdot |g(x)|\; d\,\mathbb{L}(\varphi,\epsilon)(x) \Big)^{1-1/p_v} \leq \mathfrak{C}_1 \cdot \| e^{(v)} \|_{p_v},$$

with $e^{(1)} := f$, $e^{(2)} := h$. By referring to Lemma 6.2 and 6.5, we see there is $\mathfrak{C}_2 > 0$ such that we have for $r \in \mathbb{R}$:

$$\| R^*(p,\, \mathbb{L}(\varphi,\epsilon),\, r)(f) \|_p \leq \mathfrak{C}_2 \cdot \|f\|_p,$$

$$\| \widetilde{R}^*(p,\, \lambda,\, \mathbb{L}(\varphi,\epsilon),\, r)(h) \|_q \leq \mathfrak{C}_2 \cdot \|h\|_q.$$

Now the lemma follows by first applying the triangle inequality, and then Hölder's inequality.

Lemma 9.6. *Let* $\varphi \in (0,\, \pi/2]$, $\epsilon \in [0,\, \infty)$, $j,\, k \in \{1,\, 2,\, 3\}$, $\lambda \in \mathbb{C}\backslash(-\infty,\, 0]$, $\tau \in \{-1,\, 1\}$, $p \in (1,\, 4/3)$, $q \in (1,\, 2]$, $f \in L^p(\partial\mathbb{L}(\varphi,\epsilon))^3$, $h \in L^q(\partial\mathbb{L}(\varphi,\epsilon))^3$.

Take $g \in C^1(\mathbb{R}^3)$ *with* $g\,|\,\mathbb{B}_3(0,\, 1) = 1$, $supp(g) \subset \mathbb{B}_3(0,\, 2)$. *For* $n \in \mathbb{N}$, $x \in \mathbb{R}^3$, *we set* $g_n(x) := g((1/n)\cdot x)$.

Put $\Omega := \mathbb{L}(\varphi,\epsilon)$ *if* $\tau = 1$, *and* $\Omega := \mathbb{R}^3\backslash\overline{\mathbb{L}(\varphi,\epsilon)}$ *if* $\tau = -1$. *It follows:*

$$\sup_{r\in(0,\infty)} \int_\Omega \big| V_j(\partial\mathbb{L}(\varphi,\epsilon))(f)(x + \tau\cdot(0,0,r)) \big|$$
$$\cdot \Big| -\delta_{jk} \cdot Q(\partial\mathbb{L}(\varphi,\epsilon))(f)(x + \tau\cdot(0,0,r))$$
$$+ \partial/\partial x_k\Big(V_j(\partial\mathbb{L}(\varphi,\epsilon))(f)(x + \tau\cdot(0,0,r)) \Big)$$
$$+ \partial/\partial x_j\Big(V_k(\partial\mathbb{L}(\varphi,\epsilon))(f)(x + \tau\cdot(0,0,r)) \Big) \Big| \cdot |D_k g_n(x)|\; dx$$
$$\longrightarrow 0 \qquad (n\to\infty);$$

$$\sup_{r\in(0,\infty)} \int_\Omega \big| \widetilde{V}_j^{\lambda}(\partial\mathbb{L}(\varphi,\epsilon))(h)(x + \tau\cdot(0,0,r)) \big|$$
$$\cdot \Big| -\delta_{jk} \cdot Q(\partial\mathbb{L}(\varphi,\epsilon))(h)(x + \tau\cdot(0,0,r))$$
$$+ \partial/\partial x_k\Big(\widetilde{V}_j^{\lambda}(\partial\mathbb{L}(\varphi,\epsilon))(h)(x + \tau\cdot(0,0,r)) \Big)$$
$$+ \partial/\partial x_j\Big(\widetilde{V}_k^{\lambda}(\partial\mathbb{L}(\varphi,\epsilon))(h)(x + \tau\cdot(0,0,r)) \Big) \Big| \cdot |D_k g_n(x)|\; dx$$
$$\longrightarrow 0 \qquad (n\to\infty).$$

Proof: For brevity, we set for $\xi,\, \eta \in \mathbb{R}^2$ with $\xi \neq \eta$, $r \in \mathbb{R}$:

$$K^{(1)}(r)(\xi,\eta) := \Big(\mathcal{D}_{jkl}(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)) \cdot J^{(\varphi,\epsilon)}(\eta) \Big)_{1\leq l\leq 3},$$

$$L^{(1)}(r)(\xi,\eta) := \Big(E_{jl}(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)) \cdot J^{(\varphi,\epsilon)}(\eta) \Big)_{1\leq l\leq 3}.$$

Let $K^{(2)}(r)(\xi, \eta)$ and $L^{(2)}(r)(\xi, \eta)$ be defined in the same way as $K^{(1)}(r)(\xi, \eta)$ and $L^{(1)}(r)(\xi, \eta)$, but with the functions $\mathcal{D}_{jkl}$ and E_{jl} replaced by $\widetilde{\mathcal{D}}^{\lambda}_{jkl}$ and $\widetilde{E}^{\lambda}_{jl}$, respectively. Furthermore, if $r \in (0, \infty)$, $n \in \mathbb{N}$, we set

$$\begin{aligned} A_1(r, n) \quad := \quad & \int_{\Omega} \Big| V_j(\partial \mathbb{L}(\varphi, \epsilon))(f)\,(x + \tau \cdot (0,0,r)) \Big| \\ & \cdot \Big| -\delta_{jk} \cdot Q(\partial \mathbb{L}(\varphi, \epsilon))(f)\,(x + \tau \cdot (0,0,r)) \\ & + \partial/\partial x_k \Big(V_j(\partial \mathbb{L}(\varphi, \epsilon))(f)\,(x + \tau \cdot (0,0,r)) \Big) \\ & + \partial/\partial x_j \Big(V_k(\partial \mathbb{L}(\varphi, \epsilon))(f)\,(x + \tau \cdot (0,0,r)) \Big) \Big| \cdot |D_k g_n(x)| \; dx . \end{aligned}$$

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The expression $A_2(r, n)$ is to be ~~declared~~ according to the same scheme, but with $V(\partial \mathbb{L}(\varphi, \epsilon))(f)$, $Q(\partial \mathbb{L}(\varphi, \epsilon))(f)$ replaced by $\widetilde{V}^{\lambda}(\partial \mathbb{L}(\varphi, \epsilon))(h)$, $Q(\partial \mathbb{L}(\varphi, \epsilon))(h)$, respectively. We finally set $\Phi^{(1)} := f \circ \gamma^{(\varphi, \epsilon)}$, $\Phi^{(2)} := h \circ \gamma^{(\varphi, \epsilon)}$, as well as $J := (0, \infty)$ in the case $\tau = 1$, and $J := (-\infty, 0)$ if $\tau = -1$.

Due to (3.1), (3.8), (3.11), Lemma 3.5 and Corollary 9.1, we obtain for $v \in \{1, 2\}$, $r \in (0, \infty)$, $n \in \mathbb{N}$:

$$\begin{aligned} A_v(r, n) \quad = \quad & \int_J \int_{\mathbb{R}^2} \Big| \Big(L^{(v)}(s + \tau \cdot r) \otimes \Phi^{(v)} \Big)(\xi) \Big| \\ & \cdot \Big| \Big(K^{(v)}(s + \tau \cdot r) \otimes \Phi^{(v)} \Big)(\xi) \Big| \cdot \big| D_k g(T^{(\varphi, \epsilon)}(\xi, s)) \big| \; d\xi \; ds . \end{aligned}$$

Note that $supp(D_k g_n) \subset \mathbb{B}_3(0, 2 \cdot n)$ for $n \in \mathbb{N}$, so that

$$|\gamma^{(\varphi, \epsilon)}(\xi) + (0, 0, s)| \;\le\; 2 \cdot n \quad \text{for } (\xi, s) \in \mathbb{R}^2 \times \mathbb{R} \text{ with } D_k g_n(T^{(\varphi, \epsilon)}(\xi, s)) \neq 0 .$$

Now we conclude from Lemma 3.4, with the abbreviation $\delta := 8 \cdot \sin^{-1}(\varphi)$:

$$|\xi| \le 2 \cdot n, \quad |s| \le \delta \cdot n \qquad \text{for } n \in \mathbb{N},\ (\xi, s) \in \mathbb{R}^2 \times \mathbb{R} \text{ with } D_k g_n(T^{(\varphi, \epsilon)}(\xi, s)) \neq 0 .$$

Hence we have for $r \in (0, \infty)$, $n \in \mathbb{N}$, $v \in \{1, 2\}$, with $\mathfrak{C}_1 := |D_k g|_0$:

$$\begin{aligned} A_v(r, n) \quad = \quad & \int_{J \cap (-\delta \cdot n,\, \delta \cdot n)} \int_{\mathbb{B}_2(0,\, 2 \cdot n)} \Big| \Big(L^{(v)}(s + \tau \cdot r) \otimes \Phi^{(v)} \Big)(\xi) \Big| \\ & \cdot \Big| \Big(K^{(v)}(s + \tau \cdot r) \otimes \Phi^{(v)} \Big)(\xi) \Big| \cdot \big| D_k g_n(T^{(\varphi, \epsilon)}(\xi, s)) \big| \; d\xi \; ds \\ \le \quad & \mathfrak{C}_1 \cdot (1/n) \cdot \int_{J \cap (-\delta \cdot n,\, \delta \cdot n)} \int_{\mathbb{B}_2(0,\, 2 \cdot n)} \Big| \Big(L^{(v)}(s + \tau \cdot r) \otimes \Phi^{(v)} \Big)(\xi) \Big| \\ & \cdot \Big| \Big(K^{(v)}(s + \tau \cdot r) \otimes \Phi^{(v)} \Big)(\xi) \Big| \; d\xi \; ds . \end{aligned} \tag{9.27}$$

Now let us consider the case $v = 1$. From (9.27), we obtain by Hölder's inequality, for $r \in (0, \infty)$, $n \in \mathbb{N}$:

$$\begin{aligned} A_1(r, n) \quad \le \quad & \mathfrak{C}_1 \cdot (1/n) \cdot \int_{J \cap (-\delta \cdot n,\, \delta \cdot n)} \big\| L^{(1)}(s + \tau \cdot r) \otimes \Phi^{(1)} \big\|_{1/(1/p - 1/2)} \\ & \cdot \big\| K^{(1)}(s + \tau \cdot r) \otimes \Phi^{(1)} \big\|_{1/(3/2 - 1/p)} \; ds . \end{aligned} \tag{9.28}$$

Put $\alpha_1 := 4/p - 3$. Using the fact that $p \in (1, 4/3)$, we get

$$(3/2 - 1/p)^{-1}, \;\; (1/p - 1/2)^{-1} \;\in\; (1, \infty); \qquad (3/2 - 1/p)^{-1} > p; \qquad \alpha_1 \in (0, 1);$$

$$3/2 - 1/p = 1/p - \alpha_1/2 .$$

We further define $\mathfrak{C}_2 := 12 \cdot \pi^{-1} \cdot \sin^{-2}(\varphi) \cdot |J^{(\varphi, \epsilon)}|_0$. Then we find for $\xi, \eta \in \mathbb{R}^2$ with $\xi \neq \eta$, $s \in J$, $r \in (0, \infty)$:

$$\begin{aligned} |K^{(1)}(s + \tau \cdot r)(\xi, \eta)| \;&\le\; \mathfrak{C}_2 \cdot (|\xi - \eta| + |s + \tau \cdot r|)^{-2} \\ &\le\; \mathfrak{C}_2 \cdot |s + \tau \cdot r|^{-\alpha_1} \cdot |\xi - \eta|^{-2 + \alpha_1} \;\le\; \mathfrak{C}_2 \cdot |s|^{-\alpha_1} \cdot |\xi - \eta|^{-2 + \alpha_1}, \end{aligned}$$

where the first inequality follows from Lemma 3.4. Concerning the last one, we point out that by the choice of J, we have $|s + \tau \cdot r| \ge |s| > 0$ for $s \in J$, $r \in (0, \infty)$. Now we conclude from Theorem 9.2, for $s \in J$, $r \in (0, \infty)$:

$$\big\| K^{(1)}(s + \tau \cdot r) \otimes \Phi^{(1)} \big\|_{1/(3/2 - 1/p)} \;\le\; \mathfrak{C}_3 \cdot |s|^{-\alpha_1}, \tag{9.29}$$

where $\mathfrak{C}_3 := \mathfrak{C}_2 \cdot C_{32}(p, \alpha_1) \cdot \|\Phi^{(1)}\|_p$. On the other hand, referring to Lemma 3.4, we find for $r, s \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^2$ with $\xi \neq \eta$:

$$|L^{(1)}(s + r)(\xi, \eta)| \;\le\; \mathfrak{C}_4 \cdot |\xi - \eta|^{-1},$$

with the abbreviation $\mathfrak{C}_4 := 3 \cdot (4\pi)^{-1} \cdot |J^{(\varphi, \epsilon)}|_0$. Now Theorem 9.2 yields for $r, s \in \mathbb{R}$:

$$\big\| L^{(1)}(s + r) \otimes \Phi^{(1)} \big\|_{1/(1/p - 1/2)} \;\le\; \mathfrak{C}_5, \tag{9.30}$$

with $\mathfrak{C}_5 := C_{32}(p, 1) \cdot \mathfrak{C}_4 \cdot \|\Phi^{(1)}\|_p$. Inequalities (9.29) and (9.30) are inserted into (9.28). Since $\alpha_1 < 1$, it follows for $n \in \mathbb{N}$, $r \in (0, \infty)$:

$$A_1(r, n) \quad \le \quad \mathfrak{C}_6 \cdot (1/n) \cdot \int_{J \cap (-\delta \cdot n,\, \delta \cdot n)} |s|^{-\alpha_1} \, ds \quad \le \quad \mathfrak{C}_7 \cdot n^{-\alpha_1},$$

with $\mathfrak{C}_6 := \mathfrak{C}_1 \cdot \mathfrak{C}_3 \cdot \mathfrak{C}_5$, $\mathfrak{C}_7 := 2 \cdot \mathfrak{C}_6 \cdot (1 - \alpha_1)^{-1} \cdot \delta^{1 - \alpha_1}$. Thus we have shown:

$$\sup_{r \in (0, \infty)} A_1(r, n) \to 0 \quad (n \to \infty).$$

Now we turn to the case $v = 2$. Applying Hölder's inequality on the right-hand side of (9.27), we obtain for $r \in (0, \infty)$, $n \in \mathbb{N}$,

$$\begin{aligned} & A_2(r, n) \\ \le\; & \mathfrak{C}_1 \cdot (1/n) \cdot \int_{J \cap (-\delta \cdot n,\, \delta \cdot n)} \Big\| \Big(L^{(2)}(s + \tau \cdot r) \otimes \Phi^{(2)} \Big) \Big| \mathbb{B}_2(0,\, 2 \cdot n) \Big\|_2 \\ & \cdot \Big\| \Big(K^{(2)}(s + \tau \cdot r) \otimes \Phi^{(2)} \Big) \Big| \mathbb{B}_2(0,\, 2 \cdot n) \Big\|_2 \; ds . \end{aligned} \tag{9.31}$$

Setting $\mathfrak{C}_8 := C_{17}(|\arg\lambda|)\cdot 48\cdot\sin^{-2}(\varphi)\cdot|J^{(\varphi,\epsilon)}|_0$, we infer from (5.15) and Lemma 3.4, for $\kappa\in[0,2]$, $r\in(0,\infty)$, $s\in J$, $\xi,\eta\in\mathbb{R}^2$ with $\xi\neq\eta$:

$$|K^{(2)}(s+\tau\cdot r)(\xi,\eta)| \tag{9.32}$$

$$\leq\ 3\cdot C_{17}(|\arg\lambda|)\cdot|J^{(\varphi,\epsilon)}|_0\cdot\big|\gamma^{(\varphi,\epsilon)}(\xi)-\gamma^{(\varphi,\epsilon)}(\eta)+(0,0,s+\tau\cdot r)\big|^{-2}$$

$$\leq\ \mathfrak{C}_8\cdot|\xi-\eta|^{-2+\kappa}\cdot|s|^{-\kappa}.$$

Recall that $|s+\tau\cdot r|\geq|s|$ for $r\in(0,\infty)$, $s\in J$. Moreover, defining

$$\mathfrak{C}_9 := 3\cdot C_{17}(|\arg\lambda|)\cdot(1\vee|\lambda|^{-1})\cdot|J^{(\varphi,\epsilon)}|_0,\qquad \mathfrak{C}_{10}:=64\cdot\mathfrak{C}_9\cdot\sin^{-3}(\varphi),$$

and applying (5.2) as well as Lemma 3.4, we find for $\sigma_1,\sigma_2\in(0,\infty)$ with $\sigma_1+\sigma_2\in[1,3]$, $r\in(0,\infty)$, $s\in J$, $\xi,\eta\in\mathbb{R}^2$ with $\xi\neq\eta$:

$$|L^{(2)}(s+\tau\cdot r)(\xi,\eta)|\ \leq\ \mathfrak{C}_9\cdot\big|\gamma^{(\varphi,\epsilon)}(\xi)-\gamma^{(\varphi,\epsilon)}(\eta)+(0,0,s+\tau\cdot r)\big|^{-\sigma_1-\sigma_2} \tag{9.33}$$

$$\leq\ \mathfrak{C}_{10}\cdot|\xi-\eta|^{-\sigma_1}\cdot|s|^{-\sigma_2}.$$

Now consider the case $q\in(1,2)$. Put $\kappa:=2/q-1$. By recurring to (9.32), Theorem 9.2, and to (9.33) with $\sigma_1=2-\kappa$, $\sigma_2=0$, we get for $n\in\mathbb{N}$, $r\in(0,\infty)$, $s\in J$:

$$\|K^{(2)}(s+\tau\cdot r)\otimes\Phi^{(2)}\|_2\ \leq\ C_{32}(p,\kappa)\cdot\mathfrak{C}_8\cdot\|\Phi^{(2)}\|_q\cdot|s|^{-\kappa};$$

$$\|L^{(2)}(s+\tau\cdot r)\otimes\Phi^{(2)}\|_2\ \leq\ C_{32}(p,\kappa)\cdot\mathfrak{C}_{10}\cdot\|\Phi^{(2)}\|_q.$$

After substituting these two inequalities into (9.31) and integrating in s, we see that $\sup_{r\in(0,\infty)}A_2(r,n)$ tends to zero for $n\to\infty$.

Finally assume $q=2$. Then we use (9.33) again, to get

$$|L^{(2)}(s+\tau\cdot r)(\xi,\eta)|$$

$$\leq\ \mathfrak{C}_{10}\cdot|s|^{-1/2}\cdot\Big(\chi_{(0,1)}(|\xi-\eta|)\cdot|\xi-\eta|^{-1}+\chi_{(1,\infty)}(|\xi-\eta|)\cdot|\xi-\eta|^{-5/2}\Big),$$

for s,r,ξ,η as in (9.33). Now Young's inequality (Lemma 4.9) yields for $n\in\mathbb{N}$, $r\in(0,\infty)$, $s\in J$:

$$\Big\|\Big(L^{(2)}(s+\tau\cdot r)\otimes\Phi^{(2)}\Big)\Big|\mathbb{B}_2(0,2\cdot n)\Big\|_2\ \leq\ \mathfrak{C}_{10}\cdot|s|^{-1/2}\cdot 6\cdot\pi\cdot\|\Phi^{(2)}\|_q. \tag{9.34}$$

Moreover, for $\kappa\in(0,1)$, and for n,r,s as before, we find

$$\Big\|\Big(K^{(2)}(s+\tau\cdot r)\otimes\Phi^{(2)}\Big)\Big|\mathbb{B}_2(0,2\cdot n)\Big\|_2 \tag{9.35}$$

$$\leq\ (4\cdot\pi)^{\kappa/2}\cdot n^{\kappa}\cdot\|K^{(2)}(s+\tau\cdot r)\otimes\Phi^{(2)}\|_{2/(1-\kappa)}$$

$$\leq\ (4\cdot\pi)^{\kappa/2}\cdot n^{\kappa}\cdot\mathfrak{C}_8\cdot C_{32}(2,\kappa)\cdot\|\Phi^{(2)}\|_q,$$

with the first estimate following from Hölder's inequality, and the second one from (9.32) and Theorem 9.2. After inserting inequalities (9.34), (9.35) into (9.31) and integrating in

s, we see there is a constant $\mathfrak{C}_{11}>0$ such that

$$\sup_{r\in(0,\infty)}A_2(r,n)\ \leq\ \mathfrak{C}_{11}\cdot(1/2-\kappa)^{-1}\cdot n^{-1/2}\cdot C_{32}(2,\kappa)$$

for $n\in\mathbb{N}$, $r\in(0,\infty)$, $\kappa\in(0,1/2)$. Thus it follows once more that $\sup_{r\in(0,\infty)}A_2(r,n)$ vanishes for $n\to\infty$.

Lemma 9.7. *Let* $\varphi\in(0,\pi/2]$, $\lambda\in\mathbb{C}\backslash(-\infty,0]$, $\epsilon\in[0,\infty)$, $j,k\in\{1,2,3\}$, $\tau\in\{-1,1\}$, $p\in(4/3,\infty)$, $f\in L^p(\partial\mathbb{L}(\varphi,\epsilon))^3$, $g\in C^0(\mathbb{R}^3)$ *with supp(g) compact.*

In the case $\tau=1$ *we set* $\Omega:=\mathbb{L}(\varphi,\epsilon)$, *and if* $\tau=-1$, *we put* $\Omega:=\mathbb{R}^3\backslash\overline{\mathbb{L}(\varphi,\epsilon)}$. *Then it holds:*

$$\int_\Omega\Big|\widetilde{V}_j^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(f)(x+\tau\cdot(0,0,r))-\widetilde{V}_j^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(f)(x)\Big|^2\cdot|g(x)|\,dx$$

$$\longrightarrow\ 0\qquad(r\downarrow 0);$$

$$\int_\Omega\Big|\partial/\partial x_j\Big(\widetilde{V}_k^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(f)(x+\tau\cdot(0,0,r))\Big)$$

$$-\ \partial/\partial x_j\Big(\widetilde{V}_k^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(f)(x)\Big)\Big|^2\cdot|g(x)|\,dx$$

$$\longrightarrow\ 0\qquad(r\downarrow 0);$$

$$\int_\Omega\Big|\partial/\partial x_j\Big(V_k(\partial\mathbb{L}(\varphi,\epsilon))(f)(x+\tau\cdot(0,0,r))\Big)$$

$$-\ \partial/\partial x_j\Big(V_k(\partial\mathbb{L}(\varphi,\epsilon))(f)(x)\Big)\Big|^2\cdot|g(x)|\,dx$$

$$\longrightarrow\ 0\qquad(r\downarrow 0);$$

Proof: Put $J:=(0,\infty)$ if $\tau=1$, and $J:=(-\infty,0)$ in the case $\tau=-1$. In addition, define for $\xi,\eta\in\mathbb{R}^2$ with $\xi\neq\eta$, $r\in\mathbb{R}$:

$$K^{(1)}(r)(\xi,\eta)\ :=\ \Big(\widetilde{E}_{jl}^\lambda(\gamma^{(\varphi,\epsilon)}(\xi)-\gamma^{(\varphi,\epsilon)}(\eta)+(0,0,r))\Big)_{1\leq l\leq 3},$$

$$K^{(2)}(r)(\xi,\eta)\ :=\ \Big(D_j\widetilde{E}_{kl}^\lambda(\gamma^{(\varphi,\epsilon)}(\xi)-\gamma^{(\varphi,\epsilon)}(\eta)+(0,0,r))\Big)_{1\leq l\leq 3},$$

$$K^{(3)}(r)(\xi,\eta)\ :=\ \Big(D_jE_{kl}(\gamma^{(\varphi,\epsilon)}(\xi)-\gamma^{(\varphi,\epsilon)}(\eta)+(0,0,r))\Big)_{1\leq l\leq 3}.$$

We further set $\Phi:=f\circ\gamma^{(\varphi,\epsilon)}$, and

$$A_v(r)\ :=\ \int_J\int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\Big(K^{(v)}(s+\tau\cdot r)(\xi,\eta)-K^{(v)}(s)(\xi,\eta)\Big)$$

$$\cdot\Phi(\eta)\cdot J^{(\varphi,\epsilon)}(\eta)\,d\eta\Big|^2\cdot(g\circ T^{(\varphi,\epsilon)})(\xi,s)\,d\xi\,ds$$

for $r \in (0,\infty)$, $v \in \{1, 2, 3\}$. Now fix $v \in \{1, 2, 3\}$. We shall show that

$$A_v(r) \to 0 \qquad (r \downarrow 0). \tag{9.36}$$

Due to (3.1), (3.8), (3.11), Lemma 3.5 and Corollary 9.1, the lemma then follows.

In order to prove (9.36), take $R > 0$ with $supp(g) \subset \mathbb{B}_2(0, R) \times (-R, R)$. Then we have for $r \in (0, \infty)$:

$$|A_v(r)| \le |g|_0 \cdot \int_{-R}^{R} \int_{\mathbb{B}_2(0,R)} \Big(\int_{\mathbb{R}^2} | K^{(v)}(s+\tau\cdot r)(\xi,\eta) - K^{(v)}(s)(\xi,\eta) | \cdot |\Phi(\eta)| \, d\eta \Big)^2 \, d\xi \, ds. \tag{9.37}$$

In the case $p < 2$, we set $t := 2$, $\alpha := 2/p - 1$, and if $p \ge 2$, we put $t := 6\cdot p/5$, $\alpha := 1/(3\cdot p)$. Then, recalling the assumption $p > 4/3$, we find in any case:

$$\alpha \in (0, 2/p), \quad t \ge 2, \quad \alpha\cdot t < 1, \quad 1/t = 1/p - \alpha/2. \tag{9.38}$$

Now we apply Hölder's inequality on the right-hand side of (9.37), to obtain for $r \in (0, \infty)$:

$$|A_v(r)| \le \mathfrak{C}_1 \cdot \Big(\int_{-R}^{R} \int_{\mathbb{B}_2(0,R)} \Big(\int_{\mathbb{R}^2} | K^{(v)}(s+\tau\cdot r)(\xi,\eta) - K^{(v)}(s)(\xi,\eta) | \cdot |\Phi(\eta)| \, d\eta \Big)^t \, d\xi \, ds \Big)^{2/t}, \tag{9.39}$$

where we have set $\mathfrak{C}_1 := |g|_0 \cdot (2\cdot\pi\cdot R^3)^{1/2-1/t}$. By Lemma 5.4 and 3.4, we may find constants $\mathfrak{C}_2, \mathfrak{C}_3 > 0$ such that it holds for $r \in [0,\infty)$, $s \in J$, $\xi, \eta \in \mathbb{R}^2$ with $\xi \ne \eta$:

$$\begin{aligned} | K^{(v)}(s+\tau\cdot r)(\xi,\eta) | &\le \mathfrak{C}_2 \cdot | \gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0, 0, s+\tau\cdot r) |^{-2} \\ &\le \mathfrak{C}_3 \cdot (|\xi-\eta| + |s+\tau\cdot r|)^{-2} \le \mathfrak{C}_3 \cdot |\xi-\eta|^{-2+\alpha} \cdot |s|^{-\alpha}, \end{aligned} \tag{9.40}$$

where the last inequality follows by the choice of J. In fact, the set J was defined in such a way as to admit the inequality $|s + \tau\cdot r| \ge |s| > 0$ for $s \in J$, $r \in (0, \infty)$. Due to (9.38) and Theorem 9.2, we obtain

$$\int_{-R}^{R} |s|^{-\alpha\cdot t} \cdot \int_{\mathbb{B}_2(0,R)} \Big(\int_{\mathbb{R}^2} |\xi-\eta|^{-2+\alpha} \cdot |\Phi(\eta)| \, d\eta \Big)^t \, d\xi \, ds < \infty. \tag{9.41}$$

This relation implies there is a set $\mathcal{N} \subset \mathbb{R}^3$ of measure zero such that for $(\xi, s) \in \mathbb{B}_2(0, R) \times (-R, R)$ with $(\xi, s) \notin \mathcal{N}$, the function $F_{(\xi,s)} : \mathbb{R}^2 \mapsto [0, \infty)$, defined by

$$F_{(\xi,s)}(\eta) := |s|^{-\alpha} \cdot |\xi-\eta|^{-2+\alpha} \cdot |\Phi(\eta)| \qquad \text{for } \eta \in \mathbb{R}^2,$$

is integrable. By Lebesgue's theorem on dominated convergence, it then follows from (9.40), for any (ξ, s) belonging to $(\mathbb{B}_2(0, R) \times (-R, R)) \backslash \mathcal{N}$:

$$\int_{\mathbb{R}^2} | K^{(v)}(s+\tau\cdot r)(\xi,\eta) - K^{(v)}(s)(\xi,\eta) | \cdot |\Phi(\eta)| \, d\eta \longrightarrow 0 \qquad (r \downarrow 0). \tag{9.42}$$

We know from (9.41) that the function $G : \mathbb{B}_2(0, R) \times (-R, R) \mapsto [0, \infty)$, with

$$G(\xi, s) := |s|^{-\alpha\cdot t} \cdot \Big(\int_{\mathbb{R}^2} |\xi-\eta|^{-2+\alpha} \cdot |\Phi(\eta)| \, d\eta \Big)^t \qquad \text{for } \xi \in \mathbb{B}_2(0, R),\ s \in (-R, R),$$

is integrable. Now the convergence result in (9.36) follows by recurring to (9.42), (9.40), and once again to Lebesgue's theorem on dominated convergence.

Theorem 9.3. *Let* $\varphi \in (0, \pi/2]$, $\sigma \in \{-1, 1\}$, $\epsilon \in [0, \infty)$, $p \in (1, 4/3)$, $s \in (4/3, 2)$, $f \in L^p(\partial\mathbb{L}(\varphi,\epsilon)) \cap L^s(\partial\mathbb{L}(\varphi,\epsilon))$ *with* $\Pi^*(\sigma, p, \mathbb{L}(\varphi,\epsilon))(f) = 0$. *Then it follows* $f = 0$.

Proof: Take $\tau \in \{-1, 1\}$. In the case $\tau = 1$, we set $\Omega := \mathbb{L}(\varphi,\epsilon)$, and if $\tau = -1$, we put $\Omega := \mathbb{R}^3 \backslash \overline{\mathbb{L}(\varphi,\epsilon)}$. In addition, $n^{(\Omega)}$ is to denote the outward unit normal to Ω. Take $g \in C^1(\mathbb{R}^3)$ with

$$im(g) \subset \mathbb{R}, \qquad g(x) \ge 0 \text{ for } x \in \mathbb{R}^3, \qquad g\,|\,\mathbb{B}_3(0,1) = 1, \qquad supp(g) \subset \mathbb{B}_3(0, 2).$$

If $n \in \mathbb{N}$, $x \in \mathbb{R}^3$, put $g_n(x) := g((1/n)\cdot x)$.

Let $r \in (0, \infty)$. By (3.11) and Corollary 9.1, the mapping $F_r : -\tau\cdot(0,0,r) + \Omega \longmapsto \mathbb{C}$, with

$$F_r(x) := \overline{V}(\partial\mathbb{L}(\varphi,\epsilon))(f)(x + \tau\cdot(0,0,r)) \qquad \text{for } x \in -\tau\cdot(0,0,r) + \Omega,$$

is a C^∞-function on the open set $-\tau\cdot(0,0,r) + \Omega$, which contains $\overline{\Omega}$ as a subset. Moreover, as mentioned in Corollary 9.2, we have $\Delta F_r(x) = 0$ for $x \in \Omega$. Using these facts, as well as Lemma 9.7 and the Divergence theorem, we obtain for $n \in \mathbb{N}$:

$$\begin{aligned} &\int_\Omega \Big| \nabla_x \Big(\overline{V}(\partial\mathbb{L}(\varphi,\epsilon))(f)(x) \Big) \Big|^2 \cdot g_n(x) \, dx \\ &= \lim_{r\downarrow 0} \int_\Omega \Big| \nabla_x \Big(\overline{V}(\partial\mathbb{L}(\varphi,\epsilon))(f)(x + \tau\cdot(0,0,r)) \Big) \Big|^2 \cdot g_n(x) \, dx \\ &= \lim_{r\downarrow 0} \Big[- \int_\Omega \Big(\overline{V}(\partial\mathbb{L}(\varphi,\epsilon))(f)(x + \tau\cdot(0,0,r)) \Big)^{cc} \\ &\qquad \cdot \sum_{k=1}^{3} \partial/\partial x_k \Big(\overline{V}(\partial\mathbb{L}(\varphi,\epsilon))(f)(x + \tau\cdot(0,0,r)) \Big) \cdot D_k g_n(x) \, dx \\ &\quad + \int_{\partial\mathbb{L}(\varphi,\epsilon)} \Big(\overline{V}(\partial\mathbb{L}(\varphi,\epsilon))(f)(x + \tau\cdot(0,0,r)) \Big)^{cc} \\ &\qquad \cdot \sum_{k=1}^{3} \partial/\partial x_k \Big(\overline{V}(\partial\mathbb{L}(\varphi,\epsilon))(f)(x + \tau\cdot(0,0,r)) \Big) \\ &\qquad \cdot n_k^{(\Omega)}(x) \cdot g_n(x) \, d\mathbb{L}(\varphi,\epsilon)(x) \Big]. \end{aligned} \tag{9.43}$$

Concerning the reference to Lemma 9.7, a remark is in order. In fact, setting $g^{(v)} := (\delta_{jv} \cdot f)_{1 \le j \le 3}$ for $v \in \{1, 2, 3\}$, we get

$$\overline{V}(\partial \mathbb{L}(\varphi, \epsilon))(f) = 2 \cdot \sum_{v=1}^{3} V_v(\partial \mathbb{L}(\varphi, \epsilon))(g^{(v)}). \tag{9.44}$$

Thus it makes sense to refer to Lemma 9.6 for the first equation in (9.43). The references we shall cite later in this proof are to be understood in a similar way: They are justified due to (9.44), and because it follows from (5.9):

$$3 \cdot \Pi^*(-\tau, p, \mathbb{L}(\varphi, \epsilon))(f) = \sum_{v=1}^{3} \Lambda_v^*(-\tau, p, \mathbb{L}(\varphi, \epsilon))(g^{(v)}), \tag{9.45}$$

$$3 \cdot D_k\Big(\overline{V}(\partial \mathbb{L}(\varphi, \epsilon))(f) \,\Big|\, \mathbb{R}^3 \backslash \partial \mathbb{L}(\varphi, \epsilon)\Big) = \sum_{v=1}^{3} (D_v G_k^{(v)} + D_k G_v^{(v)} - \delta_{vk} \cdot H^{(v)}) \tag{9.46}$$

for $k \in \{1, 2, 3\}$, where we used the abbreviations

$$G^{(v)} := V(\partial \mathbb{L}(\varphi, \epsilon))(g^{(v)}) \,|\, \mathbb{R}^3 \backslash \partial \mathbb{L}(\varphi, \epsilon), \quad H^{(v)} := Q(\partial \mathbb{L}(\varphi, \epsilon))(g^{(v)}).$$

Let us transform the right-hand side of (9.43) by using these equations. First we note that the outward unit normal to $\mathbb{R}^3 \backslash \overline{\mathbb{L}(\varphi, \epsilon)}$ coincides with the inward unit normal $-n^{(\varphi, \epsilon)}$ of $\mathbb{L}(\varphi, \epsilon)$. Thus, taking account of (9.45), (9.46), (9.14) and Lemma 9.5, we see that the second summand on the right-hand side of (9.43) converges for $\tau \downarrow 0$, so that the first summand must have the same property. But due to the preceding references, we even know the limit value of the second summand when τ tends to 0 from above. Inserting the corresponding result into (9.43), we obtain

$$\begin{aligned}
&\int_\Omega \Big| \nabla_x \Big(\overline{V}(\partial \mathbb{L}(\varphi, \epsilon))(f)(x) \Big) \Big|^2 \cdot g_n(x)\, dx \\
&= \lim_{\tau \downarrow 0} \int_\Omega - \Big(\overline{V}(\partial \mathbb{L}(\varphi, \epsilon))(f)(x + \tau \cdot (0,0,r)) \Big)^{cc} \\
&\qquad \cdot \sum_{k=1}^{3} \partial/\partial x_k \Big(\overline{V}(\partial \mathbb{L}(\varphi, \epsilon))(f)(x + \tau \cdot (0,0,r)) \Big) \cdot D_k g_n(x)\, dx \\
&\quad + \int_{\partial \mathbb{L}(\varphi, \epsilon)} \Big(\overline{V}(\partial \mathbb{L}(\varphi, \epsilon))(f)(x) \Big)^{cc} \\
&\qquad \cdot (-\tau) \cdot \Pi^*(-\tau, p, \mathbb{L}(\varphi, \epsilon))(f)(x) \cdot g_n(x)\, d\mathbb{L}(\varphi, \epsilon)(x).
\end{aligned}$$

Now we are going to estimate the right-hand side of the preceding equation. In a first step, we observe

$$\begin{aligned}
&\int_\Omega \Big| \nabla_x \Big(\overline{V}(\partial \mathbb{L}(\varphi, \epsilon))(f)(x) \Big) \Big|^2 \cdot g_n(x)\, dx \qquad (9.47) \\
&\le \sum_{k=1}^{3} \sup_{\tau \in (0, \infty)} \int_\Omega \Big| \overline{V}(\partial \mathbb{L}(\varphi, \epsilon))(f)(x + \tau \cdot (0,0,r)) \Big| \\
&\qquad \cdot \Big| \partial/\partial x_k \Big(\overline{V}(\partial \mathbb{L}(\varphi, \epsilon))(f)(x + \tau \cdot (0,0,r)) \Big) \Big| \cdot |D_k g_n(x)|\, dx
\end{aligned}$$

$$\begin{aligned}
&+ \int_{\partial \mathbb{L}(\varphi, \epsilon)} \Big| \overline{V}(\partial \mathbb{L}(\varphi, \epsilon))(f)(x) \Big| \\
&\qquad \cdot \Big| \Pi^*(-\tau, p, \mathbb{L}(\varphi, \epsilon))(f)(x) \Big| \cdot g_n(x)\, d\mathbb{L}(\varphi, \epsilon)(x).
\end{aligned}$$

For $x \in \mathbb{R}^3$, we have $g_n(x) \uparrow 1 \quad (n \to \infty)$. Combining this fact with (9.47), (9.45), (9.44) and Lemma 9.6, we obtain

$$\begin{aligned}
&\int_\Omega \Big| \nabla_x \Big(\overline{V}(\partial \mathbb{L}(\varphi, \epsilon))(f)(x) \Big) \Big|^2 dx \qquad (9.48) \\
&\le \int_{\partial \mathbb{L}(\varphi, \epsilon)} \Big| \overline{V}(\partial \mathbb{L}(\varphi, \epsilon))(f)(x) \Big| \cdot \Big| \Pi^*(-\tau, p, \mathbb{L}(\varphi, \epsilon))(f)(x) \Big| \, d\mathbb{L}(\varphi, \epsilon)(x).
\end{aligned}$$

Now consider the case $\sigma = 1$, that is, $\Pi^*(1, p, \mathbb{L}(\varphi, \epsilon))(f) = 0$. Then it follows from (9.48), with $\Omega = \mathbb{R}^3 \backslash \overline{\mathbb{L}(\varphi, \epsilon)}$:

$$\int_{\mathbb{R}^3 \backslash \overline{\mathbb{L}(\varphi, \epsilon)}} \Big| \nabla_x \Big(\overline{V}(\partial \mathbb{L}(\varphi, \epsilon))(f)(x) \Big) \Big|^2 dx = 0,$$

so that

$$\nabla_x \Big(\overline{V}(\partial \mathbb{L}(\varphi, \epsilon))(f)(x) \Big) = 0 \quad \text{for } x \in \mathbb{R}^3 \backslash \overline{\mathbb{L}(\varphi, \epsilon)}.$$

Thus the function $\overline{V}(\partial \mathbb{L}(\varphi, \epsilon)) \,|\, \mathbb{R}^3 \backslash \overline{\mathbb{L}(\varphi, \epsilon)}$ must be constant. On the other hand, setting

$$\mathfrak{C}_1 := (4 \cdot \pi)^{-1} \cdot |J^{(\varphi, \epsilon)}|_0, \quad \mathfrak{C}_2 := \mathfrak{C}_1 \cdot 4 \cdot \sin^{-1}(\varphi), \quad \mathfrak{C}_3 := \mathfrak{C}_2 \cdot \| f \circ \gamma^{(\varphi, \epsilon)} \|_p,$$

we find for $r \in \mathbb{R}$:

$$\begin{aligned}
&\Big| \overline{V}(\partial \mathbb{L}(\varphi, \epsilon))(f)(\gamma^{(\varphi, \epsilon)}(0) + (0,0,r)) \Big| \qquad (9.49) \\
&\le \mathfrak{C}_1 \cdot \int_{\mathbb{R}^2} |(\gamma^{(\varphi, \epsilon)}(0) - \gamma^{(\varphi, \epsilon)}(\eta) + (0,0,r))|^{-1} \cdot |(f \circ \gamma^{(\varphi, \epsilon)})(\eta)|\, d\eta \\
&\le \mathfrak{C}_2 \cdot \int_{\mathbb{R}^2} (|r| + |\eta|)^{-1} \cdot |(f \circ \gamma^{(\varphi, \epsilon)})(\eta)|\, d\eta \\
&\le \mathfrak{C}_3 \cdot \Big(\int_{\mathbb{R}^2} (|r| + |\eta|)^{-1/(1-1/p)}\, d\eta \Big)^{1-1/p} \\
&= \mathfrak{C}_3 \cdot \Big(\int_{\mathbb{R}^2} (1 + |\eta|)^{-1/(1-1/p)}\, d\eta \Big)^{1-1/p} \cdot |r|^{1-2/p},
\end{aligned}$$

where the first inequality follows from (3.8), and the second one from Lemma 3.4. Since $p < 2$, we have $1 - 2/p < 0$. Thus estimate (9.49) yields

$$\overline{V}(\partial \mathbb{L}(\varphi, \epsilon))(f)(\gamma^{(\varphi, \epsilon)}(0) + (0,0,r)) \longrightarrow 0 \quad (|r| \to \infty).$$

On the other hand, we observe

$$\gamma^{(\varphi, \epsilon)}(0) - (0,0,r) \in \mathbb{R}^3 \backslash \overline{\mathbb{L}(\varphi, \epsilon)} \quad \text{for } r \in (0, \infty);$$

see (3.11). Since the function $\overline{V}(\partial\mathbb{L}(\varphi,\epsilon))(f)$ is constant on $\mathbb{R}^3\backslash\overline{\mathbb{L}(\varphi,\epsilon)}$, as shown before, we are now able to conclude:

$$\overline{V}(\partial\mathbb{L}(\varphi,\epsilon))(f)\,\big|\,\mathbb{R}^3\backslash\overline{\mathbb{L}(\varphi,\epsilon)} \;=\; 0\,. \tag{9.50}$$

Due to (9.44) and (9.17), it follows $\overline{V}(\partial\mathbb{L}(\varphi,\epsilon))(f)(x) = 0$ for almost every $x \in \partial\mathbb{L}(\varphi,\epsilon)$. Inserting this result into (9.48), with $\Omega = \mathbb{L}(\varphi,\epsilon)$, we arrive at the equation

$$\int_{\mathbb{L}(\varphi,\epsilon)} \Big|\nabla_x\Big(\overline{V}(\partial\mathbb{L}(\varphi,\epsilon))(f)(x)\Big)\Big|^2\,dx \;=\; 0\,.$$

By an analogous reasoning as in the proof of (9.50), we infer from the preceding equation that $\overline{V}(\partial\mathbb{L}(\varphi,\epsilon))(f)\,|\,\mathbb{L}(\varphi,\epsilon)$ vanishes on $\mathbb{L}(\varphi,\epsilon)$. Recalling (9.46), (9.45) and (9.14), we may recur to Theorem 9.1, which yields $\Pi^*(-1,p,\mathbb{L}(\varphi,\epsilon))(f) = 0$. However, since

$$\Pi^*(1,p,\mathbb{L}(\varphi,\epsilon)) \;-\; \Pi^*(-1,p,\mathbb{L}(\varphi,\epsilon)) \;=\; f\,,$$

and because we assumed $\Pi^*(1,p,\mathbb{L}(\varphi,\epsilon))(f) = 0$, we finally obtain $f = 0$.

If we start with the equation $\Pi^*(-1,p,\mathbb{L}(\varphi,\epsilon)) = 0$, we first exploit (9.48) with $\Omega = \mathbb{L}(\varphi,\epsilon)$, and then proceed in an analogous way.

Lemma 9.8. *Take* $p, q \in (1,\infty)$ *with* $p < q$, $\Phi \in L^p(\mathbb{R}^2)$, $\Psi \in L^p(\mathbb{R}^2)^3$, $\tau \in \{-1, 1\}$, $\epsilon \in (0,\infty)$, $\varphi \in (0,\pi/2]$.

If $G(\tau,p,\varphi,\epsilon)(\Phi)$ *or* $G^*(\tau,p,\varphi,\epsilon)(\Phi)$ *belongs to* $L^p(\mathbb{R}^2)\cap L^q(\mathbb{R}^2)$, *then it follows* $\Phi \in L^q(\mathbb{R}^2)$.

Similarly, if $M(\tau,p,\lambda,\varphi,\epsilon)(\Psi)$ *or* $M^*(\tau,p,\lambda,\varphi,\epsilon)(\Psi)$ *is a member of the spaces* $L^p(\mathbb{R}^2)^3$ *and* $L^q(\mathbb{R}^2)^3$, *then it holds* $\Psi \in L^q(\mathbb{R}^2)^3$.

Proof: In order to show the first implication stated in the lemma, we assume that $G(\tau,p,\varphi,\epsilon)(\Phi)$ or $G^*(\tau,p,\varphi,\epsilon)(\Phi)$ belongs to $L^p(\mathbb{R}^2)\cap L^q(\mathbb{R}^2)$. For $\xi,\eta \in \mathbb{R}^2$ with $\xi \neq \eta$, we put

$$K^{(1)}(\xi,\eta) := -(4\cdot\pi)^{-1}\cdot\Big(\big(n^{(\varphi)}\circ g^{(\varphi)}\big)(\xi)\cdot\big(\gamma^{(\varphi,\epsilon)}(\xi)-\gamma^{(\varphi,\epsilon)}(\eta)\big)\Big)\cdot\big|\gamma^{(\varphi,\epsilon)}(\xi)-\gamma^{(\varphi,\epsilon)}(\eta)\big|^{-3}\cdot J^{(\varphi,\epsilon)}(\eta)\,,$$

$$K^{(2)}(\xi,\eta) := (4\cdot\pi)^{-1}\cdot\Big(\big(n^{(\varphi)}\circ g^{(\varphi)}\big)(\eta)\cdot\big(\gamma^{(\varphi,\epsilon)}(\xi)-\gamma^{(\varphi,\epsilon)}(\eta)\big)\Big)\cdot\big|\gamma^{(\varphi,\epsilon)}(\xi)-\gamma^{(\varphi,\epsilon)}(\eta)\big|^{-3}\cdot J^{(\varphi,\epsilon)}(\eta)\,.$$

Then it holds according to (6.14) and (6.15):

$$G(\tau,p,\varphi,\epsilon)(\Phi) \;=\; (\tau/2)\cdot\Phi \;+\; K^{(1)}\otimes\Phi\,,$$

$$G^*(\tau,p,\varphi,\epsilon)(\Phi) \;=\; (\tau/2)\cdot\Phi \;+\; K^{(2)}\otimes\Phi\,.$$

Due to our assumption at the beginning of the proof, there is an index $v \in \{1, 2\}$ such that

$$(\tau/2)\cdot\Phi \;+\; K^{(v)}\otimes\Phi \in L^p(\mathbb{R}^2)\cap L^q(\mathbb{R}^2)\,. \tag{9.51}$$

If $\alpha \in [0, 1]$, we may use (6.3) to find a constant $\mathfrak{C}_1(\alpha)$ with

$$|K^{(v)}(\xi,\eta)| \;\le\; \mathfrak{C}_1(\alpha)\cdot|\xi-\eta|^{-2+\alpha} \quad \text{for } \xi,\eta\in\mathbb{R}^2 \text{ with } \xi\neq\eta\,.$$

Recalling Theorem 9.2, we conclude for $s \in (p, (1/p-1/2)^{-1}]$ in the case $p < 2$, and for $s \in (p,\infty)$ if $p \ge 2$: $\|K^{(v)}\otimes\Phi\|_s < \infty$. On the other hand, since $K^{(v)}\otimes\Phi$ belongs to $L^p(\mathbb{R}^2)$ according to Lemma 6.2, it follows in the case $p < 2$:

$$K^{(v)}\otimes\Phi \;\in\; \bigcap\{\,L^s(\mathbb{R}^2) \;:\; s\in[p, (1/p-1/2)^{-1}]\,\}\,,$$

and in the case $p \ge 2$:

$$K^{(v)}\otimes\Phi \;\in\; \bigcap\{\,L^s(\mathbb{R}^2) \;:\; s\in[p,\infty)\,\}\,.$$

Hence, if $p < 2$ and $q \le (1/p-1/2)^{-1}$, or if $p \ge 2$, we see that $K^{(v)}\otimes\Phi$ is a member of $L^q(\mathbb{R}^2)$. Due to (9.51), we may now conlude $\Phi \in L^q(\mathbb{R}^2)$.

We still have to consider the case $p < 2$, $q > (1/p-1/2)^{-1}$. Then, since $p < 2$, the function $K^{(v)}\otimes\Phi$ belongs to $L^{1/(1/p-1/2)}(\mathbb{R}^2)$, as explained before. On the other hand, because $q > (1/p-1/2)^{-1} > p$, we are able to use (9.51) once more, to obtain:

$$(\tau/2)\cdot\Phi + K^{(v)}\otimes\Phi \;\in\; L^{1/(1/p-1/2)}(\mathbb{R}^2)\,.$$

Thus we have $\Phi \in L^{1/(1/p-1/2)}(\mathbb{R}^2)$. Since $q > (1/p-1/2)^{-1} > 2$, we may now recur to a case considered before, but with $(1/p-1/2)^{-1}$ in the place of p. It follows $\Phi \in L^q(\mathbb{R}^2)$.

As for the second implication claimed in the lemma, it may be proved by an analogous reasoning, the main difference being that we have to refer to (6.16), (6.17) and Lemma 6.5, instead of (6.14), (6.15) and Lemma 6.2, respectively.

Combining Theorem 9.3 with Corollary 8.3, which states that the operator $\Pi(\tau,p,\mathbb{K}(\varphi))$ is topological for $p \ge 2$, we obtain a remarkable result on $G(\tau,p,\varphi,\epsilon)$ and thus on $\Pi(\tau,p,\mathbb{L}(\varphi,\epsilon))$, for $\epsilon > 0$. In fact, these operators turn out to be one-to-one for any $p \in (1,\infty)$:

Corollary 9.3. *Let* $\varphi \in (0,\pi/2]$, $\epsilon \in (0,\infty)$, $\tau \in \{-1,1\}$, $p \in (1,\infty)$, $\Phi \in L^p(\partial\mathbb{L}(\varphi,\epsilon))$ *with* $G(\tau,p,\varphi,\epsilon)(\Phi) = 0$. *Then it follows* $\Phi = 0$.

Proof: Take $q \in (1\,,\, 4/3)$. According to Corollary 8.3, the operator $\Pi^*(-\tau,\, p,\, \mathbb{K}(\varphi))$ is topological for any $\sigma \in (0,\, \pi/2]$. Hence, by Corollary 6.6, $F^*(\tau,\, q,\, \sigma,\, 0,\, 1)$ has the same property for $\sigma \in (0,\, \pi/2]$. In particular, the latter operator is Fredholm, so that by Corollary 6.3, the mapping $F^*(\tau,\, q,\, \sigma,\, \epsilon,\, 1)$ is Fredholm as well $(\sigma \in (0,\, \pi/2])$. Now Lemma 6.17 yields:

$$index(\, F^*(\tau,\, q,\, \varphi,\, \epsilon,\, 1)\,) \quad = \quad 0\,.$$

Next we apply Corollary 6.4 to obtain that $G^*(\tau,\, q,\, \varphi,\, \epsilon)$ is Fredholm with index zero. Since $q < 4/3$, we know from Theorem 9.3 and Lemma 9.8 that $G^*(\tau,\, q,\, \varphi,\, \epsilon)$ is one-to-one. Hence $G^*(\tau,\, q,\, \varphi,\, \epsilon)$ is bijective. Since this operator is continuous as well (Lemma 6.7), it is even topological, as follows from the open mapping theorem. Because $G(\tau,\, (1-1/q)^{-1},\, \varphi,\, \epsilon)$ is adjoint to $G^*(\tau,\, q,\, \varphi,\, \epsilon)$, Lemma 8.5 yields that $G(\tau,\, (1-1/q)^{-1},\, \varphi,\, \epsilon)$ is bijective, in particular one-to-one. Here q was an arbitrary number from $(1\,,\, 4/3)$. Hence $G(\tau,\, q',\, \varphi,\, \epsilon)$ is one-to-one for any $q' \in (4,\, \infty)$.

Now we consider the number p given in the corollary. Since $G(\tau,\, p,\, \varphi,\, \epsilon)(\Phi)$ vanishes, we know from Lemma 9.8: $\Phi \in L^q(\mathbb{R}^2)$ for $q \in [p,\, \infty)$. Thus, for any such q, the function $G(\tau,\, q,\, \varphi,\, \epsilon)(\Phi)$ is well defined. Of course it coincides with $G(\tau,\, p,\, \varphi,\, \epsilon)(\Phi)$, so it must vanish too. But $G(\tau,\, q,\, \varphi,\, \epsilon)$ is one-to-one for any $q \in (4,\, \infty)$, as was shown above. Thus we obtain $\Phi = 0$.

Theorem 9.4. *Let* $\varphi \in (0,\, \pi/2]$, $\sigma \in \{-1,\, 1\}$, $\lambda \in \mathbb{C}\backslash(-\infty,\, 0]$, $\epsilon \in [0,\, \infty)$, $p \in (4/3\,,\, 2]$, $h \in L^p(\,\partial\mathbb{L}(\varphi,\, \epsilon)\,)^3$ *with* $\Gamma^*(\,\sigma,\, p,\, \lambda,\, \mathbb{L}(\varphi,\, \epsilon)\,)(h) \;=\; 0$. *It follows* $h = 0$.

