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The Stokes system in 3D-Lipschitz domains: a survey of recent results

1. Introduction.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$. By the latter assumption we mean the boundary $\partial\Omega$ may be locally described by Lipschitz continuous functions and the domain Ω does not simultaneously lie on both sides of any part of its boundary; see [9, Section 3] for more details. In the present article, we shall consider the Stokes system

$$-\nu \cdot \Delta u \ + \ \nabla \pi \ = \ f, \qquad \operatorname{div} u \ = \ 0 \tag{1.1}$$

in Ω, as well as some equations related to it. We shall prescribe either Dirichlet boundary conditions,

$$u \,|\, \partial\Omega \ = \ g, \tag{1.2}$$

or a slip condition,

$$T(u,\pi)(x) \cdot n^{(\Omega)}(x) \ = \ h \quad \text{for } x \in \partial\Omega, \tag{1.3}$$

where the function $n^{(\Omega)}$ is the outward unit normal to Ω, defined almost everywhere on $\partial\Omega$ (see [25, p. 88/89]). The term $T(u,\pi)$ is an abbreviation defined by

$$T(u,\pi) := \ (D_j u_k \ + \ D_k u_j \ - \ \delta_{jk} \cdot \pi)_{1 \le j,k \le 3},$$

with the symbols D_j, D_k denoting partial derivatives.

In Section 3, boundary value problems (1.1), (1.2) and (1.1), (1.3) will be discussed. In addition, we shall briefly address the time-dependent Stokes and Navier-Stokes system, as well as the resolvent problem related to the Stokes system (1.1). Most of the results presented in Section 3 were proved by the method of integral equations, applied within the framework of the theory of unweighted Sobolev spaces. We shall give some indications of these proofs. Finally, in Section 4, we shall concentrate on Lipschitz domains with special geometry. For such domains, the theory of partial differential equations in weighted L^p-spaces is available, and we shall attempt to give an idea of how this theory is applied to problem (1.1), (1.2).

When studying boundary value problems for the Stokes system, one is usually guided by the following rule of the thumb: The velocity part of a solution to the Stokes system has the same regularity as a solution to the Poisson equation, provided analogous boundary conditions are prescribed. Therefore it would be useful to review the L^p-theory for the Poisson equation in Lipschitz domains. However, such a review is beyond the scope of this survey. The interested reader is referred to Jerison, Kenig

[18], Costabel [4], Grisvard [15], [16], Mazja, Nasarow, Plamenewski [21], Nazarov, Plamenevsky [24].

It should be noted this survey is primarily directed at readers who are no experts in the theory of partial differential equations in non-smooth domains. We freely admit that the choice of our topics is biased by personal preferences. Still we hope to give an impression of what is going on in a field which arises much interest currently.

2. Notations. Definition of Function Spaces.

Let $\mathbb{B}_N(0, R)$ denote the N-dimensional ball with centre $x \in \mathbb{R}^N$ and with radius $\epsilon > 0$ $(N \in \mathbb{N})$. For $\alpha \in \mathbb{N}_0^N$, put $|\alpha|_* := \alpha_1 + \cdots + \alpha_N$. If $U \subset \mathbb{R}^3$, define the space $Z(U)$ by

$\left\{ \varrho : U \mapsto \mathbb{C}^3 \ : \text{ There are vectors } a, b \in \mathbb{C}^3 \text{ with } \varrho(x) = a + b \times x \text{ for } x \in U . \right\}.$

As an abbreviation for spherical coordinates in $\mathbb{R}^3$, we define for $r \in (0, \infty)$, $\sigma, \vartheta \in \mathbb{R}$:

$T(r, \sigma, \vartheta) := (\, r \cdot \cos\sigma \cdot \sin\vartheta \,,\ r \cdot \sin\sigma \cdot \sin\vartheta \,,\ \cos\vartheta\,)$.

We shall write S^2 for the unit sphere in $\mathbb{R}^3$. The symbol Δ' stands for the Laplace-Beltrami operator on S^2, that is,

$\Delta' v(\sigma, \vartheta) := \sin^{-1}(\vartheta) \cdot \partial/\partial\vartheta(\sin\vartheta \cdot D_2 v(\sigma, \vartheta)) + \sin^{-2}(\vartheta) \cdot D_1 D_1 v(\sigma, \vartheta)$

for $(\sigma, \vartheta) \in B$, where $B \subset \mathbb{R}^2$ is an open set and $v : B \mapsto \mathbb{C}$ is a twice weakly differentiable function.

Let $p \in (1, \infty)$. Suppose as in Section 1 that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain.

Let $m \in \mathbb{N}$, and denote by $W^{m,p}(\Omega)$ the usual Sobolev space of order m with exponent p. We shall write $\| \ \|_{m,p}$ for the corresponding norm, and we define $W_0^{m,p}(\Omega)$ as the completion of the set $C_0^\infty(\Omega)$ with respect to this norm. The space dual to $W_0^{1,p}(\Omega)$ will be denoted by $W^{-1,p}(\Omega)$, with norm $\| \ \|_{-1,p}$. Let $s \in (m-1, m)$, and put for $f \in W^{m-1,p}(\Omega)$:

$$I_{s,p}(\Omega)(f) := \sum_{\alpha \in \mathbb{N}_0^3,\, |\alpha|_* = m-1} \int_\Omega \int_\Omega |\, D^\alpha f(x) - D^\alpha f(y) \,| \cdot |x - y|^{-3 - p\cdot(s-m+1)} \, dx \, dy.$$

As usual, we define

$W^{s,p}(\Omega) := \{\, f \in W^{m-1,p}(\Omega) \ : \ I_{s,p}(\Omega)(f) < \infty \,\}$,

$\| f \|_{s,p} := \| f \|_{m-1,p} + (\, I_{s,p}(\Omega)(f)\,)^{1/p} \qquad \text{for } f \in W^{s,p}(\Omega).$

If $\sigma \in (0, 1]$, the definition of the space $W^{\sigma,p}(\partial\Omega)$ with norm $\| \ \|_{\sigma,p}$ should now be obvious. Set

$$L_N^p(\partial\Omega) := \left\{ \Phi \in L^p(\partial\Omega)^3 \ : \ \int_{\partial\Omega} f \cdot n^{(\Omega)} \, d\Omega = 0 \right\}.$$

For $y, z \in \mathbb{R}^3$ with $|z| = 1$, $\epsilon, \delta \in (0, \infty)$, we define the cone $K(y, z, \delta, \epsilon) \subset \mathbb{R}^3$ by

$$K(y, z, \delta, \epsilon) := \left\{ y + t \cdot b \ : \ t \in (0, \delta),\ b \in \mathbb{R}^3 \text{ with } |b| = 1,\ |b - z| < \epsilon \right\}.$$

Next we fix a non-tangential direction field $m^{(\Omega)}$ to Ω. By this we mean a function $m^{(\Omega)} : \partial\Omega \mapsto \mathbb{R}^3$ satisfying the following properties (compare [30, p. 581]):

1.) $|m^{(\Omega)}| = 1$.

2.) There is a function $\widetilde{m} \in C_0^\infty(\mathbb{R}^3)^3$ with $\widetilde{m} \,|\, \partial\Omega = m^{(\Omega)}$.

3.) There are constants $\mathcal{D}_1, \cdots, \mathcal{D}_4 \in (0,\infty)$ such that

$K(x\,, m^{(\Omega)}(x)\,, \mathcal{D}_1, \mathcal{D}_2\,) \subset \mathbb{R}^3\backslash\overline{\Omega}, \quad K(x\,, -m^{(\Omega)}(x)\,, \mathcal{D}_1, \mathcal{D}_2\,) \subset \Omega \qquad \text{for } x \in \partial\Omega;$

$\left| x + \kappa \cdot m^{(\Omega)}(x) - x' - \kappa' \cdot m^{(\Omega)}(x') \right| \geq \mathcal{D}_3 \cdot (\,|x - x'| + |\kappa - \kappa'|\,)$

for $x, x' \in \partial\Omega, \quad \kappa, \kappa' \in (-\mathcal{D}_4\,, \mathcal{D}_4)$. We refer to [9, p. 119] for some indications on how to construct such a function $m^{(\Omega)}$.

For functions $v : \Omega \mapsto \mathbb{C}$, we define the corresponding maximal function $v^* : \partial\Omega \mapsto [0,\infty)$ by

$v^*(x) := \sup\{\,|\,v(y)\,| \;:\; y \in K(\,x\,, -m^{(\Omega)}(x)\,, \mathcal{D}_1, \mathcal{D}_2\,)\,\}$.

