THE 3D STOKES SYSTEMS IN DOMAINS WITH CONICAL BOUNDARY POINTS

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We consider a bounded domain $\Omega \subset \mathbb{R}^3$ with connected boundary $\partial\Omega$. It is assumed that $\partial\Omega$ is smooth except at a point $x_0 \in \partial\Omega$. Near that point x_0, the domain Ω is to coincide with a right circular cone with vertex angle $2 \cdot \varphi$, where φ is an arbitrary number from $(0, \pi)$. We show that the Stokes system has a solution in Ω belonging to certain L^p-Sobolev spaces, with $p \geq 2$ if Dirichlet boundary conditions are prescribed, and with $p \leq 2$ for slip boundary conditions.

1 Introduction and Main Results

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with connected boundary $\partial\Omega$. Assume that $\partial\Omega$ is smooth everywhere except at a single point x_0. In a neighbourhood of this point, the domain Ω is supposed to coincide with a circular cone having vertex in x_0. Without loss of generality we may assume that x_0 coincides with the origin and the axis of the cone is directed along the x_3-axis. Let $2 \cdot \varphi$ be the vertex angle of the cone, where φ may be any value from $(0, \pi)$. Thus, setting

$$\mathbb{K}(\sigma) := \{ (\xi, r + |\xi| \cdot \cot \sigma) : \xi \in \mathbb{R}^2, r \in (0, \infty) \}, \quad \text{for } \sigma \in (0, \pi),$$

we have

$$\Omega \cap \{ x \in \mathbb{R}^3 : |x| < \epsilon \} = \mathbb{K}(\varphi) \cap \{ x \in \mathbb{R}^3 : |x| < \epsilon \}$$

for some $\epsilon > 0$. Let us consider the Stokes system

$$-\Delta u + \nabla \pi = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \tag{1}$$

with either Dirichlet or slip boundary conditions:

$$u \,|\, \partial\Omega = g, \tag{2}$$

$$T(u, \pi)(x) \cdot n^{(\Omega)}(x) = h(x) \quad \text{for } x \in \partial\Omega, \tag{3}$$

where the matrix-valued function $T(u, \pi)$ is defined by

$$T(u, \pi)_{jk} := D_j u_k + D_k u_j - \delta_{jk} \cdot \pi \quad (1 \leq j, k \leq 3).$$

Here and in the following, the function $n^{(\Omega)} : \partial\Omega \mapsto \mathbb{R}^3$ denotes the outward unit normal to Ω, and the symbols D_j, D_k denote partial derivatives.

According to Fabes, Kenig, Verchota [7], Theorem 3.9, Deuring, von Wahl [5], Lemma 5.7, the boundary value problem given in (1), (2) is solved by a pair of functions (u, π) satisfying the following relations <u>for $p = 2$</u>:

$$u \in W^{2,p}_{\text{loc}}(\Omega)^3 \cap W^{1/p-\epsilon,p}(\Omega)^3 \quad \text{for } \epsilon > 0, \quad \pi \in W^{1,p}_{\text{loc}}(\Omega), \tag{4}$$

under the assumptions $f \in L^2(\Omega)^3$, $g \in L^2(\partial\Omega)^3$ with

$$\int_{\partial\Omega} g \cdot n^{(\Omega)} \, d\Omega = 0. \tag{5}$$

Moreover, if $f \in L^2(\Omega)^3$, $g \in W^{1,2}(\partial\Omega)^3$ with (5), there is a solution (u, π) of (1), (2) satisfying the ensuing regularity conditions <u>for $p = 2$</u>:

$$u \in W^{2,p}_{\text{loc}}(\Omega)^3 \cap W^{1+1/p-\epsilon,p}(\Omega)^3,$$
$$\pi \in W^{1,p}_{\text{loc}}(\Omega) \cap W^{1/p-\epsilon,p}(\Omega) \quad \text{for } \epsilon > 0; \tag{6}$$

see [7], Theorem 4.15, and [5], loc. cit. Referring to Dahlberg, Kenig, Verchota [2], Theorem 4.6 and to [5], loc. cit., we further see there exists a solution (u, π) of (1), (3) fulfilling (6) <u>with $p = 2$</u>, provided f is given in $L^2(\Omega)^3$ and h in $L^2(\partial\Omega)^3$ with

$$\int_{\partial\Omega} h \cdot \psi \, d\Omega = 0 \quad \text{for } \psi \in Z(\partial\Omega), \tag{7}$$

where $Z(\partial\Omega)$ is defined as the set of all functions $\psi : \partial\Omega \mapsto \mathbb{R}^3$ such that

$$\psi(x) = a + b \times x \quad (x \in \partial\Omega), \quad \text{for some vectors } a, b \in \mathbb{R}^3.$$

The preceding results are valid not only for our special non-smoothly bounded domain Ω, but for any arbitrary Lipschitz domain.