Proof: Choose $\tau\,,\Omega,\, n^{(\Omega)}$ and $(g_n)_{n\in\mathbb{N}}$ as in the proof of Theorem 9.3. Recall that for any $r \in (0,\infty)$, the set $-\tau\cdot(0,0,r)+\Omega$ is open and contains $\overline{\Omega}$ as a subset. We define the mappings $F^{(r)} : -\tau\cdot(0,0,r)+\Omega \longmapsto \mathbb{C}^3$ and $G^{(r)} : -\tau(0,0,r)+\Omega \longmapsto \mathbb{C}$ by putting

$$F^{(r)}(x) \;:=\; \widetilde{V}^\lambda(\,\partial\mathbb{L}(\varphi,\, \epsilon)\,)(h)\,(x + \tau\cdot(0,0,r))\,,$$

$$G^{(r)}(x) \;:=\; Q(\,\partial\mathbb{L}(\varphi,\, \epsilon)\,)(h)\,(x + \tau\cdot(0,0,r)) \qquad \text{for } x \in -\tau\cdot(0,0,r)+\Omega\,.$$

Then $F^{(r)}$ and $G^{(r)}$ are C^∞-functions on $-\tau\cdot(0,0,r)+\Omega$, as follows from (3.11) and Corollary 9.1. According to Corollary 9.2, it holds for $r \in (0,\infty)$, $x \in \Omega$:

$$-\Delta F^{(r)}(x) \;+\; \lambda\cdot F^{(r)}(x) \;+\; \nabla G^{(r)}(x) \;\;=\;\; 0\,, \qquad \sum_{v=1}^{3} D_v F_v^{(r)}(x) \;\;=\;\; 0\,.$$

By referring to these facts, and using Lemma 9.7 and the Divergence theorem, we obtain for $n \in \mathbb{N}$:

$$\lambda\cdot\int_\Omega \sum_{j=1}^{3} |\,\widetilde{V}_j^\lambda(\,\partial\mathbb{L}(\varphi,\, \epsilon)\,)(h)\,(x)\,|^2\cdot g_n(x)\;dx \tag{9.52}$$

$$+\;(1/2)\cdot\int_\Omega \sum_{j,k=1}^{3} \Big|\,\partial/\partial x_k\Big(\,\widetilde{V}_j^\lambda(\,\partial\mathbb{L}(\varphi,\epsilon)\,)(h)\,(x)\,\Big) + \partial/\partial x_j\Big(\,\widetilde{V}_k^\lambda(\,\partial\mathbb{L}(\varphi,\epsilon)\,)(h)\,(x)\,\Big)\Big|^2\cdot g_n(x)\;dx$$

$$= \lim_{r\downarrow 0}\Big[-\int_\Omega \sum_{j,k=1}^{3}\Big(\widetilde{V}_j^\lambda(\,\partial\mathbb{L}(\varphi,\epsilon)\,)(h)\,(x+\tau\cdot(0,0,r))\Big)^{cc}$$
$$\cdot\Big(-\delta_{jk}\cdot Q(\,\partial\mathbb{L}(\varphi,\epsilon)\,)(h)\,(x+\tau\cdot(0,0,r))$$
$$+\,\partial/\partial x_k\Big(\,\widetilde{V}_j^\lambda(\,\partial\mathbb{L}(\varphi,\epsilon)\,)(h)\,(x+\tau\cdot(0,0,r))\Big)$$
$$+\,\partial/\partial x_j\Big(\,\widetilde{V}_k^\lambda(\,\partial\mathbb{L}(\varphi,\epsilon)\,)(h)\,(x+\tau\cdot(0,0,r))\Big)\Big)\cdot D_k g_n(x)\;dx$$
$$+\int_{\partial\mathbb{L}(\varphi,\epsilon)} \sum_{j,k=1}^{3}\Big(\widetilde{V}_j^\lambda(\,\partial\mathbb{L}(\varphi,\epsilon)\,)(h)\,(x+\tau\cdot(0,0,r))\Big)^{cc}$$
$$\cdot\Big(-\delta_{jk}\cdot Q(\,\partial\mathbb{L}(\varphi,\epsilon)\,)(h)\,(x+\tau\cdot(0,0,r))$$
$$+\,\partial/\partial x_k\Big(\,\widetilde{V}_j^\lambda(\,\partial\mathbb{L}(\varphi,\epsilon)\,)(h)\,(x+\tau\cdot(0,0,r))\Big)$$
$$+\,\partial/\partial x_j\Big(\,\widetilde{V}_k^\lambda(\,\partial\mathbb{L}(\varphi,\epsilon)\,)(h)\,(x+\tau\cdot(0,0,r))\Big)\Big)$$
$$\cdot\, n_k^{(\Omega)}(x)\cdot g(x)\;d\,\mathbb{L}(\varphi,\epsilon)(x)\Big].$$

Note that the outward unit normal to $\mathbb{R}^3\backslash\overline{\mathbb{L}(\varphi,\,\epsilon)}$ is identical to the inward unit normal $-n^{(\varphi,\epsilon)}$ to $\mathbb{L}(\varphi,\,\epsilon)$. Hence, by (9.14) and Lemma 9.5, the second summand on the right-hand side of (9.52) converges to a limit as $r\downarrow 0$. Thus the first summand converges as well when h tends to zero from above, and we obtain for $n \in \mathbb{N}$, applying (9.14) and Lemma 9.5 once more:

$$\lambda\cdot\int_\Omega |\,\widetilde{V}^\lambda(\,\partial\mathbb{L}(\varphi,\,\epsilon)\,)(h)\,(x)\,|^2\cdot g_n(x)\;dx \tag{9.53}$$

$$+\;(1/2)\cdot\int_\Omega \sum_{j,k=1}^{3} \Big|\,\partial/\partial x_k\Big(\,\widetilde{V}_j^\lambda(\,\partial\mathbb{L}(\varphi,\epsilon)\,)(h)\,(x)\,\Big) + \partial/\partial x_j\Big(\,\widetilde{V}_k^\lambda(\,\partial\mathbb{L}(\varphi,\epsilon)\,)(h)\,(x)\,\Big)\Big|^2\cdot g_n(x)\;dx$$

$$= -\lim_{r\downarrow 0}\int_\Omega \sum_{j,k=1}^{3}\Big(\widetilde{V}_j^\lambda(\,\partial\mathbb{L}(\varphi,\epsilon)\,)(h)\,(x+\tau\cdot(0,0,r))\Big)^{cc}$$
$$\cdot\Big(-\delta_{jk}\cdot Q(\,\partial\mathbb{L}(\varphi,\epsilon)\,)(h)\,(x+\tau\cdot(0,0,r))$$

$$+\,\partial/\partial x_k\Big(\widetilde{V}_j^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(h)\,(x+\tau\cdot(0,0,r))\Big)$$

$$+\,\partial/\partial x_j\Big(\widetilde{V}_k^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(h)\,(x+\tau\cdot(0,0,r))\Big)\Bigg)\cdot D_k g_n(x)\;dx$$

$$+\int_{\partial\mathbb{L}(\varphi,\epsilon)}\Big(\widetilde{V}^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(h)\,(x)\Big)^{cc}\cdot(-\tau)\cdot\Gamma^*(-\tau,2,\lambda,\mathbb{L}(\varphi,\epsilon))(h)\,(x)\cdot g_n(x)\;d\,\mathbb{L}(\varphi,\epsilon)\,(x)\,.$$

But in the case $\Im(\lambda)\neq 0$, the expression

$$|\Im(\lambda)|\cdot\int_\Omega\sum_{j=1}^3|\widetilde{V}^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(h)\,(x)|^2\cdot g_n(x)\;dx$$

is less than or equal to the left-hand side of (9.53). In the case $\Im(\lambda)=0$, we recall that $\lambda\in\mathbb{C}\backslash(-\infty,0]$, hence $\Re(\lambda)=\lambda>0$, so the expression

$$\lambda\cdot\int_\Omega\sum_{j=1}^3|\widetilde{V}^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(h)\,(x)|^2\cdot g_n(x)\;dx$$

coincides with the absolute value of the left-hand side of (9.53). Setting

$\gamma:=|\Im(\lambda)|$ in the case $\Im(\lambda)\neq 0$, $\gamma:=\lambda$ if $\Im(\lambda)=0$,

we see that $\gamma>0$, and we infer from (9.53), for $n\in\mathbb{N}$:

$$\gamma\cdot\int_\Omega\sum_{j=1}^3|\widetilde{V}_j^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(h)\,(x)|^2\cdot g_n(x)\;dx \tag{9.54}$$

$$\leq\sum_{j,k=1}^3\sup_{r\in(0,\infty)}\int_\Omega|\widetilde{V}_j^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(h)\,(x+\tau\cdot(0,0,r))|$$

$$\cdot\Bigg|-\delta_{jk}\cdot Q(\partial\mathbb{L}(\varphi,\epsilon))(h)\,(x+\tau\cdot(0,0,r))$$

$$+\,\partial/\partial x_k\Big(\widetilde{V}_j^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(h)\,(x+\tau\cdot(0,0,r))\Big)$$

$$+\,\partial/\partial x_j\Big(\widetilde{V}_k^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(h)\,(x+\tau\cdot(0,0,r))\Big)\Bigg|\cdot|D_k g_n(x)|\;dx$$

$$+\int_{\partial\mathbb{L}(\varphi,\epsilon)}|\widetilde{V}^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(h)\,(x)|\cdot|\Gamma^*(-\tau,2,\lambda,\mathbb{L}(\varphi,\epsilon))(h)\,(x)|\cdot g_n(x)\;d\,\mathbb{L}(\varphi,\epsilon)\,(x)\,.$$

Observe that $g_n(x)\uparrow 1\;(n\to\infty)$ for $x\in\mathbb{R}^3$. Hence we obtain from Lemma 9.6 and (9.54):

$$\gamma\cdot\int_\Omega|\widetilde{V}^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(h)\,(x)|^2\;dx \tag{9.55}$$

$$\leq\int_{\partial\mathbb{L}(\varphi,\epsilon)}|\widetilde{V}^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(h)\,(x)|\cdot|\Gamma^*(-\tau,2,\lambda,\mathbb{L}(\varphi,\epsilon))(h)\,(x)|\;d\,\mathbb{L}(\varphi,\epsilon)\,(x)\,.$$

Now suppose $\Gamma^*(1,2,\lambda,\mathbb{L}(\varphi,\epsilon))(h)=0$. Then it follows from (9.55), with $\Omega=\mathbb{R}^3\backslash\overline{\mathbb{L}(\varphi,\epsilon)}$:

$$\int_{\mathbb{R}^3\backslash\overline{\mathbb{L}(\varphi,\epsilon)}}|\widetilde{V}^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(h)\,(x)|^2\;dx\;=\;0\,.$$

This means the function $\widetilde{V}^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(h)$ vanishes on $\mathbb{R}^3\backslash\overline{\mathbb{L}(\varphi,\epsilon)}$, so that by (9.18), we have for almost every $x\in\partial\mathbb{L}(\varphi,\epsilon)$: $\widetilde{V}^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(h)\,(x)=0$. Now we conclude from (9.55), with $\Omega=\mathbb{L}(\varphi,\epsilon)$:

$$\int_{\mathbb{L}(\varphi,\epsilon)}|\widetilde{V}^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(f)\,(x)|^2\;dx\;=\;0\,,$$

so $\widetilde{V}^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(h)$ vanishes on $\mathbb{L}(\varphi,\epsilon)$ as well, and we have shown that the function $\widetilde{V}^\lambda(\partial\mathbb{L}(\varphi,\epsilon))$ is zero everywhere on $\mathbb{R}^3\backslash\partial\mathbb{L}(\varphi,\epsilon)$. Thus, by Corollary 9.1 and 9.2, the mapping $\nabla\Big(Q(\partial\mathbb{L}(\varphi,\epsilon))(h)\Big)$ must vanish too. This means the restriction $Q(\partial\mathbb{L}(\varphi,\epsilon))(h)\,|\,\mathbb{L}(\varphi,\epsilon)$ is constant. But on the other hand, we have

$$Q(\partial\mathbb{L}(\varphi,\epsilon))(h)\,(\gamma^{(\varphi,\epsilon)}(0)+(0,0,r))\;\to\;0\qquad(r\to\infty)\,.$$

This may be proved by modifying the estimate in (9.49) in an obvious way. Note that the assumption $p<2$ is not needed this time since the kernel of $Q(\partial\mathbb{L}(\varphi,\epsilon))(h)$ has a singularity which is stronger than the one appearing in (9.49). Due to the preceding results, the function $Q(\partial\mathbb{L}(\varphi,\epsilon))(h)$ must also vanish on $\mathbb{L}(\varphi,\epsilon)$, and we obtain by (9.14):

$$\widetilde{R}^*(2,\lambda,\mathbb{L}(\varphi,\epsilon),r)(h)\;=\;0\qquad\text{for } r\in(0,\infty)\,.$$

Recalling Theorem 9.1, we conclude $\Gamma^*(-1,2,\lambda,\mathbb{L}(\varphi,\epsilon))=0$. But

$$\Gamma^*(-1,2,\lambda,\mathbb{L}(\varphi,\epsilon))(h)\;-\;\Gamma^*(1,2,\lambda,\mathbb{L}(\varphi,\epsilon))(h)\;=\;h\,,$$

so that we may infer $h=0$.

When the equation $\Gamma^*(-1,2,\lambda,\mathbb{L}(\varphi,\epsilon))(h)=0$ is assumed, we may proceed in an analogous way.

Let us finally mention a consequence of Lemma 9.8 and Theorem 9.4:

Corollary 9.4. *Let* $\varphi\in(0,\pi/2]$, $\tau\in\{-1,1\}$, $\lambda\in\mathbb{C}\backslash(-\infty,0]$, $\epsilon\in(0,\infty)$, $p\in(1,2]$, $h\in L^p(\partial\mathbb{L}(\varphi,\epsilon))^3$ *with* $\Gamma^*(\tau,p,\lambda,\mathbb{L}(\varphi,\epsilon))(h)=0$. *Then it follows* $h=0$.

Chapter 10

A Representation Formula for the Operator $J(\tau, p, \lambda, \varphi, R, S)$

In the following, we shall consider the operator $J(\tau, p, \lambda, \varphi, R, S)$, which arises, as the reader may recall, by transforming the operator $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ into local coordinates, and then restricting its domain to $L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S))^3$, and its range to $L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R))^3$; see Definition 6.3. Here we shall further restrict the domain of $J(\tau, p, \lambda, \varphi, R, S)$, namely to $L^p(A(0, \epsilon)\backslash\mathbb{B}_2(0, R))^3$, where $A(0, \epsilon)$ is an abbreviation for the sector $\{r\cdot(\cos\vartheta, \sin\vartheta) : \vartheta \in (-\epsilon, \epsilon)\}$ in the plane. This restriction will be shown to coincide with a multiplier transformation, also restricted to $L^p(A(0, \epsilon)\backslash\mathbb{B}_2(0, R))^3$, plus some perturbation terms, with the multiplier transformation being applied to $H^*(\tau, p, \varphi, R, S)$ plus some other perturbation terms; see (10.19).

In Chapter 11 we shall exploit this representation of $J(\tau, p, \lambda, \varphi, R, S)$ in order to evaluate the L^p-norm of $J(\tau, p, \lambda, \varphi, R, S)$ against that of $H^*(\tau, p, \varphi, R, S)$, while keeping track on how the constants appearing in these estimates depend on the resolvent parameter λ. Afterwards, in Chapter 12, this result will allow us to derive some Fredholm properties of $J(\tau, p, \lambda, \varphi, R, S)$ from those of $H^*(\tau, p, \varphi, R, S)$. On the other hand, it will turn out that $H^*(\tau, p, \varphi, R, S)$ behaves in much the same way as $F^*(\tau, p, \varphi, R, S)$, and thus is closely connected to the operator $\Pi^*(-\tau, p, \mathbb{K}(\varphi))$. Since the behaviour of $J(\tau, p, \lambda, \varphi, R, S)$ is similar to that of $\Gamma^{(inf)}(\tau, p, \lambda, \varphi, R)$, we shall ultimately be able to show that the latter operator inherits certain features of $\Pi^*(-\tau, p, \mathbb{K}(\varphi))$ (Theorem 12.1). Note that from Chapter 8, the Fredholm poperties of $\Pi^*(-\tau, p, \mathbb{K}(\varphi))$ are known in some detail.

In order to fully exploit this knowledge, we further intend to invert the preceding line of argument, that is, we shall suppose that $\Gamma^{(inf)}(\tau, p, \lambda, \varphi, R)$ satisfies a certain property, and then obtain a contradiction by concluding that $\Pi^*(-\tau, p, \mathbb{K}(\varphi))$ has an analogous feature. This approach is based on a representation formula of $H^*(\tau, p, \varphi, R, S)$ with respect to $J(\tau, p, \lambda, \varphi, R, S)$, as given by equation (10.20), which is in some sense dual to equation (10.19) mentioned before. In fact, comparing (10.20) and (10.19), we see that the role of $J(\tau, p, \lambda, \varphi, R, S)$, X^λ, Y^λ, A^λ in one of these formulas corresponds to that of $H^*(\tau, p, \varphi, R, S)$, $\mathcal{X}^\infty$, $Y^\infty := 0$, A^∞, respectively, in the other one.

It is a principal aim of this book to discover the Fredholm properties of $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$. But by Corollary 6.5 and 6.6, this problem reduces to studying the features of the operators $\Gamma^{(ver)}(\tau, p, \lambda, \varphi, R)$ and $\Gamma^{(inf)}(\tau, p, \lambda, \varphi, R)$. As explained before, $\Gamma^{(inf)}(\tau, p, \lambda, \varphi, R)$ will turn out to behave in a similar way as $\Pi^*(-\tau, p, \mathbb{K}(\varphi))$. Concerning the operator $\Gamma^{(ver)}(\tau, p, \lambda, \varphi, R)$, which is easier to treat than $\Gamma^{(inf)}(\tau, p, \lambda, \varphi, R)$, its properties will be determined by those of $\Lambda(\tau, p, \mathbb{K}(\varphi))$, the double-layer operator related to the Stokes system (1.18). The details of this program will be elaborated in Chapter 12. We point out that the operator $\Pi^*(-\tau, p, \mathbb{K}(\varphi))$ may be used in order to construct solutions of Laplace's equation under Neumann boundary conditions ([28, p. 211-217]). Since the pressure term in (1.12) is harmonic, the preceding remarks indicate that the influence exerted by this term on $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ manifests itself by means of $\Pi^*(-\tau, p, \mathbb{K}(\varphi))$.

As should become clear by the above observations, the representation formulas (10.19) and (10.20), although very technical, are a key tool for investigating the operator $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$. In order to derive these formulas, we begin by introducing some notations:

Definition 10.1. *If* $\varphi_0 \in \mathbb{R}$, $\delta \in (0, \infty)$, *we set*

$$A(\varphi_0, \delta) := \{ r \cdot (\cos\varphi, \sin\varphi) \;:\; r \in (0,\infty),\ \varphi \in (\varphi_0 - \delta, \varphi_0 + \delta) \}.$$

Definition 10.2. *Let* $\varphi \in (0, \pi/2]$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $j, l \in \{1, 2, 3\}$, $\delta \in (0, \pi/6)$, $\xi, \eta \in \mathbb{R}^2$ *with* $\xi \neq \eta$.

In Lemma 10.4, the kernel of $J(\tau, p, \lambda, \varphi, R, S)$ *will be split up into a sum with the following summands:*

$$K^{(1)}_{jl}(\varphi, \lambda, \delta)(\xi, \eta) := -\sum_{k=1}^{3} Q^\lambda_{jkl}\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) \cdot \big(n^{(\varphi)}_k \circ g^{(\varphi)}\big)(\eta);$$

$$K^{(2)}_{jl}(\varphi, \lambda, \delta)(\xi, \eta) := -\sum_{k=1}^{3} P^\lambda_{jkl}\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) \cdot \big(n^{(\varphi)}_k \circ g^{(\varphi)}\big)(\eta) \cdot \chi_{\mathbb{R}^2\backslash A(0,3\cdot\delta)}(\eta);$$

$$\begin{aligned} &K^{(3)}_{jl}(\varphi, \lambda, \delta)(\xi, \eta) \\ &:= -\Big(\sum_{\upsilon=1}^{2} \mathcal{Y}^\lambda_\upsilon\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) \cdot \big((\xi+\eta)_\upsilon \cdot (|\xi| + |\eta|)^{-1} - \eta_\upsilon/|\eta|\big) \\ &\qquad + \mathcal{Y}^\lambda_2\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) \cdot \eta_2/|\eta| \\ &\qquad + \mathcal{Y}^\lambda_1\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) \cdot \sum_{n=1}^{\infty} \binom{1/2}{n} \cdot (-1)^n \cdot \eta_2^{2\cdot n} \cdot |\eta|^{-2\cdot n}\Big) \\ &\qquad \cdot \cot\varphi \cdot \delta_{3j} \cdot \big(n^{(\varphi)}_l \circ g^{(\varphi)}\big)(\eta) \cdot \chi_{A(0,3\cdot\delta)}(\eta); \end{aligned}$$

$$\begin{aligned} &K^{(4)}_{jl}(\varphi, \lambda, \delta)(\xi, \eta) \\ &:= -\Big(\sum_{\upsilon=1}^{2} \mathcal{X}^\lambda_\upsilon\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) \cdot \big((\xi+\eta)_\upsilon \cdot (|\xi| + |\eta|)^{-1} - \eta_\upsilon/|\eta|\big) \\ &\qquad + \mathcal{X}^\lambda_2\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) \cdot \eta_2/|\eta| \\ &\qquad + \mathcal{X}^\lambda_1\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) \cdot \sum_{n=1}^{\infty} \binom{1/2}{n} \cdot (-1)^n \cdot \eta_2^{2\cdot n} \cdot |\eta|^{-2\cdot n}\Big) \\ &\qquad \cdot \cot\varphi \cdot \delta_{3l} \cdot \big(n^{(\varphi)}_j \circ g^{(\varphi)}\big)(\eta) \cdot \chi_{A(0,3\cdot\delta)}(\eta); \end{aligned}$$

$$\begin{aligned} &K^{(5)}_{jl}(\varphi, \lambda, \delta)(\xi, \eta) \\ &:= -\Big((1-\delta_{3j}) \cdot \mathcal{Y}^\lambda_j\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) \;+\; \delta_{3j} \cdot \cot\varphi \cdot \mathcal{Y}^\lambda_1\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big)\Big) \\ &\qquad \cdot \Big(\delta_{1l} \cdot \sum_{n=1}^{\infty} \binom{1/2}{n} \cdot (-1)^n \cdot \eta_2^{2\cdot n} \cdot |\eta|^{-2\cdot n} \cdot \cos\varphi \;+\; \delta_{2l} \cdot \big(n^{(\varphi)}_2 \circ g^{(\varphi)}\big)(\eta)\Big) \\ &\qquad \cdot \chi_{A(0,3\cdot\delta)}(\eta); \end{aligned}$$

$$\begin{aligned} &K^{(6)}_{jl}(\varphi, \lambda, \delta)(\xi, \eta) \\ &:= -\Big((1-\delta_{3l}) \cdot \mathcal{X}^\lambda_l\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) \;+\; \delta_{3l} \cdot \cot\varphi \cdot \mathcal{X}^\lambda_1\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big)\Big) \\ &\qquad \cdot \Big(\delta_{1j} \cdot \sum_{n=1}^{\infty} \binom{1/2}{n} \cdot (-1)^n \cdot \eta_2^{2\cdot n} \cdot |\eta|^{-2\cdot n} \cdot \cos\varphi \;+\; \delta_{2j} \cdot \big(n^{(\varphi)}_2 \circ g^{(\varphi)}\big)(\eta)\Big) \\ &\qquad \cdot \chi_{A(0,3\cdot\delta)}(\eta); \end{aligned}$$

$$\begin{aligned} &K^{(7)}_{jl}(\varphi, \lambda, \delta)(\xi, \eta) \\ &:= -\Big((1-\delta_{3j}) \cdot \Big(\mathcal{Y}^\lambda_j\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) - \mathcal{Y}^\lambda_j\big(\xi-\eta,\ \cot\varphi\cdot(\xi-\eta)_1\big)\Big) \\ &\qquad + \cot\varphi \cdot \delta_{3j} \cdot \Big(\mathcal{Y}^\lambda_1\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) - \mathcal{Y}^\lambda_1\big(\xi-\eta,\ \cot\varphi\cdot(\xi-\eta)_1\big)\Big)\Big) \\ &\qquad \cdot \big(\delta_{1l} \cdot \cos\varphi - \delta_{3l} \cdot \sin\varphi\big) \cdot \chi_{A(0,3\cdot\delta)}(\eta); \end{aligned}$$

$$\begin{aligned} &K^{(8)}_{jl}(\varphi, \lambda, \delta)(\xi, \eta) \\ &:= -\Big((1-\delta_{3l}) \cdot \Big(\mathcal{X}^\lambda_l\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) - \mathcal{X}^\lambda_l\big(\xi-\eta,\ \cot\varphi\cdot(\xi-\eta)_1\big)\Big) \\ &\qquad + \cot\varphi \cdot \delta_{3l} \cdot \Big(\mathcal{X}^\lambda_1\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) - \mathcal{X}^\lambda_1\big(\xi-\eta,\ \cot\varphi\cdot(\xi-\eta)_1\big)\Big)\Big) \end{aligned}$$

$$\cdot \Big(\delta_{1j} \cdot \cos\varphi - \delta_{3j} \cdot \sin\varphi \Big) \cdot \chi_{A(0,\,3\cdot\delta)}(\eta)\,;$$

$$K_{jl}^{(9)}(\varphi,\,\lambda,\,\delta)(\xi,\,\eta) \quad := \quad -\bigg((1-\delta_{3j}) \cdot \mathcal{Y}_j^{\lambda}(\,\xi-\eta,\ \cot\varphi\cdot(\xi-\eta)_1\,) + \cot\varphi \cdot \delta_{3j} \cdot \mathcal{Y}_1^{\lambda}(\,\xi-\eta,\ \cot\varphi\cdot(\xi-\eta)_1\,) \bigg) \cdot \Big(\delta_{1l} \cdot \cos\varphi - \delta_{3l} \cdot \sin\varphi \Big) \cdot \chi_{A(0,\,3\cdot\delta)}(\eta)\,;$$

$$K_{jl}^{(10)}(\varphi,\,\lambda,\,\delta)(\xi,\,\eta) \quad := \quad -\bigg((1-\delta_{3l}) \cdot \mathcal{X}_l^{\lambda}(\,\xi-\eta,\ \cot\varphi\cdot(\xi-\eta)_1\,) + \cot\varphi \cdot \delta_{3l} \cdot \mathcal{X}_1^{\lambda}(\,\xi-\eta,\ \cot\varphi\cdot(\xi-\eta)_1\,) \bigg) \cdot \Big(\delta_{1j} \cdot \cos\varphi - \delta_{3j} \cdot \sin\varphi \Big) \cdot \chi_{A(0,\,3\cdot\delta)}(\eta)\,.$$

Next we introduce some terms which will allow us to represent the kernel of $H^*(\tau,\,p,\,\varphi,\,R,\,S)$ *as a sum (Lemma 10.4):*

$$L_{jl}^{(2)}(\varphi,\,\delta)(\xi,\,\eta) := -\mathcal{X}_l^{\infty}(\,g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\,) \cdot (\,n_j^{(\varphi)} \circ g^{(\varphi)}\,)(\eta) \cdot \chi_{\mathbb{R}^2\setminus A(0,\,3\cdot\delta)}(\eta)\,;$$

$$L_{jl}^{(4)}(\varphi,\,\delta)(\xi,\,\eta)$$

$$:= -\bigg(\sum_{v=1}^{2} \mathcal{X}_v^{\infty}(\,g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\,) \cdot \Big((\xi+\eta)_v \cdot (|\xi|+|\eta|)^{-1} - \eta_v/|\eta| \Big) + \mathcal{X}_2^{\infty}(\,g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\,) \cdot \eta_2/|\eta| + \mathcal{X}_1^{\infty}(\,g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\,) \cdot \sum_{n=1}^{\infty} \binom{1/2}{n} \cdot (-1)^n \cdot \eta_2^{2\cdot n} \cdot |\eta|^{-2\cdot n} \bigg) \cdot \cot\varphi \cdot \delta_{3l} \cdot (\,n_j^{(\varphi)} \circ g^{(\varphi)}\,)(\eta) \cdot \chi_{A(0,\,3\cdot\delta)}(\eta)\,;$$

$$L_{jl}^{(6)}(\varphi,\,\delta)(\xi,\,\eta)$$

$$:= -\Big((1-\delta_{3l}) \cdot \mathcal{X}_l^{\infty}(\,g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\,) + \delta_{3l} \cdot \cot\varphi \cdot \mathcal{X}_1^{\infty}(\,g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\,) \Big) \cdot \bigg(\delta_{1j} \cdot \sum_{n=1}^{\infty} \binom{1/2}{n} \cdot (-1)^n \cdot \eta_2^{2\cdot n} \cdot |\eta|^{-2\cdot n} \cdot \cos\varphi + \delta_{2j} \cdot (\,n_2^{(\varphi)} \circ g^{(\varphi)}\,)(\eta) \bigg) \cdot \chi_{A(0,\,3\cdot\delta)}(\eta)\,;$$

$$L_{jl}^{(8)}(\varphi,\,\delta)(\xi,\,\eta)$$

$$:= -\bigg((1-\delta_{3l}) \cdot \Big(\mathcal{X}_l^{\infty}(\,g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\,) - \mathcal{X}_l^{\infty}(\,\xi-\eta,\ \cot\varphi\cdot(\xi-\eta)_1\,) \Big) + \cot\varphi \cdot \delta_{3l} \cdot \Big(\mathcal{X}_1^{\infty}(\,g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\,) - \mathcal{X}_1^{\infty}(\,\xi-\eta,\ \cot\varphi\cdot(\xi-\eta)_1\,) \Big) \bigg) \cdot \Big(\delta_{1j} \cdot \cos\varphi - \delta_{3j} \cdot \sin\varphi \Big) \cdot \chi_{A(0,\,3\cdot\delta)}(\eta)\,;$$

$$L_{jl}^{(10)}(\varphi,\,\delta)(\xi,\,\eta) \quad := \quad -\bigg((1-\delta_{3l}) \cdot \mathcal{X}_l^{\infty}(\,\xi-\eta,\ \cot\varphi\cdot(\xi-\eta)_1\,) + \cot\varphi \cdot \delta_{3l} \cdot \mathcal{X}_1^{\infty}(\,\xi-\eta,\ \cot\varphi\cdot(\xi-\eta)_1\,) \bigg) \cdot \Big(\delta_{1j} \cdot \cos\varphi - \delta_{3j} \cdot \sin\varphi \Big) \cdot \chi_{A(0,\,3\cdot\delta)}(\eta)\,.$$

In Lemma 10.5 and 10.6, it will turn out that integral operators induced by the kernels $K_{jl}^{(9)}(\varphi,\,\lambda,\,\delta) + K_{jl}^{(10)}(\varphi,\,\lambda,\,\delta)$ *and* $L_{jl}^{(10)}(\varphi,\,\delta)$ $(1 \le j,l \le 3)$ *coincide with the convolution operators generated by the ensuing functions* $\mathcal{M}(\varphi,\,\lambda)$ *and* $\mathcal{M}(\varphi,\,\infty)$, *respectively, with* $\varphi \in (0,\,\pi/2]$, $\lambda \in \mathbb{C}\setminus(-\infty,\,0]$:

$$\mathcal{M}(\varphi,\,\lambda) :=$$

$$\begin{pmatrix} -\sin\varphi \cdot \cos\varphi \cdot (X_1^{\lambda}+Y_1^{\lambda}) & -\cos\varphi \cdot Y_2^{\lambda} & \sin^2(\varphi) \cdot X_1^{\lambda} - \cos^2(\varphi) \cdot Y_1^{\lambda} \\ -\cos\varphi \cdot X_2^{\lambda} & 0 & \sin\varphi \cdot X_2^{\lambda} \\ \sin^2(\varphi) \cdot Y_1^{\lambda} - \cos^2(\varphi) \cdot X_1^{\lambda} & \sin\varphi \cdot Y_2^{\lambda} & \sin\varphi \cdot \cos\varphi \cdot (X_1^{\lambda}+Y_1^{\lambda}) \end{pmatrix};$$

$$\mathcal{M}(\varphi,\,\infty) := \begin{pmatrix} -\sin\varphi \cdot \cos\varphi \cdot X_1^{\infty} & 0 & \sin^2(\varphi) \cdot X_1^{\infty} \\ -\cos\varphi \cdot X_2^{\infty} & 0 & \sin\varphi \cdot X_2^{\infty} \\ -\cos^2(\varphi) \cdot X_1^{\infty} & 0 & \sin\varphi \cdot \cos\varphi \cdot X_1^{\infty} \end{pmatrix}.$$

Before studying the properties of the preceding functions, we point out some simple but useful equations:

Lemma 10.1 *Let* $\lambda \in \mathbb{C}\setminus(-\infty,\,0]$, $\varphi \in (0,\,\pi/2]$, $\xi \in \mathbb{R}^2\setminus\{0\}$. *Let* $(G,\,H)$ *coincide with one of the pairs* $(\mathcal{Y}^{\lambda},\,Y^{\lambda})$, $(\mathcal{X}^{\lambda},\,X^{\lambda})$, $(\mathcal{X}^{\infty},\,X^{\infty})$ *of vector-valued functions. Then it holds*

$$\Big(G_j(\,\xi,\ \cot\varphi \cdot \xi_1\,) \Big)_{1\le j\le 2} = \Big(\sin\varphi \cdot (\,H_1 \circ T(\varphi)\,)(\xi),\ \ (\,H_2 \circ T(\varphi)\,)(\xi) \Big).$$

This result may be proved by some simple calculations.

Next we shall check whether the functions introduced in Defintion 10.2 can in fact be used as kernels of integral operators. It turns out that $\mathcal{M}(\varphi,\,\infty)$ and $L_{jl}^{(2\cdot v)}(\varphi,\,\delta)$, with $v \in \{1,\,\cdots,\,5\}$, only yield principal-value integrals, whereas the kernels $K_{jl}^{(v)}(\varphi,\,\lambda,\,\delta)$ $(v \in \{1,\,\cdots,\,10\})$ and $\mathcal{M}(\varphi,\,\lambda)$ lead to integrals which exist absolutely:

Lemma 10.2. *Let* $\varphi \in (0, \pi/2]$, $p \in (1, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\delta \in (0, \pi/6)$, $j, l \in \{1, 2, 3\}$, $\Phi \in L^p(\mathbb{R}^2)$. *Then the ensuing functions are well defined and belong to* $L^p(\mathbb{R}^2)$:

$K_{jl}^{(v)}(\varphi, \lambda, \delta) \otimes \Phi$ *for* $v \in \{1, \cdots, 10\}$; $L_{jl}^{(v)}(\varphi, \delta) \otimes_p \Phi$ *for* $v \in \{2, 4, 6, 8, 10\}$;

$\mathcal{M}_{jl}(\varphi, \lambda) * \Phi$, $\mathcal{M}_{jl}(\varphi, \infty) *_p \Phi$.

Proof: According to Lemma 5.5, the function $\mathcal{M}_{jl}(\varphi, \infty) *_p \Phi$ is well defined. Due to Lemma 10.1, 5.5 and Corollary 4.2, the same statement is true with respect to $L_{jl}^{(v)}(\varphi, \delta) \otimes_p \Phi$ if $v \in \{2, 8, 10\}$.

We have assumed $\delta \in (0, \pi/6)$, hence $|\eta_2/|\eta|| \leq \sin(3\cdot\delta) < 1$ for $\eta \in A(0, 3\cdot\delta)\backslash\{0\}$, so that it follows

$$\sum_{n=1}^{\infty} \left| \binom{1/2}{n} \right| \cdot |\eta_2^{2\cdot n}| \cdot |\eta|^{-2\cdot n} \quad \leq \quad \sum_{n=1}^{\infty} \left| \binom{1/2}{n} \right| \cdot \sin^{2\cdot n}(3\cdot\delta) \quad < \quad \infty. \tag{10.1}$$

Now we may conclude from (10.1) and Corollary 4.2 that the function $L_{jl}^{(v)}(\varphi, \delta) \otimes_p \Phi$ is well defined for $v \in \{4, 6\}$.

Referring to (10.1) and Lemma 5.10, 10.1, we see that the functions $\mathcal{M}_{jl}(\varphi, \lambda) \otimes \Phi$ and $K_{jl}^{(v)}(\varphi, \lambda, \delta) \otimes \Phi$ for $v \in \{2, \cdots, 10\}$ are well defined and belong to $L^p(\mathbb{R}^2)$. Lemma 5.11 implies a corresponding result for $K_{jl}^{(1)}(\varphi, \lambda, \delta) \otimes \Phi$.

In the case $\Phi \in L^p(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$, with $p, q \in (1, \infty)$, $p \neq q$, the functions $L_{jl}^{(v)}(\varphi, \delta) \otimes_p \Phi$ and $L_{jl}^{(v)}(\varphi, \delta) \otimes_q \Phi$ are of course identical. The same remark holds true with respect to $\mathcal{M}(\varphi, \infty) \otimes_p \Phi$ and $\mathcal{M}(\varphi, \infty) \otimes_q \Phi$. This observation is a special case of the result presented in the next lemma.

Lemma 10.3. *Let* $p, q \in (1, \infty)$ *with* $p \neq q$. *Take* $F_\epsilon \in L^p(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ *for* $\epsilon \in (0, \infty)$. *Furthermore, let* $F^{(p)} \in L^p(\mathbb{R}^2)$, $F^{(q)} \in L^q(\mathbb{R}^2)$, *and assume*

$$\| F_\epsilon - F^{(p)} \|_p \to 0, \qquad \| F_\epsilon - F^{(q)} \|_q \to 0 \qquad (\epsilon \downarrow 0).$$

It follows $F^{(p)} = F^{(q)}$ *almost everywhere.*

Proof: There is a sequence (ϵ_n) in $(0, \infty)$ with

$$\epsilon_n \to 0, \qquad F_{\epsilon_n}(\xi) \to F^{(p)}(\xi) \qquad (n \to \infty) \qquad \text{for almost every } \xi \in \mathbb{R}^2;$$

see [41, p. 67, Theorem 3.12]. Since $\| F_{\epsilon_n} - F^{(q)} \|_q$ vanishes for $n \to \infty$, there exists a subsequence (δ_n) of (ϵ_n) with $F_{\delta_n}(\xi) \to F^{(q)}(\xi)$ if $n \to \infty$, for almost every $\xi \in \mathbb{R}^2$. This implies $F^{(p)}(\xi) = F^{(q)}(\xi)$ for almost every $\xi \in \mathbb{R}^2$.

Due to Lemma 10.2, we may introduce the following integral operators:

Definition 10.3. *Let* $\varphi \in (0, \pi/2]$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $l \in \{1, 2, 3\}$, $\delta \in (0, \pi/6)$, $R \in (0, \infty)$, $S \in (1, \infty)$, $p \in (1, \infty)$, $\Phi \in L^p(\mathbb{R}^2)^3$. *Then we set*

$$\mathcal{A}^{(v)}(\varphi, \lambda, \delta, R, S, \Phi) \quad := \quad -\Big(\sin^{-1}(\varphi) \cdot \chi_{A(0,3\cdot\delta)} \cdot \chi_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R)} \cdot \sum_{j=1}^{3} K_{jl}^{(v)}(\varphi, \lambda, \delta) \otimes \Phi_j\Big)_{1\leq l\leq 3} \circ (T(\varphi))^{-1} \qquad \text{for } v \in \{1, \cdots, 8\};$$

$$\mathcal{A}^{(11)}(\varphi, \lambda, \delta, R, S, \Phi) \quad := \quad \Big(\sin^{-1}(\varphi) \cdot \chi_{\mathbb{R}^2\backslash A(0,3\cdot\delta)} \cdot \chi_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R)} \cdot \sum_{j=1}^{3} (K_{jl}^{(9)}(\varphi, \lambda, \delta) + K_{jl}^{(10)}(\varphi, \lambda, \delta)) \otimes \Phi_j\Big)_{1\leq l\leq 3} \circ (T(\varphi))^{-1};$$

$$\mathcal{A}^{(12)}(\varphi, \lambda, \delta, R, S, \Phi) \quad := \quad \Big(\sin^{-1}(\varphi) \cdot \chi_{\mathbb{B}_2(0,R)} \cdot \sum_{j=1}^{3} (K_{jl}^{(9)}(\varphi, \lambda, \delta) + K_{jl}^{(10)}(\varphi, \lambda, \delta)) \otimes \Phi_j\Big)_{1\leq l\leq 3} \circ (T(\varphi))^{-1};$$

$$\mathcal{B}^{(v)}(\varphi, \delta, R, S, \Phi) \quad := \quad -\Big(\sin^{-1}(\varphi) \cdot \chi_{A(0,3\cdot\delta)} \cdot \chi_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R)} \cdot \sum_{j=1}^{3} L_{jl}^{(v)}(\varphi, \delta) \otimes_p \Phi_j\Big)_{1\leq l\leq 3} \circ (T(\varphi))^{-1} \qquad \text{for } v \in \{2, 4, 6, 8\};$$

$$\mathcal{B}^{(11)}(\varphi, \delta, R, S, \Phi) \quad := \quad \Big(\sin^{-1}(\varphi) \cdot \chi_{\mathbb{R}^2\backslash A(0,3\cdot\delta)} \cdot \chi_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R)} \cdot \sum_{j=1}^{3} L_{jl}^{(10)}(\varphi, \delta) \otimes_p \Phi_j\Big)_{1\leq l\leq 3} \circ (T(\varphi))^{-1};$$

$$\mathcal{B}^{(12)}(\varphi, \delta, R, S, \Phi) := \Big(\sin^{-1}(\varphi) \cdot \chi_{\mathbb{B}_2(0,R)} \cdot \sum_{j=1}^{3} L_{jl}^{(10)}(\varphi, \delta) \otimes_p \Phi_j\Big)_{1\leq l\leq 3} \circ (T(\varphi))^{-1}.$$

Following up on our indications in Definition 10.2, we shall now describe how the kernels of $J(\tau, p, \lambda, \varphi, R, S)$ and $H^*(\tau, p, \varphi, R, S)$ are related to the functions introduced at the beginning of this chapter:

Lemma 10.4. *Let* $\varphi \in (0, \pi/2]$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\delta \in (0, \pi/6)$, $\xi, \eta \in \mathbb{R}^2$ *mit* $\xi \neq \eta$, $\eta \neq 0$, $j, l \in \{1, 2, 3\}$. *Then it holds:*

$$\sum_{k=1}^{3} \widetilde{\mathcal{D}}_{jkl}^{\lambda}(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \cdot (n_k^{(\varphi)} \circ g^{(\varphi)})(\eta) \quad = \quad \sum_{v=1}^{10} K_{jl}^{(v)}(\varphi, \lambda, \delta)(\xi, \eta); \tag{10.2}$$

$$L_l(0,\varphi)(\xi,\eta)\cdot\big(n_j^{(\varphi)}\circ g^{(\varphi)}\big)(\eta) \;=\; \sum_{v=1}^{5} L_{jl}^{(2\cdot v)}(\varphi,\delta)(\xi,\eta). \tag{10.3}$$

Proof: We obtain from (5.11):

$$\begin{aligned}\sum_{k=1}^{3} \widetilde{\mathcal{D}}^{\lambda}_{jkl}\big(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)\big)\cdot\big(n_k^{(\varphi)}\circ g^{(\varphi)}\big)(\eta) &= \sum_{v=1}^{2} K_{jl}^{(v)}(\varphi,\lambda,\delta)(\xi,\eta)\\ &\quad - \sum_{k=1}^{3} P^{\lambda}_{jkl}\big(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)\big)\cdot\big(n_k^{(\varphi)}\circ g^{(\varphi)}\big)(\eta)\cdot\chi_{A(0,3\cdot\delta)}(\eta).\end{aligned} \tag{10.4}$$

This proves (10.2) in the case $\eta\in\mathbb{R}^2\backslash A(0,3\cdot\delta)$. Therefore we assume $\eta\in A(0,3\cdot\delta)$ in the following. It holds

$$\begin{aligned}|\xi|-|\eta| &= \big(|\xi|^2+|\eta|^2\big)\cdot\big(|\xi|+|\eta|\big)^{-1} \\ &= \sum_{v=1}^{2}(\xi-\eta)_v\cdot\Big((\xi+\eta)_v\cdot\big(|\xi|+|\eta|\big)^{-1}-\eta_v\cdot|\eta|^{-1}\Big) + \sum_{v=1}^{2}(\xi-\eta)_v\cdot\eta_v\cdot|\eta|^{-1}.\end{aligned} \tag{10.5}$$

Since $\eta\in A(0,3\cdot\delta)$ and $\delta\in(0,\pi/6)$, we have $\eta_1>0$ and $\eta_2^2/|\eta|^2<1$. This implies

$$\eta_1/|\eta| \;=\; \big(1-\eta_2^2/|\eta|^2\big)^{1/2} \;=\; \sum_{n=0}^{\infty}\binom{1/2}{n}\cdot\big(-\eta_2^2/|\eta|^2\big)^n. \tag{10.6}$$

Furthermore, we observe

$$\big(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)\big)_3 \;=\; \cot\varphi\cdot\big(|\xi|-|\eta|\big). \tag{10.7}$$

Gathering up the results in (10.5) - (10.7), we arrive at these equations:

$$\begin{aligned}-\mathcal{Y}_3^{\lambda}\big(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)\big)\cdot\big(n_l^{(\varphi)}\circ g^{(\varphi)}\big)(\eta) &= K_{3l}^{(3)}(\varphi,\lambda,\delta)(\xi,\eta) - \cot\varphi\cdot\mathcal{Y}_1^{\lambda}\big(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)\big)\cdot\big(n_l^{(\varphi)}\circ g^{(\varphi)}\big)(\eta);\end{aligned}$$

$$\begin{aligned}-\mathcal{X}_3^{\lambda}\big(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)\big)\cdot\big(n_j^{(\varphi)}\circ g^{(\varphi)}\big)(\eta) &= K_{3j}^{(4)}(\varphi,\lambda,\delta)(\xi,\eta) - \cot\varphi\cdot\mathcal{X}_1^{\lambda}\big(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)\big)\cdot\big(n_j^{(\varphi)}\circ g^{(\varphi)}\big)(\eta).\end{aligned}$$

In the case $j\neq3$ and $l\neq3$, we have $K_{jl}^{(3)}(\varphi,\lambda,\delta)=0$ and $K_{jl}^{(4)}(\varphi,\lambda,\delta)=0$, respectively. Thus we find by recurring to (5.12) and (10.4):

$$\begin{aligned}\sum_{k=1}^{3}\widetilde{\mathcal{D}}^{\lambda}_{jkl}\big(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)\big)\cdot\big(n_k^{(\varphi)}\circ g^{(\varphi)}\big)(\eta) &= \sum_{v=1}^{4}K_{jl}^{(v)}(\varphi,\lambda,\delta)(\xi,\eta)\\ &\quad-\Big(\big(1-\delta_{3j}\big)\cdot\mathcal{Y}_j^{\lambda}\big(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)\big)+\delta_{3j}\cdot\cot\varphi\cdot\mathcal{Y}_1^{\lambda}\big(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)\big)\Big)\\ &\qquad\cdot\big(n_l^{(\varphi)}\circ g^{(\varphi)}\big)(\eta)\\ &\quad-\Big(\big(1-\delta_{3l}\big)\cdot\mathcal{X}_l^{\lambda}\big(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)\big)+\delta_{3l}\cdot\cot\varphi\cdot\mathcal{X}_1^{\lambda}\big(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)\big)\Big)\\ &\qquad\cdot\big(n_j^{(\varphi)}\circ g^{(\varphi)}\big)(\eta).\end{aligned}$$

Moreover, observe that $\big(n_3^{(\varphi)}\circ g^{(\varphi)}\big)(\eta)=-\sin\varphi$, and

$$\big(n_1^{(\varphi)}\circ g^{(\varphi)}\big)(\eta) \;=\; \cos\varphi\cdot\eta_1\cdot|\eta|^{-1} \;=\; \cos\varphi\cdot\sum_{n=0}^{\infty}\binom{1/2}{n}\cdot\big(-\eta_2^2/|\eta|^2\big)^n$$

(see (3.2)), so we may conclude:

$$\begin{aligned}\sum_{k=1}^{3}\widetilde{\mathcal{D}}^{\lambda}_{jkl}\big(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)\big)\cdot\big(n_k^{(\varphi)}\circ g^{(\varphi)}\big)(\eta) &= \sum_{v=1}^{6}K_{jl}^{(v)}(\varphi,\lambda,\delta)(\xi,\eta)\\ &\quad-\Big(\big(1-\delta_{3j}\big)\cdot\mathcal{Y}_j^{\lambda}\big(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)\big)+\delta_{3j}\cdot\cot\varphi\cdot\mathcal{Y}_1^{\lambda}\big(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)\big)\Big)\\ &\qquad\cdot\Big(\cos\varphi\cdot\delta_{1l}-\sin\varphi\cdot\delta_{3l}\Big)\\ &\quad-\Big(\big(1-\delta_{3l}\big)\cdot\mathcal{X}_l^{\lambda}\big(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)\big)+\delta_{3l}\cdot\cot\varphi\cdot\mathcal{X}_1^{\lambda}\big(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)\big)\Big)\\ &\qquad\cdot\Big(\cos\varphi\cdot\delta_{1j}-\sin\varphi\cdot\delta_{3j}\Big).\end{aligned}$$

This implies (10.2). Equation (10.3) may be proved in an analogous way.

Next we point out how the functions $K_{jl}^{(9)}(\varphi,\lambda,\delta)$ and $K_{jl}^{(10)}(\varphi,\lambda,\delta)$ are related to $\mathcal{M}(\varphi,\lambda)$:

Lemma 10.5. *Take* $p\in(1,\infty)$, $\varphi\in(0,\pi/2]$, $\lambda\in\mathbb{C}\backslash(-\infty,0]$, $\delta\in(0,\pi/6)$, $\Phi\in L^p(\mathbb{R}^2)^3$. *It follows*

$$\begin{aligned}&\sin^{-1}(\varphi)\cdot\Big(\sum_{j=1}^{3}\big(K_{jl}^{(9)}(\varphi,\lambda,\delta)+K_{jl}^{(10)}(\varphi,\lambda,\delta)\big)\otimes\Phi_j\Big)_{1\le l\le3}\\ &\quad=\Big(\mathcal{M}(\varphi,\lambda)*\Big(\big(\chi_{A(0,3\cdot\delta)}\cdot\Phi\big)\circ\big(T(\varphi)\big)^{-1}\Big)\Big)\circ T(\varphi).\end{aligned} \tag{10.8}$$

Proof: Let $\xi\in\mathbb{R}^2$. By Lemma 10.1 we have

$$\Big(\sin^{-1}(\varphi)\cdot\int_{\mathbb{R}^2}\sum_{j=1}^{3}K_{jl}^{(9)}(\varphi,\lambda,\delta)(\xi,\eta)\cdot\Phi_j(\eta)\,d\eta\Big)_{1\le l\le 3}$$

$$= -\int_{\mathbb{R}^2}\Big((Y_1^{\lambda}\circ T(\varphi))(\xi-\eta)\cdot(\chi_{A(0,3\cdot\delta)}\cdot\Phi_1)(\eta)$$
$$+\sin^{-1}(\varphi)\cdot(Y_2^{\lambda}\circ T(\varphi))(\xi-\eta)\cdot(\chi_{A(0,3\cdot\delta)}\cdot\Phi_2)(\eta)$$
$$+\cot\varphi\cdot(Y_1^{\lambda}\circ T(\varphi))(\xi-\eta)\cdot(\chi_{A(0,3\cdot\delta)}\cdot\Phi_3)(\eta)\Big)\,d\eta$$
$$\cdot(\cos\varphi,\,0,\,-\sin\varphi).$$

Furthermore, it holds

$$\Big(\sin^{-1}(\varphi)\cdot\int_{\mathbb{R}^2}\sum_{j=1}^{3}K_{jl}^{(10)}(\varphi,\lambda,\delta)(\xi,\eta)\cdot\Phi_j(\eta)\,d\eta\Big)_{1\le l\le 3}$$

$$= -\int_{\mathbb{R}^2}\Big(X_1^{\lambda}\circ T(\varphi),\quad \sin^{-1}(\varphi)\cdot(X_2^{\lambda}\circ T(\varphi)),\quad \cot\varphi\cdot(X_1^{\lambda}\circ T(\varphi))\Big)(\xi-\eta)$$
$$\cdot\Big(\cos\varphi\cdot(\chi_{A(0,3\cdot\delta)}\cdot\Phi_1)(\eta)-\sin\varphi\cdot(\chi_{A(0,3\cdot\delta)}\cdot\Phi_3)(\eta)\Big)\,d\eta.$$

Combining these results, we obtain

$$\Big(\sin^{-1}(\varphi)\cdot\int_{\mathbb{R}^2}\sum_{j=1}^{3}\big(K_{jl}^{(9)}(\varphi,\lambda,\delta)+K_{jl}^{(10)}(\varphi,\lambda,\delta)\big)(\xi,\eta)\cdot\Phi_j(\eta)\,d\eta\Big)_{1\le l\le 3} \tag{10.9}$$

$$=\sin^{-1}(\varphi)\cdot\int_{\mathbb{R}^2}\mathcal{M}(\varphi,\lambda)\big(T(\varphi)(\xi-\eta)\big)\cdot(\chi_{A(0,3\cdot\delta)}\cdot\Phi)(\eta)\,d\eta.$$

But the right-hand side of (10.9) is equal to

$$\int_{\mathbb{R}^2}\mathcal{M}(\varphi,\lambda)\big(T(\varphi)(\xi)-\eta\big)\cdot\Big((\chi_{A(0,3\cdot\delta)}\cdot\Phi)\circ(T(\varphi))^{-1}\Big)(\eta)\,d\eta,$$

so that (10.8) is proved.