Let $\varphi \in (0, \pi)$ and put $\mathbb{K}(\varphi) := \{\,(\,\eta\,, \cot\varphi \cdot |\eta|\,) \;:\; \eta \in \mathbb{R}^2\,\}$. This means the set $\mathbb{K}(\varphi)$ is a circular infinite cone which is symmetric to the x_3-axis, has vertex in the origin, with vertex angle $2 \cdot \varphi$. Let us fix a bounded domain U_φ which is smoothly bounded everywhere except at a single point x_0 assumed to be the origin. In a vicinity of this point, the domain U_φ is to coincide with the cone $\mathbb{K}(\varphi)$:

$$U_\varphi \cap \mathbb{B}_3(0,\epsilon_0) = \mathbb{K}(\varphi) \cap \mathbb{B}_3(0,\epsilon_0) \qquad \text{for some } \epsilon_0 \in (0,\infty). \tag{2.1}$$

Denote by B_φ the set cut out by $\mathbb{K}(\varphi)$ from the unit sphere S^2 in $\mathbb{R}^3$. The manifold B_φ with its Lebesgue surface measure may be identified with the set $[0, 2\cdot\pi) \times [0,\varphi)$ equipped with the weighted Lebesgue measure $\sin\vartheta \;\; d(\sigma,\vartheta)$. The Lebesgue space $L^p(B_\varphi)$ and the Sobolev spaces $W^{l,p}(B_\varphi)$ for $l \in \mathbb{N}$ are to be understood as spaces on $[0, 2\cdot\pi) \times [0,\varphi)$, with the usual Lebesgue measure replaced by the weighted measure just mentioned.

Let us introduce some weighted Sobolev spaces on the infinite circular cone $\mathbb{K}(\varphi)$. Take $\beta \in \mathbb{R}, \;\; l \in \mathbb{N}_0$. For any measurable function $u : \mathbb{K}(\varphi) \mapsto \mathbb{C}$ having weak derivatives up to order l, we set

$$\|\,u\,\|_{2,l,\beta} := \Big(\int_{\partial\mathbb{K}(\varphi)} \sum_{\alpha \in \mathbb{N}_0^3,\, |\alpha|_* \leq l} r^{2\cdot(\beta - l + |\alpha|_*)} \cdot |\,D^\alpha u(x)\,|^2 \;\, dx \Big)^{1/2}$$

Furthermore, we define $V^l_\beta(\,\mathbb{K}(\varphi)\,)$ as the set of all measurable functions $u : \mathbb{K}(\varphi) \mapsto \mathbb{C}$ having weak derivatives of order l and satisfying the relation $\|\,u\,\|_{2,l,\beta} < \infty$. If $l \in \mathbb{N}$, any function $u \in V^l_\beta(\,\mathbb{K}(\varphi)\,)$ has a trace $u \,|\, \partial\mathbb{K}(\varphi)$ on $\partial\mathbb{K}(\varphi)$. We set

$V^{l-1/2}_\beta(\,\mathbb{K}(\varphi)\,) := \{\,u \,|\, \partial\mathbb{K}(\varphi) \;:\; u \in V^l_\beta(\,\mathbb{K}(\varphi)\,)\,\}$.

Note that for any function $v \in V^l_\beta(\,\mathbb{K}(\varphi)\,)$, it holds $(v \circ T)(r, \cdot\,, \cdot) \in W^{l,2}(B_\varphi)$ for almost every $r \in (0,\infty)$.

3. Solutions in unweighted L^p-spaces.

In this section it is again assumed that Ω is a Lipschitz domain, as specified in Section 1. In addition, the boundary $\partial\Omega$ is supposed to be connected. It is well known that the Dirichlet problem (1.1), (1.2) for the Stokes system in Ω has a variational solution. More precisely, the following result holds for the non-homogeneous Stokes system

$$-\nu \cdot \Delta u + \nabla \pi = f, \qquad \operatorname{div} u = \tilde{f} \tag{3.1}$$

in Ω, under Dirichlet boundary conditions (1.2):

Theorem 3.1. *Let* $F \in W^{-1,2}(\Omega)^3$, $\tilde{f} \in L^2(\Omega)$, $g \in W^{1/2,2}(\partial\Omega)$ *with*

$$\int_{\partial\Omega} g \cdot n^{(\Omega)}\, d\Omega = \int_\Omega \tilde{f}\, dx.$$

Then there exists a uniquely determined pair of functions $(u, \pi) \in W^{1,2}(\Omega)^3 \times L^2(\Omega)$ *with*

$$u - u_0 \in W_0^{1,2}(\Omega)^3, \qquad \int_\Omega \pi\, dx = 0,$$

$$\int_\Omega \Big(\sum_{j,k=1}^{3} \nu \cdot D_j(u-u_0)_k \cdot D_j v_k - \pi \cdot \operatorname{div} v \Big)\, dx \tag{3.2}$$

$$= F(v) - \int_\Omega \sum_{j,k=1}^{3} \nu \cdot D_j u_{0,k} \cdot D_j v_k\, dx \qquad \text{for } v \in W_0^{1,2}(\Omega),$$

$$\int_\Omega (\operatorname{div} u - \tilde{f}) \cdot \varrho\, dx = 0 \qquad \text{for } \varrho \in L^2(\Omega) \text{ with } \int_{\partial\Omega} \varrho\, dx = 0,$$

where u_0 *may be any function with* $u_0 \in W^{1,2}(\Omega)^3$, $\operatorname{div} u_0 = 0$ *and* $u_0 \,|\, \partial\Omega = g$. *There is a constant* $C > 0$ *such that*

$$\| u \|_{1,2} + \| \pi \|_2 \leq C \cdot \Big(\| F \|_{-1,2} + \| \tilde{f} \|_2 + \| g \|_{1/2,2} \Big)$$

for $F, \tilde{f}, g, u, \pi$ *as before.*

This theorem is proved in [14, p. 56-60, 80-82] and in [29, p. 31/32]. In the case $\tilde{f} = 0$, a corresponding result holds true in L^p-spaces with $p \in [3/2, 3]$; see Shen [28, p. 802, Theorem 0.6]. Under the assumption that $\partial\Omega$ may be described by local coordinates having small Lipschitz constants, Theorem 3.1 may be generalized to any exponent $p \in (1, \infty)$. This was proved by Galdi, Simader, Sohr [12].

Now let us introduce some potential functions. We shall refer to them when indicating the proof of the results presented in the following.

For $z \in \mathbb{R}^3 \backslash \{0\}$, $j, k, l \in \{1, 2, 3\}$, $\lambda \in \mathbb{C} \backslash (-\infty, 0]$, put

$$E_{jk}(z) := (8 \cdot \pi)^{-1} \cdot \Big(\delta_{jk} \cdot |z|^{-1} + z_j \cdot z_k \cdot |z|^{-3} \Big), \qquad E_{4k}(z) := (4 \cdot \pi \cdot |z|^3)^{-1} \cdot z_k,$$

$$\overline{E}^\lambda(z) := (4 \cdot \pi \cdot |z|)^{-1} \cdot e^{-\sqrt{\lambda} \cdot |z|},$$

$$\widetilde{E}^\lambda_{jk}(z) := (4 \cdot \pi \cdot |z|)^{-1} \cdot \Big(\delta_{jk} \cdot \tilde{g}_1(\sqrt{\lambda} \cdot |z|) - z_j \cdot z_k \cdot |z|^{-2} \cdot \tilde{g}_2(\sqrt{\lambda} \cdot |z|) \Big),$$

with the functions $\widetilde{g}_1$, $\widetilde{g}_2$ defined as follows:

$$\widetilde{g}_1(r) := e^{-r} + r^{-2} \cdot (r \cdot e^{-r} + e^{-r} - 1),$$

$$\widetilde{g}_2(r) := e^{-r} + 3 \cdot r^{-2} \cdot (r \cdot e^{-r} + e^{-r} - 1) \qquad \text{for } r \in \mathbb{C}\backslash\{0\}.$$

Then the matrix-valued function $(E_{jk})_{1 \le j \le 4, 1 \le k \le 3}$ is a fundamental solution of the Stokes system (1.1) with $\nu = 1$, and the function $(\widetilde{E}^{\lambda}_{1k}, \widetilde{E}^{\lambda}_{2k}, \widetilde{E}^{\lambda}_{3k}, E_{4k})_{1 \le k \le 3}$ is a fundamental solution of the resolvent problem for the Stokes system,

$$-\Delta u + \lambda \cdot u + \nabla \pi = f, \qquad \operatorname{div} u = 0. \tag{3.3}$$