Let us briefly consider the Poisson equation

$$\Delta U = F, \tag{8}$$

under either Dirichlet or Neumann boundary conditions:

$$U \,|\, \partial\Omega = G, \tag{9}$$

$$\sum_{j=1}^{3} D_j u(x) \cdot n_j^{(\Omega)}(x) = H(x) \quad \text{for } x \in \partial\Omega. \tag{10}$$

For boundary value problem (8), (9) and (8), (10), a L^p-theory could be developed, the main points of which may be stated as follows: If $p \in [2, \infty)$, $F \in L^p(\Omega)$, $G \in L^p(\partial\Omega)$, there is a solution U of (8), (9) with

$$U \in W^{2,p}_{\text{loc}}(\Omega) \cap W^{1/p-\epsilon,p}(\Omega) \quad \text{for } \epsilon > 0;$$

see Dahlberg, Kenig [1], Theorem 4.18. Moreover, for $p \in (1, 2]$, $F \in L^p(\Omega)$, $G \in W^{1,p}(\partial\Omega)$, boundary value problem (8), (9) may be solved by a function U with

$$U \in W^{2,p}_{\text{loc}}(\Omega) \cap W^{1+1/p-\epsilon,p}(\Omega) \quad \text{for } \epsilon > 0. \tag{11}$$

This is implied by Verchota [13], Theorem 5.1. Finally, let $p \in (1, 2]$, $F \in L^p(\Omega)$, $H \in L^p(\partial\Omega)$ with

$$\int_{\partial\Omega} H \, d\Omega - \int_{\Omega} F \, dx = 0.$$

Then it may be deduced from Theorem 4.18 in [1] that problem (8), (10) has a solution U which fulfills (11). These results are also valid for arbitrary Lipschitz domains.

We see that for solutions of the Poisson equation on Lipschitz domains, a rather complete L^p-theory is available, whereas for the Stokes system, only a L^2-theory could be developed. This, admittedly, was difficult enough, but this still raises the question what to expect if $p \neq 2$. Here we shall give a partial answer to this question: Restricting ourselves to the case of our special Lipschitz domain Ω, we shall prove that among the preceding three results on solutions of the Laplace equation, two are valid for solutions of the Stokes system as well. In order to be able to state our results, we fix a non-tangential direction field $m : \partial\Omega \mapsto \mathbb{R}^3$. This function should be smooth and satisfy the ensuing conditions:

$$|m(x)| = 1, \quad x - \kappa \cdot m(x) \in \Omega, \quad x + \kappa \cdot m(x) \in \mathbb{R}^3 \backslash \overline{\Omega}$$

for $x \in \partial\Omega$, $\kappa \in (0, \mathcal{D}_1)$, and

$$\left| x + \kappa \cdot m(x) - x' - \kappa' \cdot m(x') \right| \geq \mathcal{D}_2 \cdot \left(|x - x'| + |\kappa - \kappa'| \right)$$

for $x, x' \in \partial\Omega$, $\kappa, \kappa' \in (-\mathcal{D}_1, \mathcal{D}_1)$, with certain constants $\mathcal{D}_1, \mathcal{D}_2 > 0$. Some indications on how to construct such a function are given in [12]. Now our results may be stated as follows:

Theorem 1 *Let $p \in [2, \infty)$, $f \in L^p(\Omega)^3$, $g \in L^p(\partial\Omega)^3$ with (5). Then there is a solution (u, π) of (1), (2) satisfying (4), with u taking boundary values in this sense:*

$$\int_{\partial\Omega} \Big| g(x) - u\big(x - \kappa \cdot m(x)\big)\Big|^p \, d\Omega(x) \longrightarrow 0 \qquad (\kappa \downarrow 0).$$

