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Calculating Stokes Flows Around a Sphere: Comparison of Artificial Boundary Conditions.

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SUNTO - Qui calcoliamo il moto esterno di Stikes attorno ad una sfera in $\boldsymbol{R}^3$. A questo scopo, il dominio esterno viene troncato per mezzo di una sfera di raggio opportunamente grande ed assegnamo una condizione artificiale locale sulla sfera troncante. Due diverse condizioni di questo tipo vengono confrontate.

ABSTRACT - We calculate exterior Stokes flows around a ball in R^3. To this end, we truncate the exterior domain by a large sphere and prescribe a local artificial boundary condition on the truncating sphere. Two such conditions are compared.

Let Ω be a bounded domain in R^3. Consider the Stokes system in the exterior domain $\Omega^c := R^3 \setminus \overline{\Omega}$, with a Dirichlet boundary condition on $\partial\Omega$, and a decay condition near infinity:

$$(1)\quad -\Delta u + \nabla \pi = f, \quad \operatorname{div} u = 0 \text{ in } \Omega^c, \quad u \,|\, \partial\Omega = b\,, \quad u(x) \to 0 \; (|x| \to \infty).$$

If Ω is at least Lipschitz bounded, and $b \in H^{1/2}(\partial\Omega)^3$, $f \in L^2(\Omega^c)^3$, problem (1) has a unique weak solution (u, π) such that

$$u \in L^6(\Omega^c)^3, \qquad \nabla u \in L^2(\Omega^c)^9, \qquad \pi \in L^2(\Omega^c);$$

see [6], Theorem V.2.1, Theorem II.5.1. This solution will be called «exterior flow». We are interested in numerically calculating this flow and observe

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how the error, measured in a suitable norm, depends on the refinement depth and the computational domain.

To cite some related papers, we mention [12], where the boundary element method is used to calculate exterior Stokes flows. This reference presents error estimates of approximate solutions to certain boundary integral equations. However, as concerns approximations of the exterior Stokes flow itself, the article [12] does not deal — neither theoretically nor computationally — with error estimates which are valid in a vicinity of Ω. In [16], [17], the Stokes system in 2D exterior is solved numerically, by means of a coupled finite element and boundary element method. Article [1] treats the computation of approximate solutions for the Poisson equation in 3D exterior domains. References [16], [17] and [1] present error estimates and computations of the flow field in a neighbourhood of Ω. The approach used in [1] is analogous to the one which will be exploited here (local artificial boundary conditions combined with finite element methods). As far as we know, this method was first presented by Goldstein [10], who studied the Poisson equation, and was first applied to the Stokes system in [8] and [3], but without test computations.

As starting point of this method, we consider the Stokes system in the truncated exterior domain $\Omega_R := B_R \setminus \overline{\Omega}$, for $R>0$ with $\overline{\Omega} \subset B_R$, where B_R denotes an open ball centered in the origin, with radius $R>0$. On the truncating sphere ∂B_R, we prescribe a local artificial boundary condition $\mathcal{L}_R(v_R, \varrho_R) = 0$, and thus obtain the following boundary value problem:

$$(2) \qquad -\Delta v_R + \nabla \varrho_R = f \mid \Omega_R, \quad \operatorname{div} v_R = 0 \text{ in } \Omega_R, \quad v_R \mid \partial\Omega = 0,$$
$$\mathcal{L}_R(v_R, \varrho_R)(x) = 0 \quad \text{for } x \in \partial B_R.$$

A solution (v_R, ϱ_R) of problem (2) may be considered an approximation of the exterior flow (u, π). The corresponding error («truncation error») depends on the choice of the operator $\mathcal{L}_R$. We consider two such choices,

$$\mathcal{L}_R^{(1)}(v, \varrho)(x) := 3/(2\cdot R)\cdot v(x) + \left(\sum_{k=1}^{3} (D_k v_j - (1/2)\cdot D_j v_k - \delta_{jk}\cdot\varrho)(x)\cdot(x_k/R)\right)_{1\le j\le 3} \quad \text{for } x \in \partial B_R,$$

and

$$\mathcal{L}_R^{(2)}(v, \varrho) := v(x) \quad \text{for } x \in \partial B_R,$$

where v is a vector-valued and ϱ a scalar function with the property that v, ∇v and ϱ are defined almost everywhere on ∂B_R. Note that in the case $\mathcal{L}_R = \mathcal{L}_R^{(2)}$, the condition $\mathcal{L}_R(v_R, \varrho_R) = 0$ takes the form $v_R \mid \partial B_R = 0$ (Dirichlet boundary condition). For each of our two choices of $\mathcal{L}_R$, we may write problem (2) in variational form. In fact, if $R \in (0, \infty)$ with $\overline{\Omega} \subset B_R$, we set