Further note that $\overline{E}^{\lambda}$ is a fundamental solution of the Helmholtz equation. If $j, k, l \in \{1, 2, 3\}$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, we abbreviate

$$\mathcal{D}_{jkl} := D_j E_{kl} + D_k E_{jl} - \delta_{jk} \cdot E_{4l}, \qquad \widetilde{\mathcal{D}}^{\lambda}_{jkl} := D_j \widetilde{E}^{\lambda}_{kl} + D_k \widetilde{E}^{\lambda}_{jl} - \delta_{jk} \cdot E_{4l},$$

$$\mathcal{F}_{jkl} := D_k E_{jl} - \delta_{jk} \cdot E_{4l}.$$

Take $p \in (1, \infty)$, $\tau \in \{-1, 1\}$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$ and assume that B is an open set in $\mathbb{R}^3$, with Lipschitz boundary ∂B and outward unit normal $n^{(B)}$. Then we introduce the operators

$$\Lambda(\tau, p, B), \Gamma(\tau, p, \lambda, B), \Psi(\tau, p, B) : L^p(\partial B)^3 \mapsto L^p(\partial B)^3$$

by setting for $\Phi \in L^p(\partial B)^3$, $x \in \partial\Omega$:

$$\Lambda(\tau, p, B)(\Phi)(x)$$

$$:= (\tau/2) \cdot \Phi(x) + \Big(\int_{\partial B} \sum_{j,k=1}^{3} \mathcal{D}_{jkl}(x - y) \cdot n^{(B)}_k(y) \cdot \Phi_j(y) \, dB(y) \Big)_{1 \le l \le 3},$$

and by defining $\Gamma(\tau, p, \lambda, B)(\Phi)(x)$, $\Psi(\tau, p, B)(\Phi)(x)$ by the preceding integral, but with the kernel $\mathcal{D}_{jkl}$ replaced by $\widetilde{\mathcal{D}}^{\lambda}_{jkl}$ and $\mathcal{F}_{jkl}$, respectively. Of course, these definitions only make sense if these integrals exist and belong to $L^p(\partial B)^3$ when considered as functions of $x \in \partial B$. Moreover the preceding operators should be bounded with respect to the norm of $L^p(\partial B)^3$. If the set B is the infinite circular cone $\mathbb{K}(\varphi)$ introduced in Section 2, it is shown in [7, Lemma 6.2, 6.5] that the operators $\Lambda(\tau, p, B)$ and $\Gamma(\tau, p, \lambda, B)$ satisfy these requirements. In the case $B = \Omega$, it follows by a deep-lying result by Coifman, McIntosh, Meyer [3] that all three of the operators introduced above are well defined and continuous (see [10, p. 773/774]).

We shall write $\Lambda^*(\tau, p, B)$, $\Psi^*(\tau, p, B)$ for the operator adjoint to $\Lambda(\tau, p, B)$ and $\Psi(\tau, p, B)$, respectively. For $\Phi \in L^p(\partial B)^3$, we define the double-layer potentials

$$W(B)(\Phi) : \mathbb{R}^3\backslash\partial B \mapsto \mathbb{C}^3, \qquad \Pi(B)(\Phi) : \mathbb{R}^3\backslash\partial B \mapsto \mathbb{C}$$

by setting for $x \in \mathbb{R}^3\backslash\partial B$:

$$W(B)(\Phi)(x) := \Big(\int_{\partial B} \sum_{j,k=1}^{3} \mathcal{D}_{jkl}(x - y) \cdot n^{(B)}_k(y) \cdot \Phi_j(y) \, dB(y) \Big)_{1 \le l \le 3},$$

$$\Pi(B)(\Phi)(x) := \int_{\partial B} \sum_{j,k=1}^{3} 2 \cdot D_j E_{4k}(x-y) \cdot n_k^{(B)}(y) \cdot \Phi_j(y) \; dB(y).$$

Let $M(B)(\Phi) : \mathbb{R}^3 \backslash \partial B \mapsto \mathbb{C}^3$ be defined in the same way as $W(B)(\Phi)$, except that the kernel $\mathcal{D}_{jkl}$ is to be replaced by $\mathcal{F}_{jkl}$.

Let us require from now on that B is bounded. Then we introduce the single-layer potentials

$V(\partial B)(\Phi) : \mathbb{R}^3 \mapsto \mathbb{C}^3 , \quad Q(B)(\Phi) : \mathbb{R}^3 \backslash \partial B \mapsto \mathbb{C}$

by setting for $\Phi \in L^p(\partial B)^3 , \quad x \in \mathbb{R}^3 , \quad z \in \mathbb{R}^3 \backslash \partial B :$

$$V(\partial B)(\Phi)(x) := \Big(\int_{\partial B} \sum_{k=1}^{3} E_{jk}(x-y) \cdot \Phi_k(y) \; dB(y) \Big)_{1 \le j \le 3},$$

$$Q(\partial B)(\Phi)(z) := \int_{\partial B} \sum_{k=1}^{3} E_{4k}(z-y) \cdot \Phi_k(y) \; dB(y).$$

Moreover, let $\widetilde{V}^\lambda(\partial B)(\Phi)$ be defined in the same way as $V(\partial B)(\Phi)$ except that the kernel $\mathcal{D}_{jkl}$ is to be replaced by $\widetilde{\mathcal{D}}_{jkl}^\lambda$. Finally, for $f \in L^p(B)^3$, we introduce the volume potentials $R(f) : \mathbb{R}^3 \mapsto \mathbb{C}^3 , \quad S(f) : \mathbb{R}^3 \mapsto \mathbb{C}$ by putting

$$R(f)(x) := \Big(\int_{\mathbb{R}^3} \sum_{k=1}^{3} E_{jk}(x-y) \cdot \tilde{f}_k(y) \; dy \Big)_{1 \le j \le 3},$$

$$S(f)(x) := \int_{\mathbb{R}^3} \sum_{k=1}^{3} E_{4k}(x-y) \cdot \tilde{f}_k(y) \; dy,$$

where $\tilde{f}$ denotes the zero extension of f to $\mathbb{R}^3$. The volume potential $\widetilde{R}^\lambda(f)$ is to be defined in the same way as $R(f)$, but with the kernel E_{jk} replaced by $\delta_{jk} \cdot \overline{E}^\lambda$.

The proof of the following technical lemma is simple. We refer [9, p. 130, Lemma 5.7] for the case $p = 2$.

Lemma 3.1. *Let* $p \in (1, \infty), \quad \epsilon \in (0, 1/p), \quad E \in (0, \infty)$. *Denote the identity mapping of* Ω *by* $id(\Omega)$. *Then there exists a constant* $C > 0$ *such that*

$$\Big\| \int_{\partial\Omega} K(id(\Omega), y) \cdot \Phi(y) \; d\Omega(y) \Big\|_{1/p-\epsilon, p} \le C \cdot \|\Phi\|_p$$

for $\Phi \in L^p(\partial\Omega)^3 , \quad K : \mathbb{R}^3 \times \partial\Omega \mapsto \mathbb{C}^{3\times 3}$ *measurable with* $K(\cdot, y) \in C^1(\mathbb{R}^3 \backslash \{y\})^3$ *for* $y \in \partial\Omega$,

$$|K(x,y)| \cdot |x-y|^2 + |D_l K(x,y)| \cdot |x-y|^3 \le E$$

for $x \in \mathbb{R}^3 , \quad y \in \partial\Omega , \ x \ne y , \quad 1 \le l \le 3$.

Now let us review some results on boundary value problems (1.1), (1.2) and (1.1), (1.3). We begin with the Dirichlet problem (1.1), (1.2):

Theorem 3.2. *Let $f \in L^2(\Omega)^3$, $g \in L^2_N(\partial\Omega)^3$. Then there is a solution (u, π) of (1.1), (1.2) with*

$$u \in \bigcap_{\epsilon>0} W^{1/2-\epsilon,2}(\Omega)^3 \cap W^{2,2}_{loc}(\Omega)^3\,, \quad u^* \in L^2(\partial\Omega)^3\,, \quad \pi \in W^{1,2}_{loc}(\Omega)\,. \tag{3.4}$$

and with boundary condition (1.2) satisfied in the L^2-sense, that is,

$$\int_{\partial\Omega} |\, g(x) \;-\; u(x - \kappa \cdot m^{(\Omega)}(x))\,|^2 \; d\Omega(x) \;\longrightarrow\; 0 \qquad (\kappa \downarrow 0)\,. \tag{3.5}$$

This function u is uniquely determined, and π is unique up to an additive constant. For any $\epsilon > 0$, there is a constant $C > 0$ such that

$$\|\, u\,\|_{1/2-\epsilon,2} \;\;\leq\;\; C \cdot (\,\|\, f\,\|_2 \,+\, \|\, g\,\|_2\,) \qquad \textit{for } f,\, g,\, u \textit{ as before.} \tag{3.6}$$

This theorem was established by Fabes, Kenig, Verchota [10]. In their proof, these authors look for a solution in the class of functions $\{\,(\,v(\Phi)\,,\,\varrho(\Phi)\,)\;:\;\Phi \in L^2(\partial\Omega)^3\,\}$, with

$$v(\Phi) := \Big(R(f) \,+\, M(\Phi)\Big)\Big|\Omega\,,$$

$$\varrho(\Phi) := \Big(S(f) \,+\, (1/2)\cdot\Pi(\Phi)\Big)\Big|\Omega \quad \text{for } \Phi \in L^2(\partial\Omega)^3\,,$$

where $f \in L^2(\Omega)^3$ is considered as fixed. Here we assume without loss of generality that $\nu = 1$, and we shall sometimes do the same when considering other linear problems in the following. Note that the functions $v(\Phi)$ and $\varrho(\Phi)$ are defined as the sum of a volume and a double-layer potential. We mention that the volume potentials $R(f)$, $S(f)$ do not appear in [10], but it is standard to use them in the way indicated here.