It is more difficult to show a corresponding relation between $L_{jl}^{(10)}(\varphi,\delta)$ and $\mathcal{M}(\varphi,\infty)$, since these kernels only induce principal-value integrals. In fact, when attempting to adapt the proof of Lemma 10.5 to this new situation, we had to check that the functions

$$F_\epsilon(\xi) := \int_{\mathbb{R}^2}\chi_{(\epsilon,\infty)}\Big(\big|(\sin\varphi\cdot(\xi-\eta)_1,\ (\xi-\eta)_2)\big|\Big)\cdot X_l^{\infty}(\xi-\eta)\cdot\Phi(\eta)\,d\eta$$

and

$$G_\epsilon(\xi) := \int_{\mathbb{R}^2}\chi_{(\epsilon,\infty)}(|\xi-\eta|)\cdot X_l^{\infty}(\xi-\eta)\cdot\Phi(\eta)\,d\eta,$$

($\epsilon>0$, $\xi\in\mathbb{R}^2$) both tend to the same limit for $\epsilon\downarrow 0$, with respect to the norm of the space $L^2(\mathbb{R}^2)$; see (10.15), where this relation is stated in a more formal way.

Lemma 10.6. *Let* $p\in(1,\infty)$, $\varphi\in(0,\pi/2]$, $\delta\in(0,\pi/6)$, $\Phi\in L^p(\mathbb{R}^2)^3$. *Then*

$$\sin^{-1}(\varphi)\cdot\Big(\sum_{j=1}^{3}L_{jl}^{(10)}(\varphi,\delta)\otimes_p\Phi_j\Big)_{1\le l\le 3}$$
$$=\Big(\mathcal{M}(\varphi,\infty)*_p\Big((\chi_{A(0,3\cdot\delta)}\cdot\Phi)\circ(T(\varphi))^{-1}\Big)\Big)\circ T(\varphi).$$

Proof: For brevity we put $\widetilde{\Phi}:=(\chi_{A(0,3\cdot\delta)}\cdot\Phi)\circ(T(\varphi))^{-1}$. Then we find

$$(\mathcal{M}(\varphi,\infty)*_p\widetilde{\Phi})\circ T(\varphi) \tag{10.10}$$
$$=-\sin\varphi\cdot\Big(\Big(X_1^{\infty},\ \sin^{-1}(\varphi)\cdot X_2^{\infty},\ \cot\varphi\cdot X_1^{\infty}\Big)$$
$$*_p\Big(\cos\varphi\cdot\widetilde{\Phi}_1-\sin\varphi\cdot\widetilde{\Phi}_3\Big)\Big)\circ T(\varphi).$$

On the other hand, Lemma 10.1 yields

$$\Big(\sin^{-1}(\varphi)\cdot\sum_{j=1}^{3}L_{jl}^{(10)}(\varphi,\delta)\otimes_p\Phi_j\Big)_{1\le l\le 3} \tag{10.11}$$
$$=-\Big(X_1^{\infty}\circ T(\varphi),\ \sin^{-1}(\varphi)\cdot(X_2^{\infty}\circ T(\varphi)),\ \cot\varphi\cdot(X_1^{\infty}\circ T(\varphi))\Big)$$
$$*_p\Big(\cos\varphi\cdot\chi_{A(0,3\cdot\delta)}\cdot\Phi_1-\sin\varphi\cdot\chi_{A(0,3\cdot\delta)}\cdot\Phi_3\Big).$$

Now take a sequence (ϱ_n) in $C_0^{\infty}(\mathbb{R}^2)^3$ with $\|\varrho_n-\chi_{A(0,3\cdot\delta)}\cdot\Phi\|_p\to 0$ for $n\to\infty$. Then it follows $\|\varrho_n\circ(T(\varphi))^{-1}-\widetilde{\Phi}\|_p\to 0$ $(n\to\infty)$, so we may conclude by Lemma 5.5, for $j\in\{1,2,3\}$, $l\in\{1,2\}$:

$$\Big\|X_l^{\infty}*_p\widetilde{\Phi}_j - X_l^{\infty}*_p\Big(\varrho_n\circ(T(\varphi))^{-1}\Big)_j\Big\|_p \longrightarrow 0,$$

$$\Big\|(X_l^{\infty}\circ T(\varphi))*_p(\chi_{A(0,3\cdot\delta)}\cdot\Phi_j)-(X_l^{\infty}\circ T(\varphi))*_p\varrho_{n,j}\Big\|_p \longrightarrow 0 \tag{10.12}$$

for $n\to\infty$, respectively. The first one of these relations implies

$$\Big\|(X_l^{\infty}*_p\widetilde{\Phi}_j)\circ T(\varphi)-\Big(X_l^{\infty}*_p\Big(\varrho_n\circ(T(\varphi))^{-1}\Big)_j\Big)\circ T(\varphi)\Big\|_p \longrightarrow 0 \tag{10.13}$$

if $n\to\infty$. Thus, if we are able to establish the ensuing equation for $n\in\mathbb{N}$, $j\in\{1,2,3\}$, $l\in\{1,2\}$:

$$\sin\varphi\cdot\Big(X_l^{\infty}*_p\Big(\varrho_{n,j}\circ(T(\varphi))^{-1}\Big)\Big)\circ T(\varphi) = (X_l^{\infty}\circ T(\varphi))*_p\varrho_{n,j}. \tag{10.14}$$

then we can deduce from (10.12) - (10.14) that the right-hand sides of (10.10) and (10.11) are identical, and the lemma is proved. So we still have to show (10.14). But this equation is equivalent to the following result:

$$\sin\varphi \cdot X_l^\infty *_p \Big(\varrho_{n,j} \circ (T(\varphi))^{-1}\Big) \;=\; \Big((X_l^\infty \circ T(\varphi)) *_p \varrho_{n,j}\Big) \circ (T(\varphi))^{-1}$$

for $l \in \{1, 2\}$, $j \in \{1, 2, 3\}$, $n \in \mathbb{N}$. On the other hand, we find for l, j, n as before:

$$\Big((X_l^\infty \circ T(\varphi)) *_p \varrho_{n,j}\Big) \circ (T(\varphi))^{-1}$$
$$= \sin\varphi \cdot L^p(\mathbb{R}^2) - \lim_{\epsilon\downarrow 0} \int_{\mathbb{R}^2} \chi_{(\epsilon,\infty)}\Big(\big|(T(\varphi))^{-1} - (T(\varphi))^{-1}(\eta)\big|\Big) \cdot X_l^\infty(id(\mathbb{R}^2) - \eta) \cdot \Big(\varrho_{n,j} \circ (T(\varphi))^{-1}\Big)(\eta)\; d\eta .$$

Thus we have to show for $n \in \mathbb{N}$, $l \in \{1, 2\}$, $j \in \{1, 2, 3\}$:

$$X_l^\infty *_p \Big(\varrho_{n,j} \circ (T(\varphi))^{-1}\Big) \tag{10.15}$$
$$= L^p(\mathbb{R}^2) - \lim_{\epsilon\downarrow 0} \int_{\mathbb{R}^2} \chi_{(\epsilon,\infty)}\Big(\big|(T(\varphi))^{-1} - (T(\varphi))^{-1}(\eta)\big|\Big) \cdot X_l^\infty(id(\mathbb{R}^2) - \eta) \cdot \Big(\varrho_{n,j} \circ (T(\varphi))^{-1}\Big)(\eta)\; d\eta .$$

In order to prove (10.15), we fix $j \in \{1, 2, 3\}$, $l \in \{1, 2\}$, $n \in \mathbb{N}$. For shortness we put $\widetilde{\varrho} := \varrho_{n,j} \circ (T(\varphi))^{-1}$.

Since $\mathbb{B}_2(0, \epsilon) \subset T(\varphi)(\mathbb{B}_2(0, \epsilon))$, we have

$$\chi_{\mathbb{R}^2\setminus\mathbb{B}_2(0,\epsilon)} - \chi_{T(\varphi)(\mathbb{R}^2\setminus\mathbb{B}_2(0,\epsilon))} \;=\; \chi_{T(\varphi)(\mathbb{B}_2(0,\epsilon))\setminus\mathbb{B}_2(0,\epsilon)} \qquad (\epsilon \in (0,\infty)). \tag{10.16}$$

As another abbreviation, we set for $\epsilon \in (0,\infty)$, $r \in (0,\infty)$:

$$F_\epsilon(r) := \Big\{\vartheta \in [0, \pi) \;:\; r \ge \epsilon > r \cdot \big(\sin^{-2}(\varphi) \cdot \cos^2(\vartheta) + \sin^2(\vartheta)\big)^{1/2}\Big\}.$$

Then it holds for $\epsilon \in (0,\infty)$, $\xi \in \mathbb{R}^2$:

$$\int_{T(\varphi)(\mathbb{B}_2(0,\epsilon))\setminus\mathbb{B}_2(0,\epsilon)} X_l^\infty(\eta)\; d\eta \;=\; (4\cdot\pi)^{-1} \cdot \int_{T(\varphi)(\mathbb{B}_2(0,\epsilon))\setminus\mathbb{B}_2(0,\epsilon)} \eta_l \cdot |\eta|^{-3}\; d\eta \tag{10.17}$$
$$= (4\cdot\pi)^{-1} \cdot \int_0^\infty \int_{F_\epsilon(r)\cup(\pi+F_\epsilon(r))} (\cos\vartheta,\; \sin\vartheta)_l \cdot r^{-1}\; d\vartheta\; dr \;=\; 0 .$$

Since $\widetilde{\varrho} \in C_0^\infty(\mathbb{R}^2)^3$, the constant

$$\mathfrak{C} := (4\cdot\pi)^{-1} \cdot \sup\Big\{\big|\widetilde{\varrho}(\eta) - \widetilde{\varrho}(\xi)\big| \cdot |\xi - \eta|^{-1/2} \;:\; \xi, \eta \in \mathbb{R}^2,\; \xi \ne \eta\Big\}$$

is finite. Now we find for $\epsilon \in (0,\infty)$, $\xi \in \mathbb{R}^2$:

$$\Big|\int_{\mathbb{R}^2} \chi_{T(\varphi)(\mathbb{B}_2(0,\epsilon))\setminus\mathbb{B}_2(0,\epsilon)}(\xi - \eta) \cdot X_l^\infty(\xi - \eta) \cdot \widetilde{\varrho}(\eta)\; d\eta\Big| \tag{10.18}$$
$$= \Big|\int_{T(\varphi)(\mathbb{B}_2(0,\epsilon))\setminus\mathbb{B}_2(0,\epsilon)} X_l^\infty(\sigma) \cdot \widetilde{\varrho}(\xi - \sigma)\; d\sigma\Big|$$

$$= \Big|\int_{T(\varphi)(\mathbb{B}_2(0,\epsilon))\setminus\mathbb{B}_2(0,\epsilon)} X_l^\infty(\sigma) \cdot \big(\widetilde{\varrho}(\xi - \sigma) - \widetilde{\varrho}(\xi)\big)\; d\sigma\Big|$$
$$\le \mathfrak{C} \cdot \int_{T(\varphi)(\mathbb{B}_2(0,\epsilon))\setminus\mathbb{B}_2(0,\epsilon)} |\sigma|^{-3/2}\; d\sigma$$
$$\le \mathfrak{C} \cdot \int_{\mathbb{B}_2(0,\,\sin^{-1}(\varphi)\cdot\epsilon)} |\sigma|^{-3/2}\; d\sigma \;=\; \mathfrak{C} \cdot 4\cdot\pi \cdot \big(\sin^{-1}(\varphi) \cdot \epsilon\big)^{1/2},$$

where the second equation follows from (10.17). Putting

$$\delta := \sin^{-1}(\varphi) + \sup\{\,|\eta| \;:\; \eta \in supp(\widetilde{\varrho})\,\},$$

we get for $\epsilon \in (0, 1]$:

$$\Big(\int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2} \chi_{(\epsilon,\infty)}(|\xi - \eta|) \cdot X_l^\infty(\xi - \eta) \cdot \widetilde{\varrho}(\eta)\; d\eta$$
$$- \int_{\mathbb{R}^2} \chi_{(\epsilon,\infty)}\Big(\big|(T(\varphi))^{-1}(\xi - \eta)\big|\Big) \cdot X_l^\infty(\xi - \eta) \cdot \widetilde{\varrho}(\eta)\; d\eta\Big|^p\; d\xi\Big)^{1/p}$$
$$= \Big(\int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2} \chi_{T(\varphi)(\mathbb{B}_2(0,\epsilon))\setminus\mathbb{B}_2(0,\epsilon)}(\xi - \eta) \cdot X_l^\infty(\xi - \eta) \cdot \widetilde{\varrho}(\eta)\; d\eta\Big|^p\; d\xi\Big)^{1/p}$$
$$= \Big(\int_{\mathbb{B}_2(0,\delta)}\Big|\int_{\mathbb{R}^2} \chi_{T(\varphi)(\mathbb{B}_2(0,\epsilon))\setminus\mathbb{B}_2(0,\epsilon)}(\xi - \eta) \cdot X_l^\infty(\xi - \eta) \cdot \widetilde{\varrho}(\eta)\; d\eta\Big|^p\; d\xi\Big)^{1/p}$$
$$\le \mathfrak{C} \cdot 4\cdot\pi \cdot \big(\sin^{-1}(\varphi) \cdot \epsilon\big)^{1/2} \cdot \Big(\int_{\mathbb{B}_2(0,\delta)} d\xi\Big)^{1/p},$$

with the first equation implied by (10.16), whereas the preceding inequality follows from (10.18). As for the second equation, we refer to the estimate

$$|\xi - \eta| \;\ge\; \sin^{-1}(\varphi) \;\ge\; \sin^{-1}(\varphi) \cdot \epsilon \qquad \text{for } \xi \in \mathbb{R}^2\setminus\mathbb{B}_2(0, \delta),\; \eta \in supp(\widetilde{\varrho}),\; \epsilon \in (0, 1],$$

from which we deduce $\chi_{T(\varphi)(\mathbb{B}_2(0,\epsilon))\setminus\mathbb{B}_2(0,\epsilon)}(\xi - \eta) = 0$. Thus, completing the proof of (10.15), we may conclude

$$L^p(\mathbb{R}^2) - \lim_{\epsilon\downarrow 0} \int_{\mathbb{R}^2} \chi_{(\epsilon,\infty)}(|id(\mathbb{R}^2) - \eta|) \cdot X_l^\infty(id(\mathbb{R}^2) - \eta) \cdot \widetilde{\varrho}(\eta)\; d\eta$$
$$= L^p(\mathbb{R}^2) - \lim_{\epsilon\downarrow 0} \int_{\mathbb{R}^2} \chi_{(\epsilon,\infty)}\Big(\big|(T(\varphi))^{-1} - (T(\varphi))^{-1}(\eta)\big|\Big) \cdot X_l^\infty(id(\mathbb{R}^2) - \eta) \cdot \widetilde{\varrho}(\eta)\; d\eta .$$

Theorem 10.1. *Let* $p \in (1, \infty)$, $\lambda \in \mathbb{C}\setminus(-\infty, 0]$, $\varphi \in (0, \pi/2]$, $\delta \in (0, \pi/6)$, $R \in (0,\infty)$, $S \in (1,\infty)$, $\Phi \in L^p(\mathbb{R}^2\setminus\mathbb{B}_2(0, R\cdot S))^3 \cap L^2(\mathbb{R}^2\setminus\mathbb{B}_2(0, R\cdot S))^3$, $\tau \in \{-1, 1\}$. *Then it follows for* $\xi \in A(0, 3\cdot\delta)\setminus\mathbb{B}_2(0, R)$:

$$J(\tau, p, \lambda, \varphi, R, S)(\Phi)(\xi) \tag{10.19}$$

$$= \Big[A_\tau^{\lambda,\varphi} \cdot (A_\tau^{\infty,\varphi})^{-1} \cdot \Big(\tilde{\mathcal{F}}(R)\big(\chi_{A(0,3\cdot\delta)} \cdot H^*(\tau, p, \varphi, R, S)(\Phi)\big) \circ (T(\varphi))^{-1} + \sum_{v=2,4,6,8,11,12} \mathcal{B}^{(v)}(\varphi, \delta, R, S, \Phi) \Big)^\wedge \Big]^\vee (T(\varphi)(\xi))$$

$$+ \sin^{-1}(\varphi) \cdot \sum_{v=1,\cdots,8} \Big(\sum_{j=1}^{3} K_{jl}^{(v)}(\varphi, \lambda, \delta) \otimes \Phi_j \Big)_{1\le l\le 3}(\xi).$$

Furthermore, it holds for ξ as before:

$$H^*(\tau, p, \varphi, R, S)(\Phi)(\xi) \tag{10.20}$$

$$= \Big[A_\tau^{\infty,\varphi} \cdot (A_\tau^{\lambda,\varphi})^{-1} \cdot \Big(\tilde{\mathcal{F}}(R)\big(\chi_{A(0,3\cdot\delta)} \cdot J(\tau, p, \lambda, \varphi, R, S)(\Phi)\big) \circ (T(\varphi))^{-1} + \sum_{v=1,2,3,\cdots,8,11,12} \mathcal{A}^{(v)}(\varphi, \lambda, \delta, R, S, \Phi) \Big)^\wedge \Big]^\vee (T(\varphi)(\xi))$$

$$+ \sin^{-1}(\varphi) \cdot \sum_{v=2,4,6,8} \Big(\sum_{j=1}^{3} L_{jl}^{(v)}(\varphi, \delta) \otimes_p \Phi_j \Big)_{1\le l\le 3}(\xi).$$

Proof: For shortness we set $\tilde{\Phi} := \tilde{\mathcal{F}}(R\cdot S)(\chi_{A(0,3\cdot\delta)} \cdot \Phi) \circ (T(\varphi))^{-1}$. Using (6.11) and Lemma 10.4, we find for $\xi \in \mathbb{R}^2\backslash\mathbb{B}_2(0, R)$:

$$(\tau/2) \cdot \mathcal{F}(R, S)(\Phi)(\xi) \tag{10.21}$$

$$+ \Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R\cdot S)} \sum_{j=1}^{3} \big(K_{jl}^{(9)}(\varphi, \lambda, \delta)(\xi, \eta) + K_{jl}^{(10)}(\varphi, \lambda, \delta)(\xi, \eta) \big) \cdot \Phi_j(\eta) \cdot \sin^{-1}(\varphi) \, d\eta \Big)_{1\le l\le 3}$$

$$= J(\tau, p, \lambda, \varphi, R, S)(\Phi)(\xi) - \Big(\sum_{v=1}^{8} \int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R\cdot S)} \sum_{j=1}^{3} K_{jl}^{(v)}(\varphi, \lambda, \delta)(\xi, \eta) \cdot \Phi_j(\eta) \cdot \sin^{-1}(\varphi) \, d\eta \Big)_{1\le l\le 3}.$$

Let us write $F_1(\xi)$ for the right-hand side of (10.21), with $\xi \in \mathbb{R}^2\backslash\mathbb{B}_2(0, R)$. Then it holds for such vectors ξ:

$$(\tau/2) \cdot \chi_{A(0,3\cdot\delta)}(\xi) \cdot \mathcal{F}(R, S)(\Phi)(\xi) \tag{10.22}$$

$$+ \Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R\cdot S)} \sum_{j=1}^{3} \big(K_{jl}^{(9)}(\varphi, \lambda, \delta)(\xi, \eta) + K_{jl}^{(10)}(\varphi, \lambda, \delta)(\xi, \eta) \big) \cdot \Phi_j(\eta) \cdot \sin^{-1}(\varphi) \, d\eta \Big)_{1\le l\le 3}$$

$$= \chi_{A(0,3\cdot\delta)}(\xi) \cdot F_1(\xi)$$

$$+ \chi_{\mathbb{R}^2\backslash A(0,3\cdot\delta)}(\xi) \cdot \Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R\cdot S)} \sum_{j=1}^{3} \big(K_{jl}^{(9)}(\varphi, \lambda, \delta)(\xi, \eta) + K_{jl}^{(10)}(\varphi, \lambda, \delta)(\xi, \eta) \big) \cdot \Phi_j(\eta) \cdot \sin^{-1}(\varphi) \, d\eta \Big)_{1\le l\le 3}.$$

After abbreviating the right-hand side of (10.22) by $F_2(\xi)$, for $\xi \in \mathbb{R}^2\backslash\mathbb{B}_2(0, R)$, we obtain for any $\xi \in \mathbb{R}^2$:

$$(\tau/2) \cdot \tilde{\mathcal{F}}(R)\big(\chi_{A(0,3\cdot\delta)} \cdot \mathcal{F}(R, S)(\Phi)\big)(\xi) \tag{10.23}$$

$$+ \Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R\cdot S)} \sum_{j=1}^{3} \big(K_{jl}^{(9)}(\varphi, \lambda, \delta)(\xi, \eta) + K_{jl}^{(10)}(\varphi, \lambda, \delta)(\xi, \eta) \big) \cdot \Phi_j(\eta) \cdot \sin^{-1}(\varphi) \, d\eta \Big)_{1\le l\le 3}$$

$$= \tilde{\mathcal{F}}(R)(F_2)(\xi)$$

$$+ \chi_{\mathbb{B}_2(0,R)}(\xi) \cdot \Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R\cdot S)} \sum_{j=1}^{3} \big(K_{jl}^{(9)}(\varphi, \lambda, \delta)(\xi, \eta) + K_{jl}^{(10)}(\varphi, \lambda, \delta)(\xi, \eta) \big) \cdot \Phi_j(\eta) \cdot \sin^{-1}(\varphi) \, d\eta \Big)_{1\le l\le 3}.$$

On the other hand, observe that

$$\tilde{\mathcal{F}}(R)\big(\chi_{A(0,3\cdot\delta)} \cdot \mathcal{F}(R, S)(\Phi)\big) = \tilde{\mathcal{F}}(R\cdot S)(\chi_{A(0,3\cdot\delta)} \cdot \Phi).$$

Thus, recalling Lemma 10.5, and denoting the right-hand side of (10.23) by $F_3(\xi)$ ($\xi \in \mathbb{R}^2$), we arrive at the equation

$$(\tau/2) \cdot \tilde{\mathcal{F}}(R\cdot S)(\chi_{A(0,3\cdot\delta)} \cdot \Phi) + (\mathcal{M}(\varphi, \lambda) * \tilde{\Phi}) \circ T(\varphi) = F_3,$$

which, in turn, leads to the following integral equation in $\mathbb{R}^2$:

$$(\tau/2) \cdot \tilde{\Phi} + \mathcal{M}(\varphi, \lambda) * \tilde{\Phi} = F_3 \circ (T(\varphi))^{-1}. \tag{10.24}$$

Comparing the function $F_3 \circ (T(\varphi))^{-1}$ with the notations from Definition 10.3, we see that

$$F_3 \circ (T(\varphi))^{-1} = \sum_{v=1,2,3,\cdots,8,11,12} \mathcal{A}^{(v)}(\varphi, \lambda, \delta, R, S, \Phi) + \tilde{\mathcal{F}}(R)\big(\chi_{A(0,3\cdot\delta)} \cdot J(\tau, p, \lambda, \varphi, R, S)(\Phi)\big) \circ (T(\varphi))^{-1}.$$

After applying the Fourier transform to both sides of (10.24), we conclude by Lemma 5.13:

$$A_\tau^{\lambda,\varphi} \cdot \overset{\wedge}{\tilde{\Phi}} = \Big(\sum_{v=1,2,3,\cdots,8,11,12} \mathcal{A}^{(v)}(\varphi, \lambda, \delta, R, S, \Phi) \Big)^\wedge \tag{10.25}$$

$$+ \Big(\tilde{\mathcal{F}}(R)\big(\chi_{A(0,3\cdot\delta)} \cdot J(\tau, p, \lambda, \varphi, R, S)(\Phi)\big) \circ (T(\varphi))^{-1} \Big)^\wedge.$$

We remark that due to the assumption $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R))^3 \cap L^2(\mathbb{R}^2\backslash\mathbb{B}_2(0, R))^3$, we may in fact recur to the L^2-Fourier transform in the present situation, so the use of the symbol "$\wedge$" (see Chapter 2) is justified.

Denote the right-hand of (10.25) by F_4. Since the matrix $A_\tau^{\lambda,\varphi}$ is invertible (Lemma 5.17), it follows

$$A_\tau^{\infty,\varphi} \cdot \overset{\wedge}{\widetilde{\Phi}} \;=\; A_\tau^{\infty,\varphi} \cdot \left(A_\tau^{\lambda,\varphi} \right)^{-1} \cdot F_4 . \tag{10.26}$$

Now we apply the inverse Fourier transform to both sides of (10.26). Then, recalling Lemma 5.6 and 10.3, we obtain

$$(\tau/2) \cdot \widetilde{\Phi} \;+\; \mathcal{M}(\varphi, \infty) *_p \widetilde{\Phi} \;=\; \left(A_\tau^{\infty,\varphi} \cdot \left(A_\tau^{\lambda,\varphi} \right)^{-1} \cdot F_4 \right)^{\vee} . \tag{10.27}$$

We write F_5 for the right-hand side of (10.27). Then we get for $\xi \in \mathbb{R}^2 \backslash \mathbb{B}_2(0, R)$, after inserting the definition of $\widetilde{\Phi}$ into the first summand on the left-hand side of (10.27):

$$(\tau/2) \cdot \chi_{A(0,3\cdot\delta)}(\xi) \cdot \mathcal{F}(R, S)(\Phi)(\xi) \;+\; \left(\mathcal{M}(\varphi, \infty) *_p \widetilde{\Phi} \right)\left(T(\varphi)(\xi) \right)$$
$$= \left(F_5 \circ T(\varphi) \right)(\xi) .$$

Now Lemma 10.6 implies for $\xi \in A(0, 3\cdot\delta) \setminus \mathbb{B}_2(0, R)$:

$$(\tau/2) \cdot \mathcal{F}(R, S)(\Phi)(\xi) \;+\; \sin^{-1}(\varphi) \cdot \Big(\sum_{j=1}^{3} L_{jl}^{(10)}(\varphi, \delta) \otimes_p \Phi_j \Big)_{1 \le l \le 3}(\xi)$$
$$= \left(F_5 \circ T(\varphi) \right)(\xi) ,$$

so that it follows by Lemma 10.4, for $\xi \in A(0, 3\cdot\delta) \setminus \mathbb{B}_2(0, R)$:

$$(\tau/2) \cdot \mathcal{F}(R, S)(\Phi)(\xi) \;+\; \sin^{-1}(\varphi) \cdot \Big(L_l(0, \varphi) \otimes_p \left(\left(n^{(\varphi)} \circ g^{(\varphi)} \right) \cdot \Phi \right) \Big)_{1 \le l \le 3}(\xi)$$
$$= \left(F_5 \circ T(\varphi) \right)(\xi) \;+\; \sum_{v = 2,4,6,8} \sin^{-1}(\varphi) \cdot \Big(\sum_{j=1}^{3} L_{jl}^{(v)}(\varphi, \delta) \otimes_p \Phi_j \Big)_{1 \le l \le 3}(\xi) .$$

This completes the proof of (10.20). Equation (10.19) may be shown by an analogous reasoning.

Chapter 11

L^p-Estimates of the Operator $J(\tau, p, \lambda, \varphi, R, S)$

In the following, we shall estimate the right-hand side of (10.19) and (10.20) in the norm of $L^p(A(0, \epsilon) \backslash \mathbb{B}_2(0, R))^3$ (Corollary 11.6). In this way, we shall obtain a L^p-estimate of $J(\tau, p, \lambda, \varphi, R, S)$ against $H^*(\tau, p, \varphi, R, S)$ and vice versa; see inequalities (11.43) and (11.44) in Theorem 11.1. These results are crucial for our investigation of the operator $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$, as we tried to indicate at the beginning of Chapter 10. Of course, inequalities (11.43) and (11.44) make sense only if we know the parameters which enter into their constants. Therefore we shall carefully trace constants in all the estimates derived in the following.

Our first lemma will imply that certain integral operators become small when the functions belonging to their domain and range, respectively, are defined on sets being far apart.

Lemma 11.1. *Let* $p \in (1, \infty)$, *and put* $C_{33}(p) := 2\cdot\pi \cdot (p-1)^{1-1/p}$. *Then it holds for* $\Phi \in L^p(\mathbb{R}^2)$, $R \in (0, \infty)$, $S \in (2, \infty)$:

$$\left(\int_{\mathbb{B}_2(0,R)} \left(\int_{\mathbb{R}^2 \backslash \mathbb{B}_2(0, R\cdot S)} |\xi - \eta|^{-2} \cdot |\Phi(\eta)| \, d\eta \right)^p d\xi \right)^{1/p}$$
$$\le \; C_{33}(p) \cdot \|\Phi\|_p \cdot (S-1)^{-2/p} .$$

Proof: For shortness, set $q := (1 - 1/p)^{-1}$. Then we find for Φ, R, S as in the lemma:

$$A := \left(\int_{\mathbb{B}_2(0,R)} \left(\int_{\mathbb{R}^2 \backslash \mathbb{B}_2(0, R\cdot S)} |\xi - \eta|^{-2} \cdot |\Phi(\eta)| \, d\eta \right)^p d\xi \right)^{1/p}$$
$$\le \left(\int_{\mathbb{B}_2(0,R)} \left(\int_{\mathbb{R}^2 \backslash \mathbb{B}_2(0, R\cdot S)} |\xi - \eta|^{-2\cdot q} \, d\eta \right)^{p/q} d\xi \right)^{1/p} \cdot \|\Phi\|_p$$

$$\leq \ (2\cdot\pi)^{1/q} \cdot \Big(\int_{\mathbb{B}_2(0,R)} \Big(\int_{S\cdot R}^{\infty} r \cdot (r - |\xi|)^{-2\cdot q} \, dr \Big)^{p/q} \, d\xi \Big)^{1/p} \cdot \|\Phi\|_p$$

$$\leq \ \big(2\cdot\pi \cdot S/(S-1)\big)^{1/q} \cdot \Big(\int_{\mathbb{B}_2(0,R)} \Big(\int_{S\cdot R}^{\infty} (r - |\xi|)^{-2\cdot q+1} \, dr \Big)^{p/q} \, d\xi \Big)^{1/p} \cdot \|\Phi\|_p \, .$$

Concerning the last inequality, we remark that for $r \in (S{\cdot}R, \infty)$, $\xi \in \mathbb{B}_2(0, R)$, it holds

$$r - |\xi| \ = \ (S-1)\cdot r/S + r/S - |\xi| \ \geq \ (S-1)\cdot r/S + R - |\xi|$$

$$\geq \ (S-1)\cdot r/S \, .$$

Since $S \geq 2$, that is, $S/(S-1) \leq 2$, the preceding estimate may be continued as follows:

$$A \ \leq \ (4\cdot\pi)^{1/q} \cdot (2\cdot q - 2)^{-1/q} \cdot \Big(\int_{\mathbb{B}_2(0,R)} (S\cdot R - |\xi|)^{(-2\cdot q+2)\cdot p/q} \, d\xi \Big)^{1/p} \cdot \|\Phi\|_p$$

$$\leq \ \big(2\cdot\pi \cdot (p-1)\big)^{1/q} \cdot \Big(\int_{\mathbb{B}_2(0,R)} (S\cdot R - |\xi|)^{-2} \, d\xi \Big)^{1/p} \cdot \|\Phi\|_p$$

$$\leq \ \big(2\cdot\pi \cdot (p-1)\big)^{1/q} \cdot \big((S-1)\cdot R\big)^{-2/p} \cdot \Big(\int_{\mathbb{B}_2(0,R)} d\xi \Big)^{1/p} \cdot \|\Phi\|_p$$

$$= \ C_{33}(p) \cdot (S-1)^{-2/p} \, ,$$

where the last inequality is based on the fact that $S\cdot R - |\xi| \geq (S-1)\cdot R$ for $\xi \in \mathbb{B}_2(0, R)$.

The ensuing lemma states essentially that the left-hand side in (11.1) and (11.2) becomes small whenever the vertex angle δ of the sector $A(0, \delta)$ is small too. This result will be established by estimating the left-hand side of (11.1) and (11.2) against the function $|G_l(\varphi, \sin^{1/2}(3{\cdot}\delta)) *_p \Phi|$ defined in Theorem 4.3, plus an integral operator having the kernel $|\xi - \eta|^{-1} \cdot (|\xi| + |\eta|)^{-1}$. Then the desired result follows by applying Theorem 4.3 and 4.1.

Lemma 11.2. *Let* $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$. *Then there is a constant* $C_{34}(p, \vartheta, \varphi) > 0$ *such that*

$$\Big\| \sin^{-1}(\varphi) \cdot \chi_{A(0,3\cdot\delta)} \cdot \Big(\sum_{j=1}^{3} K_{jl}^{(v)}(\varphi, \lambda, \delta) \otimes \Phi_j \Big)_{1 \leq l \leq 3} \Big\|_p \tag{11.1}$$

$$\leq \ C_{34}(p, \vartheta, \varphi) \cdot \sin^{1/2}(3\cdot\delta) \cdot \| \chi_{A(0,3\cdot\delta)} \cdot \Phi \|_p$$

for $v \in \{7, 8\}$, $\lambda \in \mathbb{C}\backslash\{0\}$ *with* $|\arg\lambda| \leq \vartheta$, $|\lambda| \geq 1$, $\Phi \in L^p(\mathbb{R}^2)^3$, $\delta \in (0, \pi/6)$.
Moreover, there exists $C_{35}(p, \varphi) > 0$ *so that the ensuing inequality is valid for* $\Phi \in L^p(\mathbb{R}^2)^3$, $\delta \in (0, \pi/6)$:

$$\Big\| \sin^{-1}(\varphi) \cdot \chi_{A(0,3\cdot\delta)} \cdot \Big(\sum_{j=1}^{3} L_{jl}^{(8)}(\varphi, \delta) \otimes_p \Phi_j \Big)_{1 \leq l \leq 3} \Big\|_p \tag{11.2}$$

$$\leq \ C_{35}(p, \varphi) \cdot \sin^{1/2}(3\cdot\delta) \cdot \| \chi_{A(0,3\cdot\delta)} \cdot \Phi \|_p \, .$$

Proof: First we introduce the constants mentioned in the lemma. To this end, we put

$$C_{34,1}(\vartheta, \varphi) := (1 \vee \cot\varphi) \cdot 32 \cdot C_{17}(\vartheta) \cdot \cot^2(\varphi) ; \quad C_{34,2}(p, \vartheta, \varphi) := C_{34,1}(\vartheta, \varphi) \cdot C_4(p) ;$$

$$C_{34}(p, \vartheta, \varphi) := \sin^{-1}(\varphi) \cdot 36 \cdot \Big(C_{34,2}(p, \vartheta, \varphi) + C_{34,1}(\vartheta, \varphi) \cdot 2\cdot\pi + (1 \vee \cot\varphi) \cdot C_7(p, \varphi) \Big) ;$$

$$C_{35,1}(\varphi) := 12 \cdot (1 \vee \cot\varphi) \cdot \pi^{-1} \cdot \cot^2(\varphi) ; \qquad C_{35,2}(p, \varphi) := C_{35,1}(\varphi) \cdot C_4(p) ;$$

$$C_{35}(p, \varphi) := 9 \cdot \sin^{-1}(\varphi) \cdot \Big(C_{35,2}(p, \varphi) + (1 \vee \cot\varphi) \cdot C_7(p, \varphi) \Big) .$$

Let λ, Φ, δ be given as in (11.1). Then we introduce some abbreviations, setting for $j, l \in \{1, 2, 3\}$, $\xi, \eta \in \mathbb{R}^2$ with $\xi \neq \eta$, $\xi \neq 0$:

$$K_{jl}^{(7,1)}(\xi, \eta) := -\Big((1 - \delta_{3j}) \cdot \big\{ \mathcal{Y}_j^{\lambda}\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) - \mathcal{Y}_j^{\lambda}\big(\xi - \eta, \cot\varphi \cdot |\xi|^{-1} \cdot \xi \cdot (\xi - \eta)\big) \big\}$$
$$+ \cot\varphi \cdot \delta_{3j} \cdot \big\{ \mathcal{Y}_1^{\lambda}\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) - \mathcal{Y}_1^{\lambda}\big(\xi - \eta, \cot\varphi \cdot |\xi|^{-1} \cdot \xi \cdot (\xi - \eta)\big) \big\} \Big)$$
$$\cdot (\delta_{1l} \cdot \cos\varphi - \delta_{3l} \cdot \sin\varphi) \cdot \chi_{A(0,3\cdot\delta)}(\eta) ;$$

$$K_{jl}^{(7,2)}(\xi, \eta) := -\Big((1 - \delta_{3j}) \cdot \big\{ \mathcal{Y}_j^{\lambda}\big(\xi - \eta, \cot\varphi \cdot |\xi|^{-1} \cdot \xi \cdot (\xi - \eta)\big) - \mathcal{Y}_j^{\lambda}\big(\xi - \eta, \cot\varphi \cdot (\xi - \eta)_1\big) \big\}$$
$$+ \cot\varphi \cdot \delta_{3j} \cdot \big\{ \mathcal{Y}_1^{\lambda}\big(\xi - \eta, \cot\varphi \cdot |\xi|^{-1} \cdot \xi \cdot (\xi - \eta)\big) - \mathcal{Y}_1^{\lambda}\big(\xi - \eta, \cot\varphi \cdot (\xi - \eta)_1\big) \big\} \Big)$$
$$\cdot (\delta_{1l} \cdot \cos\varphi - \delta_{3l} \cdot \sin\varphi) \cdot \chi_{A(0,3\cdot\delta)}(\eta) ;$$

$$K_{jl}^{(8,1)}(\xi, \eta) := -\Big((1 - \delta_{3l}) \cdot \big\{ \mathcal{X}_l^{\lambda}\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) - \mathcal{X}_l^{\lambda}\big(\xi - \eta, \cot\varphi \cdot |\xi|^{-1} \cdot \xi \cdot (\xi - \eta)\big) \big\}$$
$$+ \cot\varphi \cdot \delta_{3l}$$

$$\cdot \Big\{ \mathcal{X}_1^\lambda\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) - \mathcal{X}_1^\lambda\big(\xi - \eta,\, \cot\varphi \cdot |\xi|^{-1} \cdot \xi \cdot (\xi - \eta)\big) \Big\} \Big)$$

$$\cdot \big(\delta_{1j} \cdot \cos\varphi - \delta_{3j} \cdot \sin\varphi\big) \cdot \chi_{A(0,3\cdot\delta)}(\eta);$$

$$K_{jl}^{(8,2)}(\xi, \eta) := -\Big((1 - \delta_{3l}) \cdot \Big\{ \mathcal{X}_l^\lambda\big(\xi - \eta,\, \cot\varphi \cdot |\xi|^{-1} \cdot \xi \cdot (\xi - \eta)\big) - \mathcal{X}_l^\lambda\big(\xi - \eta,\, \cot\varphi \cdot (\xi - \eta)_1\big) \Big\}$$
$$+ \cot\varphi \cdot \delta_{3l} \cdot \Big\{ \mathcal{X}_1^\lambda\big(\xi - \eta,\, \cot\varphi \cdot |\xi|^{-1} \cdot \xi \cdot (\xi - \eta)\big) - \mathcal{X}_1^\lambda\big(\xi - \eta,\, \cot\varphi \cdot (\xi - \eta)_1\big) \Big\} \Big)$$
$$\cdot \big(\delta_{1j} \cdot \cos\varphi - \delta_{3j} \cdot \sin\varphi\big) \cdot \chi_{A(0,3\cdot\delta)}(\eta).$$

Furthermore, let $L_{jl}^{(8,1)}(\xi, \eta)$, $L_{jl}^{(8,2)}$ be defined in an analogous way as $K_{jl}^{(8,1)}(\xi, \eta)$ and $K_{jl}^{(8,2)}(\xi, \eta)$, respectively, the only difference being that the function $\mathcal{X}_1^\lambda$ is replaced by $\mathcal{X}_1^\infty$, and $\mathcal{X}_2^\lambda$ by $\mathcal{X}_2^\infty$.

For $v \in \{7, 8\}$, we have

$$\Big\| \sin^{-1}(\varphi) \cdot \chi_{A(0,3\cdot\delta)} \cdot \Big(\sum_{j=1}^{3} K_{jl}^{(v)}(\varphi, \lambda, \delta) \otimes \Phi_j \Big)_{1 \le l \le 3} \Big\|_p \tag{11.3}$$
$$\le \sin^{-1}(\varphi) \cdot \sum_{j,l=1}^{3} \Big\| \chi_{A(0,3\cdot\delta)} \cdot \big(K_{jl}^{(v)}(\varphi, \lambda, \delta) \otimes \Phi_j \big) \Big\|_p$$
$$\le \sin^{-1}(\varphi) \cdot \sum_{j,l=1}^{3} \sum_{\sigma=1}^{4} F_{j,l,\sigma}^{(v)},$$

where we used the ensuing abbreviations, for $j, l \in \{1, 2, 3\}$, $v \in \{7, 8\}$:

$$F_{j,l,1}^{(v)} := \Big(\int_{A(0,3\cdot\delta)} \Big(\int_{A(0,3\cdot\delta)} \big| K_{jl}^{(v,1)}(\xi, \eta) \cdot \Phi_j(\eta) \big| \, d\eta \Big)^p \, d\xi \Big)^{1/p};$$

$$F_{j,l,2}^{(v)} := \Big(\int_{A(0,3\cdot\delta)} \Big(\int_{A(0,3\cdot\delta)} \chi_{(0,|\lambda|^{-1/2})}(|\xi - \eta|) \cdot \big| K_{jl}^{(v,2)}(\xi, \eta) \cdot \Phi_j(\eta) \big| \, d\eta \Big)^p \, d\xi \Big)^{1/p};$$

$$F_{j,l,3}^{(7)} := \Big(\int_{A(0,3\cdot\delta)} \Big(\int_{A(0,3\cdot\delta)} \chi_{(|\lambda|^{-1/2},\infty)}(|\xi - \eta|) \cdot \big| K_{jl}^{(7,2)}(\xi, \eta) \cdot \Phi_j(\eta) \big| \, d\eta \Big)^p \, d\xi \Big)^{1/p};$$

$$F_{j,l,3}^{(8)} := \Big(\int_{A(0,3\cdot\delta)} \Big(\int_{A(0,3\cdot\delta)} \chi_{(|\lambda|^{-1/2},\infty)}(|\xi - \eta|) \cdot \big| \big(K_{jl}^{(8,2)}(\xi, \eta) - L_{jl}^{(8,2)}(\xi, \eta) \big) \cdot \Phi_j(\eta) \big| \, d\eta \Big)^p \, d\xi \Big)^{1/p};$$

$$F_{j,l,4}^{(7)} := 0;$$

$$F_{j,l,4}^{(8)} := \Big(\int_{A(0,3\cdot\delta)} \Big| \int_{A(0,3\cdot\delta)} \chi_{(|\lambda|^{-1/2},\infty)}(\xi, \eta) \cdot L_{jl}^{(8,2)}(\xi, \eta) \cdot \Phi_j(\eta) \, d\eta \Big|^p \, d\xi \Big)^{1/p}.$$

Now we are going to evaluate the preceding terms, beginning with $F_{j,l,1}^{(7)}$ and $F_{j,l,1}^{(8)}$. In fact, it holds for $j, l \in \{1, 2, 3\}$:

$$F_{j,l,1}^{(7)} \le (1 \vee \cot\varphi) \cdot \sum_{m=1}^{2} \Big(\int_{A(0,3\cdot\delta)} \Big(\int_{A(0,3\cdot\delta)} \Big| \mathcal{Y}_m^\lambda\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) - \mathcal{Y}_m^\lambda\big(\xi - \eta,\, \cot\varphi \cdot |\xi|^{-1} \cdot \xi \cdot (\xi - \eta)\big) \Big| \cdot |\Phi_j(\eta)| \, d\eta \Big)^p \, d\xi \Big)^{1/p};$$

$$F_{j,l,1}^{(8)} \le (1 \vee \cot\varphi) \cdot \sum_{m=1}^{2} \Big(\int_{A(0,3\cdot\delta)} \Big(\int_{A(0,3\cdot\delta)} \Big| \mathcal{X}_m^\lambda\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) - \mathcal{X}_m^\lambda\big(\xi - \eta,\, \cot\varphi \cdot |\xi|^{-1} \cdot \xi \cdot (\xi - \eta)\big) \Big| \cdot |\Phi_j(\eta)| \, d\eta \Big)^p \, d\xi \Big)^{1/p}.$$

Set $k_7 := 1$, $k_8 := 2$. Then it follows by Lemma 5.7, for $j, l \in \{1, 2, 3\}$, $v \in \{7, 8\}$:

$$F_{j,l,1}^{(v)} \le (1 \vee \cot\varphi) \cdot \sum_{m=1}^{2} \Big(\int_{A(0,3\cdot\delta)} \Big(\int_{A(0,3\cdot\delta)} \Big| (\xi - \eta)_m \cdot \Big[\mathcal{G}_{k_v}^\lambda\Big(|\xi - \eta|^2 + \cot^2(\varphi) \cdot \big(|\xi| - |\eta|\big)^2 \Big) - \mathcal{G}_{k_v}^\lambda\Big(|\xi - \eta|^2 + \cot^2(\varphi) \cdot \big(|\xi|^{-1} \cdot \xi \cdot (\xi - \eta)\big)^2 \Big) \Big] \cdot \Phi_j(\eta) \Big| \, d\eta \Big)^p \, d\xi \Big)^{1/p} \tag{11.4}$$
$$\le 2 \cdot (1 \vee \cot\varphi) \cdot \Big(\int_{A(0,3\cdot\delta)} \Big(\int_{A(0,3\cdot\delta)} |\xi - \eta| \cdot \Big| \int_0^1 (\mathcal{G}_{k_v}^\lambda)'\big(|\xi - \eta|^2 + \varrho(\xi, \eta, t)\big) \, dt \Big| \cdot |\Phi(\eta)| \cdot \cot^2(\varphi) \cdot \Big| \big(|\xi| - |\eta|\big)^2 - \big(|\xi|^{-1} \cdot \xi \cdot (\xi - \eta)\big)^2 \Big| \, d\eta \Big) \, d\xi \Big)^{1/p}$$

where we used the ensuing notations, for $t \in [0, 1]$, $\xi, \eta \in \mathbb{R}^2$ with $\xi \ne 0$:

$$\varrho(\xi, \eta, t) := \cot^2(\varphi) \cdot \Big(t \cdot \big(|\xi| - |\eta|\big)^2 + (1 - t) \cdot \big(|\xi|^{-1} \cdot \xi \cdot (\xi - \eta)\big)^2 \Big).$$

The last inequality in (11.4) is a consequence of the mean value theorem, which may be applied here since $|\xi - \eta|^2 + \varrho(\xi, \eta, t) \ge |\xi - \eta|^2 > 0$ for any $t \in [0, 1]$, $\xi, \eta \in \mathbb{R}^2$ with $\xi \ne 0$, $\xi \ne \eta$.

Next we refer to (5.17) and Lemma 4.3 in order to deduce from (11.4), for $v \in \{7, 8\}$, $j, l \in \{1, 2, 3\}$:

$$F_{j,l,1}^{(v)} \le C_{34,1}(\boldsymbol{\vartheta}, \varphi) \cdot \Big(\int_{A(0,3\cdot\delta)} \Big(\int_{A(0,3\cdot\delta)} |\xi - \eta|^{-1} \cdot \big(|\xi| + |\eta|\big)^{-1} \cdot |\Phi(\eta)| \, d\eta \Big)^p \, d\eta \Big)^{1/p}.$$

Now Theorem 4.1 yields for v, j, l as before:

$$F_{j,l,1}^{(v)} \le C_{34,2}(p, \vartheta, \varphi) \cdot \sin^{1/2}(3\cdot\delta) \cdot \| \chi_{A(0,3\cdot\delta)} \cdot \Phi \|_p. \tag{11.5}$$

If $t \in [0,1]$, $\xi, \eta \in \mathbb{R}^2$, we set

$$\tilde{\varrho}(\xi,\eta,t) := \cot^2(\varphi)\cdot\Big(t\cdot\big(|\xi|^{-1}\cdot\xi\cdot(\xi-\eta)\big)^2 + (1-t)\cdot(\xi-\eta)_1^2\Big).$$

Then we have for $j, l \in \{1,2,3\}$, $v \in \{7,8\}$:

$$F^{(v)}_{j,l,2} \le (1\vee\cot\varphi)\cdot\sum_{m=1}^{2}\Big(\int_{A(0,3\cdot\delta)}\Big(\int_{A(0,3\cdot\delta)}\chi_{(0,|\lambda|^{-1/2})}(|\xi-\eta|)\cdot\Big|(\xi-\eta)_m$$
$$\cdot\Big(\int_0^1(\mathcal{G}^\lambda_{k_v})'(|\xi-\eta|^2+\tilde{\varrho}(\xi,\eta,t))\,dt\Big)\cdot\Phi_j(\eta)\cdot\cot^2(\varphi)$$
$$\cdot\Big(\big(|\xi|^{-1}\cdot\xi\cdot(\xi-\eta)\big)^2-(\xi-\eta)_1^2\Big)\Big|\,d\eta\Big)^p\,d\xi\Big)^{1/p},$$

where we applied Lemma 5.7 and the mean value theorem. The use of the mean value theorem is justified because $|\xi-\eta|^2+\tilde{\varrho}(\xi,\eta,t) \ge |\xi-\eta|^2 > 0$ for $\xi, \eta \in \mathbb{R}^2$ with $\xi \ne \eta$, $t \in [0,1]$. Now it follows by (5.18) and Lemma 4.4, for $j, l \in \{1,2,3\}$, $v \in \{7,8\}$:

$$F^{(v)}_{j,l,2} \le C_{34,1}(\vartheta,\varphi)\cdot|\lambda|\cdot\Big(\int_{A(0,3\cdot\delta)}|\xi_2/|\xi||^p$$
$$\cdot\Big(\int_{A(0,3\cdot\delta)}\chi_{(0,|\lambda|^{-1/2})}(|\xi-\eta|)\cdot|\Phi(\eta)|\,d\eta\Big)^p\,d\xi\Big)^{1/p}$$
$$\le C_{34,1}(\vartheta,\varphi)\cdot|\lambda|\cdot\sin(3\cdot\delta)$$
$$\cdot\Big(\int_{A(0,3\cdot\delta)}\Big(\int_{A(0,3\cdot\delta)}\chi_{(0,|\lambda|^{-1/2})}(|\xi-\eta|)\cdot|\Phi(\eta)|\,d\eta\Big)^p\,d\xi\Big)^{1/p}.$$

Recurring to Young's inequality (Lemma 4.9), we get for j, v as before:

$$F^{(v)}_{j,l,2} \le C_{34,1}(\vartheta,\varphi)\cdot|\lambda|\cdot\sin(3\cdot\delta)\cdot\|\chi_{A(0,3\cdot\delta)}\cdot\Phi\|_p\cdot\int_{\mathbb{R}^2}\chi_{(0,|\lambda|^{-1/2})}(|\sigma|)\,d\sigma \tag{11.6}$$
$$= C_{34,1}(\vartheta,\varphi)\cdot\pi\cdot\sin(3\cdot\delta)\cdot\|\chi_{A(0,3\cdot\delta)}\cdot\Phi\|_p.$$

In order to estimate $F^{(v)}_{j,l,3}$ for $v \in \{7,8\}$, $j, l \in \{1,2,3\}$, we may proceed in an analogous way. In fact, referring to (5.16) instead of (5.18), we obtain for $v \in \{7,8\}$, $j, l \in \{1,2,3\}$, with $\mathcal{G}^\infty_1(r) := 0$ for $r \in (0,\infty)$:

$$F^{(v)}_{j,l,3} \le (1\vee\cot\varphi)\cdot\Big(\int_{A(0,3\cdot\delta)}\Big(\int_{A(0,3\cdot\delta)}\chi_{(|\lambda|^{-1/2},\infty)}(|\xi-\eta|) \tag{11.7}$$
$$\cdot\Big|(\xi-\eta)_m\cdot\int_0^1(\mathcal{G}^\lambda_{k_v}-\mathcal{G}^\infty_{k_v})'(|\xi-\eta|^2+\tilde{\varrho}(\xi,\eta,t))\,dt$$
$$\cdot\Phi_j(\eta)\cdot\cot^2(\varphi)\cdot\Big(\big(|\xi|^{-1}\cdot\xi\cdot(\xi-\eta)\big)^2-(\xi-\eta)_1^2\Big)\Big|\,d\eta\Big)^p\,d\xi\Big)^{1/p}$$
$$\le C_{34,1}(\vartheta,\varphi)\cdot|\lambda|^{-1}\cdot\Big(\int_{A(0,3\cdot\delta)}|\xi_2/|\xi||^p$$

$$\cdot\Big(\int_{A(0,3\cdot\delta)}\chi_{(|\lambda|^{-1/2},\infty)}(|\xi-\eta|)\cdot|\xi-\eta|^{-4}\cdot|\Phi(\eta)|\,d\eta\Big)^p\,d\xi\Big)^{1/p}$$
$$\le C_{34,1}(\vartheta,\varphi)\cdot|\lambda|^{-1}\cdot\sin(3\cdot\delta)\cdot\|\chi_{A(0,3\cdot\delta)}\cdot\Phi\|_p\cdot\int_{\mathbb{R}^2}\chi_{(|\lambda|^{-1/2},\infty)}(|\sigma|)\cdot|\sigma|^{-4}\,d\sigma$$
$$\le C_{34,1}(\vartheta,\varphi)\cdot\pi\cdot\sin(3\cdot\delta)\cdot\|\chi_{A(0,3\cdot\delta)}\cdot\Phi\|_p.$$

Concerning the term $F^{(8)}_{j,l,4}$, we may recur to Theorem 4.3, which yields for $j, l \in \{1,2,3\}$ (see (4.15)):

$$F^{(8)}_{j,l,4} \le (1\vee\cot\varphi)\cdot C_7(p,\varphi)\cdot\sin(3\cdot\delta)\cdot\|\chi_{A(0,3\cdot\delta)}\cdot\Phi\|_p. \tag{11.8}$$

Collecting our results, we see that (11.1) follows by combining (11.3) and (11.5) – (11.8) with the definition of $C_{34}(p,\vartheta,\varphi)$.