Moreover, for any $\epsilon > 0$, there is a constant $C > 0$ only depending on Ω, p and ϵ such that

$$\| u \|_{1/p - \epsilon, p} \le C \cdot \big(\| f \|_p + \| g \|_p \big).$$

Theorem 2 *Let $p \in (1, 2]$, $f \in L^p(\Omega)^3$, $h \in L^p(\partial\Omega)^3$ with (7). Then boundary value problem (1), (3) is solved by a pair of functions (u, π) fulfilling (6), with (3) being satisfied in the following sense:*

$$\int_{\partial\Omega} \Big| h(x) - T(u, \pi)\big(x - \kappa \cdot m(x)\big) \cdot n^{(\Omega)}(x) \Big|^p \, d\Omega(x) \longrightarrow 0 \quad (\kappa \downarrow 0).$$

If $\epsilon > 0$, there is a constant $C > 0$ only depending on Ω, p and ϵ such that

$$\| u \|_{1 + 1/p - \epsilon, p} + \| \pi \|_{1/p - \epsilon, p} \le C \cdot \big(\| f \|_p + \| h \|_p \big).$$

Concerning references dealing with the Stokes system on domains with conical boundary points, we know of [9], [10], [3], [8]. No results on solutions in L^p-spaces are stated in these references, although such results may be implied by them. However, it seems the preceding theorems do not follow from these papers. In any case, the method of proof used in these articles, that is, estimating the eigenvalues of certain operator pencils, is completely different from our approach. In fact, we recur to the method of integral equations, which was applied in [1], [2], [7] and [13] as well.

2 Proof of Theorem 1 and 2

For $j, k \in \{1, 2, 3\}$, define

$$E_{jk}(z) := (8 \cdot \pi)^{-1} \cdot \big(\delta_{jk} \cdot |z|^{-1} + z_j \cdot z_k \cdot |z|^{-3} \big),$$
$$E_{4k}(z) := (4 \cdot \pi)^{-1} \cdot z_k \cdot |z|^{-3} \quad \text{for } z \in \mathbb{R}^3 \backslash \{0\},$$
$$\mathcal{D}_{jkl} := D_j E_{kl} + D_k E_{jl} - \delta_{jk} \cdot E_{4l}.$$

This means the matrix-valued function $(E_{jk})_{1 \le j \le 4, 1 \le k \le 3}$ is a fundamental solution of the Stokes system (1).

For $q \in (1, \infty)$, $\tau \in \{-1, 1\}$, $B \subset \mathbb{R}^3$ open, we define the operators

$$\Lambda(\tau, q, B), \Lambda^*(\tau, q, B) : L^q(\partial B)^3 \mapsto L^q(\partial B)^3$$

by setting

$$\Lambda(\tau, q, B)(\Phi)(x) := (\tau/2) \cdot \Phi(x) + \Big(\int_{\partial B} \sum_{j,k=1}^{3} \mathcal{D}_{jkl}(x - y) \cdot n_k^{(B)}(y) \cdot \Phi_j(y) \, dB(y) \Big)_{1 \le l \le 3}$$

and

$$\Lambda^*(\tau, q, B)(\Phi)(x) := (\tau/2) \cdot \Phi(x) - \Big(\int_{\partial B} \sum_{j,k=1}^{3} \mathcal{D}_{jkl}(x - y) \cdot n_k^{(B)}(x) \cdot \Phi_j(y) \, dB(y) \Big)_{1 \le l \le 3},$$

where $\Phi \in L^q(\partial B)^3$, $x \in \partial B$, and $n^{(B)} : \partial B \mapsto \mathbb{R}^3$ is the outward unit normal to ∂B. Of course, these definitions only make sense if the preceding integrals exist and, when considered as a function of x, belong to $L^q(\partial B)^3$. These conditions are met if B is a bounded set with smooth boundary (see [6], Lemma 4.7, Lemma 5.1), or if $B = \Omega$ or $B = \mathbb{K}(\varphi)$ (see [4], Lemma 6.2).