For any $\Phi \in L^2(\partial\Omega)^3$, the pair of functions $(\,v(\Phi)\,,\,\varrho(\Phi)\,)$ solves the Stokes system (1.1). In addition, all such pairs $(\,v(\Phi)\,,\,\varrho(\Phi)\,)$ belong to the class of functions mentioned in (3.4). This follows by Lemma 3.1 and – concerning the relation $v(\Phi)^* \in L^2(\partial\Omega)^3$ – by the theorem in [3]; see the remarks in [10, p. 773]. Note that no difficulties arise with respect to the volume potential $R(f)$ because it may be dealt with by applying the Hardy-Littlewood-Sobolev and the Calderon-Zygmund inequalities. Also as a consequence of the results in [3], it follows the function $v(\Phi)$ takes the Dirichlet boundary value $\Psi(1,\,2,\,\Omega)(\Phi) \,+\, R(f)\,|\,\partial\Omega$, for $\Phi \in L^2(\partial\Omega)^3$. Thus, in order to satisfy boundary condition (1.2), we have to look for a function $\Phi \in L^2(\partial\Omega)^3$ with

$$\Psi(1,\,2,\,\Omega)(\Phi) \;\;=\;\; g \,-\, R(f)\,|\,\partial\Omega\,. \tag{3.7}$$

In other words, we have to solve a Fredholm integral equation of the second kind on $\partial\Omega$. This is a difficult problem since the integral over $\partial\Omega$ appearing in the definition of $\Psi(1,\,2,\,\Omega)(\Phi)$ is singular, due to the Lipschitz character of $\partial\Omega$. Still it could be shown there is a subspace $\mathfrak{D}$ of $L^2(\partial\Omega)^3$ with *codim* $\mathfrak{D} \;=\; 1$ such that the mapping

$$\gamma : \mathfrak{D} \mapsto L^2_N(\partial\Omega) \quad \text{for } \Phi \in \mathfrak{D}\,, \quad \gamma(\Phi) := \Psi(1,\,2,\,\Omega)(\Phi)$$

is bounded invertible (see [10, p. 787]). Therefore integral equation (3.7) may be uniquely solved in $\mathfrak{D}$, and there is a constant $C > 0$ with

$$\| \Phi \|_2 \quad \leq \quad C \cdot \| \Psi(1, 2, \Omega)(\Phi) \|_2 \qquad \text{for } \Phi \in \mathfrak{D}. \tag{3.8}$$

Thus, by setting $u := v(\Phi)$, $\pi := \varrho(\Phi)$, with $\Phi \in \mathfrak{D}$ satisfying (3.7), we obtain a solution of (1.1), (1.2) with properties as listed in Theorem 3.2. In particular, inequality (3.6) is valid due to Lemma 3.1 and the estimate in (3.8).

Theorem 3.2 was generalized by Shen [28] to a L^p-theory with $p \geq 2$:

Theorem 3.3. *Let $p \in [2, \infty)$, $f \in L^p(\Omega)^3$, $g \in L^p_N(\partial\Omega)$. Then there is a solution (u, π) of (1.1), (1.2) with $u \in W^{2,p}_{loc}(\Omega)^3$, $\pi \in W^{1,p}_{loc}(\Omega)$, $u^* \in L^p(\partial\Omega)^3$. This function u is uniquely determined, and π is unique up to addition of a constant.*

Shen derives this result by combining Theorem 3.2, a L^∞-estimate of Green's function for problem (1.1), (1.2), and an interpolation argument. We mention that in [28], problem (1.1), (1.2) is solved in Hölder spaces as well.

In [10], problem (1.1), (1.2) is additionally considered for more regular boundary data g:

Theorem 3.4. *Let $f \in L^2(\Omega)^3$, $g \in W^{1,2}(\partial\Omega)^3 \cap L^2_N(\partial\Omega)$. Then there is a solution (u, π) of (1.1), (1.2) such that*

$$u \in \bigcap_{\epsilon>0} W^{3/2-\epsilon,2}(\Omega)^3 \cap W^{2,2}_{loc}(\Omega)^3, \quad (\nabla u - \nabla R(f) \,|\, \partial\Omega)^* \in L^2(\partial\Omega)^3, \tag{3.9}$$

$$u^* \in L^2(\partial\Omega)^3, \quad \pi \in \bigcap_{\epsilon>0} W^{1/2-\epsilon,2}(\Omega) \cap W^{1,2}_{loc}(\Omega), \quad (\pi - S(f) \,|\, \partial\Omega)^* \in L^2(\partial\Omega),$$

For any $\epsilon \in (0, \infty)$, there is a constant $C > 0$ with

$$\| u \|_{3/2-\epsilon,2} + \left\| \pi - Vol(\Omega)^{-1} \cdot \int_\Omega \pi \, dx \right\|_{1/2-\epsilon,2} \leq C \cdot (\| f \|_2 + \| g \|_{1,2})$$

for f, g, u, π as before.

The proof of Theorem 3.4 is based on the fact that the operators $\Psi^*(1, 2, \Omega)$ and $\Psi^*(-1, 2, \Omega)$ are bounded invertible if they are restricted to appropriate subspaces of $L^2(\partial\Omega)^3$ of codimension 1; see [10, p. 785, Theorem 2.6]. These results imply the operator

$$\beta : L^2_N(\partial\Omega) \mapsto W^{1,2}(\partial\Omega)^3 \cap L^2_N(\partial\Omega), \qquad \beta(\Phi) := V(\partial\Omega)(\Phi) \,|\, \partial\Omega \quad \text{for } \Phi \in L^2_N(\partial\Omega),$$

is bounded invertible ([10, p. 792]). Thus, defining u and π as a sum of a single-layer and a volume potential,

$$u := \big(R(f) + V(\partial\Omega)(\Phi) \big) \big| \Omega, \quad \pi := \big(S(f) + Q(\partial\Omega)(\Phi) \big) \big| \Omega,$$

with Φ chosen in such a way that

$$V(\partial\Omega)(\Phi) \,|\, \partial\Omega = g - R(f) \,|\, \partial\Omega,$$

we obtain a solution (u, π) of (1.1), (1.2) exhibiting the properties required in Theorem 3.4. This follows by [10, p. 785, Theorem 4.15] and Lemma 3.1.

Now let us turn to boundary value problem (1.1), (1.3):

Theorem 3.5. *Let $f \in L^2(\Omega)^3$, $h \in L^2(\partial\Omega)^3 \cap \mathfrak{B}$, with the space $\mathfrak{B}$ defined by*

$$\mathfrak{B} := \left\{ \varrho \in L^2(\partial\Omega)^3 \; : \; \int_{\partial\Omega} \varrho \cdot \varphi \; d\Omega \;=\; 0 \quad \textit{for } \varphi \in Z(\Omega) \right\}.$$

Then there is a solution (u, π) of (1.1), (1.3) with regularity as in (3.9), and with boundary condition (1.3) satisfied in the sense that

$$\int_{\partial\Omega} | \, h(x) \;-\; T(u,\pi)(x \;-\; \kappa \cdot m^{(\Omega)}(x)\,) \cdot n^{(\Omega)}(x) \, |^2 \; d\Omega(x) \;\longrightarrow\; 0 \qquad (\kappa \downarrow 0). \quad (3.10)$$

The velocity u is uniquely determined up to addition of a function from $Z(\Omega)$, and the pressure π is unique up to a constant. For any $\epsilon > 0$, there is a constant $C > 0$ with

$$\| \nabla u \|_{1/2-\epsilon, 2} \;+\; \left\| \pi \;-\; Vol\,(\Omega)^{-1} \cdot \int_\Omega \pi \; dx \right\|_{1/2-\epsilon, 2} \;\leq\; C \cdot (\, \| f \|_2 \;+\; \| h \|_2 \,)$$

for f, h, u, π as before.

This theorem was proved by Dahlberg, Kenig, Verchota [5], who implicitly show the operator $\Lambda^*(-1, 2, \Omega)$ is bounded invertible when considered as an operator from $\mathfrak{A}$ into $\mathfrak{B}$, where $\mathfrak{A}$ is a subspace of $L^2(\partial\Omega)^3$ of codimension 6, and $\mathfrak{B}$ was defined in Theorem 3.5. On the other hand, if we take any $\Phi \in L^2(\partial\Omega)^3$, the pair of functions (u, π), with

$$u := \Big(R(f) \;+\; V(\partial\Omega)(\Phi) \Big) \Big| \Omega, \qquad \pi := \Big(S(f) \;+\; Q(\partial\Omega)(\Phi) \Big) \Big| \Omega,$$

satisfies the boundary condition

$$T(u,\pi)(x) \cdot n^{(\Omega)}(x) \;=\; -\Lambda^*(-1, 2, \Omega)(\Phi)(x) \;+\; T(\,R(f), S(f)\,)(x) \cdot n^{(\Omega)}(x)$$

for $x \in \partial\Omega$, in a sense as in (3.10). Thus an appropriate solution of (1.1), (1.3) may be obtained by choosing $\Phi \in \mathfrak{A}$ in such a way that

$$\Lambda^*(-1, 2, \Omega)(\Phi) \;=\; -h \;+\; T(\,R(f), S(f)\,) \,|\, \Omega \cdot n^{(\Omega)}.$$

Recent results by Jerison, Kenig [18, Theorem 5.15] on the Laplace equation suggest that Theorems 3.2, 3.4 and 3.5 might remain valid if the epsilon appearing in (3.4) and (3.9) is dropped. To our knowledge, however, no proof is available yet.