Proceeding in an analogous way as in (11.4), we find for $\epsilon \in (0,\infty)$, $j, l \in \{1,2,3\}$:

$$\Big(\int_{A(0,3\cdot\delta)}\Big(\int_{\mathbb{R}^2}\chi_{(\epsilon,\infty)}(|\xi-\eta|)\cdot\Big|L^{(8,1)}_{jl}(\xi,\eta)\cdot\Phi_j(\eta)\Big|\,d\eta\Big)^p\,d\xi\Big)^{1/p}$$
$$\le (1\vee\cot\varphi)\cdot(4\cdot\pi)^{-1}\cdot\sum_{m=1}^{2}\Big(\int_{A(0,3\cdot\delta)}\Big(\int_{A(0,3\cdot\delta)}\Big|(\xi-\eta)_m$$
$$\cdot\Big[\Big(|\xi-\eta|^2+\cot^2(\varphi)\cdot(|\xi|-|\eta|)^2\Big)^{-3/2}$$
$$-\Big(|\xi-\eta|^2+\cot^2(\varphi)\cdot\big(|\xi|^{-1}\cdot\xi\cdot(\xi-\eta)\big)^2\Big)^{-3/2}\Big]\cdot\Phi_j(\eta)\Big|\,d\eta\Big)^p\,d\xi\Big)^{1/p}$$
$$\le (1\vee\cot\varphi)\cdot(2\cdot\pi)^{-1}\cdot(3/2)\cdot\Big(\int_{A(0,3\cdot\delta)}\Big(\int_{A(0,3\cdot\delta)}|\xi-\eta|$$
$$\cdot\Big|\int_0^1\big(|\xi-\eta|^2+\varrho(\xi,\eta,t)\big)^{-5/2}\,dt\Big|\cdot\cot^2(\varphi)$$
$$\cdot\Big|(|\xi|-|\eta|)^2-\big(|\xi|^{-1}\cdot\xi\cdot(\xi-\eta)\big)^2\Big|\cdot|\Phi(\eta)|\,d\eta\Big)^p\,d\xi\Big)^{1/p}$$
$$\le C_{35,1}(\varphi)\cdot\Big(\int_{A(0,3\cdot\delta)}\Big(\int_{A(0,3\cdot\delta)}|\xi-\eta|^{-1}\cdot(|\xi|+|\eta|)^{-1}\cdot|\Phi(\eta)|\,d\eta\Big)^p\,d\xi\Big)^{1/p}$$
$$\le C_{35,2}(p,\varphi)\cdot\sin^{1/2}(3\cdot\delta)\cdot\|\chi_{A(0,3\cdot\delta)}\cdot\Phi\|_p,$$

where we again used the mean value theorem, Lemma 4.3 and Theorem 4.1. It follows

$$\Big\|\sin^{-1}(\varphi)\cdot\chi_{A(0,3\cdot\delta)}\cdot\Big(\sum_{j=1}^{3}L^{(8,1)}_{jl}\otimes\Phi_j\Big)_{1\le l\le 3}\Big\|_p \tag{11.9}$$
$$\le 9\cdot\sin^{-1}(\varphi)\cdot C_{35,2}(p,\varphi)\cdot\sin^{1/2}(3\cdot\delta)\cdot\|\chi_{A(0,3\cdot\delta)}\cdot\Phi\|_p.$$

According to Theorem 4.3, the function $L_{jl}^{(8,2)} *_p \Phi_j$ is well defined for $j, l \in \{1, 2, 3\}$, and satisfies the inequality

$$\| \sin^{-1}(\varphi) \cdot \chi_{A(0,3\cdot\delta)} \cdot \big(L_{jl}^{(8,2)} *_p \Phi_j \big)_{1 \le l \le 3} \|_p \tag{11.10}$$

$$\le \sin^{-1}(\varphi) \cdot 9 \cdot (1 \vee \cot\varphi) \cdot \sin(3\cdot\delta) \cdot C_7(p, \varphi) \cdot \| \chi_{A(0,3\cdot\delta)} \cdot \Phi \|_p .$$

But we have $L_{jl}^{(8)}(\varphi, \delta) = L_{jl}^{(8,1)} + L_{jl}^{(8,2)}$ for $j, l \in \{1, 2, 3\}$, so that we may deduce (11.2) from (11.9), (11.10) and the definition of $C_{35}(p, \varphi)$.

Corollary 11.1. *Take* $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$. *Then there are constants* $C_{36}(p, \vartheta, \varphi)$, $C_{37}(p, \varphi) > 0$, *so that*

$$\| \mathcal{A}^{(v)}(\varphi, \lambda, \delta, R, S, \Phi) \|_p \le C_{36}(p, \vartheta, \varphi) \cdot \sin^{1/2}(3\cdot\delta) \cdot \| \chi_{A(0,3\cdot\delta)} \cdot \Phi \|_p ,$$

$$\| \mathcal{B}^{(8)}(\varphi, \delta, R, S, \Phi) \|_p \le C_{37}(p, \varphi) \cdot \sin^{1/2}(3\cdot\delta) \cdot \| \chi_{A(0,3\cdot\delta)} \cdot \Phi \|_p$$

for $v \in \{7, 8\}$, $\lambda \in \mathbb{C}$ *with* $|\arg \lambda| \le \vartheta$, $|\lambda| \ge 1$, $R \in (0, \infty)$, $S \in (1, \infty)$, $\Phi \in L^p\big(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R\cdot S)\big)^3$, $\delta \in (0, \pi/48)$.

Proof: The corollary readily follows from Definition 10.3 and Lemma 11.2.

Next we estimate some integral operators having kernels which may be written as a product of a standard kernel, already considered in Chapter 4 or 5, times a factor which yields the term $\sin^{1/2}(3\cdot\delta)$ on the right-hand side of (11.11) and (11.12).

Lemma 11.3. *Let* $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$. *Then there is some number* $C_{38}(p, \vartheta, \varphi) > 0$ *with*

$$\Big\| \sin^{-1}(\varphi) \cdot \chi_{A(0,3\cdot\delta)} \cdot \Big(\sum_{j=1}^{3} K_{jl}^{(v)}(\varphi, \lambda, \delta) \otimes \Phi_j \Big)_{1 \le l \le 3} \Big\|_p \tag{11.11}$$

$$\le C_{38}(p, \vartheta, \varphi) \cdot \sin^{1/2}(3\cdot\delta) \cdot \| \chi_{A(0,3\cdot\delta)} \cdot \Phi \|_p$$

for $v \in \{3, 4, 5, 6\}$, $\lambda \in \mathbb{C} \backslash \{0\}$ *with* $|\arg \lambda| \le \vartheta$, $\Phi \in L^p(\mathbb{R}^2)^3$, $\delta \in (0, \pi/48)$.
Furthermore, there exists $C_{39}(p, \varphi) > 0$ *such that*

$$\Big\| \sin^{-1}(\varphi) \cdot \chi_{A(0,3\cdot\delta)} \cdot \Big(\sum_{j=1}^{3} L_{jl}^{(v)}(\varphi, \delta) \otimes_p \Phi_j \Big)_{1 \le l \le 3} \Big\|_p \tag{11.12}$$

$$\le C_{39}(p, \varphi) \cdot \sin^{1/2}(3\cdot\delta) \cdot \| \chi_{A(0,3\cdot\delta)} \cdot \Phi \|_p$$

for $v \in \{4, 6\}$, $\Phi \in L^p(\mathbb{R}^2)^3$, $\delta \in (0, \pi/48)$.

Proof: Put

$$C_{38,1} := 1 + \sum_{n=1}^{\infty} \left| \binom{1/2}{n} \right| \cdot 2^{-2\cdot n+1}; \qquad C_{38,2}(\varphi) := C_{38,1} \cdot (1 + \cot\varphi) \cdot 2;$$

$$C_{38,3}(p, \vartheta, \varphi) := C_{19}(p, \vartheta, \varphi) \cdot C_{38,2}(\varphi); \quad C_{38,4}(p, \vartheta, \varphi) := 2 \cdot \cot\varphi \cdot C_{17}(\vartheta) \cdot C_4(p);$$

$$C_{38}(p, \vartheta, \varphi) := \sin^{-1}(\varphi) \cdot 18 \cdot \big(C_{38,3}(p, \vartheta, \varphi) + C_{38,4}(p, \vartheta, \varphi) \big);$$

$$C_{39,1}(p, \varphi) := (2\cdot\pi)^{-1} \cdot \cot\varphi \cdot C_4(p);$$

$$C_{39}(p, \varphi) := \sin^{-1}(\varphi) \cdot 18 \cdot \big(C_9(p, \varphi) \cdot C_{38,2}(\varphi) + C_{39,1}(p, \varphi) \big).$$

Then assume λ, Φ, δ are given as in the lemma, and take $j, l \in \{1, 2, 3\}$. For $\mu \in \{1, 2\}$, $\eta \in \mathbb{R}^2 \backslash \{0\}$, $\xi \in \mathbb{R}^2$ with $\xi \neq \eta$, we abbreviate

$$f_\mu^{(3)}(\eta) := -\Big(\delta_{\mu 2} \cdot \eta_2/|\eta| + \delta_{\mu 1} \cdot \sum_{n=1}^{\infty} \binom{1/2}{n} \cdot (-1)^n \cdot \eta_2^{2\cdot n} \cdot |\eta|^{-2\cdot n} \Big)$$
$$\cdot \cot\varphi \cdot \delta_{3j} \cdot \big(n_l^{(\varphi)} \circ g^{(\varphi)} \big)(\eta) \cdot \chi_{A(0,3\cdot\delta)}(\eta) \cdot \Phi_j(\eta);$$

$$f_\mu^{(4)}(\eta) := -\Big(\delta_{\mu 2} \cdot \eta_2/|\eta| + \delta_{\mu 1} \cdot \sum_{n=1}^{\infty} \binom{1/2}{n} \cdot (-1)^n \cdot \eta_2^{2\cdot n} \cdot |\eta|^{-2\cdot n} \Big)$$
$$\cdot \cot\varphi \cdot \delta_{3l} \cdot \big(n_j^{(\varphi)} \circ g^{(\varphi)} \big)(\eta) \cdot \chi_{A(0,3\cdot\delta)}(\eta) \cdot \Phi_j(\eta);$$

$$f_\mu^{(5)}(\eta) := -\Big(\delta_{\mu j} \cdot (1 - \delta_{3j}) + \delta_{\mu 1} \cdot \delta_{3j} \cdot \cot\varphi \Big) \cdot \chi_{A(0,3\cdot\delta)}(\eta) \cdot \Phi_j(\eta)$$
$$\cdot \Big(\delta_{1l} \cdot \cos\varphi \cdot \sum_{n=1}^{\infty} \binom{1/2}{n} \cdot (-1)^n \cdot \eta_2^{2\cdot n} \cdot |\eta|^{-2\cdot n} + \delta_{2l} \cdot \big(n_2^{(\varphi)} \circ g^{(\varphi)} \big)(\eta) \Big);$$

$$f_\mu^{(6)}(\eta) := -\Big(\delta_{\mu l} \cdot (1 - \delta_{3l}) + \delta_{\mu 1} \cdot \delta_{3l} \cdot \cot\varphi \Big) \cdot \chi_{A(0,3\cdot\delta)}(\eta) \cdot \Phi_j(\eta)$$
$$\cdot \Big(\delta_{1j} \cdot \cos\varphi \cdot \sum_{n=1}^{\infty} \binom{1/2}{n} \cdot (-1)^n \cdot \eta_2^{2\cdot n} \cdot |\eta|^{-2\cdot n} + \delta_{2j} \cdot \big(n_2^{(\varphi)} \circ g^{(\varphi)} \big)(\eta) \Big);$$

$$g_\mu^{(3)}(\xi, \eta) := -\Big((\xi + \eta)_\mu \cdot (|\xi| + |\eta|)^{-1} - \eta_\mu \cdot |\eta|^{-1} \Big)$$
$$\cdot \cot\varphi \cdot \delta_{j3} \cdot \big(n_l^{(\varphi)} \circ g^{(\varphi)} \big)(\eta) \cdot \chi_{A(0,3\cdot\delta)}(\eta) \cdot \Phi_j(\eta);$$

$$g_\mu^{(4)}(\xi, \eta) := -\Big((\xi + \eta)_\mu \cdot (|\xi| + |\eta|)^{-1} - \eta_\mu \cdot |\eta|^{-1} \Big)$$
$$\cdot \cot\varphi \cdot \delta_{l3} \cdot \big(n_j^{(\varphi)} \circ g^{(\varphi)} \big)(\eta) \cdot \chi_{A(0,3\cdot\delta)}(\eta) \cdot \Phi_j(\eta).$$

Then it holds for $\xi \in \mathbb{R}^2$:

$$(K_{jl}^{(3)}(\varphi, \lambda, \delta) \otimes \Phi_j)(\xi) = \sum_{\mu=1}^{2} \int_{\mathbb{R}^2} \mathcal{Y}_\mu^\lambda(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \cdot (g_\mu^{(3)}(\xi, \eta) + f_\mu^{(3)}(\eta))\, d\eta; \tag{11.13}$$

$$(K_{jl}^{(4)}(\varphi, \lambda, \delta) \otimes \Phi_j)(\xi) = \sum_{\mu=1}^{2} \int_{\mathbb{R}^2} \mathcal{X}_\mu^\lambda(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \cdot (g_\mu^{(4)}(\xi, \eta) + f_\mu^{(4)}(\eta))\, d\eta; \tag{11.14}$$

$$(K_{jl}^{(5)}(\varphi, \lambda, \delta) \otimes \Phi_j)(\xi) = \sum_{\mu=1}^{2} \int_{\mathbb{R}^2} \mathcal{Y}_\mu^\lambda(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \cdot f_\mu^{(5)}(\eta)\, d\eta; \tag{11.15}$$

$$(K_{jl}^{(6)}(\varphi, \lambda, \delta) \otimes \Phi_j)(\xi) = \sum_{\mu=1}^{2} \int_{\mathbb{R}^2} \mathcal{X}_\mu^\lambda(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \cdot f_\mu^{(6)}(\eta)\, d\eta. \tag{11.16}$$

Since $\eta \in A(0, 3\cdot\delta)$, we have $|\eta_2/|\eta|| \le \sin(3\cdot\delta)$. But we assumed $\delta \le \pi/48$, so it holds $|\eta_2/|\eta|| \le 1/2$. Thus it follows for $\eta \in A(0, 3\cdot\delta)$ (see (3.2)):

$$\Big|\sum_{n=1}^{\infty} \binom{1/2}{n} \cdot (-1)^n \cdot \eta_2^{2\cdot n} \cdot |\eta|^{-2\cdot n}\Big| \le C_{38,1} \cdot \sin(3\cdot\delta);$$

$$|(n_2^{(\varphi)} \circ g^{(\varphi)})(\eta)| = |\cos\varphi \cdot \eta_2/|\eta|| \le \sin(3\cdot\delta).$$

Hence, if $\eta \in A(0, 3\cdot\delta)$, we find

$$|\eta_2/|\eta||, \quad \Big|\sum_{n=1}^{\infty} \binom{1/2}{n} \cdot (-1)^n \cdot \eta_2^{2\cdot n} \cdot |\eta|^{-2\cdot n}\Big|, \quad |(n_2^{(\varphi)} \circ g^{(\varphi)})(\eta)|$$
$$\le C_{38,1} \cdot \sin(3\cdot\delta).$$

This implies for $\eta \in \mathbb{R}^2\backslash\{0\}$, $v \in \{3, 4, 5, 6\}$, $\mu \in \{1, 2\}$:

$$|f_\mu^{(v)}(\eta)| \le C_{38,2}(\varphi) \cdot \sin(3\cdot\delta) \cdot \chi_{A(0,3\cdot\delta)}(\eta) \cdot |\Phi(\eta)|. \tag{11.17}$$

Now we apply Lemma 5.10 to obtain for $v \in \{3, 5\}$, $\kappa \in \{4, 6\}$, $\mu \in \{1, 2\}$:

$$\Big\|\int_{\mathbb{R}^2} \mathcal{Y}_\mu^\lambda(g^{(\varphi)} - g^{(\varphi)}(\eta)) \cdot f_\mu^{(v)}(\eta)\, d\eta\Big\|_p, \tag{11.18}$$
$$\Big\|\int_{\mathbb{R}^2} \mathcal{X}_\mu^\lambda(g^{(\varphi)} - g^{(\varphi)}(\eta)) \cdot f_\mu^{(\kappa)}(\eta)\, d\eta\Big\|_p$$
$$\le C_{38,3}(p, \vartheta, \varphi) \cdot \sin(3\cdot\delta) \cdot \|\chi_{A(0,3\cdot\delta)} \cdot \Phi\|_p.$$

Furthermore, if $\mu \in \{1, 2\}$, we see that

$$\Big(\int_{A(0,3\cdot\delta)} \Big|\int_{\mathbb{R}^2} \mathcal{Y}_\mu^\lambda(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \cdot g_\mu^{(3)}(\xi, \eta)\, d\eta\Big|^p\, d\xi\Big)^{1/p}, \tag{11.19}$$

$$\Big(\int_{A(0,3\cdot\delta)} \Big|\int_{\mathbb{R}^2} \mathcal{X}_\mu^\lambda(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \cdot g_\mu^{(4)}(\xi, \eta)\, d\eta\Big|^p\, d\xi\Big)^{1/p}$$
$$\le 2\cdot\cot\varphi \cdot C_{17}(\vartheta) \cdot \Big(\int_{A(0,3\cdot\delta)} \Big(\int_{A(0,3\cdot\delta)} |g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)|^{-2} \cdot |\xi - \eta| \cdot (|\xi| + |\eta|)^{-1} \cdot |\Phi(\eta)|\, d\eta\Big)^p\, d\xi\Big)^{1/p}$$
$$\le C_{38,4}(p, \vartheta, \varphi) \cdot \sin^{1/2}(3\cdot\delta) \cdot \|\chi_{A(0,3\cdot\delta)} \cdot \Phi\|_p.$$

where the first inequality follows from (5.14) and Lemma 4.1, and the second one from Theorem 4.1. After collecting the results stated in (11.13) - (11.16), (11.18), (11.19), and recalling the definition of $C_{38}(p, \vartheta, \varphi)$, we arrive at inequality (11.11).

In order to prove (11.12), we first note for $\xi \in \mathbb{R}^2$, $\epsilon \in (0, \infty)$:

$$\Big((L_{jl}^{(4)}(\varphi, \delta))_\epsilon \otimes \Phi_j\Big)(\xi) \tag{11.20}$$
$$= \sum_{\mu=1}^{2} \int_{\mathbb{R}^2} \chi_{(\epsilon,\infty)}(|\xi - \eta|) \cdot \mathcal{X}_\mu^\infty(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \cdot \Big(g_\mu^{(4)}(\xi, \eta) + f_\mu^{(4)}(\eta)\Big)\, d\eta;$$

$$\Big((L_{jl}^{(6)}(\varphi, \delta))_\epsilon \otimes \Phi_j\Big)(\xi) \tag{11.21}$$
$$= \sum_{\mu=1}^{2} \int_{\mathbb{R}^2} \chi_{(\epsilon,\infty)}(|\xi - \eta|) \cdot \mathcal{X}_\mu^\infty(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \cdot f_\mu^{(6)}(\eta)\, d\eta.$$

On the other hand, we infer from (11.17) and Corollary 4.2, if $\mu \in \{1, 2\}$, $v \in \{4, 6\}$, $\epsilon \in (0, \infty)$:

$$\Big\|\int_{\mathbb{R}^2} \chi_{(\epsilon,\infty)}(|id(\mathbb{R}^2) - \eta|) \cdot \mathcal{X}_\mu^\infty(g^{(\varphi)} - g^{(\varphi)}(\eta)) \cdot f_\mu^{(v)}(\eta)\, d\eta\Big\|_p \tag{11.22}$$
$$\le C_9(p, \varphi) \cdot C_{38,2}(\varphi) \cdot \sin(3\cdot\delta) \cdot \|\chi_{A(0,3\cdot\delta)} \cdot \Phi\|_p.$$

Moreover, if $\mu \in \{1, 2\}$, $v \in \{4, 6\}$, $\epsilon \in (0, \infty)$, it may be derived from Lemma 4.1 and Theorem 4.1 that

$$\Big(\int_{A(0,3\cdot\delta)} \Big|\int_{\mathbb{R}^2} \chi_{(\epsilon,\infty)}(|\xi - \eta|) \cdot \mathcal{X}_\mu^\infty(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \cdot g_\mu^{(v)}(\xi, \eta)\, d\eta\Big|^p\, d\xi\Big)^{1/p} \tag{11.23}$$
$$\le (2\cdot\pi)^{-1} \cdot \cot\varphi \cdot \Big(\int_{A(0,3\cdot\delta)} \Big(\int_{A(0,3\cdot\delta)} |\xi - \eta|^{-1} \cdot (|\xi| + |\eta|)^{-1} \cdot |\Phi(\eta)|\, d\eta\Big)^p\, d\xi\Big)^{1/p}$$
$$\le C_{39,1}(p, \varphi) \cdot \sin^{1/2}(3\cdot\delta) \cdot \|\chi_{A(0,3\cdot\delta)} \cdot \Phi\|_p.$$

Now inequality (11.12) follows from (11.20) - (11.23) and the definition of $C_{39}(p, \varphi)$.

Corollary 11.2. *Let* $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$. *Then there are numbers* $C_{40}(p, \vartheta, \varphi)$, $C_{41}(p, \varphi) > 0$ *with*

$$\| \mathcal{A}^{(\upsilon)}(\varphi, \lambda, \delta, R, S, \Phi) \|_p \quad \le \quad C_{40}(p, \vartheta, \varphi) \cdot \sin^{1/2}(3 \cdot \delta) \cdot \| \chi_{A(0, 3 \cdot \delta)} \cdot \Phi \|_p ,$$

$$\| \mathcal{B}^{(\mu)}(\varphi, \delta, R, S, \Phi) \|_p \quad \le \quad C_{41}(p, \varphi) \cdot \sin^{1/2}(3 \cdot \delta) \cdot \| \chi_{A(0, 3 \cdot \delta)} \cdot \Phi \|_p$$

for $\upsilon \in \{3, 4, 5, 6\}$, $\mu \in \{4, 6\}$, $\lambda \in \mathbb{C} \backslash \{0\}$ *with* $|\arg \lambda| \le \vartheta$, $R \in (0, \infty)$, $S \in (1, \infty)$, $\Phi \in L^p(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R \cdot S))^3$, $\delta \in (0, \pi/48)$.

Proof: We refer to Definition 10.3 and Lemma 11.3.

Corollary 11.3. *Take* $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$. *Then there is a constant* $C_{42}(p, \vartheta, \varphi) > 0$ *such that*

$$\Big\| \sin^{-1}(\varphi) \cdot \Big(\sum_{j=1}^{3} K_{jl}^{(1)}(\varphi, \lambda, \delta) \otimes \Phi_j \Big)_{1 \le l \le 3} \Big\|_p , \qquad \| \mathcal{A}^{(1)}(\varphi, \lambda, \delta, R, S, \Phi) \|_p$$

$$\le \; C_{42}(p, \vartheta, \varphi) \cdot (R \cdot S)^{-1} \cdot \|\Phi\|_p$$

for $\lambda \in \mathbb{C} \backslash \{0\}$ *with* $|\arg \lambda| \le \vartheta$, $|\lambda| \ge 1$, $R \in (0, \infty)$, $S \in (1, \infty)$, $\delta \in (0, \pi/6)$, $\Phi \in L^p(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R \cdot S))^3$.

Proof: Apply Lemma 5.11.

Lemma 11.4. *Take* $\delta, \delta' \in (0, \pi/2)$ *with* $\delta' < \delta$, $\xi \in A(0, \delta')$, $\eta \in \mathbb{R}^2 \backslash A(0, \delta)$. *Then it holds*

$$|\xi - \eta| \;\ge\; \sin\big((\delta - \delta')/2 \big) \cdot (|\xi| + |\eta|) .$$

Proof: There are numbers $r, s \in [0, \infty)$, $\alpha \in (-\delta', \delta')$, $\beta \in [-\pi, \pi] \backslash (-\delta, \delta)$ such that

$$\xi = r \cdot (\cos \alpha, \sin \alpha), \quad \eta = s \cdot (\cos \beta, \sin \beta).$$

Thus we find

$$|\xi - \eta|^2 \; = \; r^2 + s^2 - 2 \cdot r \cdot s \cdot \cos(\alpha - \beta) \tag{11.24}$$

$$= \; 2 \cdot (r^2 + s^2) \cdot \sin^2\big((\alpha - \beta)/2\big) \; + \; (r - s)^2 \cdot \cos(\alpha - \beta) .$$

On the other hand, if $|\alpha - \beta| \le \pi/2$, that is, $\pi/2 \ge |\alpha - \beta| \ge \delta - \delta'$, it follows

$$\cos(\alpha - \beta) \ge 0, \qquad \sin^2\big((\alpha - \beta)/2\big) \;\ge\; \sin^2\big((\delta - \delta')/2\big) .$$

Now we may conclude by the second equation in (11.24):

$$|\xi - \eta|^2 \;\ge\; 2 \cdot \sin^2\big((\delta - \delta')/2\big) \cdot \big(|\xi|^2 + |\eta|^2\big) \;\ge\; \sin^2\big((\delta - \delta')/2\big) \cdot \big(|\xi| + |\eta|\big)^2 .$$

Next consider the case $|\alpha - \beta| > \pi/2$, so that $\pi/2 < |\alpha - \beta| \le 3 \cdot \pi/2$, hence, $\cos(\alpha - \beta) \le 0$. Thus we see by the first equation in (11.24) that

$$|\xi - \eta|^2 \;\ge\; |\xi|^2 + |\eta|^2 \;\ge\; (1/2) \cdot (|\xi| + |\eta|)^2 .$$

This completes the proof of Lemma 11.4.

Next we recall that the difference $|(\mathcal{X}_l^\lambda - \mathcal{X}_l^\infty)(x)|$ may be estimated against a constant times $|\lambda|^{-1} \cdot |x|^{-4}$; see (5.13). The integral operator appearing in the next lemma is generated by the kernel $\mathcal{X}_l^\lambda - \mathcal{X}_l^\infty$, so we shall attempt to evaluate this operator by using the preceding inequality. In fact, a suitable estimate will be achieved by recurring to Lemma 11.4, which will allow us to handle the singularity $|x|^{-4}$.

Lemma 11.5. *Let* $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$. *Then there is a constant* $C_{43}(\vartheta, \varphi) > 0$ *such that*

$$\Big\| \sin^{-1}(\varphi) \cdot \chi_{A(0, 2 \cdot \delta) \backslash \mathbb{B}_2(0, R)} \cdot \Big(\sum_{j=1}^{3} \big(K_{jl}^{(2)}(\varphi, \lambda, \delta) - L_{jl}^{(2)}(\varphi, \delta) \big) \otimes_p \Phi_j \Big)_{1 \le l \le 3} \Big\|_p$$

$$\le \; C_{43}(\vartheta, \varphi) \cdot \sin^{-4}(\delta/2) \cdot R^{-2} \cdot |\lambda|^{-1} \cdot \|\Phi\|_p$$

if $p \in (1, \infty)$, $\lambda \in \mathbb{C} \backslash \{0\}$ *with* $|\arg \lambda| \le \vartheta$, $\delta \in (0, \pi/6)$, $R \in (0, \infty)$, $\Phi \in L^p(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R))^3$.

Proof: Put $C_{43,1}(\vartheta, \varphi) := 3 \cdot \pi \cdot C_{17}(\vartheta)$; $C_{43}(\vartheta, \varphi) := \sin^{-1}(\varphi) \cdot 9 \cdot C_{43,1}(\vartheta, \varphi)$.
Assume $p, \lambda, R, \Phi, \delta$ are given as in the lemma. Take $j, l \in \{1, 2, 3\}$. Then we have

$$\Big\| \chi_{A(0, 2 \cdot \delta) \backslash \mathbb{B}_2(0, R)} \cdot \Big(\big(K_{jl}^{(2)}(\varphi, \lambda, \delta) - L_{jl}^{(2)}(\varphi, \delta) \big) \otimes_p \Phi_j \Big) \Big\|_p \tag{11.25}$$

$$\le \; \sum_{k=1}^{3} \Big(\int_{A(0, 2 \cdot \delta) \backslash \mathbb{B}_2(0, R)} \Big(\int_{\mathbb{R}^2 \backslash (A(0, 3 \cdot \delta) \cup \mathbb{B}_2(0, R))} \big| (\delta_{kl} \cdot \mathcal{Y}_j^\lambda + \delta_{jk} \cdot \mathcal{X}_l^\lambda - \delta_{jk} \cdot \mathcal{X}_l^\infty)(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \big| \cdot \big| (n_k^{(\varphi)} \circ g^{(\varphi)})(\eta) \big| \cdot |\Phi(\eta)| \, d\eta \Big)^p d\xi \Big)^{1/p}$$

$$\le \; 3 \cdot C_{17}(\vartheta) \cdot |\lambda|^{-1} \cdot \Big(\int_{A(0, 2 \cdot \delta) \backslash \mathbb{B}_2(0, R)} \Big(\int_{\mathbb{R}^2 \backslash (A(0, 3 \cdot \delta) \cup \mathbb{B}_2(0, R))} |\xi - \eta|^{-4} \cdot |\Phi(\eta)| \, d\eta \Big)^p d\xi \Big)^{1/p} ;$$

with the first inequality following from Lemma 5.7, and the second one from (5.13). Next we apply Lemma 11.4 and obtain from (11.25), for $\epsilon \in (0,\infty)$:

$$\Big\| \chi_{A(0,2\cdot\delta)\backslash\mathbb{B}_2(0,R)} \cdot \Big(\big(K^{(2)}_{jl}(\varphi,\lambda,\delta) - L^{(2)}_{jl}(\varphi,\delta) \big) \otimes_p \Phi_j \Big) \Big\|_p$$

$$\le 3\cdot C_{17}(\vartheta)\cdot \sin^{-4}(\delta/2)\cdot|\lambda|^{-1} \cdot\Big(\int_{A(0,2\cdot\delta)\backslash\mathbb{B}_2(0,R)} \Big(\int_{\mathbb{R}^2\backslash(A(0,3\cdot\delta)\cup\mathbb{B}_2(0,R))} (|\xi|+|\eta|)^{-4}\cdot|\Phi(\eta)|\,d\eta\Big)^p\,d\xi\Big)^{1/p}$$

$$\le 3\cdot C_{17}(\vartheta)\cdot \sin^{-4}(\delta/2)\cdot|\lambda|^{-1} \cdot\Big(\int_{A(0,2\cdot\delta)\backslash\mathbb{B}_2(0,R)} \Big(\int_{\mathbb{R}^2\backslash(A(0,3\cdot\delta)\cup\mathbb{B}_2(0,R))} (|\xi|+|\eta|)^{-4}\,d\eta\Big)^{p-1} \cdot \int_{\mathbb{R}^2\backslash(A(0,3\cdot\delta)\cup\mathbb{B}_2(0,R))} (|\xi|+|\eta|)^{-4}\cdot|\Phi(\eta)|^p\,d\eta\,d\xi\Big)^{1/p}$$

$$\le 3\cdot C_{17}(\vartheta)\cdot \sin^{-4}(\delta/2)\cdot|\lambda|^{-1}\cdot\pi^{1-1/p}\cdot R^{-2+2/p} \cdot\Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R)} \int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R)} (|\xi|+|\eta|)^{-4}\cdot|\Phi(\eta)|^p\,d\eta\,d\xi\Big)^{1/p}$$

$$\le C_{43,1}(\vartheta,\varphi)\cdot\sin^{-4}(\delta/2)\cdot R^{-2}\cdot|\lambda|^{-1}\cdot\|\Phi\|_p,$$

where the second estimate was derived from Hölder's inequality, and the last one follows by Fubini's theorem.

The proof of the next lemma essentially amounts to an application of Lemma 11.1.

Lemma 11.6. *Let* $p\in(1,\infty)$, $\vartheta\in[0,\pi)$, $\varphi\in(0,\pi/2]$. *Then there are constants* $C_{44}(p,\vartheta,\varphi), C_{45}(p,\varphi) > 0$ *such that the following estimates are valid for* $\lambda\in\mathbb{C}\backslash\{0\}$ *with* $|\arg\lambda|\le\vartheta$, $\delta\in(0,\pi/6)$, $R\in(0,\infty)$, $S\in(2,\infty)$, $\Phi\in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0,R\cdot S))^3$:

$$\|\mathcal{A}^{(12)}(\varphi,\lambda,\delta,R,S,\Phi)\|_p,$$

$$\Big\| \sin^{-1}(\varphi)\cdot\Big(\chi_{\mathbb{B}_2(0,R)}\cdot\Big(\sum_{j=1}^3 K^{(2)}_{jl}(\varphi,\lambda,\delta)\otimes\Phi_j\Big)_{1\le l\le 3}\Big)\circ\big(T(\varphi)\big)^{-1}\Big\|_p$$

$$\le C_{44}(p,\vartheta,\varphi)\cdot(S-1)^{-2/p}\cdot\|\Phi\|_p;$$

$$\|\mathcal{B}^{(12)}(\varphi,\delta,R,S,\Phi)\|_p,$$

$$\Big\| \sin^{-1}(\varphi)\cdot\Big(\chi_{\mathbb{B}_2(0,R)}\cdot\Big(\sum_{j=1}^3 L^{(2)}_{jl}(\varphi,\delta)\otimes_p\Phi_j\Big)_{1\le l\le 3}\Big)\circ\big(T(\varphi)\big)^{-1}\Big\|_p$$

$$\le C_{45}(p,\varphi)\cdot(S-1)^{-2/p}\cdot\|\Phi\|_p.$$

Beweis: First we define the constants appearing in the lemma, setting

$C_{44,1}(p,\vartheta,\varphi) := 6\cdot(1\vee\cot\varphi)\cdot C_{17}(\vartheta)\cdot C_{33}(p)$;

$C_{44}(p,\vartheta,\varphi) := \sin^{-1-1/p}(\varphi)\cdot 9\cdot C_{44,1}(p,\vartheta,\varphi)$;

$C_{45,1}(p,\varphi) := 2\cdot(1\vee\cot\varphi)\cdot(4\cdot\pi)^{-1}\cdot C_{33}(p)$;

$C_{45}(p,\varphi) := \sin^{-1-1/p}(\varphi)\cdot 9\cdot C_{45,1}(p,\varphi)$.

Then let λ,δ,R,S,Φ be given as in the lemma, and take $j,l\in\{1,2,3\}$. It follows

$$\|\chi_{\mathbb{B}_2(0,R)}\cdot(K^{(2)}_{jl}(\varphi,\lambda,\delta)\otimes\Phi_j)\|_p$$

$$\le \Big(\int_{\mathbb{B}_2(0,R)} \Big(\int_{\mathbb{R}^2\backslash(A(0,3\cdot\delta)\cup\mathbb{B}_2(0,R\cdot S))} \sum_{k=1}^3 \Big| P^\lambda_{jkl}(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)) \cdot(n^{(\varphi)}_k\circ g^{(\varphi)})(\eta)\cdot\Phi_j(\eta)\Big|\,d\eta\Big)^p\,d\xi\Big)^{1/p}$$

$$\le 6\cdot C_{17}(\vartheta)\cdot\Big(\int_{\mathbb{B}_2(0,R)} \Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R\cdot S)} |g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)|^{-2}\cdot|\Phi(\eta)|\,d\eta\Big)^p\,d\xi\Big)^{1/p}$$

$$\le C_{44,1}(p,\vartheta,\varphi)\cdot(S-1)^{-2/p}\cdot\|\Phi\|_p,$$

with the second inequality derived from (5.12), (5.14), and the third one from (3.18) and Lemma 11.1. This lemma further yields

$$\|\chi_{\mathbb{B}_2(0,R)}\cdot(L^{(2)}_{jl}(\varphi,\delta)\otimes_p\Phi_j)\|$$

$$\le \Big(\int_{\mathbb{B}_2(0,R)} \Big(\int_{\mathbb{R}^2\backslash(A(0,3\cdot\delta)\cup\mathbb{B}_2(0,R\cdot S))} \Big| \mathcal{X}^\infty_l(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)) \cdot(n^{(\varphi)}_j\circ g^{(\varphi)})(\eta)\cdot\Phi_j(\eta)\Big|\,d\eta\Big)^p\,d\xi\Big)^{1/p}$$

$$\le (4\cdot\pi)^{-1}\cdot\Big(\int_{\mathbb{B}_2(0,R)} \Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R\cdot S)} |\xi-\eta|^{-2}\cdot|\Phi(\eta)|\,d\eta\Big)^p\,d\xi\Big)^{1/p}$$

$$\le (4\cdot\pi)^{-1}\cdot C_{33}(p)\cdot(S-1)^{-2/p}\cdot\|\Phi\|_p.$$

By a similar reasoning, the ensuing two inequalities may be deduced from (5.14) and Lemma 11.1:

$$\Big\|\chi_{\mathbb{B}_2(0,R)}\cdot\Big(\big(K^{(9)}_{jl}(\varphi,\lambda,\delta) + K^{(10)}_{jl}(\varphi,\lambda,\delta)\big)\Big)\otimes\Phi_j\Big\|_p$$

$$\le 2\cdot(1\vee\cot\varphi)\cdot\sum_{i=1}^2\Big(\int_{\mathbb{B}_2(0,R)} \Big(\int_{A(0,3\cdot\delta)\backslash\mathbb{B}_2(0,R\cdot S)} \Big(|\mathcal{Y}^\lambda_i(\xi-\eta,\cot\varphi\cdot(\xi-\eta)_1)| + |\mathcal{X}^\lambda_i(\xi-\eta,\cot\varphi\cdot(\xi-\eta)_1)|\Big)\cdot|\Phi(\eta)|\,d\eta\Big)^p\,d\xi\Big)^{1/p}$$

$$\le 4\cdot(1\vee\cot\varphi)\cdot C_{17}(\vartheta)$$

$$\cdot \Big(\int_{\mathbb{B}_2(0,R)} \Big(\int_{A(0,3\cdot\delta)\setminus\mathbb{B}_2(0,R\cdot S)} |\xi-\eta|^{-2} \cdot |\Phi(\eta)| \, d\eta \Big)^p \, d\xi \Big)^{1/p}$$

$$\leq \; C_{44,1}(p, \vartheta, \varphi) \cdot (S-1)^{-2/p} \cdot \|\Phi\|_p ;$$

$$\| \chi_{\mathbb{B}_2(0,R)} \cdot (L^{(10)}_{jl}(\varphi, \delta) \otimes_p \Phi_j) \|_p \;\; \leq \;\; C_{45,1}(p, \varphi) \cdot (S-1)^{-2/p} \cdot \|\Phi\|_p .$$

Now the lemma follows from Definition 10.3.

Up to now, we did not consider the terms $\mathcal{A}^{(\upsilon)}(\varphi, \lambda, \delta, R, S, \Phi)$ and $\mathcal{B}^{(\upsilon)}(\varphi, \delta, R, S, \Phi)$ for $\upsilon \in \{2, 11\}$. It is perhaps the most difficult task in this chapter to prove a suitable estimate for the function arising when a multiplier transformation as in Lemma 5.18 is applied to to these terms for such values of υ. We shall achieve an appropriate estimate by splitting our multiplier into a sum of two parts, with one summand supported around the origin. The other one turns out to be a C^∞-function vanishing near the origin and having derivatives which decay near infinity. The term generated by the first summand is easy to deal with (Corollary 11.4). As for the expression induced by the second summand, it will be evaluated by recurring to the decay properties of this summand. The details will be worked out in the proof of Lemma 11.10, in a situation slighly more general than the one just described. Concerning Lemma 11.8, 11.9 and Corollary 11.5, they will enable us to insert the functions $\mathcal{A}^{(\upsilon)}(\varphi, \lambda, \delta, R, S, \Phi)$ and $\mathcal{B}^{(\upsilon)}(\varphi, \delta, R, S, \Phi)$ $(\upsilon \in \{2, 11\})$ into the framework of Lemma 11.10. This point will be discussed in the proof of Lemma 11.11.

As a first step in elaborating this approach, the next lemma yields a L^p-estimate for $K^{(\upsilon)}_{jl}(\varphi, \lambda, \delta) \otimes \Phi_j$ and $L^{(\mu)}_{jl}(\varphi, \delta) \otimes_p \Phi_j$ $(\upsilon \in \{2, 9, 10\}, \mu \in \{2, 10\})$.

Lemma 11.7. *Let* $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$. *Then there exist some numbers* $C_{46}(p, \vartheta, \varphi)$, $C_{47}(p, \varphi) > 0$ *so that*

$$\Big\| \sin^{-1}(\varphi) \cdot \Big(\sum_{j=1}^{3} K^{(\upsilon)}_{jl}(\varphi, \lambda, \delta) \otimes \Phi_j \Big)_{1\leq l\leq 3} \Big\|_p \;\; \leq \;\; C_{46}(p, \vartheta, \varphi) \cdot \|\Phi\|_p$$

for $\upsilon \in \{2, 9, 10\}$, $\lambda \in \mathbb{C}\setminus\{0\}$ *with* $|\arg \lambda| \leq \vartheta$, $\Phi \in L^p(\mathbb{R}^2)^3$, $\delta \in (0, \pi/6)$;

$$\Big\| \sin^{-1}(\varphi) \cdot \Big(\sum_{j=1}^{3} L^{(\upsilon)}_{jl}(\varphi, \delta) \otimes_p \Phi_j \Big)_{1\leq l\leq 3} \Big\|_p \;\; \leq \;\; C_{47}(p, \varphi) \cdot \|\Phi\|_p$$

for $\upsilon \in \{2, 10\}$, $\Phi \in L^p(\mathbb{R}^2)^3$, $\delta \in (0, \pi/6)$.

Proof: Combine (5.12) with Lemma 10.1, 5.10, 5.5 and Corollary 4.2.

Now we may deal with the case of a multiplier with bounded support around the origin.

Corollary 11.4. *Take* $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$. *Then a constant* $C_{48}(p, \vartheta, \varphi) > 0$ *exists such that the ensuing inequalities are valid for* $\upsilon \in \{2, 11\}$, $\lambda \in \mathbb{C}\setminus\{0\}$ *with* $|\arg \lambda| \leq \vartheta$, $\delta \in (0, \pi/6)$, $R \in (0, \infty)$, $S \in (1, \infty)$, $\tau \in \{-1, 1\}$, $\Phi \in L^p(\mathbb{R}^2\setminus\mathbb{B}_2(0, R\cdot S))^3 \cap L^2(\mathbb{R}^2\setminus\mathbb{B}_2(0, R\cdot S))^3$,
$B \in \Big\{ A^{\infty,\varphi}_\tau \cdot \Big((A^{\lambda,\varphi}_\tau)^{-1} - (A^{\infty,\varphi}_\tau)^{-1} \Big), \;\; \Big(A^{\lambda,\varphi}_\tau - A^{\infty,\varphi}_\tau \Big) \cdot (A^{\infty,\varphi}_\tau)^{-1} \Big\}$:

$$\Big\| \Big(\Psi_1 \cdot B \cdot \Big(\sum_{\upsilon\in\{2,11\}} \mathcal{A}^{(\upsilon)}(\varphi, \lambda, \delta, R, S, \Phi) \Big)^{\wedge} \Big)^{\vee} \circ T(\varphi) \Big\|_p$$

$$\leq \; C_{48}(p, \vartheta, \varphi) \cdot |\lambda|^{-1/2} \cdot \|\Phi\|_p ;$$

$$\Big\| \Big(\Psi_1 \cdot B \cdot \Big(\sum_{\upsilon\in\{2,11\}} \mathcal{B}^{(\upsilon)}(\varphi, \delta, R, S, \Phi) \Big)^{\wedge} \Big)^{\vee} \circ T(\varphi) \Big\|_p$$

$$\leq \; C_{48}(p, \vartheta, \varphi) \cdot |\lambda|^{-1/2} \cdot \|\Phi\|_p .$$

(The function Ψ_1 *was introduced in Definition 5.3.)*

Proof: This corollary follows from Lemma 5.20 and 11.7.

Lemma 11.8. *If* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $\delta \in (0, \pi/6)$, $R \in (0, \infty)$, $S \in (1, \infty)$, $\Phi \in L^p(\mathbb{R}^2\setminus\mathbb{B}_2(0, R\cdot S))^3$, $j, l \in \{1, 2, 3\}$, $\lambda \in \mathbb{C}\setminus(-\infty, 0]$ *with* $|\lambda| \geq 1$, *then it follows*

$$(K^{(2)}_{jl}(\varphi, \lambda, \delta) \otimes \Phi_j) \,\big|\, A(0, 2\cdot\delta), \quad (L^{(2)}_{jl}(\varphi, \delta) \otimes_p \Phi_j) \,\big|\, A(0, 2\cdot\delta) \; \in \; C^1(A(0, 2\cdot\delta))^3 .$$

If $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$, *then there are numbers* $C_{49}(p, \vartheta, \varphi)$ *and* $C_{50}(p, \varphi)$ *in* $(0, \infty)$ *such that*

$$\Big\| \sin^{-1}(\varphi) \cdot \Big(\sum_{j=1}^{3} K^{(2)}_{jl}(\varphi, \lambda, \delta) \otimes \Phi_j \Big)_{1\leq l\leq 3} \,\Big|\, A(0, 2\cdot\delta)\setminus\overline{\mathbb{B}_2(0, R)} \Big\|_{1,p} \tag{11.26}$$

$$\leq \; C_{49}(p, \vartheta, \varphi) \cdot \sin^{-3}(\delta/2) \cdot \|\Phi\|_p ,$$

$$\Big\| \sin^{-1}(\varphi) \cdot \Big(\sum_{j=1}^{3} L^{(2)}_{jl}(\varphi, \delta) \otimes_p \Phi_j \Big)_{1\leq l\leq 3} \,\Big|\, A(0, 2\cdot\delta)\setminus\overline{\mathbb{B}_2(0, R)} \Big\|_{1,p} \tag{11.27}$$

$$\leq \; C_{50}(p, \varphi) \cdot \sin^{-3}(\delta/2) \cdot \|\Phi\|_p ,$$

for $\lambda \in \mathbb{C}$ *with* $|\arg \lambda| \leq \vartheta$, $|\lambda| \geq 1$, $\delta \in (0, \pi/6)$, $R \in (0, \infty)$, $S \in (1, \infty)$ *with* $R\cdot S \geq 1$, $\Phi \in L^p(\mathbb{R}^2\setminus\mathbb{B}_2(0, R\cdot S))^3$.

Proof: Fix $p \in (1,\infty)$, $\vartheta \in [0,\pi)$, $\varphi \in (0,\pi/2]$, and define

$C_{49,1}(\vartheta,\varphi) := 18 \cdot C_{17}(\vartheta) \cdot (1 \vee \cot\varphi)$;

$C_{49}(p,\vartheta,\varphi) := C_{46}(p,\vartheta,\varphi) + \sin^{-1}(\varphi) \cdot 54 \cdot \pi \cdot C_{49,1}(\vartheta,\varphi)$;

$C_{50,1}(\varphi) := \pi^{-1} \cdot 3 \cdot (1 \vee \cot\varphi)$; $C_{50}(p,\varphi) := C_{47}(p,\varphi) + \sin^{-1}(\varphi) \cdot 54 \cdot \pi \cdot C_{50,1}(\varphi)$.

Now take $\lambda \in \mathbb{C}$ with $|\arg\lambda| \le \vartheta$, $|\lambda| \ge 1$, $\delta \in (0,\pi/6)$, $R \in (0,\infty)$, $S \in (1,\infty)$, $\Phi \in L^p(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R\cdot S))^3$, $j, l \in \{1,2,3\}$.

For $\xi \in \mathbb{R}^2$, we observe that

$$(K_{jl}^{(2)}(\varphi,\lambda,\delta) \otimes \Phi_j)(\xi) \tag{11.28}$$

$$= -\int_{\mathbb{R}^2\backslash(A(0,3\cdot\delta)\cup\mathbb{B}_2(0,R\cdot S))} \sum_{k=1}^{3} P_{jkl}^{\lambda}(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \cdot (n_k^{(\varphi)} \circ g^{(\varphi)})(\eta) \cdot \Phi_j(\eta)\, d\eta.$$

If $\xi \in A(0, 2\cdot\delta)$, $\eta \in \mathbb{R}^2 \backslash A(0,3\cdot\delta)$, we may conclude from Lemma 11.4:

$$|\xi - \eta|^2 \ge \sin^2(\delta/2) \cdot (|\xi| + |\eta|)^2,$$

so that it follows for $\xi \in A(0,2\cdot\delta)$, $\eta \in \mathbb{R}^2\backslash(A(0,3\cdot\delta)\cup\mathbb{B}_2(0,R\cdot S))$:

$$|g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)| \ge \sin(\delta/2)\cdot(|\xi|+|\eta|) \ge \sin(\delta/2)\cdot R\cdot S > 0. \tag{11.29}$$

For $(\xi,\eta) \in \mathbb{R}^2$ with $\xi \ne \eta$, $\eta \ne 0$, we put

$$\Gamma(\xi,\eta) := \sum_{k=1}^{3} P_{jkl}^{\lambda}(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \cdot (n_k^{(\varphi)} \circ g^{(\varphi)})(\eta) \cdot \Phi_j(\eta).$$

Let us evaluate the partial derivatives of Γ. In fact, we find for $\xi \in A(0,2\cdot\delta)$, $\eta \in \mathbb{R}^2\backslash(A(0,3\cdot\delta)\cup\mathbb{B}_2(0,R\cdot S))$, $l \in \{1,2\}$:

$$|\partial/\partial\xi_l\, \Gamma(\xi,\eta)| \le (1\vee\cot\varphi)\cdot\sum_{r,k=1}^{3}|D_r P_{jkl}^{\lambda}(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta))|\cdot|\Phi(\eta)| \tag{11.30}$$

$$\le 18\cdot C_{17}(\vartheta)\cdot(1\vee\cot\varphi)\cdot|g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)|^{-3}\cdot|\Phi(\eta)|$$

$$\le C_{49,1}(\vartheta,\varphi)\cdot\sin^{-3}(\delta/2)\cdot(|\xi|+|\eta|)^{-3}\cdot|\Phi(\eta)|$$

$$\le C_{49,1}(\vartheta,\varphi)\cdot\sin^{-3}(\delta/2)\cdot|\eta|^{-3}\cdot|\Phi(\eta)|$$

with the second inequality following from (5.12), (5.14), and the third one from (11.29). On the other hand, we note

$$\int_{\mathbb{R}^2\backslash(A(0,3\cdot\delta)\cup\mathbb{B}_2(0,R\cdot S))} |\eta|^{-3}\cdot|\Phi(\eta)|\, d\eta$$

$$\le \Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R\cdot S)} |\eta|^{-3/(1-1/p)}\, d\eta\Big)^{1-1/p} \cdot \|\Phi\|_p < \infty.$$

Recalling (11.28), (11.30) and Lebesgue's theorem on dominated convergence, we see that the function $K_{jl}^{(2)}(\varphi,\lambda,\delta)\otimes\Phi_j$ is continuously differentiable on $A(0,2\cdot\delta)$. Furthermore, it follows that the function $\partial/\partial\xi_\mu\,\Gamma(\xi,\cdot)$ is integrable for any $\xi \in A(0,2\cdot\delta)$, $\mu \in \{1,2\}$, with

$$\partial/\partial\xi_\mu\,(K_{jl}^{(2)}(\varphi,\lambda,\delta)\otimes\Phi_j)(\xi) = \int_{\mathbb{R}^2\backslash(A(0,3\cdot\delta)\cup\mathbb{B}_2(0,R\cdot S))} \partial/\partial_\mu\,\Gamma(\xi,\eta)\, d\eta. \tag{11.31}$$

Thus, if $R\cdot S \ge 1$, we find for $\mu \in \{1,2\}$:

$$\Big(\int_{A(0,2\cdot\delta)\backslash\overline{\mathbb{B}_2(0,R)}} |\partial/\partial\xi_\mu\,(K_{jl}^{(2)}(\varphi,\lambda,\delta)\otimes\Phi_j)(\xi)|^p\, d\xi\Big)^{1/p} \tag{11.32}$$

$$\le C_{49,1}(\vartheta,\varphi)\cdot\sin^{-3}(\delta/2)$$
$$\cdot\Big(\int_{A(0,2\cdot\delta)\backslash\overline{\mathbb{B}_2(0,R)}}\Big(\int_{\mathbb{R}^2\backslash(A(0,3\cdot\delta)\cup\mathbb{B}_2(0,R\cdot S))}\big(\chi_{(0,1)}(|\xi-\eta|)\cdot(R\cdot S)^{-3}$$
$$+\chi_{(1,\infty)}(|\xi-\eta|)\cdot|\xi-\eta|^{-3}\big)\cdot|\Phi(\eta)|\,d\eta\Big)^p d\xi\Big)^{1/p}$$

$$\le C_{49,1}(\vartheta,\varphi)\cdot\sin^{-3}(\delta/2)\cdot\int_{\mathbb{R}^2}\big(\chi_{(0,1)}(|\sigma|) + \chi_{(1,\infty)}(|\sigma|)\cdot|\sigma|^{-3}\big)\,d\sigma\cdot\|\Phi\|_p$$

$$\le C_{49,1}(\vartheta,\varphi)\cdot\sin^{-3}(\delta/2)\cdot 3\cdot\pi\cdot\|\Phi\|_p.$$

Here we applied (11.31) and (11.30) in the first inequality, and Lemma 4.9 (Young's inequality) in the second one. Since j and l are arbitrary members of the set $\{1,2,3\}$, the estimate in (11.26) may now be deduced from (11.31), (11.32), Lemma 11.7 and the definition of $C_{49}(p,\vartheta,\varphi)$.