$$W_R^{(1)} := \{v \in H^1(\Omega_R)^3 : v \mid \partial\Omega = 0\}, \qquad W_R^{(2)} := H_0^1(\Omega_R)^3,$$

$$M_R^{(1)} := L^2(\Omega_R), \qquad M_R^{(2)} := \left\{\varrho \in L^2(\Omega_R) : \int_{\Omega_R} \varrho\, dx = 0\right\},$$

$$(3) \qquad a_R^{(1)}(v, w) := 3/(2\cdot R)\cdot \int_{\partial B_R} v\cdot w\, do_x + \int_{\Omega_R} \sum_{j,k=1}^{3} (D_j v_k \cdot D_j w_k - (1/2)\cdot D_j v_k \cdot D_k w_j)\, dx,$$

$$(4) \qquad a_R^{(2)}(v, w) := \int_{\Omega_R} \sum_{j,k=1}^{3} D_j v_k \cdot D_j w_k\, dx,$$

$$(5) \qquad \beta_R(v, \varrho) := -\int_{\Omega_R} \operatorname{div} v \cdot \varrho\, dx \quad \text{for } v, w \in H^1(\Omega_R)^3,\ \varrho \in L^2(\Omega_R),\ j \in \{1, 2\}.$$

Then the variational problem

(Va1) Find $(v, \varrho) = (v_R^{(j)}, \varrho_R^{(j)}) \in H^1(\Omega_R)^3 \times M_R^{(j)}$ such that

$$v \mid \partial B_R = 0 \text{ in the case } j = 2, \qquad v \mid \partial\Omega = b,$$
$$a_R^{(j)}(v, w) + \beta_R(w, \varrho) = \int_{\Omega_R} f\cdot w\, dx \quad \text{for } w \in W_R^{(j)},$$
$$\beta_R(v, \sigma) = 0 \quad \text{for } \sigma \in M_R^{(j)},$$

for $j \in \{1, 2\}$, is a weak form of problem (2) with $\mathcal{L}_R = \mathcal{L}_R^{(j)}$, and it admits a unique solution $(v_R^{(j)}, \varrho_R^{(j)})$ (if $j = 2$ under the additional assumption that b has mean value zero). In the case $j = 1$, $b = 0$, this result was proved in [3] [Section 4]. The case $j = 2$, $b = 0$ was treated in [7] p. 80-82 and in [2]. The (obvious) arguments for dealing with the case $b \neq 0$ may be found in [5], where an analogous problem for the Oseen system is considered.

Concerning estimates of the truncation error, let us suppose for a moment that $b = 0$ and that there is some $\varepsilon > 0$ with $f(x) = O(|x|^{-4-\varepsilon})$ for $|x| \to \infty$. Then, for $j = 1$ and for a fixed number $S \in (0, \infty)$ with $\overline{\Omega} \subset B_S$, there exists a constant $C = C(S, \Omega) > 0$ with

$$(6) \qquad \|u \mid \Omega_S - v_R^{(1)} \mid \Omega_S\|_2 \le C\cdot R^{-3/2} \quad \text{for } R \in (S, \infty).$$

For this estimate and similar ones of π and of the gradient of u, we refer to [3] Section 5. Related results were shown in [14].

If $j = 2$ (Dirichlet boundary condition on ∂B_R), it should be expected that the factor $R^{-3/2}$ in (6) has to be replaced by $R^{-1/2}$; compare [13], [14].

To test our artificial boundary conditions, we suppose from now on that Ω is a ball with center in the origin and unit radius: $\Omega = B_1$. We are interested in calculating exterior flows in the neighborhood $B_2 \setminus \overline{\Omega} = B_2 \setminus \overline{B_1}$ of B_1, so we set $S = 2$. As radius R of the truncating sphere ∂B_R, we take the values $R = S \cdot 2^J = 2^{J+1}$, with $J \in \{0, 1, 2, 3\}$. The functions f and b will be chosen later. Under these assumptions, we will calculate approximate solutions of (Va1) and consider them as approximations of the exterior Stokes flow corresponding to the given functions b and f.

Since problem (Va1) is a mixed variational problem, it may be discretized by mixed finite element methods. In order to specify such discretisations, we first have to choose a decomposition of $B_{2^{J+1}} \setminus \overline{B_1}$. In fact, we shall not triangulate this set, but polyhedrons which approximate it. The starting point for generating these polyhedrons and their triangulations is the set $[-2 \cdot 3^{-1/2}, 2 \cdot 3^{-1/2}]^3 \setminus (-3^{-1/2}, 3^{-1/2})^3$. We decompose it into six congruent hexahedrons, with vertices belonging to the set

$$\{3^{-1/2} \cdot (\sigma_1 \cdot \tau, \sigma_2 \cdot \tau, \sigma_3 \cdot \tau) : \sigma_1, \sigma_2, \sigma_3 \in \{-1, 1\}, \tau \in \{1, 2\}\}.$$