Let us remark on the time-dependent Stokes system

$$\partial/\partial t\, u(x,t) \;-\; \nu \cdot \Delta_x u(x,t) \;+\; \nabla_x \pi(x,t) \;=\; f(x,t), \qquad \mathrm{div}_x u(x,t) \;=\; 0 \quad (3.11)$$

in $\Omega \times (0, T)$, under the initial condition

$$u(x, 0) \;=\; u_0(x) \qquad \text{for } x \in \Omega,$$

and with a boundary condition on $\partial\Omega \times (0, T)$ analogous to either (1.2) or (1.3). Concerning these initial-boundary value problems, Shen [27] develops a L^2-theory which corresponds to the theory for the stationary case presented in Theorem 3.2, 3.4 and 3.5. We do not state the results from [27] here, which are somewhat more complicated than those given in Theorem 3.2, 3.4, 3.5, due to the presence of the time variable. We mention, however, that the proofs in [27] are also based on the method of integral equations, applied in a way which is analogous to the one indicated above.

Next let us consider the time-dependent, nonlinear Navier-Stokes system

$$\partial/\partial t\, u(x,t) \;-\; \nu\cdot\Delta_x u(x,t) \;+\; (\,u(x,t)\cdot\nabla_x)\,u(x,t) \;+\; \nabla_x\pi(x,t) \;=\; f(x,t)\,, \tag{3.12}$$

$$\mathrm{div}_x u(x,t) \;=\; 0$$

in $\Omega\times(0,T)$, under the initial condition

$$u(x,0) \;=\; u_0(x) \qquad \text{for } x\in\Omega\,, \tag{3.13}$$

and with a homogeneous Dirichlet boundary condition,

$$u(x,t) \;=\; 0 \qquad \text{for } (x,t)\in\partial\Omega\times(0,T)\,. \tag{3.14}$$

It is a famous result by Hopf [17] that initial-boundary value problem (3.12) - (3.14) has a weak (variational) solution. This solution is global in time, that is, the parameter $T\in(0,\infty]$ is part of the data. This leaves open the question whether problem (3.12) - (3.14) may be solved by "strong solutions", which should satisfy (3.12) pointwise. It was proved by Deuring, von Wahl [9] that under appropriate conditions on the data u_0 and f, there is some $T\in(0,\infty]$ and a strong solution (u,π) of (3.12) - (3.14) in $\Omega\times(0,T)$ such that for any $t\in(0,T)$, the pair $(\,u(\,\cdot\,,t)\,,\,\pi(\,\cdot\,,t)\,)$ belongs to the class of functions specified by (3.9). In particular, this solution is local in time, which means the parameter T is an unknown quantity, obtained as part of the solution. We remark that existence of a global strong solution is an open problem even in the case of smoothly bounded domains. However, if the data are small in a suitable sense, existence of global strong solutions is well known when problem (3.12) - (3.14) is considered on domains with regular boundary. In [9], this result is carried over to the case of Lipschitz domains.

The theory in [9] is proved by studying the resolvent problem (3.3) for $\lambda\in\mathbb{C}\backslash(-\infty,0]$, under Dirichlet boundary condition (1.2) with $g=0$. This boundary value problem may be solved by a pair of functions (u,π), with u defined as the sum of a single-layer and a volume potential:

$$u := \Big(\tilde{R}^\lambda(\,P(f)\,) \;+\; \tilde{V}^\lambda(\,\partial\Omega\,)(\Phi)\Big)\Big|\,\Omega, \qquad \pi := \Big(R(f) \;+\; Q(\,\partial\Omega\,)(\Phi)\Big)\Big|\,\Omega\,,$$

where $\Phi\in L^2_N(\partial\Omega)$ satisfies the integral equation

$$\tilde{V}^\lambda(\,\partial\Omega\,)(\Phi)\,|\,\partial\Omega \;\;=\;\; -\,\tilde{R}^\lambda(\,P(f)\,)\,|\,\partial\Omega\,, \tag{3.15}$$

and where $P(f)\in L^2(\Omega)^3$ denotes the solenoidal part and $G(f)\in L^2(\Omega)^3$ the gradient part of f; see [29, p. 15, Theorem 1.4]. A function Φ as in (3.15) exists, as may be deduced from [10, p. 792]; see [9, p. 133]. In [9], this solution (u,π) of (3.3) is estimated in way which allows to recur to a functional analytic argument yielding strong solutions of (3.12) - (3.14) in a L^2-setting. This functional analytic argument was first proposed by Fujita, Kato [11]. In the case of smoothly bounded domains, it was generalized to a L^p-framework. This extension is based on the following theorem:

Theorem 3.6. *(see [13]). Let $U\subset\mathbb{R}^3$ be a bounded domain with smooth boundary ∂U. Take $p\in(1,\infty)$. Then for any $f\in L^p(U)^3$, $\lambda\in\mathbb{C}\backslash(-\infty,0]$, there is a pair of functions $(u,\pi)\in W^{2,p}(U)^3\times W^{1,p}(U)$ solving equation (3.3) in U, under homoge-*

neous Dirichlet boundary conditions. This function u is uniquely determined, and π is unique up to addition of a constant.

Let $\vartheta \in [0, \pi)$. Then there is a constant $C > 0$ such that

$$\| u \|_p \quad \leq \quad C \cdot |\lambda|^{-1} \cdot \| f \|_p \tag{3.16}$$

for $\lambda \in \mathbb{C} \backslash \{0\}$ with $|\arg \lambda| \leq \vartheta$, and for f, u as before.

The question is open as to what may be recovered of this theorem when equation (3.3) is considered in a Lipschitz domain. It is shown in [7] that the double-layer potential $\Gamma(1, p, \lambda, \mathbb{K}(\varphi))$ related to equation (3.3) on $\mathbb{K}(\varphi)$ is not Fredholm for certain values of $p \in (1, \infty)$, $\lambda \in \mathbb{C} \backslash (-\infty, 0]$. Perhaps this fact may be exploited to prove that inequality (3.16) is not valid in certain subspaces of $L^p(\Omega)^3$ which, on the other hand, contain a uniquely determined solution of (3.3), (1.2) for $g = 0$ and for any $f \in L^p(\Omega)^3$, $\lambda \in \mathbb{C} \backslash (-\infty, 0]$.

Let us note a by-product of the results in [7] (see [8]): In the case $\Omega = U_\varphi$, with U_φ defined as in Section 2, Theorem 3.5 (Neuman-type problem) may be extended to a corresponding result in L^p-spaces with $p \in (1, 2]$, and Theorem 3.2 (Dirichlet problem) may generalized to a L^p-theory with $p \in [2, \infty)$. In particular, Theorem 3.3 may be recovered for the special domain $\Omega = U_\varphi$.

4. Solutions in weighted L^p-spaces.

Kondrat'ev [19] developed a method which allows to solve elliptic partial differential equations in domains with non-smooth boundaries exhibiting certain special geometries. To indicate a basic idea of his approach, we point out that a partial differential equation on an infinite cylinder $D \times \mathbb{R}$, with $D \subset \mathbb{R}^N$ open for some $N \in \mathbb{N}$, may be carried over by a Fourier transformation into an eigenvalue problem in D. If a solution theory is available for this eigenvalue problem, this theory may then be used for solving the original problem. Since the mapping $r \mapsto \ln r$ transforms an infinite cone into an infinite cylinder, this observation may be exploited for solving partial differential equations in cones. Examples illustrating this approach may be found in [24, Chapter 2].

Following [24, Chapter 3], let us describe how Kondrat'ev's method works out when applied to the non-homogeneous Stokes system (3.1) under Dirichlet boundary conditions (1.2).

We begin by introducing some notions related to operator-valued holomorphic mappings. Let E_1, E_2 be Banach spaces, and denote by $L(E_1, E_2)$ the space of all linear bounded operators from E_1 into E_2. Let $\lambda_0 \in \mathbb{C}$, $\epsilon \in (0, \infty)$, and assume the operator $A : \mathbb{C} \mapsto L(E_1, E_2)$ is holomorphic in $\mathbb{B}_2(0, R)$. Further assume there is a holomorphic function $\Phi : \mathbb{B}_2(\lambda_0, \epsilon) \mapsto E_1$ such that the mapping

$$\Lambda : \mathbb{B}_2(\lambda_0, \epsilon) \mapsto E_2, \quad \Lambda(\lambda) := A(\lambda)\Phi(\lambda) \quad \text{for } \lambda \in \mathbb{B}_2(\lambda_0, \epsilon),$$

vanishes in the point λ_0. Then λ_0 is called an eigenvalue of A and Φ a root function of A at the point λ_0. The multiplicity κ of the zero λ_0 of the mapping Λ is called the multiplicity of the root function Φ. There is a sequence $(\varphi_n)_{n\in\mathbb{N}_0}$ of vectors in E_1 such that

$$\Phi(\lambda) \quad = \quad \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n \cdot \varphi_n \qquad \text{for } \lambda \in \mathbb{B}_2(\lambda_0, \epsilon).$$

The vector $\varphi_0 = \Phi(\lambda_0)$ is said to be an eigenvector (or an eigenfunction, if E_2 is a function space) of A relative to the eigenvalue λ_0. The ordered set $(\varphi_0, \cdots, \varphi_{\kappa-1})$ is called a Jordan chain corresponding to the eigenvalue λ_0, and the vectors $\varphi_1, \cdots, \varphi_{\kappa-1}$ are said to be generalized eigenvectors (or generalized eigenfunctions). Let the rank of φ_0 be defined as the maximal multiplicity of all root functions Ψ of A in λ_0 with $\Psi(\lambda_0) = \varphi_0$.