Due to (11.29), we have

$$\chi_{(0,\epsilon)}(|\xi-\eta|) = 0 \quad \text{for } \epsilon \in (0,\ R\cdot S\cdot\sin(\delta/2)),\ \xi \in A(0,2\cdot\delta),$$
$$\eta \in \mathbb{R}^2\backslash(A(0,3\cdot\delta)\cup\mathbb{B}_2(0,R\cdot S)).$$

Hence, by Corollary 4.2, we conclude for $\xi \in A(0,2\cdot\delta)$, $j,l \in \{1,2,3\}$ that the integral

$$\int_{\mathbb{R}^2\backslash(A(0,3\cdot\delta)\cup\mathbb{B}_2(0,R\cdot S))} -\mathcal{X}_l^{\infty}(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta))\cdot(n_j^{(\varphi)}\circ g^{(\varphi)})(\eta)\cdot\Phi_j(\eta)\,d\eta$$

exists and equals $(L_{jl}^{(2)}(\varphi,\delta)\otimes_p\Phi_j)(\xi)$. Now the relation

$$(L_{jl}^{(2)}(\varphi,\delta)\otimes_p\Phi_j)\,\big|\,A(0,2\cdot\delta) \in C^1(A(0,2\cdot\delta))$$

as well as inequality (11.27) may be derived by arguments analogous to those used above, with the reference to (5.14) replaced by an obvious estimate of the function $\mathcal{X}_l^{\lambda}$.

Next we introduce some cut-off functions which will be needed in the following.

Definition 11.1. *Fix a function* $\tilde\zeta \in C^\infty(\mathbb{R})$ *with* $0 \le \tilde\zeta \le 1$ *and satisying the* *equations* $\tilde{\zeta}\,|\,[-3/2, 3/2] = 1$, $\tilde{\zeta}\,|\,\mathbb{R}\backslash(-5/3, 5/3) = 0$. *Then, for* $\delta \in (0, \pi/8)$, $\eta \in \mathbb{R}^2\backslash\{0\}$, *we define*

$$\zeta^{(\delta)}(\eta) := \tilde{\zeta}\,(\delta^{-1} \cdot \arcsin(\eta_2/|\eta|)),$$

where $\arcsin := (\sin|[-\pi/2, \pi/2])^{-1}$.

Take $g_0 \in C^\infty(\mathbb{R})$ *with* $g_0\,|\,[1, \infty) = 1$, $g_0\,|\,(-\infty, 1/2] = 0$. *For* $R \in (0,\infty)$, $\eta \in \mathbb{R}^2$, *we put*

$$g^R(\eta) := g_0(|\eta| - R).$$

Thus the function $\zeta^{(\delta)}$ is equal to 1 in the sector $A(0, 3{\cdot}\delta/2)$ of the plane, and vanishes outside $A(0, 2{\cdot}\delta)$, whereas g^R has the constant value 1 in $\mathbb{R}^2\backslash\mathbb{B}_2(0, R+1)$, and is equal to 0 in $\mathbb{B}_2(0, R+1/2)$. These facts and some others are stated in the next lemma.

Lemma 11.9. *Let* $\delta \in (0, \pi/8)$, $R \in (0,\infty)$. *It follows* $\zeta^{(\delta)} \in C^1(\mathbb{R}^2\backslash\{0\})$, $g^{(R)} \in C^1(\mathbb{R}^2)$, $supp(\zeta^{(\delta)}) \subset A(0, 2\cdot\delta)\backslash\{0\}$, $supp(g^{(R)}) \subset \mathbb{R}^2\backslash\mathbb{B}_2(0, R+1/2)$, $\zeta^{(\delta)}\,|\,A(0, 3\cdot\delta/2) = 1$, $g^{(R)}\,|\,\mathbb{R}^2\backslash\mathbb{B}_2(0, R+1) = 1$,

$$\sup\Big\{\,|\zeta^{(\delta)}(\eta)| + \sum_{i=1}^{2}|D_i\zeta^{(\delta)}(\eta)| \;:\; \eta \in \mathbb{R}^2\backslash\mathbb{B}_2(0, R)\Big\} \;\le\; 1 + 4\cdot|\tilde{\zeta}'|_0\cdot(\delta\cdot R)^{-1}.$$

Proof: Take $\eta \in \mathbb{R}^2\backslash\{0\}$ with $\zeta^{(\delta)}(\eta) \neq 0$. Then we find

$$|\delta^{-1}\cdot\arcsin(\eta_2/|\eta|)| \;\le\; 5/3.$$

Since $5\cdot\delta/3 < \pi/2$, it follows $|\eta_2/|\eta|| \le \sin(5{\cdot}\delta/3)$, so that $\eta \in A(0, 5{\cdot}\delta/3)$. This proves that $supp(\zeta^{(\delta)}) \subset A(0, 2{\cdot}\delta)\backslash\{0\}$.

Now assume $\eta \in A(0, 3{\cdot}\delta/2)$. Then there is $\varphi \in (-3{\cdot}\delta/2, 3{\cdot}\delta/2)$ with $\eta_2/|\eta| = \sin\varphi$. But $3{\cdot}\delta/2 < \pi/2$, so that we may conclude $\varphi = \arcsin(\eta_2/|\eta|)$, hence the number $\delta^{-1}\cdot\arcsin(\eta_2/|\eta|)$ must be a member of the interval $(-3/2, 3/2)$. Thus we see that $\zeta^{(\delta)}(\eta) = 1$ for $\eta \in A(0, 3{\cdot}\delta/2)$.

Next we observe that the function $\arcsin\,|\,(-1, 1)$ belongs to $C^1((-1,1))$. On the other hand, we have for $\eta \in \mathbb{R}^2\backslash\{0\}$ with $|\eta_2/|\eta|| \ge 2^{-1/2}$:

$$|\arcsin(\eta_2/|\eta|)| \ge \pi/4, \qquad |\delta^{-1}\cdot\arcsin(\eta_2/|\eta|)| \ge 2, \qquad \text{hence: } \zeta^{(\delta)}(\eta) = 0.$$

Thus it follows $\zeta^{(\delta)} \in C^1(\mathbb{R}^2\backslash\{0\})$. Moreover, we get for $i \in \{1, 2\}$, $\eta \in \mathbb{R}^2\backslash\{0\}$ with $|\eta_2/|\eta|| < 1/\sqrt{2}$:

$$D_i\zeta^{(\delta)}(\eta) = \tilde{\zeta}'\big(\delta^{-1}\cdot\arcsin(\eta_2/|\eta|)\big)\cdot\delta^{-1}\cdot\big(1 - \eta_2^2/|\eta|^2\big)^{-1/2}\cdot\big(\delta_{i2}\cdot|\eta|^{-1} - \eta_2\cdot\eta_i\cdot|\eta|^{-3}\big).$$

Gathering up our informations, we arrive at the result stated in the lemma.

Corollary 11.5. *For* $p \in (1,\infty)$, $\varphi \in (0, \pi/2]$, $\delta \in (0, \pi/8)$, $R \in (0,\infty)$, $S \in (1,\infty)$, $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S))^3$, $j, l \in \{1, 2, 3\}$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$ *with* $|\lambda| \ge 1$, *it holds*

$$g^{(R)}\cdot\zeta^{(\delta)}\cdot\big(K_{jl}^{(2)}(\varphi, \lambda, \delta) \otimes \Phi_j\big), \quad g^{(R)}\cdot\zeta^{(\delta)}\cdot\big(L_{jl}^{(2)}(\varphi, \delta) \otimes_p \Phi_j\big) \;\in\; C^1(\mathbb{R}^2).$$

Furthermore, if $p \in (1,\infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$, *there are constants* $C_{51}(p, \vartheta, \varphi)$ *and* $C_{52}(p, \varphi)$ *in* $(0,\infty)$ *with*

$$\Big\|\sin^{-1}(\varphi)\cdot g^{(R)}\cdot\zeta^{(\delta)}\cdot\Big(\sum_{j=1}^{3} K_{jl}^{(2)}(\varphi, \lambda, \delta)\otimes\Phi_j\Big)_{1\le l\le 3}\Big\|_{1,p}$$

$$\le\; C_{51}(p, \vartheta, \varphi)\cdot\sin^{-3}(\delta/2)\cdot\delta^{-1}\cdot\|\Phi\|_p,$$

and

$$\Big\|\sin^{-1}(\varphi)\cdot g^{(R)}\cdot\zeta^{(\delta)}\cdot\Big(\sum_{j=1}^{3} L_{jl}^{(2)}(\varphi, \delta)\otimes_p\Phi_j\Big)_{1\le l\le 3}\Big\|_{1,p}$$

$$\le\; C_{52}(p, \varphi)\cdot\sin^{-3}(\delta/2)\cdot\delta^{-1}\cdot\|\Phi\|_p$$

for $\lambda \in \mathbb{C}$ *mit* $|\arg\lambda| \le \vartheta$, $|\lambda| \ge 1$, $\delta \in (0, \pi/48)$, $R, S \in (1,\infty)$, $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S))^3$.

Proof: The corollary combines Lemma 11.8 and 11.9.

The inequality stated in the next lemma is similar to the estimate of the term G_5 in [9, p. 332]. The proof of this inequality is based on the fact that the support of the function w is seperated from the set $T(\varphi)(A(0, \delta)\backslash\mathbb{B}_2(0, R))$.

Lemma 11.10. *Let* $p \in (1,\infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$. *Then there is a constant* $C_{53}(p, \vartheta, \varphi) > 0$ *such that*

$$\Big\|\big(M\cdot\hat{w}\big)^{\vee}\circ T(\varphi)\,\big|\,A(0, \delta)\backslash\mathbb{B}_2(0, R)\Big\|_p$$

$$\le\; C_{53}(p, \vartheta, \varphi)\cdot|\lambda|^{-1/2}\cdot R^{-2}\cdot\sin^{-4}(\delta/4)\cdot\|w\|_p$$

for $\lambda \in \mathbb{C}\backslash\{0\}$ *with* $|\arg\lambda| \le \vartheta$, $\delta \in (0, \pi/8)$, $R \in (0,\infty)$, $\tau \in \{-1, 1\}$, $w \in L^p(\mathbb{R}^2)^3 \cap L^2(\mathbb{R}^2)^3$ *with* $supp(w) \subset T(\varphi)\big(\mathbb{R}^2\backslash(A(0, 3\cdot\delta/2) \cup \mathbb{B}_2(0, R))\big)$,

$$M \in \Big\{(1 - \Psi_1)\cdot A_\tau^{\infty,\varphi}\cdot\big((A_\tau^{\lambda,\varphi})^{-1} - (A_\tau^{\infty,\varphi})^{-1}\big),$$
$$(1 - \Psi_1)\cdot(A_\tau^{\lambda,\varphi} - A_\tau^{\infty,\varphi})\cdot(A_\tau^{\infty,\varphi})^{-1}\Big\}.$$

(The function Ψ_1 *was introduced in Definition 5.3.)*

Proof: We set

$C_{53,1}(\vartheta) := 25^3 \cdot C_{25}(\vartheta) \cdot (C_{14})^2; \qquad C_{53,2}(\vartheta, \varphi) := 4 \cdot C_{53,1}(\vartheta) \cdot \sin^{-4}(\varphi);$

$C_{53,3}(p, \vartheta, \varphi) := C_{53,2}(\vartheta, \varphi) \cdot \sin^{-1/p}(\varphi) \cdot \pi \cdot (p-1)^{1-2/p};$

$C_{53}(p, \vartheta, \varphi) := 9 \cdot C_{53,3}(p, \vartheta, \varphi).$

Now let $\lambda, \delta, R, \tau, w, M$ be given as in the lemma. Take $j, k \in \{1, 2, 3\}$. Note that $M_{jk} \cdot \Psi_t \in C_0^\infty(\mathbb{R}^2)$ for any $t \in \mathbb{N}$.

If $t \in \mathbb{N}$, $a \in \mathbb{N}_0^2$ with $|a|_* = 4$, $\varrho \in \mathbb{R}^2 \backslash \{0\}$, we obtain from Lemma 5.1 and 5.18:

$$|D^a(M_{jk} \cdot \Psi_t)(\varrho)| \quad \leq \quad C_{53,1}(\vartheta) \cdot |\lambda|^{-1/2} \cdot |\varrho|^{-3} \cdot \chi_{(1,\infty)}(|\varrho|).$$

It follows for $t \in \mathbb{N}$, $\gamma \in \mathbb{R}^2$:

$$\left||\gamma|^4 \cdot (M_{jk} \cdot \Psi_t)^\wedge(\gamma)\right| = \left|\Big(\sum_{r,s=1}^{2} D_r^2 D_s^2(M_{jk} \cdot \Psi_t)\Big)^\wedge(\gamma)\right| \tag{11.33}$$
$$\leq \sum_{r,s=1}^{2} \int_{\mathbb{R}^2} \left|D_r^2 D_s^2(M_{jk} \cdot \Psi_t)(\varrho)\right| d\varrho$$
$$\leq 4 \cdot C_{53,1}(\vartheta) \cdot |\lambda|^{-1/2} \cdot \int_{\mathbb{R}^2 \backslash \mathbb{B}_2(0,1)} |\varrho|^{-3}\, d\varrho \quad \leq \quad 8 \cdot \pi \cdot C_{53,1}(\vartheta) \cdot |\lambda|^{-1/2}.$$

Now we obtain for $t \in \mathbb{N}$, $\xi \in \mathbb{R}^2$ by referring to (5.44):

$$\left|\Big(M_{jk} \cdot \Psi_t \cdot (\Psi_n \cdot w_k)^\wedge\Big)^\vee(\xi)\right| \tag{11.34}$$
$$\leq (2\cdot\pi)^{-1} \cdot \int_{\mathbb{R}^2} \left|\Psi_n(\eta) \cdot w_k(\eta) \cdot (M_{jk} \cdot \Psi_t)^\wedge(\eta - \xi)\right| d\eta$$
$$= (2\cdot\pi)^{-1} \cdot \int_{\mathbb{R}^2} |\Psi_n(\eta) \cdot w_k(\eta)| \cdot |\xi - \eta|^{-4} \cdot \left||\xi-\eta|^4 \cdot (M_{jk} \cdot \Psi_t)^\wedge(\xi - \eta)\right| d\eta$$
$$\leq 4 \cdot C_{53,1}(\vartheta) \cdot |\lambda|^{-1/2} \cdot \int_{\mathbb{R}^2} |w_k(\eta)| \cdot |\xi - \eta|^{-4}\, d\eta,$$

where we applied (11.33) in the last inequality. Let $\eta \in \mathbb{R}^2$ with $w_k(\eta) \neq 0$. Then we observe that η belongs to the set $T(\varphi)\Big(\mathbb{R}^2 \backslash (A(0, 3\cdot\delta/2) \cup \mathbb{B}_2(0, R))\Big)$. Hence there is some $\eta' \in \mathbb{R}^2 \backslash (A(0, 3\cdot\delta/2) \cup \mathbb{B}_2(0, R))$ with $\eta = T(\varphi)(\eta')$. Take $\xi \in A(0,\delta) \backslash \mathbb{B}_2(0, R)$. Then we obtain by Lemma 11.4:

$$|T(\varphi)(\xi) - \eta| = |T(\varphi)(\xi - \eta')| = |(\sin^{-1}(\varphi) \cdot (\xi - \eta')_1,\ (\xi - \eta')_2)|$$
$$\geq |\xi - \eta'| \geq \sin(\delta/4) \cdot (|\xi| + |\eta'|) \geq \sin(\delta/4) \cdot \sin\varphi \cdot (|\xi| + |\eta|).$$

In the last inequality, we used the equation $\eta' = \big(T(\varphi)\big)^{-1}(\eta)$. Inserting the preceding estimate into the right-hand side of (11.34), we find for $\varrho \in T(\varphi)(A(0,\delta) \backslash \mathbb{B}_2(0, R))$ and for $n, t \in \mathbb{N}$:

$$\left|\Big(M_{jk} \cdot \Psi_t \cdot (\Psi_n \cdot w_k)^\wedge\Big)^\vee(\varrho)\right|$$
$$\leq C_{53,2}(\vartheta, \varphi) \cdot \sin^{-4}(\delta/4) \cdot |\lambda|^{-1/2}$$
$$\cdot \int_{\mathbb{R}^2 \backslash T(\varphi)(A(0, 3\cdot\delta/2) \cup \mathbb{B}_2(0,R))} \Big(|(T(\varphi))^{-1}(\varrho)| + |\eta|\Big)^{-4} \cdot |w_k(\eta)|\, d\eta.$$

Now we may apply Lemma 5.22, which yields

$$\left\|(M_{jk} \cdot \hat{w}_k)^\vee \,\middle|\, T(\varphi)(A(0,\delta) \backslash \mathbb{B}_2(0, R))\right\|_p$$
$$\leq C_{53,2}(\vartheta, \varphi) \cdot \sin^{-4}(\delta/4) \cdot |\lambda|^{-1/2}$$
$$\cdot \Bigg(\int_{T(\varphi)(A(0,\delta) \backslash \mathbb{B}_2(0,R))} \Big(\int_{\mathbb{R}^2 \backslash T(\varphi)(\mathbb{B}_2(0,R))} \Big(|(T(\varphi))^{-1}(\varrho)| + |\eta|\Big)^{-4} \cdot |w_k(\eta)|\, d\eta\Big)^p d\varrho\Bigg)^{1/p}$$
$$= C_{53,2}(\vartheta, \varphi) \cdot \sin^{-1/p}(\varphi) \cdot \sin^{-4}(\delta/4) \cdot |\lambda|^{-1/2}$$
$$\cdot \Bigg(\int_{A(0,\delta) \backslash \mathbb{B}_2(0,R)} \Big(\int_{\mathbb{R}^2 \backslash T(\varphi)(\mathbb{B}_2(0,R))} (|\xi| + |\eta|)^{-4} \cdot |w_k(\eta)|\, d\eta\Big)^p d\xi\Bigg)^{1/p}$$
$$\leq C_{53,2}(\vartheta, \varphi) \cdot \sin^{-1/p}(\varphi) \cdot \sin^{-4}(\delta/2) \cdot |\lambda|^{-1/2}$$
$$\cdot \Bigg(\int_{\mathbb{R}^2 \backslash \mathbb{B}_2(0,R)} |\xi|^{-2\cdot p} \cdot \Big(\int_{\mathbb{R}^2 \backslash \mathbb{B}_2(0,R)} |\eta|^{-2} \cdot |w_k(\eta)|\, d\eta\Big)^p d\xi\Bigg)^{1/p}$$
$$\leq C_{53,3}(p, \vartheta, \varphi) \cdot \sin^{-4}(\delta/4) \cdot |\lambda|^{-1/2} \cdot R^{-2} \cdot \|w\|_p.$$

This inequality implies the lemma.

Now we shall make use of Lemma 11.10, with w replaced by certain functions related to $\mathcal{A}^{(v)}(\varphi, \lambda, \delta, R, S, \Phi)$ and $\mathcal{B}^{(v)}(\varphi, \delta, R, S, \Phi)$, for $v \in \{2, 11\}$.

Lemma 11.11. *Let $p \in (1,\infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$. Then there is some constant $C_{54}(p, \vartheta, \varphi) > 0$ such that*

$$\left\|\big(M \cdot \hat{G}\big)^\vee \circ T(\varphi) \,\middle|\, A(0,\delta) \backslash \mathbb{B}_2(0, R)\right\|_p \tag{11.35}$$
$$\leq C_{54}(p, \vartheta, \varphi) \cdot \Big(\sin^{-4}(\delta/4) \cdot \delta^{-1} \cdot |\lambda|^{-1/2} + (S-1)^{-2/p}\Big) \cdot \|\Phi\|_p$$

for $G \in \bigcup_{v \in \{2,11\}} \Big\{\mathcal{A}^{(v)}(\varphi, \lambda, \delta, R, S, \Phi),\ \mathcal{B}^{(v)}(\varphi, \delta, R, S, \Phi)\Big\}$, $\lambda \in \mathbb{C}$ with $|\arg\lambda| \leq \vartheta$, $|\lambda| \geq 1$, $\delta \in (0, \pi/8)$, $R, S \in (2,\infty)$, $\tau \in \{-1, 1\}$, $\Phi \in L^p(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R\cdot S))^3 \cap L^2(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R\cdot S))^3$,

$$M \in \Big\{ (1-\Psi_1)\cdot A_\tau^{\infty,\varphi} \cdot \big((A_\tau^{\lambda,\varphi})^{-1} - (A_\tau^{\infty,\varphi})^{-1} \big), \quad (1-\Psi_1)\cdot (A_\tau^{\lambda,\varphi} - A_\tau^{\infty,\varphi}) \cdot (A_\tau^{\infty,\varphi})^{-1} \Big\}.$$

Proof: Define

$$C_{54,1}(p,\vartheta,\varphi) := C_{53}(p,\vartheta,\varphi)\cdot \sin^{-1/p}(\varphi)\cdot \max\{ C_{46}(p,\vartheta,\varphi), C_{47}(p,\varphi)\};$$

$$C_{54,2}(p,\vartheta,\varphi) := C_{27}(p,\vartheta)\cdot \max\{ C_{44}(p,\vartheta,\varphi), C_{45}(p,\varphi)\};$$

$$C_{54,3}(p,\vartheta,\varphi) := 2\cdot C_{29}(p,\vartheta)\cdot \max\{ C_{51}(p,\vartheta,\varphi), C_{52}(p,\varphi)\};$$

$$C_{54,4}(p,\vartheta,\varphi) := \sum_{\upsilon=1}^{3} C_{54,\upsilon}(p,\vartheta,\varphi); \qquad C_{54}(p,\vartheta,\varphi) := 3\cdot C_{54,4}(p,\vartheta,\varphi),$$

and let $\lambda, \delta, R, S, \tau, \Phi, M$ be given as in the lemma. For shortness we set

$$F_1 := -\sin^{-1}(\varphi)\cdot \Big((1-\zeta^{(\delta)})\cdot \chi_{A(0,3\cdot\delta)}\cdot \chi_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R)} \cdot \sum_{j=1}^{3} (K_{jl}^{(2)}(\varphi,\lambda,\delta)\otimes\Phi_j)\Big)_{1\le l\le 3} \circ (T(\varphi))^{-1};$$

$$F_2 := -\sin^{-1}(\varphi)\cdot \Big(\zeta^{(\delta)}\cdot (1-g^{(R)})\cdot \chi_{A(0,3\cdot\delta)}\cdot \chi_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R)} \cdot \sum_{j=1}^{3} (K_{jl}^{(2)}(\varphi,\lambda,\delta)\otimes\Phi_j)\Big)_{1\le l\le 3} \circ (T(\varphi))^{-1};$$

$$F_3 := -\sin^{-1}(\varphi)\cdot \Big(\zeta^{(\delta)}\cdot g^{(R)}\cdot \chi_{A(0,3\cdot\delta)}\cdot \chi_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R)} \cdot \sum_{j=1}^{3} (K_{jl}^{(2)}(\varphi,\lambda,\delta)\otimes\Phi_j)\Big)_{1\le l\le 3} \circ (T(\varphi))^{-1}.$$

Furthermore, we define F_4, F_5, F_6 in the same way as F_1, F_2, F_3, respectively, up to a single difference: The function $K_{jl}^{(2)}(\varphi,\lambda,\delta)\otimes\Phi_j$ is to be replaced by $L_{jl}^{(2)}(\varphi,\delta)\otimes_p\Phi_j$, for $j, l \in \{1,2,3\}$.

We point out the following equations, which are an immediate consequence of Definition 10.3:

$$\mathcal{A}^{(2)}(\varphi,\lambda,\delta,R,S,\Phi) = F_1 + F_2 + F_3; \tag{11.36}$$

$$\mathcal{B}^{(2)}(\varphi,\delta,R,S,\Phi) = F_4 + F_5 + F_6.$$

According to Lemma 11.7, we have $F_1 \in L^p(\mathbb{R}^2)^3 \cap L^2(\mathbb{R}^2)^3$. Recalling the properties of $\zeta^{(\delta)}$ (Lemma 11.9), we further note that

$$supp(F_1) \subset T(\varphi)\big(\mathbb{R}^2\backslash(A(0,3\cdot\delta/2)\cup\mathbb{B}_2(0,R))\big).$$

Now it may be deduced from Lemma 11.10 and Lemma 11.7:

$$\Big\| (M\cdot \widehat{F_1})^\vee \circ T(\varphi) \,\Big|\, A(0,\delta)\backslash\mathbb{B}_2(0,R)\Big\|_p \le C_{54,1}(p,\vartheta,\varphi)\cdot|\lambda|^{-1/2}\cdot\sin^{-4}(\delta/4)\cdot\|\Phi\|_p.$$

On the other hand, due to the poperties of $g^{(R)}$ (see Lemma 11.9), we get for $\xi\in\mathbb{R}^2$:

$$|F_2(\xi)| \le \Big| \Big(\sin^{-1}(\varphi)\cdot\chi_{\mathbb{B}_2(0,R+1)}\cdot\sum_{j=1}^{3}(K_{jl}^{(2)}(\varphi,\lambda,\delta)\otimes\Phi_j)\Big)_{1\le l\le 3} \circ (T(\varphi))^{-1}(\xi)\Big|. \tag{11.37}$$

Thus, using Lemma 5.19, we arrive at the estimate

$$\Big\| (M\cdot \widehat{F_2})^\vee \circ T(\varphi) \,\Big|\, A(0,\delta)\backslash\mathbb{B}_2(0,R)\Big\|_p \le \|(M\cdot\widehat{F_2})^\vee\|_p \le C_{27}(p,\vartheta)\cdot\|F_2\|_p$$

$$\le C_{54,2}(p,\vartheta,\varphi)\cdot(R\cdot S/(R+1) - 1)^{-2/p}\cdot\|\Phi\|_p,$$

where the last inequality is a consequence of (11.37) and Lemma 11.6, with S replaced by $R\cdot S/(R+1)$ in the latter reference. On the other hand, Corollary 11.5 and Lemma 5.21 yield

$$\Big\| (M\cdot \widehat{F_3})^\vee \circ T(\varphi) \,\Big|\, A(0,\delta)\backslash\mathbb{B}_2(0,R)\Big\|_p \le \|(M\cdot\widehat{F_3})^\vee\|_p \le C_{29}(p,\vartheta)\cdot|\lambda|^{-1/2}\cdot\sum_{i=1}^{2}\|D_iF_3\|_p$$

$$\le C_{54,3}(p,\vartheta,\varphi)\cdot|\lambda|^{-1/2}\cdot\sin^{-3}(\delta/2)\cdot\delta^{-1}\cdot\|\Phi\|_p.$$

By combining the preceding estimates with (11.36), we infer

$$\Big\| \big(M\cdot(\mathcal{A}^{(2)}(\varphi,\lambda,\delta,R,S,\Phi))^\wedge\big)^\vee \circ T(\varphi)\,\Big|\, A(0,\delta)\backslash\mathbb{B}_2(0,R)\Big\|_p$$

$$\le C_{54,4}(p,\vartheta,\varphi) \cdot\Big(\sin^{-4}(\delta/4)\cdot\delta^{-1}\cdot|\lambda|^{-1/2} + (R\cdot S/(R+1) - 1)^{-2/p}\Big)\cdot\|\Phi\|_p.$$

Finally, we note that

$$(R\cdot S/(R+1) - 1)^{-1} = (R+1)/(R\cdot(S-1)-1) \le (R+1)/((R-1)\cdot(S-1)) \le 3/(S-1),$$

where the first inequality is valid since $S\ge 2$, $R\ge 1$, and the second one holds because $R\ge 2$. Now inequality (11.35) follows for $G = \mathcal{A}^{(2)}(\varphi,\lambda,\delta,R,S,\Phi)$. If $G = \mathcal{B}^{(2)}(\varphi,\delta,R,S,\Phi)$, we may argue in an analogous way. In addition, we observe that

$supp(\mathcal{A}^{(11)}(\varphi,\lambda,\delta,R,S,\Phi))$, $\quad supp(\mathcal{B}^{(11)}(\varphi,\delta,R,S,\Phi))$

$$\subset \; T(\varphi)\Big(\mathbb{R}^2 \backslash \big(A(0, 3\cdot\delta/2) \cup \mathbb{B}_2(0, R) \big)\Big).$$

Recall that by Lemma 10.2, we have

$$\mathcal{A}^{(11)}(\varphi, \lambda, \delta, R, S, \Phi), \quad \mathcal{B}^{(11)}(\varphi, \delta, R, S, \Phi) \quad \in \; L^p(\mathbb{R}^2)^3 \cap L^2(\mathbb{R}^2)^3.$$

Hence we infer from Lemma 11.10 and 11.7:

$$\Big\| \Big(M \cdot \big(\mathcal{A}^{(11)}(\varphi, \lambda, \delta, R, S, \Phi)\big)^{\wedge}\Big)^{\vee} \circ T(\varphi) \,\Big|\, A(0, \delta)\backslash \mathbb{B}_2(0, R) \Big\|_p ,$$

$$\Big\| \Big(M \cdot \big(\mathcal{B}^{(11)}(\varphi, \delta, R, S, \Phi)\big)^{\wedge}\Big)^{\vee} \circ T(\varphi) \,\Big|\, A(0, \delta)\backslash \mathbb{B}_2(0, R) \Big\|_p$$

$$\leq \; 2 \cdot C_{54,1}(p, \vartheta, \varphi) \cdot |\lambda|^{-1/2} \cdot \sin^{-4}(\delta/4) \cdot \|\Phi\|_p .$$

Thus, in the case $G \in \{ \mathcal{A}^{(11)}(\varphi, \lambda, \delta, R, S, \Phi), \; \mathcal{B}^{(11)}(\varphi, \lambda, \delta, R, S, \Phi) \}$, inequality (11.35) is valid too.

Now we collect our previous results to obtain

Corollary 11.6. *Let* $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$. *Then there is a constant* $C_{55}(p, \vartheta, \varphi) > 0$, *such that for* $\lambda \in \mathbb{C}$ *with* $|\arg \lambda| \leq \vartheta$, $|\lambda| \geq 1$, *and for* $\delta \in (0, \pi/48)$, $R, S \in (2, \infty)$, $\Phi \in L^p(\mathbb{R}^2\backslash \mathbb{B}_2(0, R\cdot S))^3 \cap L^2(\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S))^3$, $\tau \in \{-1, 1\}$, *the ensuing inequalities are valid:*

$$\Big\| \chi_{A(0,\delta)} \cdot J(\tau, p, \lambda, \varphi, R, S)(\Phi) \Big\|_p$$

$$\leq \; C_{55}(p, \vartheta, \varphi) \cdot \Big(\| H^*(\tau, p, \varphi, R, S)(\Phi) \|_p \; + \; \delta^{1/2} \cdot \| \chi_{A(0, 3\cdot\delta)} \cdot \Phi \|_p + \big(S^{-(1 \wedge (2/p))} \; + \; \delta^{-5} \cdot |\lambda|^{-1/2} \big) \cdot \|\Phi\|_p \Big);$$

$$\Big\| \chi_{A(0,\delta)} \cdot H^*(\tau, p, \varphi, R, S)(\Phi) \Big\|_p$$

$$\leq \; C_{55}(p, \vartheta, \varphi) \cdot \Big(\| J(\tau, p, \lambda, \varphi, R, S)(\Phi) \|_p \; + \; \delta^{1/2} \cdot \| \chi_{A(0, 3\cdot\delta)} \cdot \Phi \|_p + \big(S^{-(1 \wedge (2/p))} \; + \; \delta^{-5} \cdot |\lambda|^{-1/2} \big) \cdot \|\Phi\|_p \Big).$$

Proof: We estimate the right-hand side of (10.19) and (10.20) by using Lemma 5.19, 11.2, 11.3, 11.5, 11.6, 11.11, and Corollary 11.1 – 11.4. The present corollary then follows if we note that $S/2 \leq S - 1$ for $S \in (2, \infty)$, and

$$\sin\sigma \;\leq\; \sigma, \quad (1/2)\cdot\sigma \;\leq\; \sin\sigma \qquad \text{for } \sigma \in (0, \pi/4).$$

It should be obvious that in the preceding corollary, we may replace the sector $A(0, \delta)$ by $A(\gamma, \delta)$, for any $\gamma \in [0, 2\cdot\pi)$. We shall check this fact in the next lemma:

Lemma 11.12. *Take* $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\varphi \in (0, \pi/2]$, $\lambda \in \mathbb{C}$ *with* $|\arg \lambda| \leq \vartheta$, $|\lambda| \geq 1$, $\delta \in (0, \pi/48)$, $R, S \in (2, \infty)$, $\gamma \in [0, 2\cdot\pi)$, $\tau \in \{-1, 1\}$, $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S))^3 \cap L^2(\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S))^3$. *Then it holds*

$$\Big\| \chi_{A(\gamma,\delta)} \cdot J(\tau, p, \lambda, \varphi, R, S)(\Phi) \Big\|_p$$

$$\leq \; C_{55}(p, \vartheta, \varphi) \cdot \Big(\| H^*(\tau, p, \varphi, R, S)(\Phi) \|_p \; + \; \delta^{1/2} \cdot \| \chi_{A(\gamma, 3\cdot\delta)} \cdot \Phi \|_p + \big(S^{-(1 \wedge (2/p))} \; + \; \delta^{-5} \cdot |\lambda|^{-1/2} \big) \cdot \|\Phi\|_p \Big);$$

$$\Big\| \chi_{A(\gamma,\delta)} \cdot H^*(\tau, p, \varphi, R, S)(\Phi) \Big\|_p$$

$$\leq \; C_{55}(p, \vartheta, \varphi) \cdot \Big(\| J(\tau, p, \lambda, \varphi, R, S)(\Phi) \|_p \; + \; \delta^{1/2} \cdot \| \chi_{A(\gamma, 3\cdot\delta)} \cdot \Phi \|_p + \big(S^{-(1 \wedge (2/p))} \; + \; \delta^{-5} \cdot |\lambda|^{-1/2} \big) \cdot \|\Phi\|_p \Big).$$

Proof: Set $\tilde{B} := \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}$, $\quad B := \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Define the rotation $f_{\tilde{B}} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ by $f_{\tilde{B}}(\eta) := \tilde{B} \cdot \eta$ für $\eta \in \mathbb{R}^2$. Then fix $\xi, \eta \in \mathbb{R}^2$ with $\xi \neq \eta$. It holds

$$\Big(\sum_{j,k=1}^{3} \widetilde{\mathcal{D}}^{\lambda}_{jkl}\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) \cdot \big(n_k^{(\varphi)} \circ g^{(\varphi)}\big)(\eta) \cdot \Phi_j(\eta) \Big)_{1 \leq l \leq 3} \tag{11.38}$$

$$= \; B \cdot \Big(\sum_{j,k=1}^{3} \widetilde{\mathcal{D}}^{\lambda}_{jkl}\Big(B \cdot \big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big)\Big) \cdot \big(n_k^{(\varphi)} \circ g^{(\varphi)}\big)(\tilde{B}^T \cdot \eta) \cdot \big(B^T \cdot \Phi\big)_j(\eta) \Big)_{1 \leq l \leq 3}.$$

In order to prove this result, the left-hand side of (11.38) is split up into a sum according to the way the function $\widetilde{\mathcal{D}}^{\lambda}_{jkl}$ was written as a sum in (5.11), (5.12). Then three summands arise, which are transformed as indicated by the following example:

$$\Big(\sum_{j,k=1}^{3} \delta_{kl} \cdot \mathcal{Y}^{\lambda}_j\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) \cdot \big(n_k^{(\varphi)} \circ g^{(\varphi)}\big)(\eta) \cdot \Phi_j(\eta) \Big)_{1 \leq l \leq 3}$$

$$= \; \mathcal{G}^{\lambda}_1\big(|g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)|^2\big) \cdot \Big(\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) \cdot \Phi(\eta) \Big) \cdot \big(n^{(\varphi)} \circ g^{(\varphi)}\big)(\eta)$$

$$= \mathcal{G}_1^\lambda\Big(\big|B^T\cdot(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta))\big|^2\Big)$$
$$\cdot\Big[\Big(B^T\cdot(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta))\Big)\cdot\Big(B^T\cdot\Phi(\eta)\Big)\Big]\cdot B\cdot(n^{(\varphi)}\circ g^{(\varphi)})(\tilde{B}^T\cdot\eta)$$
$$= B\cdot\Big(\sum_{j,k=1}^{3}\delta_{kl}\cdot\mathcal{Y}_j^\lambda\Big(B^T\cdot(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta))\Big)\cdot(n_k^{(\varphi)}\circ g^{(\varphi)})(\tilde{B}^T\cdot\eta)\cdot(B^T\cdot\Phi)_j(\eta)\Big)_{1\le l\le 3}$$

where the first of the preceding equations, as well as the third one, follows from Lemma 5.7. Concerning the second one, observe that it holds due to (3.2):

$$B^T\cdot(n^{(\varphi)}\circ g^{(\varphi)})(\eta) = B^T\cdot(\cos\varphi\cdot\eta_1/|\eta|,\ \cos\varphi\cdot\eta_2/|\eta|,\ -\sin\varphi)$$
$$= (\cos\varphi\cdot|\eta|^{-1}\cdot\tilde{B}^T\cdot\eta,\ -\sin\varphi) = (n^{(\varphi)}\circ g^{(\varphi)})(\tilde{B}^T\cdot\eta).$$

Referring to equation (11.38) and (6.11), we obtain by an application of the substitution rule:

$$\Big\|\chi_{A(\gamma,\delta)}\cdot J(\tau,p,\lambda,\varphi,R,S)(\Phi)\Big\|_p \tag{11.39}$$
$$= \Big\|\chi_{A(0,\delta)}\cdot J(\tau,p,\lambda,\varphi,R,S)\big((B^T\cdot\Phi)\circ f_{\tilde{B}}\big)\Big\|_p.$$

When checking this result, note that

$$\tilde{B}^T\cdot(\cos(\gamma+\varrho),\ \sin(\gamma+\varrho)) = (\cos\varrho,\ \sin\varrho); \tag{11.40}$$
$$B^T\cdot(g^{(\varphi)}(\xi)-g^{(\varphi)}(\eta)) = g^{(\varphi)}(\tilde{B}^T\cdot\xi)-g^{(\varphi)}(\tilde{B}^T\cdot\eta)$$

for $\varrho\in[-\pi,\pi]$, $t\in(0,\infty)$, $\xi,\eta\in\mathbb{R}^2$. Using (6.10), and proceeding as in the proof of (11.39), we find

$$\Big\|\chi_{A(\gamma,\delta)}\cdot H^*(\tau,p,\varphi,R,S)(\Phi)\Big\|_p \tag{11.41}$$
$$= \Big\|\chi_{A(0,\delta)}\cdot H^*(\tau,p,\varphi,R,S)\big((B^T\cdot\Phi)\circ f_{\tilde{B}}\big)\Big\|_p$$

Finally, equation (11.40) implies

$$\Big\|\chi_{A(0,3\cdot\delta)}\cdot\big((B^T\cdot\Phi)\circ f_{\tilde{B}}\big)\Big\|_p = \|\chi_{A(\gamma,3\cdot\delta)}\cdot\Phi\|_p; \tag{11.42}$$
$$\|(B^T\cdot\Phi)\circ f_{\tilde{B}}\|_p = \|\Phi\|_p.$$

Combining Corollary 11.6 with (11.39), (11.41) and (11.42) yields the lemma.

Now we are in a position to prove the main result of this chapter:

Theorem 11.1. *Take $p\in(1,\infty)$, $\vartheta\in[0,\pi)$, $\varphi\in(0,\pi/2]$. Then there exists a constant $C_{56}(p,\vartheta,\varphi)>0$ such that for $\lambda\in\mathbb{C}$ with $|\arg\lambda|\le\vartheta$, $|\lambda|\ge 1$, $N\in\mathbb{N}$ with $N>24$, $R,S\in(2,\infty)$, $\tau\in\{-1,1\}$, $\Phi\in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0,R\cdot S))^3$, the following inequalities hold true:*

$$\|J(\tau,p,\lambda,\varphi,R,S)(\Phi)\|_p \tag{11.43}$$
$$\le C_{56}(p,\vartheta,\varphi)\cdot\Big(N\cdot\|H^*(\tau,p,\varphi,R,S)(\Phi)\|_p + \Big(N^{-1/2}+N\cdot S^{-(1\wedge(2/p))}+N^6\cdot|\lambda|^{-1/2}\Big)\cdot\|\Phi\|_p\Big);$$

$$\|H^*(\tau,p,\varphi,R,S)(\Phi)\|_p \tag{11.44}$$
$$\le C_{56}(p,\vartheta,\varphi)\cdot\Big(N\cdot\|J(\tau,p,\lambda,\varphi,R,S)(\Phi)\|_p + \Big(N^{-1/2}+N\cdot S^{-(1\wedge(2/p))}+N^6\cdot|\lambda|^{-1/2}\Big)\cdot\|\Phi\|_p\Big).$$

We mention a consequence of these inequalities which will be made precise in the proof of Theorem 12.1 and 12.2: If N, S and λ are chosen in a suitable way, all the terms on the right-hand side of (11.43) and (11.44) become small, except the expressions $N\cdot\|H^*(\tau,p,\varphi,R,S)(\Phi)\|_p$ and $N\cdot\|J(\tau,p,\lambda,\varphi,R,S)(\Phi)\|_p$, respectively. In this sense, we have estimated $J(\tau,p,\lambda,\varphi,R,S)$ against $H^*(\tau,p,\varphi,R,S)$, and vice versa.

Proof: Define $C_{56}(p,\vartheta,\varphi):=9\cdot(\pi/2)\cdot C_{55}(p,\vartheta,\varphi)$, and let $\lambda, N, R, S, \tau, \Phi$ be given as in the lemma. Assume in addition that Φ belongs to $L^2(\mathbb{R}^2\backslash\mathbb{B}_2(0,R\cdot S))^3$.

Since $N>24$, we have $\pi/(2\cdot N)<\pi/48$. Hence Lemma 11.12 may be applied with $\delta=\pi/(2\cdot N)$ and $\gamma=(k+1/2)\cdot\pi/N$, for $k\in\{0,\cdots,N-1\}$. It follows

$$\|J(\tau,p,\varphi,R,S)(\Phi)\|_p^p = \sum_{k=0}^{N-1}\Big\|\chi_{A((k+1/2)\cdot\pi/N,\ \pi/(2\cdot N))}\cdot J(\tau,p,\varphi,R,S)(\Phi)\Big\|_p^p$$
$$\le \sum_{k=0}^{N-1}(C_{55}(p,\vartheta,\varphi))^p\cdot 3^{p-1}\cdot\Big(\|H^*(\tau,p,\varphi,R,S)(\Phi)\|_p^p + (\pi\cdot(2\cdot N)^{-1})^{p/2}\cdot\|\chi_{A((k+1/2)\cdot\pi/N,\ 3\cdot\pi/(2\cdot N))}\cdot\Phi\|_p^p + \Big(S^{-(1\wedge(2/p))}+(2\cdot N/\pi)^5\cdot|\lambda|^{-1/2}\Big)^p\cdot\|\Phi\|_p^p\Big)$$
$$= (C_{55}(p,\vartheta,\varphi))^p\cdot 3^{p-1}\cdot\Big(N\cdot\|H^*(\tau,p,\varphi,R,S)(\Phi)\|_p^p + (\pi\cdot(2\cdot N)^{-1})^{p/2}\cdot\sum_{k=0}^{N-1}\|\chi_{A((k+1/2)\cdot\pi/N,\ 3\cdot\pi/(2\cdot N))}\cdot\Phi\|_p^p$$

$$+ N \cdot \left(S^{-(1 \wedge (2/p))} + (2 \cdot N/\pi)^5 \cdot |\lambda|^{-1/2} \right)^p \cdot \|\Phi\|_p^p \Big),$$

where we used Lemma 11.12 in the first inequality. Next we note that

$$\sum_{k=0}^{N-1} \| \chi_{A((k+1/2)\cdot\pi/N,\ 3\cdot\pi/(2\cdot N))} \cdot \Phi \|_p^p$$

$$= \sum_{k=0}^{N-1} \sum_{i \in \{-1,0,1\}} \| \chi_{A((k+1/2-i)\cdot\pi/N,\ \pi/(2\cdot N))} \cdot \Phi \|_p^p = 3 \cdot \|\Phi\|_p^p .$$

Combining the preceding results yields inequality (11.43). Recall that we assumed $\Phi \in L^2(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R \cdot S))^3$. But this condition can be removed by means of an approximation argument based on Lemma 5.12, Corollary 4.2 and (6.11).

As for the estimate in (11.44), it may be established by analogous arguments.

Chapter 12

Fredholm Properties of the Operator $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$

In the following, we shall put to use our previous results – many of them unrelated up to now – in order to find out the Fredholm properties of $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$. It will turn out that $\Gamma^{(inf)}(\tau, p, \lambda, \varphi, R)$ is closely connected to $\Pi^*(-\tau, p, \mathbb{K}(\varphi))$ (Theorem 12.1, 12.2), $\Gamma^{(ver)}(\tau, p, \lambda, \varphi, R)$ to $\Lambda^{(ver)}(\tau, p, \varphi, R)$ (Lemma 12.6), and finally $\Lambda^{(ver)}(\tau, p, \varphi, R)$ to $\Lambda(\tau, p, \mathbb{K}(\varphi))$ (Theorem 12.3). Moreover, we shall recall Corollary 6.5 and 6.6, which imply that $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ is Fredholm if and only if the operators $\Gamma^{(inf)}(\tau, p, \lambda, \varphi, R)$ and $\Gamma^{(ver)}(\tau, p, \lambda, \varphi, R)$ have the same property. By combining these results, it will be possible to link the Fredholm properties of $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ with those of the operators $\Pi^*(-\tau, p, \mathbb{K}(\varphi))$ and $\Lambda(\tau, p, \mathbb{K}(\varphi))$; see Theorem 1.5. Further consequences include a result on non-regularity of the operator $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ (Corollary 12.7), as well as the fact that inequality (1.24) cannot hold for certain values of p (Corollary 12.9).

We begin our discussion by checking how $J(\tau, p, \lambda, \varphi, R, S)$ is transformed when acting on a translation.

Lemma 12.1. *Take* $p \in (1, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\varphi \in (0, \pi/2]$, $R \in [0, \infty)$, $\mu \in (0, \infty)$, $S \in [1, \infty)$, $\tau \in \{-1, 1\}$, $\Phi \in L^p\left(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R \cdot S)\right)^3$.

For brevity we set $I_1 := id\left(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R \cdot S/\mu)\right)$, $I_2 := id\left(\mathbb{R}^2 \backslash \mathbb{B}_2(0, R)\right)$. *Then it holds*

$$J(\tau, p, \lambda, \varphi, R, S)(\Phi)$$

$$= \Big(J(\tau, p, \mu^2 \cdot \lambda, \varphi, R/\mu, S)\left(\Phi \circ (\mu \cdot I_1)\right) \Big) \circ ((1/\mu) \cdot I_2).$$

This means in particular that

$$\| J(\tau, p, \lambda, \varphi, R, S)(\Phi) \|_p = \mu^{2/p} \cdot \left\| J(\tau, p, \mu^2 \cdot \lambda, \varphi, R/\mu, S)\left(\Phi \circ (\mu \cdot I_1)\right) \right\|_p .$$

Proof: The lemma follows by the substitution rule. Note that

$$\mu^2 \cdot \widetilde{\mathcal{D}}^{\lambda}_{jkl}(\mu \cdot z) = \widetilde{\mathcal{D}}^{\mu^2 \cdot \lambda}_{jkl}(z) \qquad \text{for } z \in \mathbb{R}^3\backslash\{0\}.$$

Lemma 12.2 *Let B be a Banach space, $A : B \mapsto B$ a linear bounded operator, and $A^* : B \mapsto B$ the adjoint of A. Furthermore, assume that A is Fredholm with index 0, and suppose that A or A^* is one-to-one. Then A and A^* are both topological.*

Proof: The operator A^* is bounded too ([29, p. 154]). In addition, according to [29, p. 234, Theorem 5.13], A^* is Fredholm with index 0. Since A or A^* is one-to-one, it follows now that A or A^* is bijective. Thus we obtain by the open mapping theorem that A or A^* is topological. But then Lemma 8.5 yields that both operators A and A^* are topological.

Theorem 12.1. *Let $p \in (1, \infty)$, $\varphi_0 \in (0, \pi/2]$, $\tau \in \{-1, 1\}$, and assume the operator $\Pi^*(-\tau, p, \mathbb{K}(\varphi))$ is Fredholm for any $\varphi \in [\varphi_0, \pi/2]$.*

Then, for $R \in (0, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, the operator $\Gamma^{(inf)}(\tau, p, \lambda, \varphi_0, R)$ is also Fredholm with index 0.

Proof: By Corollary 6.6 and our assumptions in the theorem, we know that $F^*(\tau, p, \varphi, 0, 1)$ is Fredholm for any $\varphi \in [\varphi_0, \pi/2]$. Corollary 6.3 then yields that $F^*(\tau, p, \varphi, 1, 1)$ is Fredholm if $\varphi \in [\varphi_0, \pi/2]$. This means by Lemma 6.17:

$$index(F^*(\tau, p, \varphi, 1, 1)) = 0 \qquad \text{for } \varphi \in [\varphi_0, \pi/2].$$

Now fix $\varphi \in [\varphi_0, \pi/2]$. Due to our preceding result on $F^*(\tau, p, \varphi, 1, 1)$, and because of Corollary 6.4, we may conclude that $G^*(\tau, p, \varphi, 1)$ is also a Fredholm operator with index 0.

The operator $G(\tau, p/(p-1), \varphi, 1)$ is adjoint to $G^*(\tau, p, \varphi, 1)$ and, in addition, it is one-to-one and bounded (Corollary 9.3, Lemma 6.7). Thus we deduce from Lemma 12.2 that $G^*(\tau, p, \varphi, 1)$ is topological. This implies there is a constant $\mathfrak{C}_1 > 0$ with

$$\|\Phi\|_p \;\le\; \mathfrak{C}_1 \cdot \| G^*(\tau, p, \varphi, 1)(\Phi) \|_p \qquad \text{for } \Phi \in L^p(\mathbb{R}^2). \tag{12.1}$$

Defining $\mathfrak{C}_2 := \sin^{-1}(\varphi) \cdot C_1 \cdot (4 \cdot \pi)^{-1}$, $\mathfrak{C}_3 := \mathfrak{C}_2 \cdot C_{33}(p)$, we conclude from (6.14), Lemma 3.2 and 11.1, for $R \in (0, \infty)$, $S \in [2, \infty)$, $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R \cdot S))$:

$$\left\| G^*(\tau, p, \varphi, 1)\big(\widetilde{\mathcal{F}}(R \cdot S)(\Phi)\big) \,\middle|\, \mathbb{B}_2(0, R) \right\|_p \tag{12.2}$$

$$\le \mathfrak{C}_2 \cdot \left(\int_{\mathbb{B}_2(0,R)} \left(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S)} |\xi - \eta|^{-2} \cdot |\Phi(\eta)| \, d\eta \right)^p d\xi \right)^{1/p}$$

$$\le \mathfrak{C}_3 \cdot (S-1)^{-2/p} \cdot \|\Phi\|_p .$$

Put $\mathfrak{C}_4 := (2 \cdot \mathfrak{C}_1 \cdot \mathfrak{C}_3)^{p/2} + 2$. Then we get for $R \in [1, \infty)$, $S \in [\mathfrak{C}_4, \infty)$, $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R \cdot S))$:

$$\|\Phi\|_p \le \mathfrak{C}_1 \cdot \left\| G^*(\tau, p, \varphi, 1)\big(\widetilde{\mathcal{F}}(R \cdot S)(\Phi)\big) \right\|_p$$

$$= \mathfrak{C}_1 \cdot \| F^*(\tau, p, \varphi, R, S)(\Phi) \|_p + \mathfrak{C}_1 \cdot \left\| G^*(\tau, p, \varphi, 1)\big(\widetilde{\mathcal{F}}(R \cdot S)(\Phi)\big) \,\middle|\, \mathbb{B}_2(0, R) \right\|_p$$

$$\le \mathfrak{C}_1 \cdot \| F^*(\tau, p, \varphi, R, S)(\Phi) \|_p + \mathfrak{C}_1 \cdot \mathfrak{C}_3 \cdot (S-1)^{-2/p} \cdot \|\Phi\|_p$$

$$\le \mathfrak{C}_1 \cdot \| F^*(\tau, p, \varphi, R, S)(\Phi) \|_p + (1/2) \cdot \|\Phi\|_p ,$$

where the first line of this estimate is valid due to (12.1), the second one follows from (6.8), (6.14), (3.6) and (3.7), and the third one is a consequence of (12.2). Hence we have shown for $R \in [1, \infty)$, $S \in [\mathfrak{C}_4, \infty)$, $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R \cdot S))$:

$$\|\Phi\|_p \le 2 \cdot \mathfrak{C}_1 \cdot \| F^*(\tau, p, \varphi, R, S)(\Phi) \|_p . \tag{12.3}$$

Thus, if $R \in [1, \infty)$, $S \in [\mathfrak{C}_4, \infty)$, the operator $F^*(\tau, p, \varphi, R, S)$ is one-to-one and has closed range. This means in particular that $F^*(\tau, p, \varphi, R, S)$ has property F_+ (see Chapter 2).

Now fix $R_0 \in (2, \infty)$ and $S_0 \in [\mathfrak{C}_4, \infty)$. We are going to prove that the operator $H^*(\tau, p, \varphi, R_0, S_0)$ is one-to-one and has closed range too.