Let $p \in (1, \infty)$ be fixed from now on. We define the volume potentials $R(f)$, $S(f)$ by setting

$$R(f)(x) := \Big(\int_\Omega \sum_{k=1}^{3} E_{jk}(x - y) \cdot f_k(y) \, dy \Big)_{1 \le j \le 3},$$

$$S(f)(x) := \int_\Omega \sum_{k=1}^{3} E_{4k}(x - y) \cdot f_k(y) \, dy \quad \text{for } f \in L^p(\Omega)^3, \ x \in \mathbb{R}^3.$$

According to [6], Satz 1.4, it holds

$$R(f) \,|\, \Omega \in W^{2,p}(\Omega)^3, \quad S(f) \,|\, \Omega \in W^{1,p}(\Omega),$$

and there is a constant $C = C(\Omega, p) > 0$ such that

$$\big\| R(f) \,|\, \Omega \big\|_{2,p} + \big\| S(f) \,|\, \Omega \big\|_{1,p} \le C \cdot \|f\|_p \qquad \text{for } f \in L^p(\Omega)^3. \tag{12}$$

This implies that

$$\big\| R(f) \,|\, \partial\Omega \big\|_{1,p} \le C \cdot \|f\|_p \qquad \text{for } f \in L^p(\Omega)^3, \tag{13}$$

with some constant $C = C(\Omega, p) > 0$. By [6], Satz 1.4, it further holds

$$-\Delta R(f) + \nabla S(f) = f, \quad \operatorname{div} R(f) = 0.$$

For $\Phi \in L^p(\partial\Omega)^3$, we define the single-layer potentials $V(\Phi)$, $Q(\Phi)$ by

$$V(\Phi)(x) := \Big(\int_{\partial\Omega} \sum_{k=1}^{3} E_{jk}(x-y) \cdot \Phi_k(y) \; d\Omega(y) \Big)_{1 \le j \le 3},$$

$$Q(\Phi)(x) := \int_{\partial\Omega} \sum_{k=1}^{3} E_{4k}(x-y) \cdot \Phi_k(y) \; d\Omega(y) \qquad (x \in \mathbb{R}^3 \backslash \partial\Omega).$$

Moreover, we introduce the double-layer potentials $W(\Phi)$, $\Pi(\Phi)$ by

$$W(\Phi)(x) := \Big(\int_{\partial\Omega} \sum_{j,k=1}^{3} \mathcal{D}_{jkl}(x-y) \cdot n_k^{(\Omega)}(y) \cdot \Phi_j(y) \; d\Omega(y) \Big)_{1 \le l \le 3},$$

$$\Pi(\Phi)(x) := \int_{\partial\Omega} 2 \cdot \sum_{j,k=1}^{3} D_j E_{4k}(x-y) \cdot n_k^{(\Omega)}(y) \cdot \Phi_j(y) \; d\Omega(y)$$

for $x \in \mathbb{R}^3 \backslash \partial\Omega$. Note that for $(A, B) \in \{ (V(\Phi), Q(\Phi)), (W(\Phi), \Pi(\Phi)) \}$, we have

$$A_j, B \in C^\infty(\mathbb{R}^3 \backslash \partial\Omega) \; (1 \le j \le 3), \qquad -\Delta A + \nabla B = 0, \quad \operatorname{div} A = 0.$$

Moreover, for any $\epsilon > 0$, it holds

$$V(\Phi) \,|\, \Omega \in W^{1+1/p-\epsilon, p}(\Omega)^3, \qquad W(\Phi)_j \,|\, \Omega, \; Q(\Phi) \,|\, \Omega \in W^{1/p-\epsilon, p}(\Omega)$$

for $j \in \{1, 2, 3\}$, $\Phi \in L^p(\partial\Omega)^3$. In addition, there is a constant $C = C(\Omega, p, \epsilon) > 0$ with

$$\| V(\Phi) \,|\, \Omega \|_{1+1/p-\epsilon, p} + \| W(\Phi) \,|\, \Omega \|_{1/p-\epsilon, p} + \| Q(\Phi) \,|\, \Omega \|_{1/p-\epsilon, p} \le C \cdot \|\Phi\|_p \quad \text{for } \Phi \in L^p(\partial\Omega)^3. \tag{14}$$