Set $J := \mathit{dim\,kernel}\, A(\lambda_0)$ and assume that $J < \infty$ and the rank of all eigenvectors of A corresponding to λ_0 is finite. Let $\varphi^{(0,1)}, \cdots, \varphi^{(0,J)}$ be a system of eigenvectors such that for $j \in \{1, \cdots, J\}$, the rank of $\varphi^{(0,j)}$ is maximal among the ranks of all eigenvectors belonging to any direct complement in $\mathit{kernel}\, A(\lambda_0)$ of the space spanned by the vectors $\varphi^{(0,1)}, \cdots, \varphi^{(0,j-1)}$. If for each $j \in \{1, \cdots, J\}$ the vectors $\varphi^{(0,j)}, \cdots, \varphi^{(\kappa_j-1,j)}$ form a Jordan chain, for some $\kappa_j \in \mathbb{N}$, then the set

$$\left\{ \left(\varphi^{(0,j)}, \cdots, \varphi^{(\kappa_j-1,j)}\right) \; : \; j \in \{1, \cdots, J\} \right\}$$

is said to be a canonical system of Jordan chains corresponding to the eigenvalue λ_0.

Let us write the non-homogeneous Stokes system (3.1) under Dirichlet boundary conditions in a way which is compatible with the notations in [24, p. 5/6]. To this end we set

$$L_{lj}(\xi) := |\xi|^2 \cdot \delta_{lj}, \quad L_{l4}(\xi) := i \cdot \xi_l, \quad L_{4j} := -i \cdot \xi_j, \quad L_{44} := 0, \quad B_{lm}(\xi) := \delta_{lm}$$

for $\xi \in \mathbb{R}^3$, $l, j, k \in \{1, 2, 3\}$, $m \in \{1, \cdots, 4\}$. Put $L := (L_{lj})_{1 \le l, j \le 4}$. Obviously the functions L_{lj}, B_{kj} are homogeneous polynomials with $\det L(\xi) \neq 0$ for $\xi \in \mathbb{R}^3$, $\xi \neq 0$. Moreover, using the abbreviation

$$D_x := (-i \cdot D_1, -i \cdot D_2, -i \cdot D_3)$$

as in [24, p. 4], we have

$$\sum_{j=1}^{4} L_{lj}(D_x)(u_j) \;=\; -\Delta u_l + D_l u_4, \qquad \sum_{j=1}^{4} L_{4j}(D_x)(u_j) \;=\; -\operatorname{div}(u_1, u_2, u_3),$$

$$\sum_{j=1}^{4} B_{lj}(D_x)(u_j) \;=\; u_l$$

for $l \in \{1, 2, 3\}$. Here the unknown functions were denoted by $u_1, \cdots, u_4$, that is, u_4 stands for the pressure. Thus we see that with the preceding definition of L and B, problem $\{L(D_x), B(D_x)\}$ from [24, p. 5/6] corresponds to the non-homogeneous Stokes problem (3.1) under Dirichlet boundary conditions (1.2). This problem is elliptic in the sense of [24, p. 5/6], that is, elliptic in the sense of Agmon, Douglis, Nirenberg [1]. This fact is explained in [29, p. 33/34]. Note that for

$t_m = 2, \quad s_m = 0, \quad \sigma_m = -2 \quad \text{for } m \in \{1, 2, 3\}, \quad t_4 = 1, \quad s_4 = -1,$

it holds $ord\, L_{lj} \le s_l + t_j, \quad ord\, B_{kj} \le \sigma_k + t_j$ for $l, j \in \{1, \cdots, 4\}, \quad k \in \{1, 2, 3\}$, where $ord\, L_{lj}$, $ord\, B_{kj}$ denotes the degree of the polynomial L_{lj} and B_{kj}, respectively.

Let $\varphi \in (0, \pi)$ and set for $\beta \in \mathbb{R}$

$$\mathcal{D}^0_\beta(\mathbb{K}(\varphi)) := V^2_\beta(\mathbb{K}(\varphi))^3 \times V^1_\beta(\mathbb{K}(\varphi)),$$

$$\mathcal{R}^0_\beta(\mathbb{K}(\varphi)) := V^0_\beta(\mathbb{K}(\varphi))^3 \times V^1_\beta(\mathbb{K}(\varphi)) \times V^{3/2}_\beta(\mathbb{K}(\varphi))^3$$

We further define for $\sigma, \vartheta \in \mathbb{R}$:

$$A(\sigma, \vartheta) := \begin{pmatrix} \cos\sigma \cdot \sin\vartheta & \sin\sigma \cdot \sin\vartheta & \cos\vartheta \\ -\sin\sigma \cdot \sin\vartheta & \cos\sigma \cdot \sin\vartheta & 0 \\ \cos\sigma \cdot \cos\vartheta & \sin\sigma \cdot \cos\vartheta & -\sin\vartheta \end{pmatrix}.$$

Then it holds for $w \in V^1_\beta(\mathbb{K}(\varphi))$, $r \in (0, \infty)$, $\sigma \in \mathbb{R}$, $\vartheta \in (-\varphi, \varphi)$:

$$(r \cdot D_1, D_2, D_3)(w \circ T)(r, \sigma, \vartheta) = r \cdot A(\sigma, \vartheta) \cdot (\nabla w)(T(r, \sigma, \vartheta)).$$

Now we may write the operators $L(D_x)$, $B(D_x)$ in spherical coordinates; compare [24, p. 64, (3.5.9)]. To this end we put

$$\mathbb{L}_{lj}((\sigma, \vartheta), (\alpha, \beta), \lambda)$$

$$:= \delta_{lj} \cdot \Big(\lambda^2 - i \cdot \lambda + \sin^{-1}(\vartheta) \cdot (-i \cdot \cos\vartheta \cdot \beta + \sin\vartheta \cdot \beta^2) + \sin^{-2}(\vartheta) \cdot \alpha^2\Big),$$

$$\mathbb{L}_{l4}((\sigma, \vartheta), (\alpha, \beta), \lambda) := i \cdot \Big(A(\sigma, \vartheta)^{-1} \cdot (\lambda, \alpha, \beta)\Big)_l,$$

$$\mathbb{L}_{4j}((\sigma, \vartheta), (\alpha, \beta), \lambda) := -i \cdot \Big(A(\sigma, \vartheta)^{-1} \cdot (\lambda, \alpha, \beta)\Big)_j, \qquad \mathbb{L}_{44} := 0,$$

$\mathbb{B}_{km}((\sigma, \vartheta), (\alpha, \beta), \lambda) := \delta_{km}$ for $k, j, l \in \{1, 2, 3\}$, $m \in \{1, \cdots, 4\}$, $(\sigma, \vartheta) \in B_\varphi$, $\alpha, \beta \in \mathbb{R}$. We further set

$$D_\omega := (-i \cdot \partial/\partial\sigma, -i \cdot \partial/\partial\vartheta), \qquad (r \cdot \partial/\partial r)^2 f(r) := r \cdot \partial/\partial r(r \cdot \partial/\partial r f(r))$$

for $f : I \mapsto \mathbb{C}$ weakly differentiable, with $I \subset \mathbb{R}$ an interval. Then we obtain for $j, l \in \{1, 2, 3\}$, $m \in \{1, \cdots, 4\}$, $r \in (0, \infty)$, $\beta, \sigma \in \mathbb{R}$, $\vartheta \in (-\varphi, \varphi)$, $u = (u_1, \cdots, u_4) \in \mathcal{D}^0_\beta(\mathbb{K}(\varphi))$:

$$L_{lj}(D_x)(u)(T(r, \sigma, \vartheta)) = r^{-2} \cdot \mathbb{L}_{lj}((\sigma, \vartheta), D_\omega, -i \cdot r \cdot \partial/\partial r)(u \circ T)(r, \sigma, \vartheta),$$

$$L_{l4}(D_x)(u)(T(r, \sigma, \vartheta)) = r^{-1} \cdot \mathbb{L}_{l4}((\sigma, \vartheta), D_\omega, -i \cdot r \cdot \partial/\partial r)(u \circ T)(r, \sigma, \vartheta),$$

$$L_{4j}(D_x)(u)(T(r, \sigma, \vartheta)) = r^{-1} \cdot \mathbb{L}_{4j}((\sigma, \vartheta), D_\omega, -i \cdot r \cdot \partial/\partial r)(u \circ T)(r, \sigma, \vartheta),$$

$$B_{km}(D_x)(u)(T(r, \sigma, \vartheta)) = \mathbb{B}_{km}((\sigma, \vartheta), D_\omega, -i \cdot r \cdot \partial/\partial r)(u \circ T)(r, \sigma, \vartheta).$$