Due to Lemma 6.8, we see that $H^*(\tau, p, \varphi, R_0, S_0)$ must be one-to-one since $F^*(\tau, p, \varphi, R_0, S_0)$ has the same property. This leaves us to show that $H^*(\tau, p, \varphi, R, S)$ has closed range. Therefore we take a sequence $(g_k)_{k\in\mathbb{N}}$ in the range of $H^*(\tau, p, \varphi, R_0, S_0)$, and we assume that $(g_k)_{k\in\mathbb{N}}$ converges with respect to the norm of the space $L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R_0))^3$. For $k \in \mathbb{N}$, choose

$$\Phi_k \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R_0 \cdot S_0))^3 \quad \text{with} \quad g_k = H^*(\tau, p, \varphi, R_0, S_0)(\Phi_k).$$

Then the sequence

$$\Big(\big(n^{(\varphi)} \circ g^{(\varphi)}\big) \cdot H^*(\tau, p, \varphi, R_0, S_0)(\Phi_k) \Big)_{k\in\mathbb{N}}$$

tends to a limit with respect to the norm of $L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R_0))$. Hence, by Lemma 6.8, the same must be true for

$$\Big(F^*(\tau, p, \varphi, R_0, S_0)\Big(\big(n^{(\varphi)} \circ g^{(\varphi)}\big) \cdot \Phi_k \Big) \Big)_{k\in\mathbb{N}}.$$

Now inequality (12.3) yields that the sequence $\Big(\big(n^{(\varphi)} \circ g^{(\varphi)}\big) \cdot \Phi_k \Big)_{k\in\mathbb{N}}$ converges in the space $L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R_0 \cdot S_0))$. Recurring to Corollary 4.2, we obtain convergence of

$$\Big(L_j(0, \varphi) \otimes_p \Big(\big(n^{(\varphi)} \circ g^{(\varphi)}\big) \cdot \Phi_k \Big) \Big)_{k\in\mathbb{N}}$$

in $L^p(\mathbb{R}^2)$, for any $j \in \{1, 2, 3\}$. Thus, by the definition of $H^*(\tau, p, \varphi, R_0, S_0)$, it is clear the sequence $(\Phi_k)_{k\in\mathbb{N}}$ tends to a limit function in the space $L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R_0\cdot S_0))^3$. Since $H^*(\tau, p, \varphi, R_0, S_0)$ is bounded (see Corollary 4.2), it follows

$$L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R_0))^3 - \lim_{k\to\infty} g_k \quad \in \quad im\Big(H^*(\tau, p, \varphi, R_0, S_0)\Big).$$

Thus we have shown that $H^*(\tau, p, \varphi, R_0, S_0)$ has closed range. Furthermore, this operator is bounded and one-to-one, as explained above. Therefore, we are now able to apply the open mapping theorem, which yields existence of a constant $\mathfrak{C}_5 > 0$ with

$$\|\Phi\|_p \quad \le \quad \mathfrak{C}_5 \cdot \|\, H^*(\tau, p, \varphi, R_0, S_0)(\Phi)\,\|_p \qquad \text{for } \Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R_0\cdot S_0))^3.$$

Of course this implies

$$\|\Phi\|_p \quad \le \quad \mathfrak{C}_5 \cdot \|\, H^*(\tau, p, \varphi, R_0, S)(\Phi)\,\|_p \tag{12.4}$$

for $S \in [S_0, \infty)$, $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R_0\cdot S))^3$.

Next we fix $\vartheta \in (-\pi, \pi)$. Then, applying (12.4) and Theorem 11.1, we find for $\mu \in [1, \infty)$, $N \in \mathbb{N}$ with $N \ge 24$, $S \in [S_0, \infty)$, $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R_0\cdot S))^3$:

$$\begin{aligned}\|\Phi\|_p \quad \le \quad & \mathfrak{C}_6 \cdot \Big(N \cdot \|\, J(\tau, p, \mu\cdot e^{i\cdot\vartheta}, \varphi, R_0, S)(\Phi)\,\|_p \\ & + \big(N^{-1/2} + N\cdot S^{-(1\wedge(2/p))} + N^6\cdot\mu^{-1/2}\big)\cdot\|\Phi\|_p\Big),\end{aligned} \tag{12.5}$$

with $\mathfrak{C}_6 := \mathfrak{C}_5 \cdot C_{56}(p, |\vartheta|, \varphi)$. Define $\mathfrak{C}_7 := \max\{24,\, 16\cdot(\mathfrak{C}_6)^2\} + 1$, and let N denote the unique integer belonging to the interval $[\mathfrak{C}_7 - 1, \mathfrak{C}_7)$. It follows from (12.5), for μ, S, Φ as before:

$$\begin{aligned}\|\Phi\|_p \quad \le \quad & (1/4)\cdot\|\Phi\|_p + \mathfrak{C}_6\cdot\mathfrak{C}_7\cdot\|\, J(\tau, p, \mu\cdot e^{i\cdot\vartheta}, \varphi, R_0, S)(\Phi)\,\|_p \\ & + \Big(C_6\cdot\mathfrak{C}_7\cdot S^{-(1\wedge(2/p))} + C_6\cdot(\mathfrak{C}_7)^6\cdot\mu^{-1/2}\Big)\cdot\|\Phi\|_p.\end{aligned}$$

Next choose $S_1 \in \mathbb{R}$ with $S_1 \ge S_0$, $S_1 \ge (4\cdot\mathfrak{C}_7\cdot\mathfrak{C}_6)^{1/(1\wedge(2/p))}$, and put

$$\mu_0 := \big(\mathfrak{C}_6\cdot(\mathfrak{C}_7)^6\cdot 4\big)^2 + 1, \qquad \mathfrak{C}_8 := \mathfrak{C}_6\cdot\mathfrak{C}_7.$$

It follows for $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R_0\cdot S_1))^3$:

$$\|\Phi\|_p \quad \le \quad (3/4)\cdot\|\Phi\|_p + \mathfrak{C}_8\cdot\|\, J(\tau, p, \mu_0\cdot e^{i\cdot\vartheta}, \varphi, R_0, S_1)(\Phi)\,\|_p,$$

that is,

$$\|\Phi\|_p \quad \le \quad 4\cdot\mathfrak{C}_8\cdot\|\, J(\tau, p, \mu_0\cdot e^{i\cdot\vartheta}, \varphi, R_0, S_1)(\Phi)\,\|_p.$$

Hence the operator $J(\tau, p, \mu_0\cdot e^{i\cdot\vartheta}, \varphi, R_0, S_1)$ is one-to-one, with closed range. In particular, it has property F_+. Taking account of Corollary 6.1, we see that $J(\tau, p, \mu_0\cdot e^{i\cdot\vartheta}, \varphi, R_0\cdot S_1, 1)$ is F_+ too. Recalling Lemma 12.1, 6.11 and Corollary 6.2, we conclude that $J(\tau, p, t\cdot e^{i\cdot\vartheta}, \varphi, R, 1)$ is F_+ for any $R, t \in (0, \infty)$.

But ϑ was arbitrarily chosen in $(-\pi, \pi)$. Thus the operator $J(\tau, p, \lambda, \varphi, R, 1)$ must be F_+ for any $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $R \in (0, \infty)$. Since φ is an arbitrary member of the interval $[\varphi_0, \pi/2]$, it follows by Lemma 6.17:

$$index(J(\tau, p, \lambda, \varphi_0, R, 1)) \quad = \quad 0 \qquad \text{for } \lambda \in \mathbb{C}\backslash(-\infty, 0],\ R \in (0, \infty),$$

that is, the operator $J(\tau, p, \lambda, \varphi_0, R, 1)$ is Fredholm with index 0. Now Corollary 6.6 yields the conclusion of the theorem.

Corollary 12.1. *Take* $p \in (1, 2]$, $\varphi \in (0, \pi/2]$, $\tau \in \{-1, 1\}$. *Then, for any* $R \in (0, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, *the operator* $\Gamma^{(inf)}(\tau, p, \lambda, \varphi, R)$ *is Fredholm with index* 0.

Proof: According to Corollary 8.3, the operator $\Pi^*(-\tau, p, \mathbb{K}(\varphi))$ is topological for any $\varphi \in (0, \pi/2]$. In particular, it is Fredholm with index 0. Thus the corollary follows from Theorem 12.1.

Corollary 12.2. *Let* $\varphi \in (0, \pi/2]$, $\tau \in \{-1, 1\}$, $\epsilon \in (0, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$. *Then the operators* $\Gamma(\tau, 2, \lambda, \mathbb{L}(\varphi, \epsilon))$ *and* $\Gamma^*(\tau, 2, \lambda, \mathbb{L}(\varphi, \epsilon))$ *are topological.*

Proof: Due to Corollary 12.1, the operator $\Gamma^{(inf)}(\tau, 2, \lambda, \varphi, \epsilon)$ is Fredholm with index 0. This means by Corollary 6.6 that $J(\tau, 2, \lambda, \varphi, \epsilon, 1)$ has the same property. Thus, recalling Corollary 6.4, we see that $M(\tau, 2, \lambda, \varphi, \epsilon)$ is Fredholm with index 0, and we conclude from Corollary 6.6 that $\Gamma(\tau, 2, \lambda, \mathbb{L}(\varphi, \epsilon))$ is an operator of the same type.

On the other hand, the operator $\Gamma^*(\tau, 2, \lambda, \mathbb{L}(\varphi, \epsilon))$ is one-to-one, bounded, and adjoint to $\Gamma(\tau, 2, \lambda, \mathbb{L}(\varphi, \epsilon))$. The first of these facts follows from Theorem 9.4, and the second one from (6.17) and Lemma 6.7. The third one is obvious. But now the conclusion of Corollary 12.2 follows from Lemma 12.2.

Lemma 12.3. *Take* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $\mathfrak{C}, R \in (0, \infty)$, $S \in [1, \infty)$, $\tau \in \{-1, 1\}$, *and assume*

$$\|\Phi\|_p \quad \le \quad \mathfrak{C}\cdot\|\, H^*(\tau, p, \varphi, R, S)(\Phi)\,\|_p \qquad \textit{for } \Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S))^3.$$

Suppose that T *belongs to* $(0, \infty)$ *and satisfies the inequality*

$$T \quad \ge \quad \big((3/\pi)\cdot C_{33}(p)\cdot\mathfrak{C}\cdot\sin^{-1}(\varphi)\big)^{p/2} + 2,$$

where $C_{33}(p)$ *was introduced in Lemma 11.1. Then it follows*

$$\|\Psi\|_p \le 2\cdot\mathfrak{C}\cdot \| F^*(\tau, p, \varphi, R, S\cdot T)(\Psi)\|_p \quad \textit{for } \Psi\in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S\cdot T)).$$

Proof: If $\eta\in\mathbb{R}^2\backslash\{0\}$, we set

$$b^{(1)}(\eta) := (\sin\varphi\cdot\eta_1/|\eta|,\ \sin\varphi\cdot\eta_2/|\eta|,\ \cos\varphi),$$

$$b^{(2)}(\eta) := (-\eta_2/|\eta|,\ \eta_1/|\eta|,\ 0), \qquad b^{(3)}(\eta) := (n^{(\varphi)}\circ g^{(\varphi)})(\eta).$$

For any $\eta\in\mathbb{R}^2\backslash\{0\}$, the tuple $(b^{(j)}(\eta))_{1\le j\le 3}$ is an orthonormal basis of $\mathbb{R}^3$, as follows by (3.2).

Take $\Psi\in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S\cdot T))$. For $j\in\{1, 2\}$, $\xi\in\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S)$, we define

$$\Phi^{(j)}(\xi) := -2\cdot\tau\cdot\sin^{-1}(\varphi)\cdot\Big(\big(L_l(0,\varphi)\otimes_p\Psi\big)_{1\le l\le 3}(\xi)\cdot b^{(j)}(\xi)\Big)\cdot b^{(j)}(\xi).$$

Furthermore, for $\xi\in\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S)$, we set

$$\Phi^{(3)}(\xi) := 0, \qquad \text{if } \xi\in\mathbb{B}_2(0, R\cdot S\cdot T)\backslash\mathbb{B}_2(0, R\cdot S),$$

$$\Phi^{(3)}(\xi) := \Psi(\xi)\cdot(n^{(\varphi)}\circ g^{(\varphi)})(\xi), \qquad \text{if } \xi\in\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S\cdot T).$$

Then we find

$$\|\Psi\|_p = \Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R\cdot S\cdot T)}|\Psi(\xi)|^p\,d\xi\Big)^{1/p} = \Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R\cdot S)}|\Phi^{(3)}(\xi)|^p\,d\xi\Big)^{1/p} \tag{12.6}$$

$$\le \Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R\cdot S)}\Big|\sum_{j=1}^{3}\Phi^{(j)}(\xi)\Big|^p\,d\xi\Big)^{1/p} \le \mathfrak{C}\cdot\Big\|H^*(\tau, p, \varphi, R, S)\Big(\sum_{j=1}^{3}\Phi^{(j)}\Big)\Big\|_p,$$

with the first inequality holding true since $(b^{(j)})_{1\le j\le 3}$ is an orthonormal basis of $\mathbb{R}^3$, and the second inequality being implied by the assumptions in the lemma. For brevity we put

$$F := H^*(\tau, p, \varphi, R, S)\Big(\sum_{j=1}^{3}\Phi^{(j)}\Big).$$

It holds for $\xi\in\mathbb{R}^2\backslash\mathbb{B}_2(0, R)$:

$$|F(\xi)| \le \sum_{j=1}^{3}|F(\xi)\cdot b^{(j)}(\xi)|,$$

so that it follows by (12.6):

$$\|\Psi\|_p \le \mathfrak{C}\cdot\sum_{j=1}^{3}\Big\|F\cdot\big(b^{(j)}\,|\,\mathbb{R}^2\backslash\mathbb{B}_2(0, R)\big)\Big\|_p. \tag{12.7}$$

On the other hand, recalling the definition of $H^*(\tau, p, \varphi, R, S)$ and $\Phi^{(j)}$, we see that

$$F = (\tau/2)\cdot\sum_{j=1}^{3}\mathcal{F}(R, S)(\Phi^{(j)}) + \sin^{-1}(\varphi)\cdot\big(L_l(0,\varphi)\otimes_p\Psi\big)_{1\le l\le 3}\,|\,\mathbb{R}^2\backslash\mathbb{B}_2(0, R).$$

By this equation and the choice of $\Phi^{(1)}$, $\Phi^{(2)}$, we infer for $\xi\in\mathbb{R}^2\backslash\mathbb{B}_2(0, R)$, $j\in\{1, 2\}$:

$$F(\xi)\cdot b^{(j)}(\xi)$$

$$= \chi_{\mathbb{B}_2(0,R\cdot S)\backslash\mathbb{B}_2(0,R)}(\xi)\cdot\sin^{-1}(\varphi)\cdot\Big(\big(L_l(0,\varphi)\otimes_p\Psi\big)_{1\le l\le 3}(\xi)\cdot b^{(j)}(\xi)\Big).$$

In addition, recalling Lemma 6.8 and the definition of $\Phi^{(3)}$, we get for $\xi\in\mathbb{R}^2\backslash\mathbb{B}_2(0, R)$:

$$F(\xi)\cdot b^{(3)}(\xi) = F(\xi)\cdot(n^{(\varphi)}\circ g^{(\varphi)})(\xi)$$

$$= (\tau/2)\cdot\mathcal{F}(R, S\cdot T)(\Psi)(\xi)$$

$$+ \sin^{-1}(\varphi)\cdot\Big(\big(L_l(0,\varphi)\otimes_p\Psi\big)_{1\le l\le 3}(\xi)\cdot(n^{(\varphi)}\circ g^{(\varphi)})(\xi)\Big)$$

$$= (n^{(\varphi)}\circ g^{(\varphi)})(\xi)\cdot H^*(\tau, p, \varphi, R, S\cdot T)\big(\Psi\cdot(n^{(\varphi)}\circ g^{(\varphi)})\big)(\xi)$$

$$= F^*(\tau, p, \varphi, R, S\cdot T)(\Psi)(\xi).$$

Now it follows by (12.7):

$$\|\Psi\|_p \le \mathfrak{C}\cdot\|F^*(\tau, p, \varphi, R, S\cdot T)(\Psi)\|_p \tag{12.8}$$

$$+ 2\cdot\mathfrak{C}\cdot\sin^{-1}(\varphi)\cdot\Big\|\big(L_l(0,\varphi)\otimes_p\Psi\big)_{1\le l\le 3}\,\Big|\,\mathbb{B}_2(0, R\cdot S)\backslash\mathbb{B}_2(0, R)\Big\|_p.$$

On the other hand, Lemma 11.1 yields

$$\Big\|\big(L_l(0,\varphi)\otimes_p\Psi\big)_{1\le l\le 3}\,\Big|\,\mathbb{B}_2(0, R\cdot S)\backslash\mathbb{B}_2(0, R)\Big\|_p$$

$$\le 3\cdot(4\cdot\pi)^{-1}$$

$$\cdot\Big(\int_{\mathbb{B}_2(0,R\cdot S)\backslash\mathbb{B}_2(0,R)}\Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R\cdot S\cdot T)}|\xi-\eta|^{-2}\cdot|\Psi(\eta)|\,d\eta\Big)^p\,d\xi\Big)^{1/p}$$

$$\le 3\cdot(4\cdot\pi)^{-1}\cdot C_{33}(p)\cdot(T-1)^{-2/p}\cdot\|\Psi\|_p.$$

This means by the choice of T:

$$2\cdot\mathfrak{C}\cdot\sin^{-1}(\varphi)\cdot\Big\|\big(L_l(0,\varphi)\otimes_p\Psi\big)_{1\le l\le 3}\,\Big|\,\mathbb{B}_2(0, R\cdot S)\backslash\mathbb{B}_2(0, R)\Big\|_p \le (1/2)\cdot\|\Psi\|_p.$$

Combining this result with inequality (12.8) yields the estimate claimed in the lemma.

Lemma 12.4. *Take* $p\in(1,\infty)$, $\varphi\in(0,\pi/2]$, $\mathfrak{C}\in(0,\infty)$, $\tau\in\{-1, 1\}$, *and assume*

$$\|\Phi\|_p \le \mathfrak{C}\cdot\|G^*(\tau, p, \varphi, 1)(\Phi)\|_p \quad \textit{for } \Phi\in L^p(\mathbb{R}^2).$$

It follows for $\epsilon\in(0,\infty)$, $\Phi\in L^p(\mathbb{R}^2)$:

$$\|\Phi\|_p \le \mathfrak{C}\cdot\|G^*(\tau, p, \varphi, \epsilon)(\Phi)\|_p.$$

Proof: Apply (6.14), Lemma 3.2 and the substitution rule.

Lemma 12.5. *Let* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $\tau \in \{-1, 1\}$, $\Phi \in L^p(\mathbb{R}^2)$. *Then it holds*

$$\| G^*(\tau, p, \varphi, \epsilon)(\Phi) - F^*(\tau, p, \varphi, 0, 1)(\Phi) \|_p \longrightarrow 0 \qquad (\epsilon \downarrow 0).$$

Proof: By Lemma 6.7 we may choose a constant $\mathfrak{C} > 0$ such that we get for $\Psi \in L^p(\mathbb{R}^2)$:

$$\| F^*(\tau, p, \varphi, 0, 1)(\Psi) \|_p, \quad \| G^*(\tau, p, \varphi, 1)(\Psi) \|_p \quad \leq \quad \mathfrak{C} \cdot \|\Psi\|_p. \tag{12.9}$$

Then it follows by Lemma 12.4:

$$\| G^*(\tau, p, \varphi, \epsilon)(\Psi) \|_p \quad \leq \quad \mathfrak{C} \cdot \|\Psi\|_p \qquad \text{for } \Psi \in L^p(\mathbb{R}^2), \ \epsilon \in (0, \infty). \tag{12.10}$$

For shortness we put for $\epsilon \in (0, \infty)$, $\xi, \eta \in \mathbb{R}^2$ with $\xi \neq \eta$:

$$K^{(1)}(\epsilon)(\xi, \eta) := (4\cdot\pi)^{-1} \cdot \Big((n^{(\varphi,\epsilon)} \circ \gamma^{(\varphi,\epsilon)})(\xi) \cdot (\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta)) \Big) \cdot | \gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) |^{-3} \cdot J^{(\varphi,\epsilon)}(\eta) \cdot \Phi(\eta),$$

$$K^{(2)}(\xi, \eta) := (4\cdot\pi)^{-1} \cdot \Big((n^{(\varphi)} \circ g^{(\varphi)})(\xi) \cdot (g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \Big) \cdot | g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta) |^{-3} \cdot \sin^{-1}(\varphi) \cdot \Phi(\eta).$$

Then we obtain from (3.6), (6.8) and (6.14), for $\epsilon \in (0, \infty)$:

$$\| G^*(\tau, p, \varphi, \epsilon)(\Phi) - F^*(\tau, p, \varphi, 0, 1)(\Phi) \|_p \quad \leq \quad \sum_{j=1}^{5} I_j(\epsilon),$$

where we used the abbreviations

$$I_1(\epsilon) := \left(\int_{\mathbb{R}^2 \setminus \mathbb{B}_2(0,\epsilon)} \left| \int_{\mathbb{B}_2(0,\epsilon)} K^{(1)}(\epsilon)(\xi, \eta)\, d\eta \right|^p d\xi \right)^{1/p},$$

$$I_2(\epsilon) := \left(\int_{\mathbb{R}^2 \setminus \mathbb{B}_2(0,\epsilon)} \left| \int_{\mathbb{B}_2(0,\epsilon)} K^{(2)}(\xi, \eta)\, d\eta \right|^p d\xi \right)^{1/p},$$

$$I_3(\epsilon) := \left(\int_{\mathbb{B}_2(0,\epsilon)} \left| \int_{\mathbb{R}^2} K^{(2)}(\xi, \eta)\, d\eta \right|^p d\xi \right)^{1/p},$$

$$I_4(\epsilon) := \left(\int_{\mathbb{B}_2(0,\epsilon)} \left| \int_{\mathbb{B}_2(0,2\cdot\epsilon)} K^{(1)}(\epsilon)(\xi, \eta)\, d\eta \right|^p d\xi \right)^{1/p},$$

$$I_5(\epsilon) := \left(\int_{\mathbb{B}_2(0,\epsilon)} \left| \int_{\mathbb{R}^2 \setminus \mathbb{B}_2(0,2\cdot\epsilon)} K^{(1)}(\epsilon)(\xi, \eta)\, d\eta \right|^p d\xi \right)^{1/p}.$$

Recalling (6.14), (6.8), and applying (12.10), (12.9), we arrive at the ensuing estimates, which hold true for $\epsilon \in (0, \infty)$:

$$I_1(\epsilon) \quad \leq \quad \left\| G^*(\tau, p, \varphi, \epsilon)\big(\chi_{\mathbb{B}_2(0,\epsilon)} \cdot \Phi\big) \right\|_p \quad \leq \quad \mathfrak{C} \cdot \| \chi_{\mathbb{B}_2(0,\epsilon)} \cdot \Phi \|_p,$$

$$I_2(\epsilon) \quad \leq \quad \left\| F^*(\tau, p, \varphi, 0, 1)\big(\chi_{\mathbb{B}_2(0,\epsilon)} \cdot \Phi\big) \right\|_p \quad \leq \quad \mathfrak{C} \cdot \| \chi_{\mathbb{B}_2(0,\epsilon)} \cdot \Phi \|_p.$$

Since $\Phi \in L^p(\mathbb{R}^2)$, it it follows by Lebesgue's theorem on dominated convergence that $\| \chi_{\mathbb{B}_2(0,\epsilon)} \cdot \Phi \|_p$ tends to 0 for $\epsilon \downarrow 0$. This implies $I_1(\epsilon) \to 0$, $I_2(\epsilon) \to 0$ for $\epsilon \downarrow 0$.

Referring to (6.8) again, we see that $I_3(\epsilon) = \| F^*(\tau, p, \varphi, 0, 1)(\Phi) \,|\, \mathbb{B}_2(0, \epsilon) \|_p$ for $\epsilon \in (0, \infty)$. But the function $F^*(\tau, p, \varphi, 0, 1)(\Phi)$ belongs to the space $L^p(\mathbb{R}^2)$, as implied by the definition of $F^*(\tau, p, \varphi, 0, 1)$. Therefore, applying Lebesgue's theorem once more, we obtain

$$\| F^*(\tau, p, \varphi, 0, 1)(\Phi) \,|\, \mathbb{B}_2(0, \epsilon) \|_p \to 0 \quad (\epsilon \downarrow 0), \qquad \text{that is,} \qquad I_3(\epsilon) \to 0 \quad (\epsilon \downarrow 0).$$

Furthermore, using (6.14), we find for $\epsilon \in (0, \infty)$:

$$I_4(\epsilon) = \left\| G^*(\tau, p, \varphi, \epsilon)\big(\chi_{\mathbb{B}_2(0,2\cdot\epsilon)} \cdot \Phi\big) \,\middle|\, \mathbb{B}_2(0, \epsilon) \right\|_p \leq \left\| G^*(\tau, p, \varphi, \epsilon)\big(\chi_{\mathbb{B}_2(0,2\cdot\epsilon)} \cdot \Phi\big) \right\|_p \leq \mathfrak{C} \cdot \| \chi_{\mathbb{B}_2(0,2\cdot\epsilon)} \cdot \Phi \|_p.$$

The last inequality is valid due to (12.10). Since $\Phi \in L^p(\mathbb{R}^2)$, it follows $I_4(\epsilon) \to 0 \ (\epsilon \downarrow 0)$.

Putting $\mathfrak{C}_1 := (4\cdot\pi)^{-1} \cdot C_1 \cdot \sin^{-1}(\varphi)$, we infer from Lemma 3.2, for $\epsilon \in (0, \infty)$:

$$I_5(\epsilon) \quad \leq \quad \mathfrak{C}_1 \cdot \left(\int_{\mathbb{B}_2(0,\epsilon)} \left(\int_{\mathbb{R}^2 \setminus \mathbb{B}_2(0,2\cdot\epsilon)} |\xi - \eta|^{-2} \cdot |\Phi(\eta)|\, d\eta \right)^p d\xi \right)^{1/p}.$$

If $\epsilon \in (0, \infty)$, $\xi \in \mathbb{B}_2(0, \epsilon)$, $\eta \in \mathbb{R}^2 \setminus \mathbb{B}_2(0, 2\cdot\epsilon)$, it holds $|\xi - \eta| \geq (1/4) \cdot (|\xi| + |\eta|)$, so we get for $\epsilon \in (0, \infty)$:

$$I_5(\epsilon) \quad \leq \quad 4 \cdot \mathfrak{C}_1 \cdot \left(\int_{\mathbb{B}_2(0,\epsilon)} \left(\int_{\mathbb{R}^2} |\xi - \eta|^{-1} \cdot (|\xi| + |\eta|)^{-1} \cdot |\Phi(\eta)|\, d\eta \right)^p d\xi \right)^{1/p} \tag{12.11}$$

Due to Theorem 4.1, there is a set $\mathcal{N} \subset \mathbb{R}^2$ of measure zero such that

$$\int_{\mathbb{R}^2} |\xi - \eta|^{-1} \cdot (|\xi| + |\eta|)^{-1} \cdot |\Phi(\eta)|\, d\eta \quad < \quad \infty \qquad \text{for } \xi \in \mathbb{R}^2 \setminus \mathcal{N}.$$

Hence the preceding integral defines a function F mapping $\mathbb{R}^2 \setminus \mathcal{N}$ into $[0, \infty)$. By Theorem 4.1, this function belongs to $L^p(\mathbb{R}^2 \setminus \mathcal{N})$. Using Lebesue's theorem again, we see that $\| F \,|\, \mathbb{B}_2(0, \epsilon) \|_p$ tends to 0 for $\epsilon \downarrow 0$. Now (12.11) implies $I_5(\epsilon) \to 0$ for $\epsilon \downarrow 0$. This completes the proof of the lemma.

Theorem 12.2. *Take* $p \in (1, \infty)$, $\vartheta \in (-\pi, \pi)$, $\tau \in \{-1, 1\}$, $\varphi_0 \in (0, \pi/2]$, *and assume that for any* $\varphi \in [\varphi_0, \pi/2]$, *there are numbers* $\mu, R \in (0, \infty)$ *such that* $\Gamma^{(inf)}(\tau, p, \mu \cdot e^{i\cdot\vartheta}, \varphi, R)$ *is a Fredholm operator. Then* $\Pi^*(-\tau, p, \mathbb{K}(\varphi_0))$ *is topological.*

Proof: From Corollary 6.6 and the assumptions in the theorem, we deduce that for any $\varphi \in [\varphi_0, \pi/2]$, there are positive reals μ, R such that $J(\tau, p, \mu \cdot e^{i\cdot\vartheta}, \varphi, R, 1)$ is Fredholm. If follows by Lemma 12.1, 6.11 and Corollary 6.2 that $J(\tau, p, e^{i\cdot\vartheta}, \varphi, 1, 1)$ is Fredholm for any $\varphi \in [\varphi_0, \pi/2]$. Now Lemma 6.17 yields

$$index(J(\tau, p, e^{i\cdot\vartheta}, \varphi, 1, 1)) \quad = \quad 0 \qquad \text{for } \varphi \in [\varphi_0, \pi/2].$$

Next we fix $\varphi \in [\varphi_0, \pi/2]$. Corollary 6.4 implies that $M(\tau, p, e^{i\cdot\vartheta}, \varphi, 1)$ is a Fredholm operator with index 0. This operator is even topological, as follows by distinguishing between the cases $p < 2$ and $p \geq 2$. In fact, if $p \geq 2$, we refer to Corollary 9.4, which yields that $M^*(\tau, p/(p-1), e^{i\cdot\vartheta}, \varphi, 1)$ is one-to-one. But the operator $M(\tau, p, e^{i\cdot\vartheta}, \varphi, 1)$ is adjoint to $M^*(\tau, p/(p-1), e^{i\cdot\vartheta}, \varphi, 1)$ (see (6.16), (6.17)), hence, by Lemma 12.2, $M(\tau, p, e^{i\cdot\vartheta}, \varphi, 1)$ must be topological. In order to deal with the case $p < 2$, consider a function $\Psi \in L^p(\mathbb{R}^2)^3$ having the property that $M(\tau, p, e^{i\cdot\vartheta}, \varphi, 1)(\Psi)$ vanishes. Then, according to Lemma 9.8, we have $\Psi \in L^2(\mathbb{R}^2)^3$, so Corollary 12.2 yields $\Psi = 0$. Hence $M(\tau, p, e^{i\cdot\vartheta}, \varphi, 1)$ is one-to-one, and it follows by Lemma 12.2 that $M(\tau, p, e^{i\cdot\vartheta}, \varphi, 1)$ is topological. We conclude that in any case, there is a number $\mathfrak{C}_1 > 0$ with

$$\|\Phi\|_p \quad \leq \quad \mathfrak{C}_1 \cdot \| M(\tau, p, e^{i\cdot\vartheta}, \varphi, 1)(\Phi) \|_p \qquad \text{for } \Phi \in L^p(\mathbb{R}^2)^3. \tag{12.12}$$

If $R \in (0, \infty)$, $S \in [1, \infty)$, $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S))^3$, it holds by (6.16):

$$\Big\| M(\tau, p, e^{i\cdot\vartheta}, \varphi, 1)\big(\widetilde{\mathcal{F}}(R\cdot S)(\Phi)\big) \,\Big|\, \mathbb{B}_2(0, R) \Big\|_p \tag{12.13}$$

$$\leq \Big(\int_{\mathbb{B}_2(0,R)} \Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R\cdot S)} \Big| \sum_{j,k=1}^{3} \widetilde{\mathcal{D}}^{\lambda}_{jkl}(\gamma^{(\varphi,1)}(\xi) - \gamma^{(\varphi,1)}(\eta)) \cdot (n_k^{(\varphi,1)} \circ \gamma^{(\varphi,1)})(\eta) \cdot \Phi_j(\eta) \cdot J^{(\varphi,1)}(\eta) \Big| \, d\eta \Big)^p d\xi \Big)^{1/p}$$

$$\leq \mathfrak{C}_2 \cdot \Big(\int_{\mathbb{B}_2(0,R)} \Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R\cdot S)} |\xi - \eta|^{-2} \cdot |\Phi(\eta)| \, d\eta \Big)^p d\xi \Big)^{1/p},$$

with $\mathfrak{C}_2 := 9 \cdot |J^{(\varphi,1)}|_0 \cdot C_{17}(\vartheta)$, where the last inequality is implied by (5.15).

Setting $\mathfrak{C}_3 := \mathfrak{C}_2 \cdot C_{33}(p)$, we obtain from (12.13) and Lemma 11.1, if $R \in (0, \infty)$, $S \in [2, \infty)$, $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S))^3$:

$$\Big\| M(\tau, p, e^{i\cdot\vartheta}, \varphi, 1)\big(\widetilde{\mathcal{F}}(R\cdot S)(\Phi)\big) \,\Big|\, \mathbb{B}_2(0, R) \Big\|_p \quad \leq \quad \mathfrak{C}_3 \cdot (S-1)^{-2/p} \cdot \|\Phi\|_p. \tag{12.14}$$

Put $\mathfrak{C}_4 := (2 \cdot \mathfrak{C}_1 \cdot \mathfrak{C}_3)^{p/2} + 2$. Then we find for $R \in [1, \infty)$, $S \in [\mathfrak{C}_4, \infty)$, $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S))^3$:

$$\|\Phi\|_p \quad \leq \quad \mathfrak{C}_1 \cdot \Big\| M(\tau, p, e^{i\cdot\vartheta}, \varphi, 1)\big(\widetilde{\mathcal{F}}(R\cdot S)(\Phi)\big) \Big\|_p$$

$$\leq \quad \mathfrak{C}_1 \cdot \| J(\tau, p, e^{i\cdot\vartheta}, \varphi, R, S)(\Phi) \|_p$$

$$+ \quad \mathfrak{C}_1 \cdot \Big\| M(\tau, p, e^{i\cdot\vartheta}, \varphi, 1)\big(\widetilde{\mathcal{F}}(R\cdot S)(\Phi)\big) \,\Big|\, \mathbb{B}_2(0, R) \Big\|_p$$

$$\leq \quad \mathfrak{C}_1 \cdot \| J(\tau, p, e^{i\cdot\vartheta}, \varphi, R, S)(\Phi) \|_p \; + \; \mathfrak{C}_1 \cdot \mathfrak{C}_3 \cdot (S-1)^{-2/p} \cdot \|\Phi\|_p$$

$$\leq \quad \mathfrak{C}_1 \cdot \| J(\tau, p, e^{i\cdot\vartheta}, \varphi, R, S)(\Phi) \|_p \; + \; (1/2) \cdot \|\Phi\|_p,$$

where the first inequality follows from (12.12), and the second one from (12.14). As for the one equation appearing in the preceding estimate, it may be derived from (6.11), (6.16), (3.6) and (3.7). Thus we have for $R \in [1, \infty)$, $S \in [\mathfrak{C}_4, \infty)$, $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S))^3$:

$$\|\Phi\|_p \quad \leq \quad 2 \cdot \mathfrak{C}_1 \cdot \| J(\tau, p, e^{i\cdot\vartheta}, \varphi, R, S)(\Phi) \|_p. \tag{12.15}$$

Taking $R \in [1, \infty)$, $S \in [\mathfrak{C}_4, \infty)$, $\mu \in (0, \infty)$, $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S/\mu))^3$, and using the abbreviation $I := id(\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S))$, we now obtain

$$\|\Phi\|_p \quad = \quad \mu^{-2/p} \cdot \| \Phi \circ ((1/\mu) \cdot I) \|_p \tag{12.16}$$

$$\leq \quad 2 \cdot \mu^{-2/p} \cdot \mathfrak{C}_1 \cdot \Big\| J(\tau, p, e^{i\cdot\vartheta}, \varphi, R, S)\big(\Phi \circ ((1/\mu) \cdot I)\big) \Big\|_p$$

$$= \quad 2 \cdot \mathfrak{C}_1 \cdot \| J(\tau, p, \mu^2 \cdot e^{i\cdot\vartheta}, \varphi, R/\mu, S)(\Phi) \|_p,$$

with the preceding inequality implied by (12.15). The last equation follows by Lemma 12.1. Let us evaluate the right-hand side of (12.16) by using Theorem 11.1. For this purpose, we take $\mu \in [2, \infty)$, $S \in [\mathfrak{C}_4, \infty)$, $N \in \mathbb{N}$ with $N \geq 24$, and $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, \mu\cdot S))^3$. Then, applying (12.16) with $R := \mu^2$, as well as Theorem 11.1 with $R := \mu$, $\lambda := \mu^2 \cdot e^{i\cdot\vartheta}$, we obtain the following estimate:

$$\|\Phi\|_p \quad \leq \quad 2 \cdot \mathfrak{C}_1 \cdot C_{56}(p, |\vartheta|, \varphi) \cdot \Big(N \cdot \| H^*(\tau, p, \varphi, \mu, S)(\Phi) \|_p + \big(N^{-1/2} + N \cdot S^{-(1 \wedge (2/p))} + N^6 \cdot \mu^{-1} \big) \cdot \|\Phi\|_p \Big). \tag{12.17}$$

Put $\mathfrak{C}_5 := 2 \cdot \mathfrak{C}_1 \cdot C_{56}(p, |\vartheta|, \varphi)$, $\mathfrak{C}_6 := \max\{24, 16 \cdot (\mathfrak{C}_5)^2\} + 1$. Let N denote the uniquely determined integer belonging to the interval $[\mathfrak{C}_6 - 1, \mathfrak{C}_6)$. Then it follows from (12.17), for $\mu \in [2, \infty)$, $S \in [\mathfrak{C}_4, \infty)$, $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, \mu\cdot S))^3$:

$$\|\Phi\|_p \quad \leq \quad (1/4) \cdot \|\Phi\|_p + \mathfrak{C}_5 \cdot \mathfrak{C}_6 \cdot \| H^*(\tau, p, \varphi, \mu, S)(\Phi) \|_p + \big(C_5 \cdot \mathfrak{C}_6 \cdot S^{-(1 \wedge (2/p))} + C_5 \cdot (\mathfrak{C}_6)^6 \cdot \mu^{-1} \big) \cdot \|\Phi\|_p. \tag{12.18}$$

Choose $S_0 \in \mathbb{R}$ so large that that $S_0 \geq \mathfrak{C}_4$, $S_0 \geq (4 \cdot \mathfrak{C}_6 \cdot \mathfrak{C}_5)^{1/(1 \wedge (2/p))}$, and define $\mu_0 := \max\{2, 4 \cdot \mathfrak{C}_5 \cdot (\mathfrak{C}_6)^6\}$. Then we infer from (12.18), if $\Phi \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, \mu_0 \cdot S_0))^3$:

$$\|\Phi\|_p \quad \leq \quad (3/4) \cdot \|\Phi\|_p + \mathfrak{C}_5 \cdot \mathfrak{C}_6 \cdot \| H^*(\tau, p, \varphi, \mu_0, S_0)(\Phi) \|_p,$$

so we obtain, with $\mathfrak{C}_7 := 4 \cdot \mathfrak{C}_5 \cdot \mathfrak{C}_6$:

$$\|\Phi\|_p \;\le\; \mathfrak{C}_7 \cdot \| H^*(\tau, p, \varphi, \mu_0, S_0)(\Phi) \|_p \,. \tag{12.19}$$

Put $T_0 := \big((3/\pi) \cdot C_{33}(p) \cdot \mathfrak{C}_7 \cdot \sin^{-1}(\varphi) \big)^{p/2} + 2$. Then we conclude from (12.19) and Lemma 12.3:

$$\|\Psi\|_p \;\le\; 2 \cdot \mathfrak{C}_7 \cdot \| F^*(\tau, p, \varphi, \mu_0, S_0 \cdot T_0)(\Psi) \|_p \qquad \text{for } \Psi \in L^p(\mathbb{R}^2 \backslash \mathbb{B}_2(0, \mu_0 \cdot S_0 \cdot T_0)) \,.$$

This means the operator $F^*(\tau, p, \varphi, \mu_0, S_0 \cdot T_0)$ is one-to-one and has closed range. In particular, it has property F_+ (Chapter 2). Thus Corollary 6.1 implies that $F^*(\tau, p, \varphi, \mu_0 \cdot S_0 \cdot T_0, 1)$ is F_+ too. But then, due to Corollary 6.2, $F^*(\tau, p, \varphi, 1, 1)$ must be an operator of the same type. Recalling that φ was chosen arbitratily in $[\varphi_0, \pi/2]$, we are now able to apply Lemma 6.17, which yields

$$index(F^*(\tau, p, \varphi, 1, 1)) \;=\; 0 \qquad \text{for } \varphi \in [\varphi_0, \pi/2] \,.$$

Therefore $F^*(\tau, p, \varphi, 1, 1)$ is a Fredholm operator with index 0 $(\varphi \in (0, \pi/2])$.

Now let us fix $\varphi \in [\varphi_0, \pi/2]$ once more. Corollary 6.4 shows that $G^*(\tau, p, \varphi, 1)$ is a Fredholm operator with index 0. On the other hand, we know from Corollary 9.3 that $G(\tau, p/(p-1), \varphi, 1)$ is one-to-one. But the latter operator is adjoint to $G^*(\tau, p, \varphi, 1)$, and both these operators are bounded (Lemma 6.7). Now we may refer to Lemma 12.2 which yields that $G^*(\tau, p, \varphi, 1)$ is topological. So there is some $\mathfrak{C}_8 > 0$ with

$$\|\Psi\|_p \;\le\; \mathfrak{C}_8 \cdot \| G^*(\tau, p, \varphi, 1)(\Psi) \|_p \qquad \text{for } \Psi \in L^p(\mathbb{R}^2) \,.$$

Lemma 12.4 implies

$$\|\Psi\|_p \;\le\; \mathfrak{C}_8 \cdot \| G^*(\tau, p, \varphi, \epsilon)(\Psi) \|_p \qquad \text{for } \epsilon \in (0, \infty) \,,\; \Psi \in L^p(\mathbb{R}^2) \,.$$

This means by Lemma 12.5:

$$\|\Psi\|_p \;\le\; \mathfrak{C}_8 \cdot \| F^*(\tau, p, \varphi, 0, 1)(\Psi) \|_p \qquad \text{for } \Psi \in L^p(\mathbb{R}^2) \,.$$

Therefore the operator $F^*(\tau, p, \varphi, 0, 1)$ is one-to-one, with closed range. Since φ is an arbitrary member of the interval $[\varphi_0, \pi/2]$, it follows from Lemma 6.17:

$$index(F^*(\tau, p, \varphi_0, 0, 1)) \;=\; 0 \,,$$

so $F^*(\tau, p, \varphi_0, 0, 1)$ must be a Fredholm operator. But this mapping was already proved to be one-to-one, hence it is even topological; see Lemma 12.2. Thus the operator $\Pi^*(-\tau, p, \mathbb{K}(\varphi_0))$ must be topological too.

Lemma 12.6. *Let* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $\tau \in \{-1, 1\}$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $R \in (0, \infty)$.

Then the operator $L(\tau, p, \lambda, \varphi, R)$ *is Fredholm if and only if* $A(\tau, p, \varphi, R, 1)$ *has the same property.*

By Corollary 6.6, this result may also be stated in the following way : $\Gamma^{(ver)}(\tau, p, \lambda, \varphi, R)$ *is Fredholm if and only if* $\Lambda^{(ver)}(\tau, p, \varphi, R)$ *is Fredholm too.*

Proof: Define an operator $W : L^p(\mathbb{B}_2(0, R))^3 \mapsto L^p(\mathbb{B}_2(0, R))^3$ by setting

$$W(\Phi)(\xi) \;:=\; \Big(\int_{\mathbb{B}_2(0,R)} \sum_{j,k=1}^{3} (D_j \overline{E}^{\lambda}_{kl} + D_k \overline{E}^{\lambda}_{jl})(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \cdot (n_k^{(\varphi)} \circ g^{(\varphi)})(\eta) \cdot \Phi_j(\eta) \, d\eta \Big)_{1 \le l \le 3}$$

for $\Phi \in L^p(\mathbb{B}_2(0, R))^3$, $\xi \in \mathbb{B}_2(0, R)$. According to Lemma 5.4, it holds for $\xi, \eta \in \mathbb{R}^2\backslash\{0\}$ with $\xi \neq \eta$, and for $j, l \in \{1, 2, 3\}$:

$$\Big| \sum_{k=1}^{3} (D_j \overline{E}^{\lambda}_{kl} + D_k \overline{E}^{\lambda}_{jl})(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \cdot (n_k^{(\varphi)} \circ g^{(\varphi)})(\eta) \Big|$$
$$\le\; 6 \cdot C_{15}(|\arg \lambda|) \cdot |\lambda|^{1/2} \cdot |\xi - \eta|^{-1} \,.$$

If follows from Lemma 6.3 that W is compact. On the other hand, we conclude from (6.22), (6.18) und (5.1):

$$L(\tau, p, \lambda, \varphi, R) \;=\; A(\tau, p, \varphi, R, 1) \;+\; W \,.$$

Now the lemma is implied by [34, p. 24, Theorem 3.4].

Corollary 12.3. *Take* $p \in (1, \infty)$, $\vartheta \in (-\pi, \pi)$, $\tau \in \{-1, 1\}$, $R \in (0, \infty)$, $\varphi_0 \in (0, \pi/2]$. *Assume that for any* $\varphi \in [\varphi_0, \pi/2]$, *there is some number* $\mu \in (0, \infty)$ *such that the operator* $\Gamma(\tau, p, \mu \cdot e^{i \cdot \vartheta}, \mathbb{K}(\varphi))$ *is Fredholm.*

Then $\Pi^*(-\tau, p, \mathbb{K}(\varphi))$ *is topological and* $\Lambda^{(ver)}(\tau, p, \varphi, R)$ *Fredholm* $(\varphi \in [\varphi_0, \pi/2])$.

Proof: Due to Corollary 6.5 and 6.6, we may conclude from our assumptions that for any $\varphi \in [\varphi_0, \pi/2]$, there exists some $\mu \in (0, \infty)$ such that the operators $\Gamma^{(inf)}(\tau, p, \mu \cdot e^{i \cdot \vartheta}, \varphi, R)$ and $\Gamma^{(ver)}(\tau, p, \mu \cdot e^{i \cdot \vartheta}, \varphi, R)$ are both Fredholm. But then Theorem 12.2 yields that $\Pi^*(-\tau, p, \mathbb{K}(\varphi))$ is topological, and Lemma 12.6 implies that $\Lambda^{(ver)}(\tau, p, \varphi, R)$ is Fredholm, for $\varphi \in [\varphi_0, \pi/2]$.

Corollary 12.4. *Take* $p \in (1, \infty)$, $\tau \in \{-1, 1\}$, $\varphi_0 \in (0, \pi/2]$. *Suppose there is a number* $R \in (0, \infty)$ *such that* $\Lambda^{(ver)}(\tau, p, \varphi, R)$ *is Fredholm for any* $\varphi \in [\varphi_0, \pi/2]$. *Furthermore, assume that* $\Pi^*(-\tau, p, \mathbb{K}(\varphi))$ *is topological for* $\varphi \in [\varphi_0, \pi/2]$.

Then, if $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\varphi \in [\varphi_0, \pi/2]$, *the operator* $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ *is Fredholm with index* 0.

Proof: According to Theorem 12.1, the operator $\Gamma^{(inf)}(\tau, p, \lambda, \varphi, R)$ is Fredholm with index 0 if $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\varphi \in [\varphi_0, \pi/2]$. By Lemma 12.6 and our assumptions, we know that $\Gamma^{(ver)}(\tau, p, \lambda, \varphi, R)$ is Fredholm ($\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\varphi \in [\varphi_0, \pi/2]$). Now Corollary 12.4 follows from Corollary 6.5, 6.6 and Lemma 6.17.

Lemma 12.7. *Let* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $S \in [1, \infty)$, $\Phi \in L^p(\mathbb{R}^2)^3$. *It follows:*

$$\left\| \tilde{G}(R \cdot S)\big(A(\tau, p, \varphi, R, S)(\Phi \,|\, \mathbb{B}_2(0, R))\big) - A(\tau, p, \varphi, \infty, 1)(\Phi) \right\|_p \longrightarrow 0,$$

$$\left\| \tilde{G}(R \cdot S)\big(A^*(\tau, p, \varphi, R, S)(\Phi \,|\, \mathbb{B}_2(0, R))\big) - A^*(\tau, p, \varphi, \infty, 1)(\Phi) \right\|_p \longrightarrow 0$$

for R *tending to infinity.*

Proof: We are going to show that the first conclusion follows from (6.18). The second result may be deduced from (6.19) by proceeding in an analogous way.

If $\xi, \eta \in \mathbb{R}^2\backslash\{0\}$ with $\xi \neq \eta$, we set

$$f(\xi, \eta) := \Big(\sum_{j,k=1}^{3} \mathcal{D}_{jkl}\big(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)\big) \cdot \big(n_k^{(\varphi)} \circ g^{(\varphi)}\big)(\eta) \cdot \Phi_j(\eta) \cdot \sin^{-1}(\varphi) \Big)_{1 \le l \le 3}.$$

Then, recurring to (6.18), we obtain for $R \in (0, \infty)$:

$$\left\| \tilde{G}(R \cdot S)\big(A(\tau, p, \varphi, R, S)(\Phi \,\big|\, \mathbb{B}_2(0, R))\big) - A(\tau, p, \varphi, \infty, 1)(\Phi) \right\|_p \le \sum_{j=1}^{3} I_j(R),$$

where we used the abbreviations

$$I_1(R) := \Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S)} \Big| (\tau/2) \cdot \Phi(\xi) + \int_{\mathbb{R}^2} f(\xi, \eta)\, d\eta \Big|^p \, d\xi \Big)^{1/p},$$

$$I_2(R) := \Big(\int_{\mathbb{B}_2(0, R\cdot S)\backslash \mathbb{B}_2(0, R)} \Big| (\tau/2) \cdot \Phi(\xi) + \int_{\mathbb{R}^2\backslash\mathbb{B}_2(0, R)} f(\xi, \eta)\, d\eta \Big|^p \, d\xi \Big)^{1/p},$$

$$I_3(R) := \Big(\int_{\mathbb{B}_2(0, R)} \Big| \int_{\mathbb{R}^2\backslash\mathbb{B}_2(0, R)} f(\xi, \eta)\, d\eta \Big|^p \, d\xi \Big)^{1/p}.$$

According to (6.18), it holds

$$I_1(R) = \| A(\tau, p, \varphi, \infty, 1)(\Phi) \,|\, \mathbb{R}^2\backslash\mathbb{B}_2(0, R\cdot S) \|_p \quad \text{for } R \in (0, \infty).$$

Since the function $A(\tau, p, \varphi, \infty, 1)(\Phi)$ belongs to $L^p(\mathbb{R}^2)^3$, we may use Lebesgue's theorem on dominated convergence to obtain $I_1(R) \to 0$ for $R \to \infty$. Furthermore, it holds for $R \in (0, \infty)$:

$$I_2(R) = \left\| A(\tau, p, \varphi, \infty, 1)\big(\chi_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R)} \cdot \Phi\big) \,\big|\, \mathbb{B}_2(0, R\cdot S)\backslash \mathbb{B}_2(0, R) \right\|_p$$

$$\le \| A(\tau, p, \varphi, \infty, 1)\big(\chi_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R)} \cdot \Phi\big) \|_p.$$

Due to Lemma 6.7, there exists some $\mathfrak{C} > 0$ such that

$$\| A(\tau, p, \varphi, \infty, 1)(\Psi) \|_p \le \mathfrak{C} \cdot \|\Psi\|_p \quad \text{for } \Psi \in L^p(\mathbb{R}^2)^3.$$

Therefore we have

$$I_2(R) \le \mathfrak{C} \cdot \| \Phi \,|\, \mathbb{R}^2\backslash\mathbb{B}_2(0, R) \|_p \quad \text{for } R \in (0, \infty).$$

But Φ is contained in $L^p(\mathbb{R}^2)^3$, so we may apply Lebesgue's theorem once more, which yields $I_2(R) \to 0$ $(R \to \infty)$. Finally, we observe for $R \in (0, \infty)$:

$$I_3(R) = \left\| A(\tau, p, \varphi, \infty, 1)\big(\chi_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R)} \cdot \Phi\big) \,\big|\, \mathbb{B}_2(0, R) \right\|_p$$

$$\le \| A(\tau, p, \varphi, \infty, 1)\big(\chi_{\mathbb{R}^2\backslash\mathbb{B}_2(0,R)} \cdot \Phi\big) \|_p \le \mathfrak{C} \cdot \| \Phi \,|\, \mathbb{R}^2\backslash\mathbb{B}_2(0, R) \|_p.$$

Thus, referring to Lebesgue's theorem again, we conclude $I_3(R) \to 0$ $(R \to \infty)$.

Collecting our results, we arrive at the conclusion of the lemma.

Lemma 12.8. *Take* $p \in (1, \infty)$, *and put* $C_{57}(p) := 4 \cdot \pi \cdot (p-1)^{-1/p}$. *Then it holds for* $S \in [2, \infty)$, $\Phi \in L^p(\mathbb{B}_2(0, 1))$:

$$\Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0, S)} \Big(\int_{\mathbb{B}_2(0,1)} |\xi - \eta|^{-2} \cdot |\Phi(\eta)| \, d\eta \Big)^p d\xi \Big)^{1/p}$$

$$\le C_{57}(p) \cdot S^{-2\cdot(1-1/p)} \cdot \|\Phi\|_p.$$

Proof: This inequality may be established by slightly modifying the proof of Lemma 11.1.

Lemma 12.9. *Let* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $S \in [1, \infty)$, $\mathfrak{C} \in (0, \infty)$ *with*

$$\|\Phi\|_p \le \mathfrak{C} \cdot \| A(\tau, p, \varphi, 1, S)(\Phi) \|_p \quad \text{for } \Phi \in L^p(\mathbb{B}_2(0, 1))^3,$$

or

$$\|\Phi\|_p \le \mathfrak{C} \cdot \| A^*(\tau, p, \varphi, 1, S)(\Phi) \|_p \quad \text{for } \Phi \in L^p(\mathbb{B}_2(0, 1))^3.$$

If the first estimate is true, it follows

$$\|\Phi\|_p \le \mathfrak{C} \cdot \| A(\tau, p, \varphi, R, S)(\Phi) \|_p \quad \text{for } R \in (0, \infty),\ \Phi \in L^p(\mathbb{B}_2(0, R))^3,$$

and in case the second inequality is valid, it holds

$$\|\Phi\|_p \le \mathfrak{C} \cdot \| A^*(\tau, p, \varphi, R, S)(\Phi) \|_p \quad \text{for } R \in (0, \infty),\ \Phi \in L^p(\mathbb{B}_2(0, R))^3.$$

Proof: Combine (6.18), (6.19) and (3.2) with the substitution rule.

Theorem 12.3. *Let* $p \in (1, \infty)$, $\varphi_0 \in (0, \pi/2]$, $\tau \in \{-1, 1\}$. *Assume that for any* $\varphi \in [\varphi_0, \pi/2]$, *there is some number* $R \in (0, \infty)$ *such that* $\Lambda^{(ver)}(\tau, p, \varphi, R)$ *has property* F_+. *Then the operator* $\Lambda(\tau, p, \mathbb{K}(\varphi))$ *is topological for* $\varphi \in [\varphi_0, \pi/2]$.