The latter result was proved in [5], Lemma 5.7, for the case $p = 2$. A generalization to the case $p \neq 2$ is immediate. As for the behaviour of our boundary potentials near $\partial\Omega$, the following results ("jump relations") hold true:

$$\int_{\partial\Omega} \Big| \Lambda(1, p, \Omega)(\Phi)(x) - W(\Phi)(x - \kappa \cdot m(x)) \Big|^p \; d\Omega(x) \longrightarrow 0,$$

$$\int_{\partial\Omega} \Big| \Lambda^*(-1, p, \Omega)(\Phi)(x) - T(V(\Phi), Q(\Phi))(x - \kappa \cdot m(x)) \cdot n^{(\Omega)}(x) \Big|^p \; d\Omega(x) \longrightarrow 0$$

for $\kappa \downarrow 0$, if $\Phi \in L^p(\partial\Omega)^3$. These relations are well-known and may, for example, be deduced from [6], Satz 4.1, Lemma 4.8, combined with [4], Theorem 9.1.

Now we define for $f \in L^p(\Omega)^3$, $\Phi \in L^p(\partial\Omega)^3$

$$u(f, \Phi) := (R(f) + W(\Phi)) \,|\, \Omega, \quad \pi(f, \Phi) := (S(f) + \Pi(\Phi)) \,|\, \Omega,$$

$$v(f, \Phi) := (R(f) + V(\Phi)) \,|\, \Omega, \quad \varrho(f, \Phi) := (S(f) + Q(\Phi)) \,|\, \Omega.$$

Collecting our previous results, we find that the pair of functions

$$(u, \pi) = (u(f, \Phi), \pi(f, \Phi))$$

satisfies the regularity conditions stated in (4), and

$$(u, \pi) = (v(f, \Phi), \varrho(f, \Phi))$$

those in (6), provided $f \in L^p(\Omega)^3$, $\Phi \in L^p(\partial\Omega)^3$. Moreover, both of these pairs (u, π) solve the Stokes system (1), and for $\epsilon > 0$, there is a constant $C = C(\Omega, p, \epsilon) > 0$ with

$$\| v(f, \Phi) \,|\, \Omega \|_{1+1/p-\epsilon, p} + \| u(f, \Phi) \,|\, \Omega \|_{1/p-\epsilon, p} + \| \varrho(f, \Phi) \,|\, \Omega \|_{1/p-\epsilon, p} \le C \cdot (\|f\|_p + \|\Phi\|_p),$$

as follows from (12) and (14). We finally observe that

$$\int_{\partial\Omega} \Big| \Lambda(1, p, \Omega)(\Phi)(x) + R(f)(x) - u(f, \Phi)(x - \kappa \cdot m(x)) \Big|^p \; d\Omega(x) \longrightarrow 0,$$

$$\int_{\partial\Omega} \Big| \Lambda^*(-1, p, \Omega)(\Phi)(x) + T(R(f), S(f))(x) \cdot n^{(\Omega)}(x) - T(v(f, \Phi), \varrho(f, \Phi))(x - \kappa \cdot m(x)) \cdot n^{(\Omega)}(x) \Big|^p \; d\Omega(x) \longrightarrow 0$$

for $\kappa \downarrow 0$. Thus, referring to (13), we see that Theorem 1 and 2 are now reduced to the ensuing claims pertaining to certain integral operators on $\partial\Omega$:

Theorem 3 *Set* $L^p_n(\partial\Omega) := \{ g \in L^p(\partial\Omega)^3 \,:\, g$ *fulfills (5)*$\}$. *Then, if* $p \geq 2$, *there is a subspace* F_p *of* $L^p(\partial\Omega)^3$ *with codimension 1 such that the mapping*

$$A_p : F_p \mapsto L^p_n(\partial\Omega), \quad A_p(\Phi) := \Lambda(1, p, \Omega)(\Phi),$$

is bounded invertible.

Theorem 4 *Set* $L^p_\Psi(\partial\Omega) := \{ h \in L^p(\partial\Omega)^3 \,:\, h$ *fullfils (7)*$\}$. *Then, if* $p \leq 2$, *there is a subspace* G_p *of* $L^p(\partial\Omega)^3$ *with codimension 6 such that the mapping*

$$B_p : G_p \mapsto L^p_\Psi(\partial\Omega), \quad B_p(\Phi) := \Lambda^*(-1, p, \Omega)(\Phi),$$

is bounded invertible.