In particular, the operators $L_{lj}(D_x)$, $B_{kj}(D_x)$ are model operators in the sense of [24, p. 64]. Now recall the definition of the set B_φ in Section 2 and put

$$\mathcal{D}^0\mathcal{H}(B_\varphi) := W^{2,2}(B_\varphi)^3 \times W^{1,2}(B_\varphi),$$

$$\mathcal{R}^0\mathcal{H}(B_\varphi) := L^2(B_\varphi)^3 \times W^{1,2}(B_\varphi) \times W^{3/2,2}(\partial B_\varphi)^3.$$

Furthermore, we introduce the mappings

$\mathcal{A}_\varphi : \mathbb{C} \longmapsto L\Big(\mathcal{D}^0\mathcal{H}(B_\varphi)\,,\ \mathcal{R}^0\mathcal{H}(B_\varphi) \Big)\,,$

$\mathcal{A}'_\varphi : \mathbb{C} \longmapsto L\Big(\mathcal{D}^0\mathcal{H}(B_\varphi)\,,\ L^2(B_\varphi)^3 \times W^{1,2}(B_\varphi) \Big)\,,$

$\mathcal{L}_\varphi : \mathbb{C} \longmapsto L\Big(W_0^{1,2}(B_\varphi)^3 \times L^2(B_\varphi)\,,\ W^{-1,2}(B_\varphi)^3 \times L^2(B_\varphi) \Big)$

in the following way: First define

$$\mathcal{A}_\varphi(\lambda)_l(v)(\sigma,\vartheta) := \sum_{j=1}^{3} \mathbb{L}_{lj}\big((\sigma,\vartheta)\,,\ D_\omega\,,\ \lambda - 2\cdot i \,\big)(v_j)(\sigma,\vartheta)$$
$$+\ \mathbb{L}_{l4}\big((\sigma,\vartheta)\,,\ D_\omega\,,\ \lambda - i \,\big)(v_4)(\sigma,\vartheta)\,,$$

$$\mathcal{A}_\varphi(\lambda)_m(v)(\sigma',\vartheta') := \sum_{j=1}^{4} \mathbb{B}_{m-4\,,j}\big((\sigma',\vartheta')\,,\ D_\omega\,,\ \lambda - i\cdot t_j \,\big)(v_j)(\sigma',\vartheta') \;=\; v_{m-4}(\sigma',\vartheta')\,,$$

for $\lambda \in \mathbb{C}\,,\ l \in \{1, \cdots, 4\}\,,\ m \in \{5, 6, 7\}\,,\ v = (v_1, \cdots, v_4) \in \mathcal{D}^0\mathcal{H}(B_\varphi)\,,\ (\sigma,\vartheta) \in B_\varphi\,,\ (\sigma',\vartheta') \in \partial B_\varphi$. Next set

$\mathcal{A}'_\varphi(\lambda)_l := \mathcal{A}_\varphi(\lambda)_l \qquad \text{for } \lambda \in \mathbb{C}\,,\ l \in \{1, \cdots, 4\}$

and finally define

$$\mathcal{L}_\varphi(\lambda)_l(v) := (\lambda^2+\lambda)\cdot v_l \;+\; \Delta' v_l \;-\; \Big(A^{-1}\cdot\big((\lambda-1)\cdot v_4,\, D_1 v_4,\, D_2 v_4 \big) \Big)_l\,,$$

$$\mathcal{L}_\varphi(\lambda)_4(v) := \sum_{j=1}^{3} \Big(A^{-1}\cdot\big(\lambda\cdot v_j,\, D_1 v_j,\, D_2 v_j \big) \Big)_j$$

for $\lambda \in \mathbb{C}\,,\ l \in \{1, 2, 3\}\,,\ v = (v_1, \cdots, v_4) \in \Big(W^{2,2}(B_\varphi)^3 \cap W_0^{1,2}(B_\varphi)^3 \Big) \times W^{1,2}(B_\varphi)$.
It should now be obvious how to define $\mathcal{L}_\varphi(\lambda)(v)$ if $v \in W_0^{1,2}(B_\varphi)^3 \times L^2(B_\varphi)$. The mappings $\mathcal{A}_\varphi\,, \mathcal{A}'_\varphi$ and $\mathcal{L}_\varphi$ are called "operator pencils".

Clearly, for $v \in \mathcal{D}^0\mathcal{H}(B_\varphi)\,,\ \lambda \in \mathbb{C}$, it holds $\mathcal{A}_\varphi(\lambda)(v) = 0$ if and only if $\mathcal{A}'_\varphi(\lambda)(v) = 0$ and $(v_1, v_2, v_3)\,|\,\partial B_\varphi = 0$.

Furthermore, if $\mathcal{A}'_\varphi(\lambda)(v) = 0$ for some $\lambda \in \mathbb{C},\ v \in \mathcal{D}^0\mathcal{H}(B_\varphi)$ with $(v_1, v_2, v_3)\,|\,\partial B_\varphi = 0$, then it follows $\mathcal{L}_\varphi(2-\lambda/i)(v) = 0$.

For $\beta \in \mathbb{R}$, define the mapping $\mathfrak{A}_\varphi^{(\beta)} : \mathcal{D}_\beta^0(\mathbb{K}(\varphi)) \mapsto \mathcal{R}_\beta^0(\mathbb{K}(\varphi))$ by setting

$$\mathfrak{A}_{\varphi,l}^{(\beta)}(v)(x) := \sum_{j=1}^{4} L_{lj}(D_x)(v)(x)\,, \qquad \mathfrak{A}_{\varphi,k+4}^{(\beta)}(v)(y) := \sum_{j=1}^{4} B_{kj}(D_x)(v)(y)$$

for $l \in \{1, \cdots, 4\}\,,\ k \in \{1, 2, 3\}\,,\ v \in \mathcal{D}_\beta^0(\mathbb{K}(\varphi))\,,\ x \in \mathbb{K}(\varphi)\,,\ y \in \partial\mathbb{K}(\varphi)$.

Now we are able to state an existence results for the non-homogeneous Stokes system (3.1) in $\mathbb{K}(\varphi)$, under Dirichlet boundary conditions. In fact, it holds according to [24, p. 66, Theorem 3.5.1]:

Theorem 4.1. *Let $\beta \in \mathbb{R}$. Then the mapping $\mathfrak{A}_\varphi^{(\beta)}$ is an isomorphism provided no eigenvalues of the operator pencil $\mathcal{A}_\varphi$ are situated on the line $\mathbb{R} + i\cdot(\beta+3/2) \subset \mathbb{C}$.*

Thus, for any number $\beta \in \mathbb{R}$ which satisfies the condition of Theorem 4.1, problem (3.1), (1.2) may be uniquely solved in the space $\mathcal{D}^0_\beta(\mathbb{K}(\varphi))$, for any data $(f_1, f_2, f_3, \tilde{f}, g_1, g_2, g_3) \in \mathcal{R}^0_\beta(\mathbb{K}(\varphi))$. Of course, Theorem 4.1 is only as good as our knowledge of the eigenvalues of $\mathcal{A}_\varphi$. This is a reason why these eigenvalues are the subject of intensive study; see Maz'ya, Plamenevskii [22], [23], Dauge [6], Koz'lov, Maz'ya, Schwab [20]. These studies are based on two facts. First, it is well known that the set of all eigenvalues of $\mathcal{A}_\varphi$ is discrete in $\mathbb{C}$; see [24, p. 11, Theorem 1.2.1]. A second important information is given by the following theorem on asymptotic expansions:

Theorem 4.2 *(see [24, p. 68/69, Theorem 3.5.6]). Let $\beta, \gamma \in \mathbb{R}$ with $\beta \neq \gamma$. Assume*

$$(f_1, f_2, f_3, \tilde{f}, g_1, g_2, g_3) \in \mathcal{R}^0_\beta(\mathbb{K}(\varphi)) \cap \mathcal{R}^0_\gamma(\mathbb{K}(\varphi)). \tag{4.1}$$

Suppose the lines $\mathbb{R} + i\cdot(\beta+3/2)$ and $\mathbb{R} + i\cdot(\gamma+3/2)$ do not contain any eigenvalue of the pencil $\mathcal{A}_\varphi$. Let $\lambda_1, \cdots, \lambda_N$, with $N \in \mathbb{N}$, denote the different eigenvalues of $\mathcal{A}_\varphi$ between these lines.

For $v \in \{1, \cdots, N\}$, set $J_v := \dim \operatorname{kernel} \mathcal{A}_\varphi(\lambda_v)$. It holds $J_v < \infty$, and the rank of all eigenvectors of $\mathcal{A}_\varphi$ corresponding to λ_v is finite $(1 \le v \le N)$.