Conversely, if $\Lambda(\tau, p, \mathbb{K}(\varphi_0))$ *is topological, then* $\Lambda^{(ver)}(\tau, p, \varphi_0, R)$ *must be Fredholm for any* $R \in (0, \infty)$.

Analogous relations are valid between $\Lambda^{*(ver)}(\tau, p, \varphi, R)$ *and* $\Lambda^*(\tau, p, \mathbb{K}(\varphi))$.

Proof: We shall show that $\Lambda^{(ver)}(\tau, p, \varphi, R)$ and $\Lambda(\tau, p, \mathbb{K}(\varphi))$ are linked as claimed in the theorem. A corresponding relation between $\Lambda^{*(ver)}(\tau, p, \varphi, R)$ and $\Lambda^*(\tau, p, \mathbb{K}(\varphi))$ may be proved by the same arguments. We start from the assumption that for any $\varphi \in [\varphi_0, \pi/2]$, there is some $R \in (0, \infty)$ such that $\Lambda^{(ver)}(\tau, p, \varphi, R)$ is a F_+-operator, and hence (Corollary 6.6), the mapping $A(\tau, p, \varphi, R, 1)$ has the same property.

Let $\varphi \in [\varphi_0, \pi/2]$ be fixed for the time being. Due to our assumption, we may apply Corollary 6.2, which yields that $A(\tau, p, \varphi, 1, 1)$ has property F_+. On the other hand, recalling Theorem 8.1, we know there is some $q \in (1, p]$ such that $\Lambda(\tau, q, \mathbb{K}(\varphi))$ is topological. Hence, by (6.20), the operator $A(\tau, q, \varphi, \infty, 1)$ is topological too, so there is a number $\mathfrak{C}_1 > 0$ with

$$\|\Phi\|_q \quad \leq \quad \mathfrak{C}_1 \cdot \| A(\tau, q, \varphi, \infty, 1)(\Phi) \|_q \qquad \text{for } \Phi \in L^q(\mathbb{R}^2)^3. \tag{12.20}$$

Choose $S \in [2, \infty)$ so large that

$$S \quad \geq \quad \left(\mathfrak{C}_1 \cdot C_{57}(q) \cdot 27/(2\cdot\pi) \cdot \sin^{-1}(\varphi) \right)^{1/(2-2/q)}, \tag{12.21}$$

where $C_{57}(q)$ was introduced in Lemma 12.8. Since $A(\tau, p, \varphi, 1, 1)$ is F_+, we conclude from Corollary 6.1 that $A(\tau, p, \varphi, 1, S)$ has the same property. This means in particular the operator $A(\tau, p, \varphi, 1, S)$ has closed range. We are going to show that in addition, the operator $A(\tau, p, \varphi, 1, S)$ is one-to-one. To this end, assume that Φ is a function from $L^p(\mathbb{B}_2(0, 1))^3$ satisfying the equation $A(\tau, p, \varphi, 1, S)(\Phi) \; = \; 0$. Since the number q appearing in (12.20) was chosen as a member of the set $(1, p]$, we know that $\Phi \in L^q(\mathbb{B}_2(0, 1))^3$, so the function $A(\tau, q, \varphi, 1, S)(\Phi)$ vanishes too. This implies

$$\left(A(\tau, q, \varphi, \infty, 1)(\widetilde{G}(1)(\Phi)) \right) | \mathbb{B}_2(0, S) \quad = \quad 0,$$

and we may conclude by (12.20) and the choice of Φ:

$$\begin{aligned} \|\Phi\|_q \quad &= \quad \| \widetilde{G}(1)(\Phi) \|_q \\ &\leq \quad \mathfrak{C}_1 \cdot \left\| \left(A(\tau, q, \varphi, \infty, 1)(\widetilde{G}(1)(\Phi)) \right) | \mathbb{R}^2 \backslash \mathbb{B}_2(0, S) \right\|_q. \end{aligned} \tag{12.22}$$

But for $\xi \in \mathbb{R}^2 \backslash \mathbb{B}_2(0, S)$, we have

$$\begin{aligned} A(\tau, q, \varphi, \infty, 1)(\widetilde{G}(1)(\Phi))(\xi) \quad = \quad & \Big(\int_{\mathbb{B}_2(0,1)} \sum_{j,k=1}^{3} \mathcal{D}_{jkl}(g^{(\varphi)}(\xi) - g^{(\varphi)}(\eta)) \\ & \cdot (n_k^{(\varphi)} \circ g^{(\varphi)})(\eta) \cdot \Phi_j(\eta) \cdot \sin^{-1}(\varphi) \; d\eta \Big)_{1 \leq l \leq 3}, \end{aligned}$$

and it follows by (12.22) and (5.9):

$$\begin{aligned} \|\Phi\|_q \quad \leq \quad & \mathfrak{C}_1 \cdot 27 \cdot (4\cdot\pi)^{-1} \cdot \sin^{-1}(\varphi) \\ & \cdot \left(\int_{\mathbb{R}^2 \backslash \mathbb{B}_2(0,S)} \left(\int_{\mathbb{B}_2(0,1)} |\xi - \eta|^{-2} \cdot \Phi(\eta) \; d\eta \right)^q d\xi \right)^{1/q}. \end{aligned}$$

Now Lemma 12.8 yields

$$\|\Phi\|_q \quad \leq \quad \mathfrak{C}_1 \cdot 27 \cdot (4\cdot\pi)^{-1} \cdot C_{57}(q) \cdot S^{-2\cdot(1-1/q)} \cdot \sin^{-1}(\varphi) \cdot \|\Phi\|_q.$$

Due to the choice of S in (12.21), the preceding estimate implies $\|\Phi\|_q \leq (1/2) \cdot \|\Phi\|_q$, so Φ must vanish. Hence the operator $A(\tau, p, \varphi, 1, S)$ is one-to-one. Furthermore, this operator is bounded (see Lemma 6.7), and it has closed range, as proved before. Therefore, we may apply the open mapping theorem to obtain a constant $\mathfrak{C}_2 > 0$ with

$$\|\Phi\|_p \quad \leq \quad \mathfrak{C}_2 \cdot \| A(\tau, p, \varphi, 1, S)(\Phi) \|_p \qquad \text{for } \Phi \in L^p(\mathbb{B}_2(0, 1))^3.$$

It follows from Lemma 12.9:

$$\|\Phi\|_p \quad \leq \quad \mathfrak{C}_2 \cdot \| A(\tau, p, \varphi, R, S)(\Phi) \|_p \qquad \text{for } R \in (0, \infty), \; \Phi \in L^p(\mathbb{B}_2(0, R))^3.$$

This means for $R \in (0, \infty)$, $\Phi \in L^p(\mathbb{R}^2)^3$:

$$\| \Phi | \mathbb{B}_2(0, R) \|_p \quad \leq \quad \mathfrak{C}_2 \cdot \left\| \widetilde{G}(R \cdot S) \left(A(\tau, p, \varphi, R, S)(\Phi | \mathbb{B}_2(0, R)) \right) \right\|_p. \tag{12.23}$$

On the other hand, by recurring to Lebesgue's theorem on dominated convergence, we find that

$$\| \Phi | \mathbb{B}_2(0, R) \|_p \to \|\Phi\|_p \quad (R \to \infty) \qquad \text{for } \Phi \in L^p(\mathbb{R}^2)^3.$$

Thus we infer from (12.23) and Lemma 12.7:

$$\|\Phi\|_p \quad \leq \quad \mathfrak{C}_2 \cdot \| A(\tau, p, \varphi, \infty, 1)(\Phi) \|_p \qquad \text{for } \Phi \in L^p(\mathbb{R}^2)^3.$$

This proves that $A(\tau, p, \varphi, \infty, 1)$ is one-to-one and has closed range, and thus has property F_+. Recalling that φ is an arbitrary member of the set $[\varphi_0, \pi/2]$, we now conclude by Lemma 6.17:

$$\mathit{index}(A(\tau, p, \varphi, \infty, 1)) \quad = \quad 0 \qquad \text{for } \varphi \in [\varphi_0, \pi/2].$$

Since $A(\tau, p, \varphi, \infty, 1)$ was shown to be one-to-one, it follows that $A(\tau, p, \varphi, \infty, 1)$ is bijective ($\varphi \in [\varphi_0, \pi/2]$). On the other hand, the operator $A(\tau, p, \varphi, \infty, 1)$ is bounded (Lemma 6.7), so we may use the open mapping theorem once more, to obtain that $A(\tau, p, \varphi, \infty, 1)$ – and hence $\Lambda(\tau, p, \mathbb{K}(\varphi))$ – is topological ($\varphi \in (0, \pi/2]$).

Conversely, if $\Lambda(\tau, p, \mathbb{K}(\varphi_0))$ is assumed to be topological, then it is clear by Corollary 6.3 and 6.6 that $\Lambda^{(ver)}(\tau, p, \varphi_0, R)$ is Fredholm for any $R \in (0, \infty)$.

Now we are able to state Corollary 12.3 and 12.4 in a somewhat different way.

Corollary 12.5. *Let* $p \in (1, \infty)$, $\vartheta \in (-\pi, \pi)$, $\tau \in \{-1, 1\}$, $\varphi_0 \in (0, \pi/2]$, *and assume that for any* $\varphi \in [\varphi_0, \pi/2]$, *there exists some* $\mu \in (0, \infty)$ *such that* $\Gamma(\tau, p, \mu \cdot e^{i \cdot \vartheta}, \mathbb{K}(\varphi))$ *is Fredholm.*

Then the operators $\Lambda(\tau, p, \mathbb{K}(\varphi))$ *and* $\Pi^*(-\tau, p, \mathbb{K}(\varphi))$ *are topological for* $\varphi \in [\varphi_0, \pi/2]$.

Proof: Combine Corollary 12.3 with Theorem 12.3.

Corollary 12.6. *Take* $p \in (1, \infty)$, $\tau \in \{-1, 1\}$, $\varphi_0 \in (0, \pi/2]$. *Suppose the mappings* $\Pi^*(-\tau, p, \mathbb{K}(\varphi))$ *and* $\Lambda(\tau, p, \mathbb{K}(\varphi))$ *are topological for any* $\varphi \in [\varphi_0, \pi/2]$.

Then, for $\varphi \in [\varphi_0, \pi/2]$, $\lambda \in \mathbb{C} \backslash (-\infty, 0]$, *the operator* $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ *is Fredholm with index* 0.

Proof: The corollary is an immediate consequence of Corollary 12.4 and Theorem 12.3.

Corollary 12.7. *There are numbers* $p \in (2, \infty)$, $\varphi \in (0, \pi/2)$, $\vartheta \in (-\pi, \pi)$ *such that for any* $\mu \in (0, \infty)$, *the operator* $\Gamma(1, p, \mu \cdot e^{i \cdot \vartheta}, \mathbb{K}(\varphi))$ *is <u>not</u> Fredholm.*

If $\tau \in \{-1, 1\}$, *there are parameters* $p \in (1, 2)$, $\varphi \in (0, \pi/2)$, $\vartheta \in (-\pi, \pi)$ *so that* $\Gamma(\tau, p, \mu \cdot e^{i \cdot \vartheta}, \mathbb{K}(\varphi))$ *is also not Fredholm for* $\mu \in (0, \infty)$.

Proof: Combine Corollary 12.5 with Theorem 8.2.

At the end of this chapter, we are going to draw some conclusions concerning the estimate of $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ proposed in (1.24).

Lemma 12.10. *Let* $p \in (1, \infty)$, $\tau \in \{-1, 1\}$, $\vartheta \in [0, \pi)$, $\mu \in (0, \infty)$, $\varphi \in (0, \pi/2]$. *Assume that* $\Gamma(\tau, p, \mu \cdot e^{i \cdot \sigma}, \mathbb{K}(\varphi))$ *is topological for* $\sigma \in [-\vartheta, \vartheta]$. *Then there is some* $\mathfrak{C} > 0$ *such that it holds for* $f \in L^p(\partial \mathbb{K}(\varphi))^3$, $\lambda \in \mathbb{C} \backslash \{0\}$ *with* $|\arg \lambda| \leq \vartheta$:

$$\|f\|_p \quad \leq \quad \mathfrak{C} \cdot \| \Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))(f) \|_p .$$

Proof: Due to (6.13), the operator $J(\tau, p, \mu \cdot e^{i \cdot \sigma}, \varphi, 0, 1)$ must be topological for $\sigma \in [-\vartheta, \vartheta]$. Thus, for any $\sigma \in [-\vartheta, \vartheta]$, there is some number $\mathfrak{C}_1(\sigma) > 0$ with

$$\|\Phi\|_p \quad \leq \quad \mathfrak{C}_1(\sigma) \cdot \| J(\tau, p, \mu \cdot e^{i \cdot \sigma}, \varphi, 0, 1)(\Phi) \|_p \qquad \text{for } \Phi \in L^p(\mathbb{R}^2)^3 .$$

In addition, by Lemma 6.16, there exists a number $\mathfrak{C}_2 > 0$ such that

$$\| J(\tau, p, \mu \cdot e^{i \cdot \sigma}, \varphi, 0, 1)(\Phi) \; - \; J(\tau, p, \mu \cdot e^{i \cdot \varrho}, \varphi, 0, 1)(\Phi) \|_p \quad \leq \quad \mathfrak{C}_2 \cdot |\sigma - \varrho| \cdot \|\Phi\|_p$$

for $\Phi \in L^p(\mathbb{R}^2)^3$, $\sigma, \varrho \in [-\delta, \delta]$. Now, applying the Heine-Borel theorem, we may find a further constant $\mathfrak{C}_3 > 0$ with

$$\|\Phi\|_p \quad \leq \quad \mathfrak{C}_3 \cdot \| J(\tau, p, \mu \cdot e^{i \cdot \sigma}, \varphi, 0, 1)(\Phi) \|_p \qquad \text{for } \Phi \in L^p(\mathbb{R}^2)^3, \; \sigma \in [-\delta, \delta] .$$

This means by Lemma 12.1, for $\Phi \in L^p(\mathbb{R}^2)^3$, $\lambda \in \mathbb{C} \backslash \{0\}$ with $|\arg \lambda| \leq \vartheta$:

$$\|\Phi\|_p \quad \leq \quad \mathfrak{C}_3 \cdot \| J(\tau, p, \lambda, \varphi, 0, 1)(\Phi) \|_p .$$

Due to (6.13), this inequality implies the lemma.

Corollary 12.8. *Let* $p \in (1, \infty)$, $\vartheta \in [0, \pi)$, $\varphi_0 \in (0, \pi/2]$, $\tau \in \{-1, 1\}$. *Assume that* $\Pi^*(-\tau, p, \mathbb{K}(\varphi))$ *is topological for any* $\varphi \in [\varphi_0, \pi/2]$. *In addition, suppose there is some* $R > 0$ *such that* $\Lambda^{(ver)}(\tau, p, \varphi, R)$ *has property* F_+ *for* $\varphi \in [\varphi_0, \pi/2]$. *Finally, let* $\Gamma^*(\tau, p, \lambda, \mathbb{K}(\varphi_0))$ *be one-to-one for* $\lambda \in \mathbb{C} \backslash \{0\}$ *with* $|\arg \lambda| \leq \vartheta$.

(Instead of the second assumption, it may be required that $\Lambda(\tau, p, \mathbb{K}(\varphi))$ *is topological for* $\varphi \in [\varphi_0, \pi/2]$; *see Theorem 12.3.)*

Then $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi_0))$ *is topological for* $\lambda \in \mathbb{C} \backslash \{0\}$ *with* $|\arg \lambda| \leq \vartheta$, *and there is some* $\mathfrak{C} > 0$ *such that*

$$\|f\|_p \quad \leq \quad \mathfrak{C} \cdot \| \Gamma(\tau, p, \lambda, \mathbb{K}(\varphi_0))(f) \|_p$$

for $f \in L^p(\partial \mathbb{K}(\varphi))^3$, $\lambda \in \mathbb{C} \backslash \{0\}$ *with* $|\arg \lambda| \leq \vartheta$.

Proof: Corollary 12.4 or 12.6 implies the operator $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi_0))$ is Fredholm with index 0 if $\lambda \in \mathbb{C} \backslash (-\infty, 0]$ with $|\arg \lambda| \leq \vartheta$. On the other hand, we assumed that $\Gamma^*(\tau, p, \lambda, \mathbb{K}(\varphi_0))$ is one-to-one $(\lambda \in \mathbb{C} \backslash \{0\}$ with $|\arg \lambda| \leq \vartheta)$. Since $\Gamma^*(\tau, p, \lambda, \mathbb{K}(\varphi_0))$ is adjoint to $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi_0))$, the latter operator must be topological, as follows by Lemma 12.2 $(\lambda \in \mathbb{C} \backslash \{0\}$ with $|\arg \lambda| \leq \vartheta)$. Thus the corollary is implied by Lemma 12.10.

Corollary 12.9. *Let* $\lambda \in \mathbb{C} \backslash (-\infty, 0]$, $\tau \in \{-1, 1\}$.

Then there are reals $p \in (1, 2)$, $\varphi \in (0, \pi/2)$, *as well as a sequence* (f_n) *in* $L^p(\partial \mathbb{K}(\varphi))^3$ *such that*

$$\| \Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))(f_n) \|_p \;\leq\; 1 \qquad \text{for } n \in \mathbb{N}, \qquad \|f_n\|_p \to \infty \quad (n \to \infty). \tag{12.24}$$

In the case $\tau = 1$, *we may further find parameters* $p \in (2, \infty)$, $\varphi \in (0, \pi/2)$, *as well* *as a sequence* (f_n) *in* $L^p(\partial\mathbb{K}(\varphi))^3$ *such that the statement in (12.24) holds true once more.*

Proof: Suppose the first part of the lemma to be false. Then, for any $p \in (1, 2)$, $\varphi \in (0, \pi/2)$, there is a number $\mathfrak{C}_1(p, \varphi) > 0$ such that

$$\|f\|_p \leq \mathfrak{C}_1(p, \varphi) \cdot \|\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))(f)\|_p \quad \text{for } f \in L^p(\partial\mathbb{K}(\varphi))^3.$$

Hence the operator $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ is F_+ for $p \in (1, 2)$, $\varphi \in (0, \pi/2)$. But then, due to Corollary 6.6, the mapping $J(\tau, p, \lambda, \varphi, 0, 1)$ has the same property. Now Lemma 6.17 yields

$$index(J(\tau, p, \lambda, \varphi, 0, 1) = 0 \quad \text{for } p \in (1, 2), \ \varphi \in (0, \pi/2].$$

This means in particular that $J(\tau, p, \lambda, \varphi, 0, 1)$ is Fredholm for $p \in (1, 2)$, $\varphi \in (0, \pi/2]$. Thus, by Corollary 6.6, an analogous result holds true for $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$. But then Corollary 12.5 implies that $\Lambda(\tau, p, \mathbb{K}(\varphi))$ is topological for any $p \in (1, 2)$, $\varphi \in (0, \pi/2]$, so we have arrived at a contradiction to Theorem 8.2. This proves the first part of the corollary.

Now consider the case $\tau = 1$, and assume that the second part of the lemma is false. By repeating the previous arguments, we see that $\Gamma(1, p, \lambda, \mathbb{K}(\varphi))$ is Fredholm for $p \in (2, \infty)$, $\varphi \in (0, \pi/2]$. Now it follows from Corollary 12.5 that $\Pi^*(-1, p, \mathbb{K}(\varphi))$ is topological for φ, p as before. However, the last result contradicts Theorem 8.2, so we have established the second part of Corollary 12.9.

Chapter 13

Further Results Based on the L^2-Theory for the Stokes System in Bounded Lipschitz Domains

In the present chapter, we shall complete our L^p-theory for the the double-layer operator $\Lambda(\tau, p, \mathbb{K}(\varphi))$, which is related to the Stokes system (1.18) in $\mathbb{K}(\varphi)$ (see Chapter 1). As the reader may recall, we could show in Chapter 8 that for any $\tau \in \{-1, 1\}$, the operator $\Lambda(\tau, p, \mathbb{K}(\varphi))$ is not topological for certain values of $p \in (1, 2)$ and $\varphi \in (0, \pi/2)$ (Theorem 8.2). Furthermore, when $\varphi \in (0, \pi/2)$ and $\tau \in \{-1, 1\}$ are kept fixed, the corresponding set of exceptional values of $p \in (1, 2)$ is at most countable (Theorem 8.1). Here we intend to prove that $\Lambda(\tau, p, \mathbb{K}(\varphi))$ is topological if $p \in [2, \infty)$, $\varphi \in (0, \pi/2]$ and $\tau \in \{-1, 1\}$; see Theorem 13.1. The proof of this theorem is based on two inequalities established by Shen [43, p. 364, Lemma 5.2.11 (ii); p. 369, Lemma 5.3.7, with $\tau = 0$]. These results concern the gradient of a single-layer potential related to the Stokes system (1.18) in a bounded Lipschitz domain. Shen proves that the boundary value of this gradient may be estimated in L^2-norms against Dirichlet and Neumann boundary data.

In the case p=2, we shall deduce from Theorem 13.1 that $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ is topological and satisfies inequality (1.24) (see Corollary 13.3). This result, in turn, will lead to solutions of the resolvent problem (1.12) in $\mathbb{K}(\varphi)$, under Dirichlet boundary conditions (Corollary 13.4). Finally, by recurring to Theorem 8.1, it will be shown the two inequalities from [43] mentioned before cannot both be generalized to L^q-estimates with $q \geq 2$ (Theorem 13.2).

We begin our study by considering a bounded domain Ω in $\mathbb{R}^3$, with connected Lipschitz boundary. Then we can choose $k(\Omega) \in \mathbb{N}$, $\alpha(\Omega) \in (0, \infty)$, orthonormal matrices $A_1^{(\Omega)}, \cdots, A_{k(\Omega)}^{(\Omega)} \in \mathbb{R}^{3\times 3}$, vectors $C_1^{(\Omega)}, \cdots, C_{k(\Omega)}^{(\Omega)} \in \mathbb{R}^3$, and Lipschitz continuous functions $a_1^{(\Omega)}, \cdots, a_{k(\Omega)}^{(\Omega)}$ mapping $[-\alpha(\Omega), \alpha(\Omega)]^2$ into $\mathbb{R}$ such that the following properties hold true (compare [23, p. 269/270, 305/306]):

If for $i \in \{1, \cdots, k(\Omega)\}$, $\gamma \in (0, 1]$, the sets $\Delta^\gamma, \Lambda_i^\gamma, U_i$ are defined by

$$\Delta^{\gamma} := \big(-\gamma\cdot\alpha(\Omega)\,,\ \gamma\cdot\alpha(\Omega)\big)^2,$$

$$\Lambda_i^{\gamma} := \Big\{ A_i^{(\Omega)}\cdot\big(\eta\,,\ a_i^{(\Omega)}(\eta)\big) \ +\ C_i^{(\Omega)} \ :\ \eta\in\Delta^{\gamma}\Big\},$$

$$U_i := \Big\{ A_i^{(\Omega)}\cdot\big(\eta\,,\ a_i^{(\Omega)}(\eta)+r\big) \ +\ C_i^{(\Omega)} \ :\ \eta\in\Delta^1,\ r\in\big(-\alpha(\Omega)\,,\ \alpha(\Omega)\big)\Big\},$$

and if the function $H^{(i)}:\Delta^1\times\big(-\alpha(\Omega),\,\alpha(\Omega)\big)\mapsto U_i$ is introduced by

$$H^{(i)}(\eta,r) := A_i^{(\Omega)}\cdot\big(\eta\,,\ a_i^{(\Omega)}(\eta)+r\big)\ +\ C_i^{(\Omega)} \qquad \text{for } \eta\in\Delta^1,\ r\in\big(-\alpha(\Omega)\,,\ \alpha(\Omega)\big),$$

then we may require

$$U_i\cap\Omega \ =\ H^{(i)}\Big(\Delta^1\times\big(-\alpha(\Omega)\,,\,0\big)\Big),\qquad U_i\cap(\mathbb{R}^3\backslash\overline{\Omega}) \ =\ H^{(i)}\Big(\Delta^1\times\big(0\,,\,\alpha(\Omega)\big)\Big),$$

$$U_i\cap\partial\Omega \ =\ \Lambda_i^1 \qquad \text{for } i\in\{1,\cdots,k(\Omega)\},$$

$$\partial\Omega \ =\ \bigcup_{i=1}^{k(\Omega)}\Lambda_i^{1/4}. \tag{13.1}$$

A Lipschitz continuous function defined on an open set is differentiable on its domain except on a set of measure zero ([40, p. 108], [35, p. 88/89]). This implies that the outward unit normal $n^{(\Omega)}(x)$ to Ω exists at almost every point $x\in\partial\Omega$ ([35, p. 88/89]).

It may be shown there is a constant $\mathcal{D}_1>0$ with

$$\Big| H^{(i)}(\rho)\ +\ \kappa\cdot A_i^{(\Omega)}\cdot(0,0,1)\ -\ H^{(i)}(\eta)\ -\ \kappa'\cdot A_i^{(\Omega)}\cdot(0,0,1)\Big| \tag{13.2}$$
$$\geq\ \mathcal{D}_1\cdot\big(|\rho-\eta|\ +\ |\kappa-\kappa'|\big)$$

for $\rho,\eta\in\Delta^1$, $\kappa,\kappa'\in\big(-\alpha(\Omega),\alpha(\Omega)\big)$, $i\in\{1,\cdots,k(\Omega)\}$; compare [13, p. 57, (2.26)]. Let $i\in\{1,\cdots,k(\Omega)\}$. From (13.2), it follows that the mapping $H^{(i)}$ is one-to-one. Thus $H^{(i)}$ is a bijective Lipschitz function. Its inverse may be computed easily, and turns out to be Lipschitz continuous too. Moreover, the Jacobian $(dH^{(i)}/dX)(\eta,r)$ of $H^{(i)}$ exists at almost every point $(\eta,r)\in\Delta^1\times\big(-\alpha(\Omega)\,,\,\alpha(\Omega)\big)$, and it holds

$$\det\big(dH^{(i)}/dX\big)(\eta,r)\ =\ 1 \qquad \text{for a.e. } (\eta,r)\in\Delta^1\times\big(-\alpha(\Omega)\,,\,\alpha(\Omega)\big),\ \ i\in\{1,\cdots,k(\Omega)\}.$$

Thus we obtain by the substitution rule:

$$\int_{U_i} f(x)\,dx \ =\ \int_{-\alpha(\Omega)}^{\alpha(\Omega)}\int_{\Delta^1}(f\circ H^{(i)})(\eta,s)\ d\eta\,ds \qquad \text{for } f\in L^1(U_i). \tag{13.3}$$

Concerning an elementary proof of this formula, we refer to Rudin [41, p. 173, 8.26], where the substitution rule is shown for differentiable transformations with a continuous inverse. Rudin's proof also carries through for transformations which still have a continuous inverse, but are only differentiable almost everywhere. Since the latter properties are satisfied by $H^{(i)}$ ([40, p. 108], [35, p. 88/89]), we see that formula (13.3) is implied by the arguments in [41].

For $i\in\{1,\cdots,k(\Omega)\}$, the mapping

$$h^{(i)}:\Delta^1\mapsto\Lambda_i^1,\qquad h^{(i)}(\eta) := A_i^{(\Omega)}\cdot\big(\eta\,,\ a_i^{(\Omega)}(\eta)\big)\ +\ C_i^{(\Omega)}\quad(\eta\in\Delta^1)$$

is a bijective Lipschitz function with range in $\partial\Omega$, and its Jacobian $(dh^{(i)}/dX)(\eta)$ has maximal rank, for almost every $\eta\in\Delta^1$. In particular, it is a parametric representation of $\partial\Omega$. Set

$$J^{(i)}(\eta) := \Big(1\ +\ \sum_{r=1}^{2}|D_r h^{(i)}(\eta)|^2\Big)^{1/2} \qquad \text{for a.e. } \eta\in\Delta^1,\ i\in\{1,\cdots,k(\Omega)\}.$$

Then $J^{(i)}$ is the element of surface area related to $h^{(i)}$, so it holds for any integrable function $f:\partial\Omega\mapsto\mathbb{C}$:

$$\int_{\Lambda_i^1} f\,d\Omega \ =\ \int_{\Delta^1}(f\circ h^{(i)})\cdot J^{(i)}\,d\eta. \tag{13.4}$$

Because of (13.1), we may choose functions $\Psi_1^{(\Omega)},\cdots,\Psi_{k(\Omega)}^{(\Omega)}\in C_0^\infty(\mathbb{R}^3)$ with

$$0\leq\Psi_i^{(\Omega)}\leq 1,\ \ supp(\Psi_i^{(\Omega)})\subset U_i^{1/4}\ \text{ for } i\in\{1,\cdots,k(\Omega)\},\qquad \sum_{i=1}^{k(\Omega)}\Psi_i^{(\Omega)}|\partial\Omega\ =\ 1.$$

Define

$$\widetilde{m}^{(\Omega)}(x) := \sum_{i=1}^{k(\Omega)}\Psi_i^{(\Omega)}(x)\ \cdot\ A_i^{(\Omega)}\cdot(0,0,1) \quad \text{for } x\in\partial\Omega.$$

Then $\widetilde{m}^{(\Omega)}$ is a C^∞-function on $\mathbb{R}^3$ and does not vanish on $\partial\Omega$. In particular, there exists some number $\delta(\Omega)>0$ such that $m^{(\Omega)}(x)\neq 0$ for any $x\in\mathbb{R}^3$ with $dist(x,\partial\Omega)\leq\delta(\Omega)$. Thus we may choose a function $\widetilde{\Psi}^{(\Omega)}\in C_0^\infty(\mathbb{R}^3)$ such that $\widetilde{\Psi}^{(\Omega)}(x)=1$ for $x\in\mathbb{R}^3$ with $dist(x,\partial\Omega)<\delta(\Omega)/2$, and

$$supp(\Psi^{(\Omega)})\subset\{\,x\in\mathbb{R}^3\ :\ dist(x,\partial\Omega)<\delta(\Omega)\,\}.$$

Define the function $m^{(\Omega)}:\mathbb{R}^3\mapsto\mathbb{R}^3$ by

$$m^{(\Omega)}(x) := \widetilde{\Psi}^{(\Omega)}(x)\ \cdot\ |\widetilde{m}(x)|^{-1}\ \cdot\ \widetilde{m}(x) \qquad \text{for } x\in\partial\Omega \text{ with } dist(x,\partial\Omega)<\delta(\Omega),$$

and by $m^{(\Omega)}(x):=0$ else. Then it is clear that $m^{(\Omega)}$ belongs to $C_0^\infty(\mathbb{R}^3)^3$, and $|m^{(\Omega)}(x)|=1$ for $x\in\mathbb{R}^3$ with $dist(x,\partial\Omega)<\delta(\Omega)/2$. Furthermore, $m^{(\Omega)}$ is a "nontangential" direction with respect to $\partial\Omega$. To make this statement more precise, we define a cone $K(y,z,\delta,\epsilon)\subset\mathbb{R}^3$, for $\epsilon,\delta\in(0,\infty)$, $y,z\in\mathbb{R}^3$ with $|z|=1$, by setting

$$K(y,z,\delta,\epsilon) := \Big\{\,y+t\cdot b\ :\ t\in(0,\delta),\ b\in\mathbb{R}^3 \text{ with } |b|=1,\ |b-z|<\epsilon\Big\}.$$

Then there are constants $\mathcal{D}_2,\mathcal{D}_3,\mathcal{D}_4,\mathcal{D}_5\in(0,\infty)$ with

$$K(x,m^{(\Omega)}(x),\mathcal{D}_2,\mathcal{D}_3)\subset\mathbb{R}^3\backslash\overline{\Omega},\quad K(x,-m^{(\Omega)}(x),\mathcal{D}_2,\mathcal{D}_3)\subset\Omega \quad \text{for } x\in\partial\Omega, \tag{13.5}$$

$$\big|x\ +\ \kappa\cdot m(x)\ -\ x'\ -\ \kappa'\cdot m(x')\big|\ \geq\ \mathcal{D}_5\cdot\big(|x\ -\ x'|\ +\ |\kappa-\kappa'|\big) \tag{13.6}$$

for $x,x'\in\partial\Omega$, $\kappa,\kappa'\in(-\mathcal{D}_4,\mathcal{D}_4)$. We remark that an indication on how to construct the function $m^{(\Omega)}$ is given in [35, p. 246]. As for a detailed proof of (13.5) and (13.6), it is by no means trivial, but on the other hand, it does not represent any fundamental difficulty, and we leave it to the reader.

Inequality (13.6) implies there is another constant $\mathcal{D}_6>0$ such that

$$|x - y - \kappa \cdot m^{(\Omega)}(y)| \;\geq\; \mathcal{D}_6 \cdot |x - y| \qquad \text{for } x \in \overline{\Omega},\; y \in \partial\Omega,\; \kappa \in [0, \mathcal{D}_4), \tag{13.7}$$

$$|x - y + \kappa \cdot m^{(\Omega)}(y)| \;\geq\; \mathcal{D}_6 \cdot |x - y| \qquad \text{for } x \in \mathbb{R}^3\backslash\Omega,\; y \in \partial\Omega,\; \kappa \in [0, \mathcal{D}_4). \tag{13.8}$$

We further point out that for any $\alpha \in (-\infty, 2)$, it holds

$$\sup\Big\{ \int_{\partial\Omega} |x - y|^{-\alpha} \; d\Omega\,(y) \;:\; x \in \partial\Omega \Big\} \;<\; \infty. \tag{13.9}$$

For $p \in (1, \infty)$, $\Phi \in L^p(\partial\Omega)^3$, we define the single-layer potentials $V(\partial\Omega)(\Phi)$ and $Q(\partial\Omega)(\Phi)$ by setting

$$V(\partial\Omega)(\Phi)(x) := \Big(\int_{\partial\Omega} \sum_{k=1}^{3} E_{jk}(x - y) \cdot \Phi_k(y) \; d\Omega\,(y) \Big)_{1 \leq j \leq 3} \tag{13.10}$$

for $x \in \mathbb{R}^3\backslash\partial\Omega$ and for a.e. $x \in \partial\Omega$, and

$$Q(\partial\Omega)(\Phi)(x) := \int_{\partial\Omega} \sum_{k=1}^{3} E_{4k}(x - y) \cdot \Phi_k(y) \; d\Omega\,(y) \qquad \text{for } x \in \mathbb{R}^3\backslash\partial\Omega.$$

Note that the integral on the right-hand side of (13.10) exists for almost every $x \in \partial\Omega$, as may be shown by (13.9) and Hölder's inequality; compare the proof of Lemma 4.9. If $x \in \mathbb{R}^3\backslash\partial\Omega$, existence of the preceding integrals is obvious.

If $\Phi \in L^p(\partial\Omega)^3$ for some $p \in (1, \infty)$, then the mappings

$$u := V(\partial\Omega)(\Phi)\,|\,\mathbb{R}^3\backslash\partial\Omega \qquad \text{and} \qquad \pi := Q(\partial\Omega)(\Phi)\,|\,\mathbb{R}^3\backslash\partial\Omega$$

are C^∞-functions, and it holds

$$D^a u_j(x) \;=\; \int_{\partial\Omega} \sum_{k=1}^{3} D^a E_{jk}(x - y) \cdot \Phi_k(y) \; d\Omega\,(y),$$

$$D^a \pi(x) \;=\; \int_{\partial\Omega} \sum_{k=1}^{3} D^a E_{4k}(x - y) \cdot \Phi_k(y) \; d\Omega\,(y),$$

for $x \in \mathbb{R}^3\backslash\partial\Omega$, $j \in \{1, 2, 3\}$. This follows by Lebesgue's theorem on dominated convergence; compare the proof of Lemma 9.3 and Corollary 9.1. Due to the properties of the kernels E_{jk} $(1 \leq j, k \leq 3)$, we may conclude that the pair of functions (u, π) is a solution of the Stokes system (1.18) in $\mathbb{R}^3\backslash\partial\Omega$.

For $\kappa \in (0, \mathcal{D}_4)$, we put

$$\Omega^{(-1,\kappa)} := \{ x \in \mathbb{R}^3 \;:\; dist(x, \mathbb{R}^3\backslash\Omega) < \mathcal{D}_5 \cdot \kappa \},$$

$$\Omega^{(1,\kappa)} := \{ x \in \mathbb{R}^3 \;:\; dist(x, \overline{\Omega}) < \mathcal{D}_5 \cdot \kappa \},$$

with $\mathcal{D}_4, \mathcal{D}_5$ from (13.6). Note that for $\kappa \in (0, \mathcal{D}_4)$, $\tau \in \{-1, 1\}$, the set $\Omega^{(\tau,\kappa)}$ is open, with $\overline{\Omega} \subset \Omega^{(1,\kappa)}$ and $\mathbb{R}^3\backslash\Omega \subset \Omega^{(-1,\kappa)}$.

Next, for p, Φ as in (13.10), and for $\kappa \in (0, \mathcal{D}_4)$, $\tau \in \{-1, 1\}$, we define the functions $V^{(\tau,\kappa)}$, $Q^{(\tau,\kappa)}$ on $\Omega^{(\tau,\kappa)}$ by setting

$$V^{(\tau,\kappa)}(\Phi)(x) := \Big(\int_{\partial\Omega} \sum_{k=1}^{3} E_{jk}(x - y - \tau \cdot \kappa \cdot m^{(\Omega)}(y)) \cdot \Phi_k(y) \; d\Omega\,(y) \Big)_{1 \leq j \leq 3}$$

$$Q^{(\tau,\kappa)}(\Phi)(x) := \int_{\partial\Omega} \sum_{k=1}^{3} E_{4k}(x - y - \tau \cdot \kappa \cdot m^{(\Omega)}(y)) \cdot \Phi_k(y) \; d\Omega\,(y)$$

for $x \in \Omega^{(\tau,\kappa)}$. As a consequence of (13.6) and Lebesgue's theorem on dominated convergence, we obtain $V_j^{(\tau,\kappa)}$, $Q^{(\tau,\kappa)} \in C^\infty(\Omega^{(\tau,\kappa)})$ if $\tau \in \{-1, 1\}$, $\kappa \in (0, \mathcal{D}_4)$, $j \in \{1, 2, 3\}$, and if we assume $\Phi \in L^p(\partial\Omega)^3$ for some $p \in (1, \infty)$. Moreover, it follows

$$D^a V_j^{(\tau,\kappa)}(\Phi)(x) \;=\; \int_{\partial\Omega} \sum_{k=1}^{3} D^a E_{jk}(x - y - \tau \cdot \kappa \cdot m^{(\Omega)}(y)) \cdot \Phi_j(y) \; d\Omega\,(y),$$

$$D^a Q^{(\tau,\kappa)} \;=\; \int_{\partial\Omega} \sum_{k=1}^{3} D^a E_{4k}(x - y - \tau \cdot \kappa \cdot m^{(\Omega)}(y)) \cdot \Phi_j(y) \; d\Omega\,(y),$$

for $x \in \Omega^{(\tau,\kappa)}$, $a \in \mathbb{N}_0^3$. In particular, the pair of functions $(V^{(\tau,\kappa)}(\Phi), Q^{(\tau,\kappa)}(\Phi))$ solves the Stokes system (1.18) in $\Omega^{(\tau,\kappa)}$.

For $\tau \in \{-1, 1\}$, $p \in (1, \infty)$, we introduce bounded operators $\Lambda(\tau, p, \Omega)$, $\Lambda^*(\tau, p, \Omega)$ from the space $L^p(\partial\Omega)^3$ into itself by setting for $\Phi \in L^p(\partial\Omega)^3$:

$$\Lambda(\tau, p, \Omega)(\Phi) \;:=\; (\tau/2) \cdot \Phi$$
$$+\; L^p(\partial\Omega)^3 - \lim_{\epsilon\downarrow 0} \Big(\int_{\partial\Omega} \chi(|I - y|) \cdot \sum_{j,k=1}^{3} \mathcal{D}_{jkl}(I - y) \cdot \Phi_j(y) \cdot n_k^{(\Omega)}(y) \; d\Omega\,(y) \Big)_{1 \leq l \leq 3},$$

$$\Lambda^*(\tau, p, \Omega)(\Phi) \;:=\; (\tau/2) \cdot \Phi$$
$$-\; L^p(\partial\Omega)^3 - \lim_{\epsilon\downarrow 0} \Big(\int_{\partial\Omega} \chi(|I - y|) \cdot \sum_{k,l=1}^{3} \mathcal{D}_{jkl}(I - y) \cdot \Phi_l(y) \cdot n_k^{(\Omega)} \; d\Omega\,(y) \Big)_{1 \leq j \leq 3},$$

where I denotes the identical mapping of $\partial\Omega$. It is known by [4] that the preceding operators are well defined and bounded; compare the remarks by [21, p. 773/774] and [47, p. 578-582]. For more details we refer to [6].

We shall need another consequence of the results in [4]. In fact, as stated in [21, p. 773/774], for any $j, k \in \{1, 2, 3\}$, $\tau \in \{-1, 1\}$, $p \in (1, \infty)$, $\Phi \in L^p(\partial\Omega)^3$, there is a function $B_{jk}(\tau, p, \Omega)(\Phi)$ such that

$$\int_{\partial\Omega} \Big| D_j\Big(V_k(\partial\Omega)(\Phi)\,\big|\,\mathbb{R}^3\backslash\partial\Omega \Big)(x + \tau \cdot \kappa \cdot m^{(\Omega)}(x)) \;-\; B_{jk}(\tau, p, \Omega)(\Phi)(x) \Big|^p \; d\Omega\,(x)$$
$$\longrightarrow 0 \quad (\kappa \downarrow 0),$$

and a function $C(\tau, p, \Omega)(\Phi) \in L^p(\partial\Omega)$ with

$$\int_{\partial\Omega} |Q(\partial\Omega)(\Phi)(x + \tau \cdot \kappa \cdot m^{(\Omega)}(x)) \;-\; C(\tau, p, \Omega)(\Phi)(x)|^p \; d\Omega\,(x) \longrightarrow 0 \quad (\kappa \downarrow 0).$$

In the preceding relations, the convergence of $V(\partial\Omega)(\Phi)$ and $Q(\partial\Omega)(\Phi)$ is "nontangential" ([21, p. 772]), as required in [21, p. 773/774]. This follows from (13.5).

According to the trace theorem in [21, p. 774, (0.9), (0.10)], we have the following "jump relation":

$$\sum_{k=1}^{3}\Big(B_{jk}(\tau,\,p,\,\Omega)(\Phi) \;+\; B_{kj}(\tau,\,p,\,\Omega)(\Phi) \;-\; \delta_{jk}\cdot Q(\tau,\,p,\,\Omega)(\Phi)\Big)\cdot n_k^{(\Omega)} \tag{13.11}$$

$$= \; -\Lambda_j^*(\tau,\,p,\,\Omega)(\Phi)$$

for $j\in\{1,\,2,\,3\}$, $p\in(1,\,\infty)$, $\Phi\in L^p(\partial\Omega)^3$. This result is an analogue of Theorem 9.1, where the domain $\mathbb{K}(\varphi)$ was considered instead of Ω. We note another consequence of [21, p.774, (0.9), (0.10)], namely

$$\int_{\partial\Omega}\sum_{j,\,k=1}^{3}\mathcal{D}_{jkl}(x+\tau\cdot\kappa\cdot m^{(\Omega)}(x)-y)\cdot n_k^{(\Omega)}(y)\cdot\Phi_j(y)\;d\Omega\,(y) \tag{13.12}$$

$$\longrightarrow\; \Lambda_l(-\tau,\,p,\,\Omega)(\Phi)(y)\quad(\kappa\downarrow 0)$$

for $l\in\{1,\,2,\,3\}$, $p\in(1,\,\infty)$, $\Phi\in L^p(\partial\Omega)^3$, and for almost every $x\in\partial\Omega$.

Finally, we shall need a third consequence of [21, p. 774, (0.9)]. In fact, for $k,\,l\in\{1,\,2,\,3\}$, and for $p,\,\Phi$ as before, it holds

$$B_{kl}(1,\,p,\,\Omega)(\Phi)\;-\;\sum_{j=1}^{3}n_j^{(\Omega)}\cdot B_{jl}(1,\,p,\,\Omega)(\Phi)\cdot n_k^{(\Omega)} \tag{13.13}$$

$$=\;B_{kl}(-1,\,p,\,\Omega)(\Phi)\;-\;\sum_{j=1}^{3}n_j^{(\Omega)}\cdot B_{jl}(-1,\,p,\,\Omega)(\Phi)\cdot n_k^{(\Omega)}.$$

Hence, concerning the sum of derivatives of $V_l(\partial\Omega)(\Phi)$ the limit of which is considered on the left- and right-hand side of (13.13), it does not matter whether this limit is taken inside Ω or outside $\overline{\Omega}$. On the other hand, adding $Q(\partial\Omega)(\Phi)$ to certain derivatives of $V(\partial\Omega)(\Phi)$ yields a function having the property that its trace from inside Ω does not coincide with its trace from outside $\overline{\Omega}$. This is shown by equation (13.11) and motivates the term "jump relation".

We shall denote the limit function appearing on the left- and right-hand side of (13.13) by $T_{kl}(p,\,\Omega)(\Phi)$, for $k,\,l\in\{1,\,2,\,3\}$, $p\in(1,\,\infty)$, $\Phi\in L^p(\partial\Omega)^3$.

The choice of derivatives in (13.13) allows us to interpret $T_{kl}(p,\,\Omega)(\Phi)$ in the following way: For almost every $x\in\partial\Omega$, the vector $(T_{kl}(p,\,\Omega)(\Phi)(x))_{1\le k\le 3}$ coincides with the projection of

$$\lim_{\kappa\to\infty}\nabla\Big(V_l(\partial\Omega)(\Phi)\,\Big|\,\mathbb{R}^3\backslash\partial\Omega\Big)(x+\kappa\cdot m^{(\Omega)}(x))$$

on the tangential plane to $\partial\Omega$ in x $(l\in\{1,\,2,\,3\})$. This is the reason why the vector $(T_{kl}(p,\,\Omega)(\Phi)(x))_{1\le k\le 3}$ is called "tangential derivative" (of $V_l(\partial\Omega)(\Phi)$ in x).

In the following, we would like to show that $\Lambda^*(-1,\,p,\,\Omega)$ and $\Lambda(-1,\,p,\,\Omega)$ are Fredholm operators for $p\in(1,\,2]$. However, we shall only succeed in proving this point for domains which have one conical boundary point and are smoothly bounded everywhere else, but this result will be enough for our purposes. After all, we are only going to study

the preceding operators in order to obtain information on the behaviour of the mapping $\Lambda(\tau,\,q,\,\mathbb{K}(\varphi))$, for $q\in[2,\,\infty)$. With this aim in mind, we can of course restrict ourselves to the kind of domain just mentioned. Still, we shall stick to our general Lipschitz domain Ω as long as this is possible without additional effort.

Our first lemma should be compared to Lemma 9.7.

Lemma 13.1. *Let* $R>0$ *with* $\overline{\Omega}\subset\mathbb{B}_3(0,\,R)$, $p\in(4/3,\,\infty)$, $\Phi\in L^p(\partial\Omega)^3$, $j,\,l\in\{1,\,2,\,3\}$. *Then it holds*

$$\int_{\mathbb{B}_3(0,\,R)\backslash\overline{\Omega}}\Big|\,D_lV_j^{(-1,\,\kappa)}(\Phi)(x)\;-\;\partial/\partial x_l\Big(V_j(\partial\Omega)(\Phi)(x)\Big)\Big|^2\;dx\;\longrightarrow\;0\quad(\kappa\downarrow 0), \tag{13.14}$$

$$\int_{\Omega}\Big|\,D_lV_j^{(1,\,\kappa)}(\Phi)(x)\;-\;\partial/\partial x_l\Big(V_j(\partial\Omega)(\Phi)(x)\Big)\Big|^2\;dx\;\longrightarrow\;0\quad(\kappa\downarrow 0). \tag{13.15}$$

Proof: By (13.8), there is a constant $\mathfrak{C}_1>0$ such that

$$|\,D_lE_{jk}(x-y+\tau\cdot\kappa\cdot m^{(\Omega)}(y))\,|\quad\le\quad\mathfrak{C}_1\cdot|x-y|^{-2} \tag{13.16}$$

for $x\in\mathbb{R}^3\backslash\Omega$, $y\in\partial\Omega$, $\kappa\in(0,\,\mathcal{D}_4)$, $k\in\{1,\,2,\,3\}$. We shall show that

$$\mathcal{A}:=\int_{\mathbb{B}_3(0,\,R)\backslash\overline{\Omega}}\Big(\int_{\partial\Omega}|x-y|^{-2}\cdot|\Phi(y)|\;d\Omega\,(y)\Big)^2\;dx\quad<\quad\infty.$$

Then the relation in (13.14) follows from (13.16) and Lebesgue's theorem on dominated convergence. The claim in (13.15) may be proved by a similar reasoning, but with a reference to (13.7) instead of (13.8).

In order to show that $\mathcal{A}<\infty$, we set $t:=2$, $\alpha:=2/p-1$ in the case $p<2$, and $t:=6\cdot p/5$, $\alpha:=1/(3\cdot p)$ if $p\ge 2$. Then, due to the assumption $p>4/3$, the relations in (9.38) hold true. Putting $\mathfrak{C}_2:=(4\cdot\pi\cdot R^3)^{1-2/t}$, we obtain by Hölder's inequality:

$$\mathcal{A}\quad\le\quad\mathfrak{C}_2\cdot\Big(\int_{\mathbb{B}_3(0,\,R)\backslash\overline{\Omega}}\Big(\int_{\partial\Omega}|x-y|^{-2}\cdot|\Phi(y)|\;d\Omega\,(y)\Big)^t\;dx\Big)^{2/t}$$

$$\le\quad\mathfrak{C}_2\cdot k(\Omega)\cdot\sum_{k=1}^{k(\Omega)}(A_i+B_i),$$

where we used the abbreviations

$$A_i:=\Big(\int_{U_i\cap(\mathbb{B}_3(0,\,R)\backslash\overline{\Omega})}\Big(\int_{\Lambda_i^{1/2}}|x-y|^{-2}\cdot|\Phi(y)|\;d\Omega(y)\Big)^t\;dx\Big)^{2/t},$$

$$B_i:=\Big(\int_{\mathbb{B}_3(0,\,R)\backslash(\overline{\Omega}\cup U_i)}\Big(\int_{\Lambda_i^{1/2}}|x-y|^{-2}\cdot|\Phi(y)|\;d\Omega(y)\Big)^t\;dx\Big)^{2/t}.$$

for $i\in\{1,\,\cdots,\,k(\Omega)\}$. Obviously, we have $B_i<\infty$ $(i\in\{1,\,\cdots,\,k(\Omega)\})$. On the other hand, we obtain from (13.3), (13.4) and (13.2), for $i\in\{1,\,\cdots,\,k(\Omega)\}$:

$$A_i \;\le\; \Big(\int_{\alpha(\Omega)}^{\alpha(\Omega)} \int_{\Delta^1} \Big(\int_{\Delta^{1/2}} |H^{(i)}(\varrho, r) - h^{(i)}(\eta)|^{-2} \cdot |\Phi \circ h^{(i)}(\eta)| \cdot J^{(i)}(\eta) \; d\eta \Big)^t \; d\varrho \, dr \Big)^{2/t}$$

$$\le\; \mathfrak{C}_3 \cdot \Big(\int_{\alpha(\Omega)}^{\alpha(\Omega)} \int_{\Delta^1} \Big(\int_{\Delta^{1/2}} |\varrho - \eta|^{-2+\alpha} \cdot |r|^{-\alpha} \cdot |\Phi \circ h^{(i)}(\eta)| \; d\eta \Big)^t \; d\varrho \, dr \Big)^{2/t}$$

with the abbreviation $\mathfrak{C}_3 := (\mathcal{D}_1)^{-4} \cdot \big(\sup\{J^{(i)}(\eta) \;:\; \eta \in \Delta^1\}\big)^2$. Now we integrate in r, and then apply Theorem 9.2 (Hardy-Littlewood-Sobolev inequality). Recalling (9.38) and our assumption $\Phi \in L^p(\partial\Omega)^3$, it follows $\mathcal{A}_i < \infty$ $(i \in \{1, \cdots, k(\Omega)\})$. This completes the proof of Lemma 13.1.

Next we note a consequence of inequality (13.6) and the mean value theorem. In fact, it holds for $j, k \in \{1, 2, 3\}$, $\tau \in \{-1, 1\}$, $p \in (1, \infty)$, $\Phi \in L^p(\partial\Omega)^3$:

$$\int_{\partial\Omega} \Big| D_j\Big(V_k(\partial\Omega)(\Phi) \,\big|\, \mathbb{R}^3\backslash\partial\Omega \Big)(x - \tau \cdot \kappa \cdot m^{(\Omega)}(x)) \;-\; D_j\Big(V_k^{(\tau,\kappa)}(\Phi) \Big)(x) \Big|^p \; d\Omega(x)$$
$$\longrightarrow 0 \quad (\kappa \downarrow 0),$$

compare the arguments in [13, p. 171/172]. Now it follows from (13.11), for j, τ, p, Φ as before:

$$\int_{\partial\Omega} \Big| \Lambda_j^*(\tau, p, \Omega)(\Phi)(x) \;+\; \sum_{k=1}^{3} \Big(D_j\Big(V_k^{(-\tau,\kappa)}(\Phi) \Big) + D_k\Big(V_j^{(-\tau,\kappa)}(\Phi) \Big) \qquad (13.17)$$
$$- \;\delta_{jk} \cdot Q^{(-\tau,\kappa)}(\Phi) \Big)(x) \cdot n_k^{(\Omega)}(x) \Big|^p \; d\Omega(x) \;\longrightarrow\; 0 \quad (\kappa \downarrow 0).$$

Furthermore, using Hölder's inequality as in the proof of Lemma 4.9, we may deduce from (13.9), (13.6) and Lebesgue's theorem on dominated convergence:

$$\int_{\partial\Omega} \Big| V_j(\partial\Omega)(\Phi)(x) - V_j^{(\tau,\kappa)}(\Phi)(x) \Big|^p \; d\Omega(x) \longrightarrow 0 \qquad \text{for } \kappa \downarrow 0, \qquad (13.18)$$

$$\int_{\partial\Omega} \Big| V_j(\partial\Omega)(\Phi)(x) - V_j(\partial\Omega)(\Phi)(x + \kappa \cdot m^{(\Omega)}(x)) \Big|^p \; d\Omega(x) \qquad (13.19)$$
$$\longrightarrow 0 \quad (\kappa \to 0),$$

if $j \in \{1, 2, 3\}$, $\tau \in \{-1, 1\}$, $p \in (1, \infty)$, $\Phi \in L^p(\partial\Omega)^3$.