In order to establish these theorems, we first remark that the operators

$$\Lambda(\tau, q, \Omega) \text{ and } \Lambda^*(\tau, q, \Omega) \quad (\tau \in \{-1, 1\}, q \in (1, \infty))$$

are bounded; see [4], Lemma 6.2. Furthermore, the operator $\Lambda(\tau, p, \mathbb{K}(\sigma))$ is Fredholm for $\tau \in \{-1, 1\}$, $q \in [2, \infty)$, $\sigma \in (0, \pi)$; see [4], Theorem 13.1, and note that the mappings

$$\Lambda(\tau, q, \mathbb{K}(\sigma)) \text{ and } \Lambda(-\tau, q, \mathbb{K}(\pi - \sigma)) \quad (\sigma \in (0, \pi/2], q \in (1, \infty))$$

have the same Fredholm properties. Define the operators

$$A(\tau, q, \sigma, r), A^*(\tau, q, \sigma, r) : L^q(\mathbb{B}(r))^3 \mapsto L^q(\mathbb{B}(r))^3$$

by setting

$$A(\tau, q, \sigma, r)(\Phi)(\xi) := (\tau/2) \cdot \Phi(\xi) + \Big(\int_{\mathbb{B}(r)} \sin^{-1}(\sigma) \cdot \sum_{j,k=1}^{3} \mathcal{D}_{jkl}(g^{(\sigma)}(\xi) - g^{(\sigma)}(\eta)) \cdot (n_k^{(\sigma)} \circ g^{(\sigma)})(\eta) \cdot \Phi_j(\eta) \, d\eta \Big)_{1 \leq l \leq 3},$$

and

$$A^*(\tau, q, \sigma, r)(\Phi)(\xi) := (\tau/2) \cdot \Phi(\xi) - \Big(\int_{\mathbb{B}(r)} \sin^{-1}(\sigma) \cdot \sum_{j,k=1}^{3} \mathcal{D}_{jkl}(g^{(\sigma)}(\xi) - g^{(\sigma)}(\eta)) \cdot (n_k^{(\sigma)} \circ g^{(\sigma)})(\xi) \cdot \Phi_j(\eta) \, d\eta \Big)_{1 \leq l \leq 3},$$

for $\tau \in \{-1, 1\}$, $\sigma \in (0, \pi)$, $r > 0$, $q \in (1, \infty)$, $\Phi \in L^q(\mathbb{B}(r))^3$, $\xi \in \mathbb{B}(r)$, with

$$g^{(\sigma)}(\eta) := (\eta_1, \eta_2, |\eta| \cdot \cot\sigma) \quad \text{for } \eta \in \mathbb{R}^2, \qquad \mathbb{B}(r) := \{\xi \in \mathbb{R}^2 : |\xi| < r\},$$

and $n^{(\sigma)}$ the outward unit normal to $\mathbb{K}(\sigma)$. According to [4], Theorem 12.3, Corollary 6.6, the operator $A(\tau, q, \sigma, r)$ is Fredholm for $q \in [2, \infty)$ and for τ, σ, r as before. Since the operators

$$A(\tau, q, \sigma, r), \quad A^*(\tau, (1 - 1/q)^{-1}, \sigma, r)$$

are adjoint, the mapping $A^*(\tau, q, \sigma, r)$ must be Fredholm for $q \in (1, 2]$. Now Lemma 6.17 in [4] yields

$$\operatorname{index} A^*(\tau, q, \sigma, r) = 0 \quad \text{for } \tau \in \{-1, 1\}, \ q \in (1, 2], \ \sigma \in (0, \pi), \ r > 0.$$

This in turn implies that $\Lambda^*(\tau, q, \Omega)$ is Fredholm with index 0 ($\tau \in \{-1, 1\}$, $q \in (1, 2]$); see Lemma 13.7 in [4] and its proof. But the operators

$$\Lambda^*(\tau, q, \Omega) \quad \text{and} \quad \Lambda(\tau, (1 - 1/q)^{-1}, \Omega) \tag{15}$$

are adjoint, hence we may conclude $\Lambda(\tau, q, \Omega)$ is Fredholm with index 0, for $\tau \in \{-1, 1\}$, $q \geq 2$. It is well known that