For $v \in \{1, \cdots, N\}$, $j \in \{1, \cdots, J_v\}$, choose $\kappa(v,j) \in \mathbb{N}$ and functions $\Phi_v^{(0,j)}, \cdots, \Phi_v^{(\kappa(v,j)-1,j)} \in \mathcal{D}^0\mathcal{H}(B_\varphi)$ such that the set

$$\left\{ \left(\Phi_v^{(0,j)}, \cdots, \Phi_v^{(\kappa(v,j)-1,j)}\right) \;:\; j \in \{1, \cdots, J_v\} \right\}$$

is a canonical system of Jordan chains corresponding to the eigenvalue λ_v. For brevity, set

$$u_v^{(k,j)}(r,\sigma,\vartheta) := \left(r^{t_l + i\cdot\lambda_v} \cdot \sum_{s=0}^{k} (s!)^{-1} \cdot (i\cdot \ln r)^s \cdot \Phi_v^{(k-s,j)}(\sigma,\vartheta) \right)_{1\le l\le 4}$$

with $v \in \{1, \cdots, N\}$, $j \in \{1, \cdots, J_v\}$, $k \in \{0, \cdots, \kappa(j, v) - 1\}$. These functions solve problem (3.1), (1.2) with $\Omega = \mathbb{K}(\varphi)$, $f = 0$, $\tilde{f} = 0$, $g = 0$. They are called "power solutions" of (3.1) corresponding to the eigenvalue λ_v. Let

$$u = (u_1, \cdots, u_4) \in \mathcal{D}^0_\beta(\mathbb{K}(\varphi)), \quad v = (v_1, \cdots v_4) \in \mathcal{D}^0_\gamma(\mathbb{K}(\varphi))$$

be the uniquely determined solution of (3.1), (1.2) in $\mathcal{D}^0_\beta(\mathbb{K}(\varphi))$ and $\mathcal{D}^0_\gamma(\mathbb{K}(\varphi))$, respectively, for the data given in (4.1). Then there are uniquely determined numbers $c_v^{(k,j)} \in \mathbb{C}$ $(1 \le v \le N,\ 1 \le j \le J_v,\ 0 \le k \le \kappa^{(j,v)} - 1)$ such that it holds for $r \in (0,\infty)$, $\sigma \in \mathbb{R}$, $\vartheta \in (-\varphi, \varphi)$:

$$u(T(r,\sigma,\vartheta)) = \sum_{v=1}^{N} \sum_{j=1}^{J_v} \sum_{k=0}^{\kappa(j,v)-1} c_v^{(k,j)} \cdot u_v^{(k,j)}(r,\sigma,\vartheta) + v(T(r,\sigma,\vartheta)).$$

This theorem has many interesting applications. Here we only point out a consequence for the study of the eigenvalues of $\mathcal{A}_\varphi$. In fact, if a number $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}_\varphi$, Theorem 4.2 implies there is a power solution of (3.1) corresponding to this eigenvalue. This fact is the starting point of, for example, the articles [23] and [20].

Let us indicate some results from these articles. It is shown in [23] that the strip

$\{\lambda \in \mathbb{C} \ : \ |\Re\lambda + 1/2| \leq 1/2\}$ does not contain any eigenvalue of the operator pencil $\mathcal{L}_\varphi$. Thus the pencil $\mathcal{A}'_\varphi$, and hence $\mathcal{A}_\varphi$, has no eigenvalues in the strip $\{\lambda \in \mathbb{C} \ : \ |\Im\lambda - 5/2| \leq 1/2\}$. Therefore Theorem 4.1 implies the mapping $\mathfrak{A}_\varphi^{(1)}$ is an isomorphism. Moreover, according to [20, p. 70, 96], there is an angle $\alpha^* \approx 0.6665 \cdot \pi$ such that in the case $\varphi < \alpha^*$, there are no eigenvalues of the pecil $\mathcal{L}_\varphi$ in the strip $\{\lambda \in \mathbb{C} \ : \ -1 \leq \Re\lambda \leq 1/2\}$. As a consequence, the pencil $\mathcal{A}_\varphi$ has no eigenvalues in the strip $\{\lambda \in \mathbb{C} \ : \ 3/2 \leq \Im\lambda \leq 3\}$. Now Theorem 4.1 yields the operator $\mathfrak{A}_\varphi^{(0)}$ is an isomorphism if $\varphi < \alpha^*$.

For an application of these results, take $f \in L^2(U_\varphi)^3$, with the domain U_φ introduced in Section 2. Let $(u_\varphi, \pi_\varphi) \in W_0^{1,2}(U_\varphi)^3 \times L^2(U_\varphi)$ be the solution of variational problem (3.2), for $\Omega = U_\varphi$ (Theorem 3.1). In other words, the pair (u_φ, π_φ) is the weak solution in L^2 of problem (3.1), (1.2). Using the theory of the Stokes system on smooth domains (see [2]) and the fact that $\mathfrak{A}_\varphi^{(1)}$ is an isomorphism, we may conclude by a localization argument that the functions u_φ and π_φ have weak derivatives up to order 2 and 1, respectively, with

$$\int_{\mathbb{K}(\varphi)\cap\mathbb{B}_3(0,\epsilon_0)} |x|^2 \cdot \left(|\,D^2 u_\varphi(x)\,|^2 + |\,\nabla\pi_\varphi(x)\,|^2 \right) \, dx \quad < \quad \infty,$$

where ϵ_0 was introduced in (2.1). In addition, we recall that if $\varphi < \alpha^*$, the operator $\mathfrak{A}_\varphi^{(0)}$ is an isomorphism. This implies by another localization argument that $u_\varphi \in W^{2,2}(U_\varphi)^3$, $\pi_\varphi \in W^{1,2}(U_\varphi)$ in the case $\varphi < \alpha^*$; see [20, p. 70].

The preceding theory may be generalized in several respects. For example, results on higher regularity may be included, as is done systematically in [24]. Futhermore, there are variants of the preceding theory pertaining to Hölder spaces and to L^p-spaces with $p \neq 2$ (see [24, Section 3.6] for example). In [24, Chapter 4], Theorems 4.1 and 4.2 are extended to domains Ω having a finite number of non-regular boundary points with the property that a vicinity of each of these points is diffeomorphic to $\mathbb{K}(\varphi) \cap \mathbb{B}_2(0, \epsilon)$, for some $\varphi \in (0, \pi/2]$, $\epsilon > 0$. Moreover, one may admit certain cones which cut out non-smooth domains from the sphere S^2; see [6], [20]. It is further possible to treat domains with edges ([24, Chapter 8]) and domains with outlets to infinity ([24, Chapter 5], [26]). Finally some results carry over to the stationary, nonlinear Navier-Stokes equation ([20], [26]).

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Partial regularity and weighted energy estimates of global weak solutions of the Navier–Stokes system

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be an exterior domain or the whole space $\Omega = \mathbb{R}^3$. Given an external force $f = (f_1, f_2, f_3)$ and an initial value u_0 consider the instationary Navier-Stokes system

$$\begin{gathered} u_t - \Delta u + u \cdot \nabla u + \nabla p = f\,, \quad \operatorname{div} u = 0 \\ u(0) = u_0\,, \quad u_{|_{\partial\Omega}} = 0 \end{gathered} \tag{1.1}$$

where the fourth equation is absent when $\Omega = \mathbb{R}^3$; here $u = (u_1, u_2, u_3)$ denotes the unknown velocity field and p a corresponding pressure. It is well known that under weak assumptions on u_0 and f the Galerkin approximation method yields a global weak solution of (1.1) satisfying the energy inequality

$$\frac{1}{2}\,\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2 d\tau \le \frac{1}{2}\,\|u_0\|_2^2 + \int_0^t (f, u)_\Omega\, d\tau \tag{1.2}$$

for all $t \ge 0$, see e.g. [6, 10]. Using a different approximation procedure [7, 9] one obtains a so-called *suitable weak solution* satisfying the strong or *generalized energy inequality*

$$\frac{1}{2}\,\|u(t)\|_2^2 + \int_s^t \|\nabla u\|_2^2\, d\tau \le \frac{1}{2}\,\|u(s)\|_2^2 + \int_s^t (f, u)_\Omega\, d\tau \tag{1.3}$$

for a.a. $s > 0$, including $s = 0$, and all $t \ge s$.

In [3] Sohr and the author introduced a weighted version of (1.3) by using the radial weight function $r^\alpha = |x|^\alpha$, $0 \le \alpha \le 1$, for an exterior domain Ω with $0 \notin \overline{\Omega}$. Assuming that u_0 and f have finite norms in certain function spaces with weight r^α there exists a suitable weak solution u of (1.1) such that

$$\begin{aligned} &\frac{1}{2}\,\|r^{\alpha/2}u(t)\|_2^2 + \frac{1-\alpha}{1+\alpha}\int_s^t \|r^{\alpha/2}\nabla u\|_2^2\, d\tau \\ &\le \frac{1}{2}\,\|r^{\alpha/2}u(s)\|_2^2 + \int_s^t (r^\alpha f, u)_\Omega\, d\tau + E(u_0, f, \alpha)\,|t-s|^\beta\,. \end{aligned}$$

Here $\beta = 0$ for $0 \le \alpha < \frac{1}{2}$, but $\beta > 0$ for $\alpha \ge \frac{1}{2}$. Moreover for $\alpha = 1$ the term $\int_s^t \|r^{\alpha/2}\nabla u\|_2^2\, d\tau$ is shown to be bounded by $c|t-s|^{1/2}$; cf. Lemma 8.2 in [1] for the case