Next we introduce a finitely dimensional function space which will play an important role in the following. To this end, take an arbitrary subset B of $\mathbb{R}^3$ and define the constant functions $\mathcal{E}_B, \mathcal{N}_B : B \mapsto \mathbb{R}$ by $\mathcal{E}_B(x) := 1$, $\mathcal{N}_B(x) := 0$ for $x \in B$. Then we set

$$Z(B) \;:=\; \Big\{ \Big(-\big(id(B)\big)_3, \mathcal{N}_B, \big(id(B)\big)_1 \Big), \Big(\mathcal{N}_B, -\big(id(B)\big)_3, \big(id(B)\big)_2 \Big),$$
$$\Big(-\big(id(B)\big)_2, \big(id(B)\big)_1, \mathcal{N}_B \Big), (\mathcal{E}_B, \mathcal{N}_B, \mathcal{N}_B), (\mathcal{N}_B, \mathcal{E}_B, \mathcal{N}_B), (\mathcal{N}_B, \mathcal{N}_B, \mathcal{E}_B) \Big\}.$$

Note that if B is a domain in $\mathbb{R}^3$, we have

$$Z(B) \;=\; \big\{ f \in C^1(B)^3 \;:\; D_i f_j + D_j f_i = 0 \text{ for } i, j \in \{1, 2, 3\} \big\}. \qquad (13.20)$$

A proof of this well known result may be found in [13, p. 173-178].

Lemma 13.2. *Let* $p \in (4/3, \infty)$. *Then it holds* $dim\Big(kern\big(\Lambda^*(-1, p, \Omega)\big)\Big) \le 6$.

Proof: We shall adapt the method of proof used in [30, p. 62/63] and [13, p. 189/190]. To this end, we assume there are seven functions $\Psi^{(1)}, \cdots, \Psi^{(7)} \in kern\big(\Lambda^*(-1, p, \Omega)\big)$ being linearly independent. Take $k \in \{1, \cdots, 7\}$, and recall that for any $\kappa \in (0, \mathcal{D}_4)$, the functions $u_j^{(\kappa)} := V_j^{(1,\kappa)}(\Psi^{(k)})$ $(j \in \{1, 2, 3\})$ and $\pi^{(\kappa)} := Q^{(1,\kappa)}(\Psi^{(k)})$ belong to $C^\infty(\Omega^{(1,\kappa)})$, and the pair of functions $(u^{(\kappa)}, \pi^{(\kappa)})$ solves the Stokes problem (1.18) in $\Omega^{(1,\kappa)}$. Using these facts, and recalling that $\overline{\Omega} \subset \Omega^{(1,\kappa)}$, we obtain by the Divergence theorem:

$$(1/2) \cdot \sum_{j,k=1}^{3} \int_\Omega \Big(D_j u_k^{(\kappa)} + D_k u_j^{(\kappa)} \Big) \cdot \Big(D_j u_k^{(\epsilon)} + D_k u_j^{(\epsilon)} \Big) \; dx$$
$$= \int_{\partial\Omega} \sum_{j,k=1}^{3} \Big(D_j u_k^{(\kappa)} + D_k u_j^{(\kappa)} - \delta_{jk} \cdot \pi^{(\kappa)} \Big)(x) \cdot u_j^{(\epsilon)}(x) \cdot n_k^{(\Omega)}(x) \; d\Omega(x)$$

for $\kappa, \epsilon \in (0, \mathcal{D}_4)$. We remark that the Divergence theorem is of course valid when a bounded domain with Lipschitz boundary is considered; see [35, p. 121, Théorème 1.1]. Now we first let κ and then ϵ tend to zero. Applying (13.17) and Lemma 13.1, and using the fact that $\Lambda^*(-1, p, \Omega)(\Psi^{(k)})$ vanishes, we find

$$\int_\Omega \sum_{j,k=1}^{3} \Big(\partial/\partial x_j \big(V_k(\partial\Omega)(\Phi)(x) \big) + \partial/\partial x_k \big(V_j(\partial\Omega)(\Phi)(x) \big) \Big)^2 \; dx \;=\; 0.$$

From this equation and (13.20), we deduce $V(\partial\Omega)(\Phi) \,|\, \Omega \in Z(\Omega)$ for $k \in \{1, \cdots, 7\}$. Since $dim\, Z(\Omega) = 6$, there must be numbers $\lambda_1, \cdots, \lambda_7 \in \mathbb{C}$ with

$$\sum_{k=1}^{7} \lambda_k \cdot V(\partial\Omega)(\Psi^{(k)}) \,|\, \Omega \;=\; 0 \quad \text{and} \quad (\lambda_1, \cdots, \lambda_7) \ne 0.$$

Setting $\Psi := \sum_{k=1}^{7} \lambda_k \cdot \Psi^{(k)}$, it follows $\Psi \in L^p(\partial\Omega)^3$, $V(\partial\Omega)(\Psi) \,|\, \Omega = 0$, so we may conclude from (13.19) that $V(\partial\Omega)(\Psi)$ vanishes almost everywhere on $\partial\Omega$.

Next we are going to exploit the fact that $v^{(\kappa)} := V^{(-1,\kappa)}(\Psi)$ and $\varrho^{(\kappa)} := Q^{(-1,\kappa)}(\Psi)$ are C^∞-functions on $\Omega^{(-1,\kappa)}$, and that the pair of functions $(v^{(\kappa)}, \varrho^{(\kappa)})$ solves the Stokes system (1.18) in $\Omega^{(-1,\kappa)}$ $(\kappa \in (0, \mathcal{D}_4))$. Furthermore, we recall the relation $\mathbb{R}^3\backslash\Omega \subset \Omega^{(-1,\kappa)}$. It is due to these facts that for $R \in (0, \infty)$ with $\overline{\Omega} \subset \mathbb{B}_3(0, R)$, and for $\kappa, \epsilon \in (0, \mathcal{D}_4)$, we may use the Divergence theorem to get the ensuing equation:

$$(1/2)\cdot\sum_{j,k=1}^{3}\int_{\mathbb{B}_3(0,R)\backslash\overline{\Omega}}\Big(D_jv_k^{(\kappa)}+D_kv_j^{(\kappa)}\Big)\cdot\Big(D_jv_k^{(\epsilon)}+D_kv_j^{(\epsilon)}\Big)\,dx$$

$$=-\int_{\partial\Omega}\sum_{j,k=1}^{3}\Big(D_jv_k^{(\kappa)}+D_kv_j^{(\kappa)}-\delta_{jk}\cdot\varrho^{(\kappa)}\Big)(x)\cdot v_j^{(\epsilon)}(x)\cdot n_k^{(\Omega)}(x)\,d\Omega(x)$$

$$+\int_{\partial\mathbb{B}_3(0,R)}\sum_{j,k=1}^{3}\Big(D_jv_k^{(\kappa)}+D_kv_j^{(\kappa)}-\delta_{jk}\cdot\varrho^{(\kappa)}\Big)(x)\cdot v_j^{(\epsilon)}(x)\cdot(x_k/|x|)\,do_x\,.$$

On both sides of this equation, we first let ϵ and then κ tend to zero. After that, we let R grow to infinity. By recurring to (13.18), (13.20) and Lemma 13.1, and using the fact that $V(\partial\Omega)(\Psi)$ vanishes on $\partial\Omega$, we obtain that $V(\partial\Omega)(\Psi)\,|\,\mathbb{R}^3\backslash\overline{\Omega}$ belongs to $Z(\Phi)$.

On the other hand, the term $V(\partial\Omega)(\Psi)(x)$ decays for $|x|\to\infty$, as is clear by the definition of $V(\partial\Omega)$. Since $Z(\mathbb{R}^3\backslash\overline{\Omega})$ contains no function with this property except the zero function, it follows that $V(\partial\Omega)(\Psi)$ vanishes everywhere on $\mathbb{R}^3\backslash\overline{\Omega}$. Thus we infer from (1.18) that $Q(\partial\Omega)(\Psi)$ is constant. But the latter function also decays for large values of $|x|$, and hence must vanish everywhere on $\mathbb{R}^3\backslash\overline{\Omega}$. Now we infer from (13.11): $\Lambda^*(1,p,\Omega)(\Psi)=0$. On the other hand, we know by the choice of Ψ that $\Lambda^*(-1,p,\Omega)(\Psi)$ vanishes too. Thus it follows $\Psi=0$. Since the functions $\Psi^{(1)},\cdots,\Psi^{(7)}$ are linearly independent, we finally arrive at the equation $\lambda_1=\cdots=\lambda_7=0$. But this is a contradiction to the choice of $\lambda_1,\cdots,\lambda_7$, so the lemma is proved.

Lemma 13.3. *Let $p\in(1,\infty)$. Then it holds $\Lambda(-1,p,\Omega)(\Psi)=0$ for $\Psi\in Z(\partial\Omega)$. This means in particular: $dim\Big(kern(\Lambda(-1,p,\Omega))\Big)\geq 6$.*

Proof: For $\kappa\in(0,\mathcal{D}_4)$, $\Psi\in Z(\partial\Omega)$, $l\in\{1,2,3\}$, $x\in\partial\Omega$, we obtain by the Divergence theorem:

$$\int_{\partial\Omega}\sum_{j,k=1}^{3}\mathcal{D}_{jkl}(x+\kappa\cdot m^{(\Omega)}(x)-y)\cdot n_k^{(\Omega)}(y)\cdot\Psi_j(y)\,d\Omega(y)\quad=\quad 0\,.$$

Hence the lemma follows from (13.12).

The next lemma recalls that weakly singular integral operators on $\partial\Omega$ are compact.

Lemma 13.4. *Let $K:\partial\Omega\times\partial\Omega\mapsto\mathbb{C}^3$ be a measurable function. Assume there is some constant $\mathfrak{C}>0$ and a number $\alpha\in(-2,\infty)$ such that $|K(x,y)|\leq\mathfrak{C}\cdot|x-y|^{-\alpha}$ for $x,y\in\partial\Omega$. Take $p\in(1,\infty)$, and let B be a measurable subset of $\partial\Omega$.*

Then the operator $\mathcal{K}:L^p(B)^3\mapsto L^p(B)^3$, defined by

$$\mathcal{K}(\Phi)(x):=\Big(\int_B\sum_{k=1}^{3}K_{jk}(x-y)\cdot\Phi(y)\,d\Omega(y)\Big)_{1\leq l\leq 3}$$

for $\Phi\in L^p(B)^3$ and for a.e. $x\in B$, is well defined, bounded, and compact.

Proof: Using (13.9), we may adapt the proof of Lemma 6.3 and 4.9 to the present situation. This means the lemma is reduced to the fact that Hille-Tamarkin operators are compact; see [28].

In the proof of the next lemma, we shall use the results from [43] mentioned at the beginning of this chapter.

Lemma 13.5. *The operator $\Lambda^*(-1,2,\Omega)$ has property F_+.*

Proof: By [43, p. 369, Lemma 5.3.7, with $\tau=0$], there is a constant $\mathfrak{C}_1>0$ such that

$$\sum_{j,k=1}^{3}\|B_{jk}(-1,2,\Omega)(\Phi)\|_2 \tag{13.21}$$
$$\leq\ \mathfrak{C}_1\cdot\Big(\|\Lambda^*(-1,2,\Omega)(\Phi)\|_2+\|V(\partial\Omega)(\Phi)\,|\,\partial\Omega\|_2\Big)$$

for $\Phi\in L^2(\partial\Omega)^3$. Furthermore, referring to [43, p. 364, Lemma 5.2.11 (ii), with $\tau=0$], we see there is another constant $\mathfrak{C}_2>0$ with

$$\sum_{j,k=1}^{3}\Big(\|B_{jk}(+1,2,\Omega)(\Phi)\|_2+\|C(1,2,\Omega)(\Phi)\|_2\Big) \tag{13.22}$$
$$\leq\ \mathfrak{C}_2\cdot\Big(\sum_{k,l=1}^{3}\|T_{kl}(2,\Omega)(\Phi)\|_2+\|V(\partial\Omega)(\Phi)\,|\,\partial\Omega\|_2\Big)$$

for $\Phi\in L^2(\partial\Omega)^3$. We remark that the right-hand side of (13.22) may be replaced by $\|V(\partial\Omega)(\Phi)\,|\,\partial\Omega\|_{1,2}$ times a constant only depending on Ω; see [47, p. 580]. Thus it is the Dirichlet boundary value of the functions $V(\partial\Omega)(\Phi)\,|\,\Omega$ and $V(\partial\Omega)(\Phi)\,|\,\mathbb{R}^3\backslash\overline{\Omega}$ (see (13.19)) which appears on the right-hand side of (13.22).

Since $\Phi=\Lambda^*(1,2,\Omega)(\Phi)-\Lambda^*(-1,2,\Omega)(\Phi)$, and because of (13.22), (13.11), we conclude for $\Phi\in L^2(\partial\Omega)^3$:

$$\|\Phi\|_2$$
$$\leq\ 2\cdot\mathfrak{C}_2\cdot\Big(\sum_{k,l=1}^{3}\|T_{kl}(2,\Omega)(\Phi)\|_2+\|V(\partial\Omega)(\Phi)\,|\,\partial\Omega\|_2\Big)+\|\Lambda^*(-1,2,\Omega)(\Phi)\|_2\,.$$

Recalling the definition of $T_{kl}(2,\Omega)$ (see the remark following (13.13)), and using (13.21), we may infer from the preceding inequality, for $\Phi\in L^2(\partial\Omega)^3$:

$$\|\Phi\|_2\ \leq\ \mathfrak{C}_3\cdot\Big(\|\Lambda^*(-1,2,\Omega)(\Phi)\|_2+\|V(\partial\Omega)(\Phi)\,|\,\partial\Omega\|_2\Big), \tag{13.23}$$

with $\mathfrak{C}_3:=2\cdot\mathfrak{C}_2\cdot(4\cdot\mathfrak{C}_1+1)+1$. But the operator

$$\mathcal{K} : L^2(\partial\Omega)^3 \mapsto L^2(\partial\Omega)^3, \qquad \mathcal{K}(\Phi) := V(\partial\Omega)(\Phi)\,|\,\partial\Omega \quad \text{for } \Phi \in L^2(\partial\Omega)^3$$

is compact, as follows from Lemma 13.4. By combining this fact with inequality (13.23) and [34, p. 18, Lemma 2.1], we may conclude that $\Lambda^*(-1, 2, \Omega)$ is a F_+-operator.

Before restricting our considerations to more special Lipschitz domains, let us point out some trivial facts, which are mentioned here because it may be useful after all to take note of them:

Lemma 13.6. *Let* $\tau \in \{-1, 1\}$, $p, q \in (1, \infty)$ *with* $p < q$. *Then it holds*

$$L^q(\partial\Omega)^3 \subset L^p(\partial\Omega)^3, \qquad \Lambda^*(\tau, p, \Omega)(\Phi) = \Lambda^*(\tau, q, \Omega)(\Phi) \quad \textit{for } \Phi \in L^q(\partial\Omega)^3,$$

$$kern(\Lambda^*(\tau, q, \Omega)) \subset kern(\Lambda^*(\tau, p, \Omega)), \qquad im(\Lambda^*(\tau, q, \Omega)) \subset im(\Lambda^*(\tau, p, \Omega)).$$

Corresponding relations are valid for $\Lambda(\tau, q, \Omega)$ *and* $\Lambda(\tau, p, \Omega)$.

Now we define domains Ω_φ, $\widetilde{\Omega}_\varphi$ having one conical boundary point, and being smoothly bounded everywhere else. We begin by choosing functions $\widetilde{\zeta}_1, \widetilde{\zeta}_2 \in C^\infty([-\pi/2, \pi/2])$ with $0 \le \widetilde{\zeta}_j \le 1$ for $j \in \{1, 2\}$,

$$\widetilde{\zeta}_1\,|\,[-\pi/2, \pi/8] = 1, \qquad \widetilde{\zeta}_1\,|\,[\pi/4, \pi/2] = 0,$$

$$\widetilde{\zeta}_2\,|\,[-\pi/2, 3\cdot\pi/8] = 0, \qquad \widetilde{\zeta}_2\,|\,[7\cdot\pi/16, \pi/2] = 1.$$

Moreover, for $j \in \{1, 2\}$, we define the function $\overline{\zeta}_j : [-3\cdot\pi/2, \pi/2] \mapsto [0, 1]$ by setting

$$\overline{\zeta}_j(\sigma) := \widetilde{\zeta}_j(\sigma) \quad \text{for } \sigma \in [-\pi/2, \pi/2], \qquad \overline{\zeta}_j(\sigma) := \widetilde{\zeta}_j(-\sigma - \pi) \quad \text{for } \sigma \in [-3\cdot\pi/2, -\pi/2].$$

Finally, let ζ_j denote the $2\cdot\pi$-periodic extension of $\overline{\zeta}_j$ to $\mathbb{R}$ ($j \in \{1, 2\}$). Note that $\zeta_j \in C^\infty(\mathbb{R})$.

Let $\varphi \in (0, \pi/2]$, put $a(\varphi) := 1 \vee (2\cot\varphi)$, and define the functions $f_\varphi, \widetilde{f}_\varphi : \mathbb{R} \mapsto \mathbb{R}$ by setting for $\sigma \in \mathbb{R}$:

$$f_\varphi(\sigma) := a(\varphi)\,\zeta_1(\sigma)\cdot\sin\sigma \;+\; (1 - \zeta_1(\sigma))\cdot\big(a(\varphi) + \zeta_2(\sigma)\cdot\cot\varphi\cdot(1 - |\cos\sigma|)\big),$$

$$\widetilde{f}_\varphi(\sigma) := a(\varphi)\,\zeta_1(\sigma)\cdot\sin\sigma \;+\; (1 - \zeta_1(\sigma))\cdot\big(a(\varphi) - \zeta_2(\sigma)\cdot\cot\varphi\cdot(1 - |\cos\sigma|)\big).$$

Observe that

$$\widetilde{f}_\varphi(\sigma) = a(\varphi) - \cot\varphi + \cos\sigma\cdot\cot\varphi \qquad \text{for } \sigma \in [7\cdot\pi/16, \pi/2],$$

$$\widetilde{f}_\varphi(\sigma) = a(\varphi) \quad \text{for } \sigma \in [\pi/4, 3\cdot\pi/8],$$

$$\widetilde{f}_\varphi(\sigma) = a(\varphi)\sin\sigma \quad \text{for } \sigma \in [-\pi/2, \pi/8],$$

with analogous equations holding true for f_φ. Now we define the sets Ω_φ and $\widetilde{\Omega}_\varphi$ by

$$\Omega_\varphi := \Big\{ -\big(a(\varphi)\cdot\cos\sigma\cdot\cos\vartheta,\ a(\varphi)\cdot\cos\sigma\cdot\sin\vartheta,\ t - a(\varphi) - \cot\varphi\big) :$$
$$\sigma \in (0, \pi/2],\ \vartheta \in [0, 2\cdot\pi),\ t \in \mathbb{R} \text{ with } -f_\varphi(\sigma) < t < f_\varphi(\sigma) \Big\},$$

$$\widetilde{\Omega}_\varphi := \Big\{ \big(a(\varphi)\cdot\cos\sigma\cdot\cos\vartheta,\ a(\varphi)\cdot\cos\sigma\cdot\sin\vartheta,\ t - a(\varphi) + \cot\varphi\big) :$$
$$\sigma \in (0, \pi/2],\ \vartheta \in [0, 2\cdot\pi),\ t \in \mathbb{R} \text{ with } -\widetilde{f}_\varphi(\sigma) < t < \widetilde{f}_\varphi(\sigma) \Big\},$$

Then Ω_φ and $\widetilde{\Omega}_\varphi$ are bounded Lipschitz domains in $\mathbb{R}^3$, with connected boundaries. In addition, their boundaries are smooth except at one singular point lying at the origin. In a neighbourhood of this point, Ω_φ coincides with the cone $\mathbb{K}(\varphi)$, and $\widetilde{\Omega}_\varphi$ with $\mathbb{K}(\pi - \varphi)$. Let us state some of these facts in a more precise form. To this end, we set

$\varrho_0 := \cos(7\cdot\pi/16)$.

Then it holds

$$0 \in g^{(\varphi)}\big(\mathbb{B}_2(0, \varrho_0)\big) \subset \partial\Omega_\varphi,$$

$$\Big\{ g^{(\varphi)}(\eta) - (0, 0, r) \,:\, \eta \in \mathbb{B}_2(0, \varrho_0),\ r \in (0, \cot\varphi) \Big\} \subset \widetilde{\Omega}_\varphi, \tag{13.24}$$

$$\Big\{ g^{(\varphi)}(\eta) + (0, 0, r) \,:\, \eta \in \mathbb{B}_2(0, \varrho_0),\ r \in (0, \infty) \Big\} \subset \mathbb{R}^3 \backslash \overline{\widetilde{\Omega}_\varphi}. \tag{13.25}$$

Moreover, for any $x \in \Omega_\varphi\backslash\{0\}$, there is an orthonormal matrix $A_x \in \mathbb{R}^{3\times3}$, a vector $C_x \in \mathbb{R}^3$, positive reals ϵ_x, ϵ'_x, and a function $a_x \in C^\infty([-\epsilon_x, \epsilon_x]^2)$ with the properties to follow: Let

$$\Delta_x := (-\epsilon_x, \epsilon_x)^2, \qquad U_x := \Big\{ A_x\cdot(\eta, a_x(\eta) + r) + C_x \,:\, \eta \in \Delta_x,\ r \in (-\epsilon'_x, \epsilon'_x) \Big\},$$

and define the function $H_x : \Delta_x \times (-\epsilon'_x, \epsilon'_x) \mapsto U_x$ by

$$H_x(\eta, r) := A_x\cdot(\eta, a_x(\eta) + r) + C_x \qquad \text{for } \eta \in \Delta_x,\ r \in (-\epsilon'_x, \epsilon'_x).$$

Then it holds

$$U_x \cap \widetilde{\Omega}_\varphi = \{ H_x(\eta, r) \,:\, \eta \in \Delta_x,\ r \in (-\epsilon'_x, 0) \} \tag{13.26}$$

$$U_x \cap \mathbb{R}^3\backslash\overline{\widetilde{\Omega}_\varphi} = \{ H_x(\eta, r) \,:\, \eta \in \Delta_x,\ r \in (0, \epsilon'_x) \}, \tag{13.27}$$

$$U_x \cap \partial\widetilde{\Omega} = \{ H_x(\eta, 0) \,:\, \eta \in \Delta_x \}. \tag{13.28}$$

In particular, the domain $\widetilde{\Omega}_\varphi$ is Lipschitz bounded, and its outward unit normal $n^{(\widetilde{\Omega}_\varphi)}$ exists at every point $x \in \widetilde{\Omega}_\varphi\backslash\{0\}$. Furthermore, we note that

$$n^{(\widetilde{\Omega}_\varphi)}(x) = -n^{(\varphi)}(x) \qquad \text{for } x \in g^{(\varphi)}(\mathbb{B}_2(0, \varrho_0)). \tag{13.29}$$

Concerning the domain Ω_φ, the same statements hold true, with the exception of (13.24), (13.25) and (13.29). In the first two of these equations, the terms $-(0, 0, r)$ and $+(0, 0, r)$ must be exchanged. As for equation (13.29), it has to be replaced by

$$n^{(\Omega_\varphi)}(x) = n^{(\varphi)}(x) \qquad \text{for } x \in g^{(\varphi)}(\mathbb{B}_2(0, \varrho_0)). \tag{13.30}$$

Using these special domains Ω_φ and $\widetilde{\Omega}_\varphi$, we shall be able to link our L^p-theory for the infinite cone $\mathbb{K}(\varphi)$ with the L^2-theory for Lipschiz domains with bounded boundary. The key result in this respect is the following lemma:

Lemma 13.7. *Let* $\varphi \in (0, \pi/2]$, $\tau \in \{-1, 1\}$, $p \in (1, \infty)$. *Then the operator* $\Lambda^*(\tau, p, \widetilde{\Omega}_\varphi)$ *is Fredholm (or* F_+*) if and only if* $A^*(-\tau, p, \varphi, \varrho_0/2, 1)$ *is Fredholm (or* F_+*). In case these operators are* F_+, *they have the same index.*

An analogous result holds true with respect to $\Lambda^*(\tau, p, \Omega_\varphi)$ *and* $A^*(\tau, p, \varphi, \varrho_0/2, 1)$.

Proof: We shall show the first part of the lemma, which is related to $\Lambda^*(\tau, p, \widetilde{\Omega}_\varphi)$ and $A^*(-\tau, p, \varphi, \varrho_0/2, 1)$. As for the second part, it may be proved by an analogous reasoning, the only difference being that we have to refer to (13.30) instead of (13.29).

We shall prove our claim by reducing it to Lemma 6.9, which will be applied with $A = \partial\widetilde{\Omega}_\varphi$, with the σ-algebra of Lebesgue-measurable subsets of $\partial\widetilde{\Omega}_\varphi$ in the place of $\mathcal{B}$, and with the surface measure on $\partial\widetilde{\Omega}_\varphi$ in the place of μ. Moreover, we take

$$K = \Lambda^*(\tau, p, \widetilde{\Omega}_\varphi), \quad A_1 = g^{(\varphi)}(\mathbb{B}_2(0, \varrho_0/2)), \quad A_2 = \partial\widetilde{\Omega}_\varphi \backslash g^{(\varphi)}(\mathbb{B}_2(0, \varrho_0/2)).$$

Then consider the operators $K^{(1)}, \cdots, K^{(4)}$ defined in Lemma 6.9. It follows by (6.19) and (13.30):

$$(K^{(1)}(f)) \circ g^{(\varphi)} \,|\, \mathbb{B}_2(0, \varrho_0/2) = A^*(-\tau, p, \varphi, \varrho_0/2, 1)(f) \quad \text{for } f \in L^p(\mathbb{B}_2(0, \varrho_0/2))^3.$$

Hence, if we can show that $K^{(2)}$, $K^{(3)}$ and $K^{(4)}$ are compact, Lemma 6.9 and 6.12 yield the first part of Lemma 13.7. But it is easy to infer from (6.1) and Lemma 6.3 that $K^{(3)}$ and $K^{(4)}$ are compact. This leaves us to deal with the operator $K^{(2)}$. To this end, we note there is a constant $\mathfrak{C}_1 > 0$ such that

$$|n^{(\widetilde{\Omega}_\varphi)}(x) \cdot (x - y)| \quad \leq \quad \mathfrak{C}_1 \cdot |x - y|^2 \quad \text{for } x, y \in \partial\widetilde{\Omega}_\varphi \backslash g^{(\varphi)}(\mathbb{B}_2(0, \varrho_2/2)). \tag{13.31}$$

To see this, observe that for any $x \in \partial\widetilde{\Omega}_\varphi \backslash \{0\}$, there exists a smooth parametric representation as in (13.26) - (13.28). Thus we may find a suitable constant $\mathfrak{C}_1$ by using some well known arguments; compare [13, p. 63/64], for example. Inequality (13.31) and equation (5.9) imply there is a constant $\mathfrak{C}_2 > 0$ with

$$\Big|\sum_{k=1}^{3} \mathcal{D}_{jkl}(x - y) \cdot n_k^{(\widetilde{\Omega}_\varphi)}(x)\Big| \quad \leq \quad |x - y|^{-1}$$

for x, y as in (13.31). Now Lemma 13.4 yields compactness of $K^{(2)}$. This completes the proof of Lemma 13.7.

Corollary 13.1. *Let* $\varphi \in (0, \pi/2]$, $\tau \in \{-1, 1\}$, $p \in (1, \infty)$. *Then the operator* $\Lambda^*(\tau, p, \widetilde{\Omega}_\varphi)$ *is Fredholm if* $\Lambda^*(-\tau, p, \mathbb{K}(\varphi))$ *is topological. Conversely, in case* $\Lambda^*(\tau, p, \widetilde{\Omega}_\varphi)$ *has property* F_+, *the operator* $\Lambda^*(-\tau, p, \mathbb{K}(\varphi))$ *must be topological.*

An analogous result holds true with respect to $\Lambda^*(\tau, p, \Omega_\varphi)$ *and* $\Lambda^*(\tau, p, \mathbb{K}(\varphi))$.

Proof: Use Lemma 13.7, Theorem 12.3 and Corollary 6.6.

The next lemma is implicitly proved in [7, p. 815-817] for any bounded domain with Lipschitz boundary. In the case of our special domains Ω_φ and $\widetilde{\Omega}_\varphi$, this lemma is an immediate consequence of the preceding results.

Lemma 13.8. *Take* $\varphi \in (0, \pi/2]$, $\Omega \in \{\Omega_\varphi, \widetilde{\Omega}_\varphi\}$. *Then the operators* $\Lambda(-1, 2, \Omega)$ *and* $\Lambda^*(-1, 2, \Omega)$ *are Fredholm with index zero. In addition, it holds*

$$kern(\Lambda(-1, 2, \Omega)) = Z(\Omega).$$

Proof: Consider the case $\Omega = \widetilde{\Omega}_\varphi$. If $\Omega = \Omega_\varphi$, we may proceed in an analogous way. By Lemma 13.5, the operator $\Lambda^*(-1, 2, \widetilde{\Omega}_\sigma)$ has property F_+ for $\sigma \in (0, \pi/2]$. Thus Lemma 13.7 implies $A^*(1, 2, \sigma, \varrho_0/2, 1)$ to be F_+ for any $\sigma \in (0, \pi/2]$. Now we may apply Lemma 6.17, which yields

$$index(A^*(1, 2, \varphi, \varrho_0/2, 1)) = 0.$$

By referring to Lemma 13.7 once more, we see that $\Lambda^*(-1, 2, \widetilde{\Omega}_\varphi)$ is Fredholm with index zero. Since the operator $\Lambda(-1, 2, \widetilde{\Omega}_\varphi)$ is adjoint to $\Lambda^*(-1, 2, \widetilde{\Omega}_\varphi)$, it must also be Fredholm with index zero. Now we recur to Lemma 13.2, 13.3 and the closed range theorem ([29, p. 234, Theorem 5.13]), to obtain

$$dim(kern(K)) = 6 \quad \text{for } K \in \{\Lambda^*(-1, 2, \widetilde{\Omega}_\varphi), \Lambda(-1, 2, \widetilde{\Omega}_\varphi)\}.$$

Thus the equation in the last line of Lemma 13.8 follows from Lemma 13.3.

Corollary 13.2. *Let* $p \in (4/3, 2]$, $q \in [2, \infty)$, $\varphi \in (0, \pi/2]$, $\Omega \in \{\Omega_\varphi, \widetilde{\Omega}_\varphi\}$. *Then it holds*

$$kern(\Lambda^*(-1, p, \Omega)) = kern(\Lambda^*(-1, 2, \Omega)), \qquad kern(\Lambda(-1, q, \Omega)) = Z(\partial\Omega),$$

$$dim(kern(K)) = 6 \quad \textit{for } K \in \{(\Lambda^*(-1, p, \Omega), \Lambda^*(-1, q, \Omega)\}.$$

Proof: This corollary readily follows from Lemma 13.8, 13.6, 13.2 and 13.3.

Lemma 13.9. *Let* $\epsilon \in (0, 2/3)$, $\varphi \in (0, \pi/2]$, $\Omega \in \{\Omega_\varphi, \widetilde{\Omega}_\varphi\}$. *Then there is some number* $p \in (4/3, 4/3 + \epsilon)$ *such that the operators* $\Lambda^*(-1, p, \Omega)$ *and* $\Lambda(-1, (1 - 1/p)^{-1}, \Omega)$ *are Fredholm.*

Proof: Let us assume $\Omega = \widetilde{\Omega}_\varphi$. The case $\Omega = \Omega_\varphi$ may be treated in an analogous way. Due to Theorem 8.1, there exists some $p \in (4/3, 4/3+\epsilon)$ such that $\Lambda^*(1, p, \mathbb{K}(\varphi))$ is topological. Hence, by Corollary 13.1, the operator $\Lambda^*(-1, p, \widetilde{\Omega}_\varphi)$ is Fredholm. Since $\Lambda(-1, (1-1/p)^{-1}, \widetilde{\Omega}_\varphi)$ is adjoint to $\Lambda^*(-1, p, \widetilde{\Omega}_\varphi)$, the lemma follows.

Lemma 13.10. *Let* $\varphi \in (0, \pi/2]$, $\Omega \in \{\Omega_\varphi, \widetilde{\Omega}_\varphi\}$, $p, q \in (1, 2)$ *with* $p < q$. *Assume that* $\Lambda^*(-1, p, \Omega)$ *is Fredholm, and*

$$kern(\Lambda^*(-1, p, \Omega)) = kern(\Lambda^*(-1, 2, \Omega)). \tag{13.32}$$

Then $\Lambda^*(-1, q, \Omega)$ *is a Fredholm operator.*

Proof: Lemma 13.6 and the assumption in (13.32) imply

$$kern(\Lambda^*(-1, p, \Omega)) = kern(\Lambda^*(-1, q, \Omega)) = \Lambda^*(-1, 2, \Omega) =: K. \tag{13.33}$$

In particular, we have $dim(K) < \infty$; see Lemma 13.8. Moreover, according to Corollary 13.2, the adjoint of $\Lambda^*(-1, p, \Omega)$ has a finitely dimensional kernel. Thus, by the closed graph theorem ([29, p. 234, Theorem 5.13]), we still have to show that $\Lambda^*(-1, q, \Omega)$ has closed range. To this end, we choose a topological complement M of K in $L^p(\partial\Omega)^3$ (see [16, p. 113, (5.9.5)]), and then define the mapping $A_p : M \mapsto im(\Lambda^*(-1, p, \Omega))$ by

$$A_p(\Phi) := \Lambda^*(-1, p, \Omega)(\Phi) \quad \text{for } \Phi \in M.$$

We point out that A_p is bijective and $M \cap L^2(\partial\Omega)^3$ is an algebraic complement of K in $L^2(\partial\Omega)^3$. Since $dim(K) < \infty$, it follows that $L^2(\partial\Omega)^3$ is even the topological sum of K and $M \cap L^2(\partial\Omega)^3$ ([16, p. 113, (5.9.3); p. 103]). Furthermore, the mapping $A_2 : M \cap L^2(\partial\Omega)^3 \mapsto im(\Lambda^*(-1, 2, \Omega))$, with

$$A_2(\Phi) := \Lambda^*(-1, 2, \Omega)(\Phi) \quad \text{for } \Phi \in M \cap L^2(\partial\Omega)^3,$$

is bijective. Now let $\alpha \in \{p, 2\}$. Since $M \cap L^\alpha(\partial\Omega)^3$ is a topological complement of K in $L^\alpha(\partial\Omega)^3$, the set $M \cap L^\alpha(\partial\Omega)^3$ must be closed in $L^\alpha(\partial\Omega)^3$ ([16, p. 103]). Moreover, the space $im(\Lambda^*(-1, \alpha, \Omega))$ is also closed in $L^\alpha(\partial\Omega)^3$. In the case $\alpha = p$, this is a consequence of our assumptions in the lemma, and if $\alpha = 2$, we refer to Lemma 13.8. Now it follows from the open mapping theorem there is a constant $\mathfrak{C}_1(\alpha) > 0$ with

$$\| (A_\alpha)^{-1}(\Phi) \|_\alpha \leq \mathfrak{C}_1(\alpha) \cdot \|\Phi\|_\alpha \quad \text{for } \Phi \in im(\Lambda^*(-1, \alpha, \Omega)). \tag{13.34}$$

According to Corollary 13.2, it holds

$$kern(\Lambda(-1, (1-1/\alpha)^{-1}, \Omega)) = Z(\partial\Omega), \tag{13.35}$$

so the closed graph theorem implies

$$\int_{\partial\Omega} f \cdot g \; d\Omega = 0 \quad \text{for } f \in Z(\partial\Omega),\ g \in im(\Lambda^*(-1, \alpha, \Omega)),$$

and we conclude

$$Z(\partial\Omega) \cap im(\Lambda^*(-1, \alpha, \Omega)) = \{0\}.$$

On the other hand, we have $Z(\partial\Omega) \subset L^\alpha(\partial\Omega)^3$ and $dim(Z(\partial\Omega)) = 6$. Moreover, we infer from (13.35) and the closed graph theorem : $codim(\Lambda^*(-1, \alpha, \Omega)) = 6$. Hence $L^\alpha(\partial\Omega)^3$ is the algebraic direct sum of $Z(\partial\Omega)$ and $im(\Lambda^*(-1, \alpha, \Omega))$. Since the space $Z(\partial\Omega)$ is finitely dimensional, this sum is even topological. ([16, p. 113, (5.9.3); p. 103]). Thus there is a constant $\mathfrak{C}_2(\alpha) > 0$ with

$$\| \Phi_1 + \Phi_2 \|_\alpha \leq \mathfrak{C}_2(\alpha) \cdot \{ \|\Phi_1\|_\alpha + \|\Phi_2\|_\alpha \} \tag{13.36}$$

for $\Phi_1 \in Z(\partial\Omega)$, $\Phi_2 \in im(\Lambda^*(-1, \alpha, \Omega))$. Next we introduce the mapping S_α as the uniquely determined linear operator acting on the space $L^\alpha(\partial\Omega)^3$, and satisfying the equations

$$S_\alpha(\Phi) = 0 \quad \text{for } \Phi \in Z(\Omega), \qquad S_\alpha(\Phi) = (A_\alpha)^{-1}(\Phi) \quad \text{for } \Phi \in im(\Lambda^*(-1, \alpha, \Omega)).$$

Because of (13.34) and (13.36), this operator is bounded. Recall that α was arbitrarily chosen in $\{p, 2\}$. We further observe that

$$S_p \,|\, L^2(\partial\Omega)^3 = S_2. \tag{13.37}$$

This equation is easily verified by using Lemma 13.6 and the definitions of S_2, S_p, A_2 and A_p. Still, this relation should not be taken as self-evident. Actually, when the infinite cone $\mathbb{K}(\varphi)$ is considered instead of the bounded domain Ω_φ or $\widetilde{\Omega}_\varphi$, an analogue of equation (13.37) cannot be established since it would contradict Theorem 8.1. This contradiction would arise by the interpolation argument we are going to use now. In fact, since $q \in (p, 2)$, we may apply the Riesz-Thorin interpolation theorem (see [2, p. 2]), which yields the relation $S_p(L^q(\partial\Omega)^3) \subset L^q(\partial\Omega)^3$, as well as existence of a constant $\mathfrak{C}_3 > 0$ with

$$\| S_p(\Psi) \|_q \leq \mathfrak{C}_3 \cdot \|\Psi\|_q \quad \text{for } \Psi \in L^q(\partial\Omega)^3.$$

By combining this result with Lemma 13.6 and the fact that $\Lambda^*(-1, q, \Omega)$ is bounded, we readily obtain that the range of $\Lambda^*(-1, q, \Omega)$ is closed.

Theorem 13.1. *Let* $\varphi_0 \in (0, \pi/2]$, $\tau \in \{-1, 1\}$, $p \in [2, \infty)$. *Then the operator* $\Lambda(\tau, p, \mathbb{K}(\varphi_0))$ *is topological.*

Proof: Consider the case $\tau = 1$. If $\tau = -1$, we only need to repeat the ensuing arguments with $\widetilde{\Omega}_{\varphi_0}$ replaced by Ω_{φ_0}.

Take $\varphi \in [\varphi_0, \pi/2]$. According to Lemma 13.8, the operator $\Lambda^*(-1, 2, \widetilde{\Omega}_\varphi)$ is Fredholm. Moreover, for any $\epsilon \in (0, 2/3)$, there is some $p \in (4/3, 4/3+\epsilon)$ such that

$\Lambda^*(-1,\, p,\, \widetilde{\Omega}_\varphi)$ is Fredholm too; see Lemma 13.9. Thus we may conclude from Lemma 13.10 that $\Lambda^*(-1,\, q,\, \widetilde{\Omega}_\varphi)$ is Fredholm for any $q \in (4/3,\, 2]$. Now Corollary 13.1 yields that $\Lambda^*(1,\, q,\, \mathbb{K}(\varphi))$ is topological for $q \in (4/3,\, 2]$. Since the operator $\Lambda(1,\, (1-1/q)^{-1},\, \mathbb{K}(\varphi))$ is adjoint to $\Lambda^*(1,\, q,\, \mathbb{K}(\varphi))$ for $q \in (1, \infty)$, we infer from Lemma 8.5 that $\Lambda(1,\, p,\, \mathbb{K}(\varphi))$ is topological for $p \in [2, 4)$. Thus, by Lemma 8.13, the operator $\Lambda(1,\, p,\, \mathbb{K}(\varphi))$ must have the same property if $p \in (4, \infty)$. Next, referring to Lemma 8.5 again, we see that $\Lambda^*(1,\, q,\, \mathbb{K}(\varphi))$ is topological for $q \in (1, 4/3)$. Recalling Theorem 12.3 and Corollary 6.6, we conclude $A^*(1,\, q,\, \varphi,\, \varrho_0/2,\, 1)$ is Fredholm for $q \in (1, 4/3)$. But φ was chosen arbitrarily in $[\varphi_0,\, \pi/2]$, so Lemma 6.17 yields

$$index(A^*(1,\, q,\, \varphi,\, \varrho_0/2,\, 1)) \;=\; 0,$$

and we may deduce from Lemma 13.7 that $\Lambda^*(-1,\, q,\, \widetilde{\Omega}_\varphi)$ is Fredholm with index zero, for $q \in (1, 4/3)$. Now fix a number $p \in (1, 4/3)$. We know from Corollary 13.2 that

$$dim\Big(kern(\Lambda(-1,\, (1-1/p)^{-1},\, \widetilde{\Omega}_\varphi))\Big) \;=\; 6.$$

Since the operator $\Lambda^*(-1,\, p,\, \widetilde{\Omega}_\varphi)$ is Fredholm, its range must have the codimension 6. On the other hand, the index of this operator is zero, as we proved above. Hence it follows

$$dim\Big(kern(\Lambda^*(-1,\, p,\, \widetilde{\Omega}_\varphi))\Big) \;=\; 6.$$

Thus we infer from Lemma 13.6 and Corollary 13.2:

$$kern(\Lambda^*(-1,\, p,\, \widetilde{\Omega}_\varphi)) \;=\; kern(\Lambda^*(-1,\, 2,\, \widetilde{\Omega}_\varphi)).$$

Now we are in a position to apply Lemma 13.10, which yields that $\Lambda^*(-1,\, 4/3,\, \widetilde{\Omega}_\varphi)$ is a Fredholm operator. Hence Corollary 13.1 and Lemma 8.5 imply $\Lambda(1,\, 4/3,\, \mathbb{K}(\varphi))$ is topological. This completes the proof of the theorem.

Corollary 13.3. *Let* $\vartheta \in [0, \pi)$, $\varphi_0 \in (0, \pi/2]$, $\tau \in \{-1, 1\}$. *Then the operator* $\Gamma(\tau,\, 2,\, \lambda,\, \mathbb{K}(\varphi_0))$ *is topological for* $\lambda \in \mathbb{C}\backslash\{0\}$ *with* $|\arg \lambda| \le \vartheta$. *Furthermore, there is a constant* $\mathfrak{C} > 0$ *such that*

$$\|f\|_2 \;\le\; \mathfrak{C} \cdot \big\|\Gamma(\tau,\, 2,\, \lambda,\, \mathbb{K}(\varphi_0))(f)\big\|_2$$

for $f \in L^2(\partial\mathbb{K}(\varphi_0))^3$, $\lambda \in \mathbb{C}\backslash\{0\}$ *with* $|\arg\lambda| \le \vartheta$.

Proof: Take $\varphi \in [\varphi_0,\, \pi/2]$. According to Theorem 13.1, the operator $\Lambda(\tau,\, 2,\, \mathbb{K}(\varphi))$ is topological. Moreover, we know by Corollary 8.3 that $\Pi^*(-\tau,\, 2,\, \mathbb{K}(\varphi))$ has the same property. Finally, Theorem 9.4 states that $\Gamma^*(\tau,\, 2,\, \lambda,\, \mathbb{K}(\varphi))$ is one-to-one $(\lambda \in \mathbb{C}\backslash\{0\}$ with $|\arg\lambda| \le \vartheta)$. Now Corollary 13.3 follows from Corollary 12.8.

Corollary 13.4. *Let* $\varphi \in (0, \pi/2]$. *For* $g \in L^2(\partial\mathbb{K}(\varphi))^3$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\tau \in$

$\{-1, 1\}$, $l \in \{1, 2, 3\}$, *we put*

$$u_l(\tau,\lambda,g)(x) := \int_{\partial\mathbb{K}(\varphi)} \sum_{j,k=1}^{3} \widetilde{\mathcal{D}}^{\lambda}_{jkl}(x-y) \cdot \Big(\big[\Gamma(\tau,\, 2,\, \lambda,\, \mathbb{K}(\varphi))\big]^{-1}(g)\Big)_j(y) \cdot n_k^{(\varphi)}(y)\; d\mathbb{K}(\varphi)(y),$$

$$\pi(\tau,\lambda,g)(x) := \int_{\partial\mathbb{K}(\varphi)} \Big(2 \sum_{j,k=1}^{3} \Big[D_j E_{4k}(x-y) - \lambda \cdot (4\cdot\pi)^{-1} \cdot \delta_{jk} \cdot \big(|x-y|^{-1} - |(0,0,1)-y|^{-1}\big)\Big) \cdot \Big(\big[\Gamma(\tau,\, 2,\, \lambda,\, \mathbb{K}(\varphi))\big]^{-1}(g)\Big)_j(y) \cdot n_k^{(\varphi)}(y)\; d\mathbb{K}(\varphi)(y),$$

with $x \in \mathbb{K}(\varphi)$ *in the case* $\tau = 1$, *and with* $x \in \mathbb{R}^3\backslash\overline{\mathbb{K}(\varphi)}$ *if* $\tau = -1$.

I) It holds for g, λ, τ *as before:*

- *a) The functions* $u_l(\tau,\lambda,g)$, $\pi(\tau,\lambda,g)$ *belong to* $C^\infty(\mathbb{K}(\varphi))$ *in the case* $\tau = 1$, *and to* $C^\infty(\mathbb{R}^3\backslash\overline{\mathbb{K}(\varphi)})$ *if* $\tau = -1$ $(l \in \{1,2,3\})$.
- *b) The pair of functions* $\big(u(\tau,\lambda,g),\; \pi(\tau,\lambda,g)\big)$ *solves the resolvent equation (1.12).*
- *c) The function* $u(\tau,\lambda,g)$ *satisfies boundary condition (1.13) in the following sense:*

$$\int_{\partial\mathbb{K}(\varphi)} \big|u(\tau,\lambda,g)(x+\tau\cdot(0,0,r)) - g(x)\big|^2\; d\mathbb{K}(\varphi)(x) \;\longrightarrow\; 0 \qquad (r \downarrow 0).$$

II) Let $\vartheta \in [0,\pi)$. *Then there is a constant* $\mathfrak{C} = \mathfrak{C}(\vartheta,\varphi) > 0$, *such that for* $\tau \in \{-1, 1\}$, $g \in L^2(\partial\mathbb{K}(\varphi))^3$, $\lambda \in \mathbb{C}\backslash\{0\}$ *with* $|\arg\lambda| \le \vartheta$, *the following estimate holds true:*

$$\sup_{r\in(0,\infty)} \Big(\int_{\partial\mathbb{K}(\varphi)} \big|u(\tau,\lambda,g)(x+\tau\cdot(0,0,r))\big|^2\; d\mathbb{K}(\varphi)(x)\Big)^{1/2} \;\le\; \mathfrak{C}\cdot\|g\|_2.$$

Proof: For I)a) and I)b), we refer to Corollary 9.1 and 9.2. The claim stated in I)c) may be deduced from Theorem 9.1 by means of a duality argument. As for part II) of the corollary, it follows from (3.1), Lemma 5.12 and Corollary 13.3.

We point out that the mappings $u(\tau,\lambda,g)$ and $\pi(\tau,\lambda,g)$ cannot be defined unless Corollary 13.3 is available. This is due to the fact that invertibility of the operator $\Gamma(\tau,\, 2,\, \lambda,\, \mathbb{K}(\varphi))$ is needed for these definitions. Furthermore, it was possible to estimate the function $u(\tau,\lambda,g)$ uniformly in λ only because an estimate of this kind was known for the operator $\Gamma(\tau,\, 2,\, \lambda,\, \mathbb{K}(\varphi))$ (Corollary 13.3).

Let us finally turn to the non-regularity result announced at the beginning of this chapter and in Chapter 1.

Theorem 13.2. *There are some numbers* $p \in (2, \infty)$, $\varphi \in (0, \pi/2)$, *as well as a sequence* (f_n) *in* $L^p(\Omega_\varphi)^3$ *such that*

$$\sum_{j,k=1}^{3} \Big(\| B_{jk}(1, p, \Omega_\varphi)(f_n) \|_p + \| C(1, p, \Omega_\varphi)(f_n) \|_p$$

$$\geq n \cdot \Big(\| V(\partial\Omega_\varphi)(f_n) \,|\, \partial\Omega_\varphi \|_p + \sum_{k,l=1}^{3} \| T_{kl}(p, \Omega_\varphi)(f_n) \|_p \Big) \quad \textit{for } n \in \mathbb{N}$$

or

$$\sum_{j,k=1}^{3} \| B_{jk}(-1, p, \Omega_\varphi)(f_n) \|_p$$

$$\geq n \cdot \Big(\| \Lambda^*(-1, p, \Omega_\varphi)(f_n) \|_p + \| V(\partial\Omega_\varphi)(f_n) \,|\, \partial\Omega_\varphi \|_p \Big) \quad \textit{for } n \in \mathbb{N}.$$

This theorem is shown by similar arguments as were used in the proof of Lemma 13.5. Still, for the convenience of the reader, we shall check in detail how these arguments work out in the present situation.

Proof of Theorem 13.2: Let us assume there are no numbers p, φ and no sequence (f_n) satisfying the properties stated in the theorem. Then, for any $p \in (2, \infty)$, $\varphi \in (0, \pi/2)$, there is a constant $\mathfrak{C}_1(p, \varphi) > 0$ such that it holds for $g \in L^p(\partial\Omega_\varphi)^3$:

$$\sum_{j,k=1}^{3} \Big(\| B_{jk}(1, p, \Omega_\varphi)(g) \|_p + \| C(1, p, \Omega_\varphi)(g) \|_p \Big) \tag{13.38}$$

$$\leq \mathfrak{C}_1(p, \varphi) \cdot \Big(\| V(\partial\Omega_\varphi)(g) \,|\, \partial\Omega_\varphi \|_p + \sum_{k,l=1}^{3} \| T_{kl}(p, \Omega_\varphi)(g) \|_p \Big)$$

and

$$\sum_{j,k=1}^{3} \| B_{jk}(-1, p, \Omega_\varphi)(g) \|_p \tag{13.39}$$

$$\leq \mathfrak{C}_1(p, \varphi) \cdot \Big(\| \Lambda^*(-1, p, \Omega_\varphi)(g) \|_p + \| V(\partial\Omega_\varphi)(g) \,|\, \partial\Omega_\varphi \|_p \Big).$$

Due to Theorem 8.2, we may choose $\varphi \in (0, \pi/2)$, $p \in (2, \infty)$ so that $\Lambda^*(-1, p, \mathbb{K}(\varphi))$ is not topological. ~~Such values of p and φ are to be kept fixed from now on.~~ Let $h \in L^p(\partial\Omega_\varphi)^3$. Then we have

$$h = \Lambda^*(1, p, \Omega_\varphi)(h) - \Lambda^*(-1, p, \Omega_\varphi)(h).$$

From this equation and (13.11), we obtain

$$\|h\|_p \leq \| \Lambda^*(-1, p, \Omega_\varphi)(h) \|_p + 2 \cdot \sum_{j,k=1}^{3} \| B_{jk}(1, p, \Omega_\varphi)(h) \|_p + 3 \cdot \| C(1, p, \Omega_\varphi)(h) \|_p.$$

Due to our assumption in (13.38), it follows

$$\|h\|_p \leq \| \Lambda^*(-1, p, \Omega_\varphi)(h) \|_p + 3 \cdot \mathfrak{C}_1(p, \varphi) \cdot \Big(\| V(\partial\Omega_\varphi)(h) \,|\, \partial\Omega_\varphi \|_p + \sum_{k,l=1}^{3} \| T_{kl}(p, \Omega_\varphi)(h) \|_p \Big), \tag{13.40}$$

On the other hand, it holds by the definition of $T_{kl}(p, \Omega_\varphi)(h)$ (see below (13.13)):

$$\sum_{k,l=1}^{3} \| T_{kl}(p, \Omega_\varphi)(h) \|_p \leq 4 \cdot \sum_{k,l=1}^{3} \| B_{kl}(-1, p, \Omega_\varphi)(h) \|_p. \tag{13.41}$$

Combining (13.40), (13.41) and the assumption in (13.39), we get

$$\|h\|_p \leq \big(1 + 4 \cdot (\mathfrak{C}_1(p, \varphi))^2 \cdot \big(\| \Lambda^*(-1, p, \Omega_\varphi)(h) \|_p + \| V(\partial\Omega_\varphi)(h) \,|\, \partial\Omega_\varphi \|_p \big),$$

where h is an arbitrary function from the space $L^p(\partial\Omega_\varphi)^3$.

Define the operator $\mathcal{K} : L^p(\partial\Omega_\varphi)^3 \mapsto L^p(\partial\Omega_\varphi)^3$ by

$$\mathcal{K}(h) := V(\partial\Omega_\varphi)(h) \,|\, \partial\Omega_\varphi \quad \text{for } h \in L^p(\partial\Omega_\varphi)^3.$$

According to Lemma 13.4, this operator is compact. Hence we may deduce from [34, p. 18, Lemma 2.1] that $\Lambda^*(-1, p, \Omega_\varphi)$ has property F_+. ~~Now~~ Corollary 13.1 yields that $\Lambda^*(-1, p, \mathbb{K}(\varphi))$ is topological – a contradiction to the choice of p and φ.

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