$$\operatorname{kernel} \Lambda^*(1, 2, \Omega) = \operatorname{span}\{n^{(\Omega)}\}, \qquad \dim\operatorname{kernel} \Lambda^*(-1, 2, \Omega) = 6,$$
$$\operatorname{kernel} \Lambda(-1, 2, \Omega) = Z(\partial\Omega), \qquad \dim\operatorname{kernel} \Lambda(1, 2, \Omega) = 1;$$

see Lemma 13.8 in [4] and Theorem 4.6 in [2]. From this we deduce the relations

$$\operatorname{kernel} \Lambda^*(1, q, \Omega) \supset \operatorname{span}\{n^{(\Omega)}\}, \qquad \dim\operatorname{kernel} \Lambda^*(-1, q, \Omega) \geq 6,$$

for $q \leq 2$, and

$$\operatorname{kernel} \Lambda(-1, q, \Omega) = Z(\partial\Omega), \qquad \dim\operatorname{kernel} \Lambda(1, q, \Omega) \leq 1;$$

for $q \geq 2$. Since the operators in (15) have index 0 for $q \in (1, 2]$, and because

$$\operatorname{index} \Lambda(1, q, \Omega) = \dim\operatorname{kernel} \Lambda(1, q, \Omega) - \dim\operatorname{kernel} \Lambda^*(1, (1 - 1/q)^{-1}, \Omega)$$

for $q \geq 2$, and

$$\operatorname{index} \Lambda^*(-1, q, \Omega) = \dim\operatorname{kernel} \Lambda^*(-1, q, \Omega) - \dim\operatorname{kernel} \Lambda(-1, (1 - 1/q)^{-1}, \Omega)$$

for $q \leq 2$, we may now conclude

$$\text{kernel}\,\Lambda^*(1, q, \Omega) = \text{span}\,\{n^{(\Omega)}\}, \quad \dim\text{kernel}\,\Lambda^*(-1, q, \Omega) = 6, \tag{16}$$

for $q \leq 2$, and

$$\text{kernel}\,\Lambda(-1, q, \Omega) = Z(\partial\Omega), \quad \dim\text{kernel}\,\Lambda(1, q, \Omega) = 1; \tag{17}$$

for $q \geq 2$. Now Theorem 3 and 4 follow by the theory of Fredholm operators.

We mention that due to (16) and (17), boundary value problems (1), (2) and (1), (3) may be solved in the exterior domain $\mathbb{R}\backslash\overline{\Omega}$ as well. This may be shown by the same arguments as used in the case of a smoothly bounded domain. We refer to [6], p. 187-196, where results from [11] were worked out in detail.

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WEIGHTED ESTIMATES FOR THE OSEEN EQUATIONS AND THE NAVIER-STOKES EQUATIONS IN EXTERIOR DOMAINS*

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In this paper we develop weighted L^q-estimates for the linear Oseen equations in $\mathbb{R}^n, n \geq 2$, and extend them by perturbation to the nonlinear case. In case $n = 3$ these estimates are used to prove decay properties of solutions with finite Dirichlet integral and nonzero velocity at infinity of the stationary Navier-Stokes equations in exterior domains. It follows that these solutions are PR-solutions in the sense of Finn. This yields a short new proof of Babenko's result[1] (for another approach see Galdi[8,9,10]) and extends it to a larger class of forces with unbounded support. Furthermore this method avoids the use of the explicit integral representation of the solution.

1 Introduction

In an exterior domain $\Omega \subseteq \mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$ of class $C^2, n \geq 2$, consider the stationary Navier-Stokes system

$$-\nu\Delta u + u \cdot \nabla u + \nabla p = f, \quad \text{div}\, u = 0 \text{ in } \Omega,$$
$$u|_\Gamma = u_\Gamma, \quad u \to u_\infty \quad \text{as } |x| \to \infty; \tag{1.1}$$

here $\nu > 0$ is a constant, $f = (f_1, \ldots, f_n) : \Omega \to \mathbb{R}^n$, $u_\Gamma : \Gamma \to \mathbb{R}^n$, and $u_\infty \in \mathbb{R}^n$ are the prescribed data while the velocity field $u = (u_1, \ldots, u_n)$ and the pressure p are the desired solutions representing a flow within Ω. It is well known[15,16] that for a suitable right-hand side (1.1) has a weak solution u with

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