When we want to perform N refinement steps, with $N \geq 1$, we begin by replacing each of these six hexahedrons by 8 smaller ones. In order to explain how these are chosen, let us consider the example of the hexahedron having vertices in the set

$$\{3^{-1/2} \cdot (\sigma, \tau, 1) : \sigma, \tau \in \{-1, 1\}\} \cup \{3^{-1/2} \cdot (\sigma, \tau, 2) : \sigma, \tau \in \{-2, 2\}\}.$$

This hexahedron is replaced by 8 smaller ones with vertices belonging to the set

$$\bigcup_{s \in \{1, 3/2, 2\}} \{3^{-1/2} \cdot (\sigma, \tau, s) : \sigma, \tau \in \{-s, s\}\} \cup$$

$$\cup \{2^{-1/2} (\sigma, \tau, s) : \sigma, \tau \in \{0, s, -s\}\} \cup \{(0, 0, r) : r \in \{1, 3/2, 2\}\},$$

and with the point $(0, 0, 3/2)$ as a common vertex. This procedure of replacing each hexahedron by 8 smaller ones is repeated N times. We obtain a set of hexahedrons decomposing a polyhedron which approximates $B_2 \setminus \overline{B_1}$. If the radius R of the truncating sphere is given by $R = 2^{J+1}$, for some $J \in \mathbb{N}$, we approximate the set $B_{2^{J+1}} \setminus \overline{B_1}$ by the union of the hexahedrons already generated, plus the union of the sets obtained by multiplying these hexahedrons by the factors $2, \ldots, 2^J$.

Finally, each hexahedron H obtained in this way is decomposed into 24 tetrahedrons, with the midpoint of H (that is, the algebraic mean of the eight vertices of H) being a common vertex of all these tetrahedrons. Moreover the midpoint of each face F of H is chosen as a common vertex of four of these tetrahedrons. If all four vertices of F are located on $\partial B_{2^{J+1}}$ or ∂B_1, respectively, this midpoint is shifted in radial direction onto ∂B_1 respectively $B_{2^{J+1}}$.

We define the polyhedron $P_{N,J}$ as the interior of the union of the closed tetrahedrons obtained in this way. These tetrahedrons constitute a decomposition of $P_{N,J}$ which is denoted by $\mathcal{C}_{N,J}$. Thus

$$\overline{P_{N,J}} = \cup \{K : K \in \mathcal{C}_{N,J}\}, \qquad \mathring{K} \cap \mathring{K}' = \emptyset \text{ for } K, K' \in \mathcal{C}_{N,J}, \qquad K \neq K',$$

where $\mathring{K}$ stands for the interior of a tetrahedron $K \in \mathcal{C}_{N,J}$. Note that we may distinguish two parts of the boundary of $P_{N,J}$: an outer boundary $\partial P_{N,J}^{(o)}$ consisting of triangular faces with vertices on $\partial B_{2^{J+1}}$, and an inner boundary $\partial P_{N,J}^{(i)}$ with vertices on ∂B_1.

We further remark that if K_1 and K_2 are two elements of $\mathcal{C}_{N,J}$, and if K_2 is twice as distant from the origin as K_1, then the diameter of K_2 is about double that of K_1. It was shown by Goldstein [10], [11] for the case of the Poisson equation that the error rates of finite element methods do not worsen when grids of this type are used. The number of gridpoints of $\mathcal{C}_{N,J}$ is bounded by $C \cdot N^3 \cdot J$, with a constant C independent of N and J; see [11], p. 166, [3], Lemma 6.3. Thus, as concerns the critical question of how the truncating radius $R = 2^{J+1}$ influences the number of gridpoints of $\mathcal{C}_{N,J}$, this influence is represented by the factor J. This means that due to the grid adaption just described, the complexity of the problem does not increase too strongly with growing values of J.

The mixed variational problem (Va1) relates to functions on $\Omega_R = B_R \setminus \overline{\Omega} = B_{2^{J+1}} \setminus \overline{B_1}$, but not to functions on the set $P_{N,J}$. Thus we discretize problem (Va1) in a way which might be considered as a non-conforming method. We choose a discretisation based on the Mini element. In fact, let $\mathcal{P}_1(K)$ denote the set of all polynomials over K of degree up to one, for a tetrahedron K in R^3. We further write b_K for the bubble function over K given by the fourth-order polynomial which may be transformed by an affine mapping into the function $x_1 \cdot x_2 \cdot x_3 \cdot (1 - x_1 - x_2 - x_3)$ on the usual standard tetrahedron in R^3. Then we put for $N, J \in \mathbb{N}_0$:

$$V_{N,J} := \{w \in C^0(\overline{P_{N,J}})^3 : w | K \in \operatorname{span}(\mathcal{P}_1(K) \cup \{b_K\})^3 \text{ for } K \in \mathcal{C}_{N,J}\},$$

$$V_{N,J}^{(1)} := \{w \in V_{N,J} : w | \partial \mathcal{P}_{N,J}^{(i)} = 0\},$$

$$V_{N,J}^{(2)} := \{w \in V_{N,J} : w | \partial P_{N,J} = 0\},$$

$$M_{N,J}^{(1)} := \{\varrho \in C^0(\overline{P_{N,J}}) : \varrho | K \in \mathcal{P}_1(K) \text{ for } K \in \mathcal{C}_{N,J}\},$$

$$M_{N,J}^{(2)} := \left\{\varrho \in C^0(\overline{P_{N,J}}) : \varrho | K \in \mathcal{P}_1(K) \text{ for } K \in \mathcal{C}_{N,J}, \int_{\mathcal{P}_{N,J}} \varrho \, dx = 0\right\}.$$

For $j \in \{1, 2\}$, $N, J \in \mathbb{N}_0$, $v, w \in H^1(P_{N,J})^3$, $\varrho \in L^2(P_{N,J})$, we define $\alpha^{(j)}_{N,J}(v, w)$ and $\beta_{N,J}(w, \sigma)$ as in (3)-(5), but with the domains of integration Ω_R, ∂B_R replaced by $P_{N,J}$ and $\partial P^{(o)}_{N,J}$, respectively.

Take $j \in \{1, 2\}$, $N, J \in \mathbb{N}_0$, and assume in addition that the boundary data are continuous. Then we consider the following mixed finite element problem:

(Va2) Find $(v, \varrho) = (v^{(j)}_{N,J}, \varrho^{(j)}_{N,J}) \in V_{N,J} \times M^{(j)}_{N,J}$ such that

$$v \,|\, \partial P^{(o)}_{N,J} = 0 \quad \text{in the case } j = 2,$$

$$v(x) = b(x) \quad \text{for each vertex } x \text{ of } P_{N,J} \text{ with } x \in \partial P^{(i)}_{N,J},$$

$$\alpha^{(j)}_{N,J}(v, w) + \beta_{N,J}(w, \varrho) = \int_{B_{N,J}} f \cdot w \, dx \quad \text{for } w \in V^{(j)}_{N,J},$$

$$\beta_{N,J}(v, \sigma) = 0 \quad \text{for } \sigma \in M^{(j)}_{N,J}.$$

This problem admits a unique solution $(v, \varrho) = (v^{(j)}_{N,J}, \varrho^{(j)}_{N,J})$; see [4], Theorem 5.1 in the case $j = 1$, $b = 0$, and [7], p. 124/125, [2] in the case $j = 2$, $b = 0$. Supplementary arguments for dealing with the case $b \neq 0$ may be found in [15], p. 173/174. We consider this solution a finite element approximation of the exterior flow (u, π). Error estimates related to this approximation are proved in [4], Section 5 under the assumptions $j = 1$, Ω a polyhedron in R^3, f as in (6), and $P_{N,J}$ chosen in such a way that the interior boundary $\partial P^{(i)}_{N,J}$ of $P_{N,J}$ coincides with $\partial\Omega$. In this situation it is shown in [4], Section 5 that there are $C > 0$, J_0, $N_0 \in \mathbb{N}$ such that for $N, J \in \mathbb{N}$ with $N \geq N_0$, $J \geq J_0$, if u is H^2-regular

$$\|u \,|\, \Omega_S - v^{(1)}_{N,J} \,|\, B_S\|_2 \leq$$

$$\leq C \cdot \left((S \cdot 2^J)^{-3/2} + (S \cdot 2^J)^{-1/2} \cdot h_{N,J} \cdot (1 + \ln(S/h_{N,J}))^{1/2} + h^2_{N,J} \right),$$

where $h_{N,J}$ denotes the mesh size near $\partial\Omega$ of the triangulations $\mathcal{C}_{N,J}$. We note that the first term on the right-hand side of (7) corresponds to the truncation error, that is, to the error due to the transition from the exterior problem (1) to problem (2) on Ω_R. The second summand arises because problem (2) on Ω_R is replaced by a variational problem on a polyhedron, and the third term represents the usual discretisation error. We further remark that the error $h^2_{N,J}$ would not diminish if the mesh size were bounded by $h_{N,J}$ not only near Ω, but *everywhere* in the truncating polyhedron $P_{N,J}$ (Goldstein [10], [11]). We repeat that estimate (7) is valid in the situation considered in [4] (Ω polyhedral, $j = 1$, $b = 0$). In the case $\Omega = B_1$ treated here, an additional error component arises because $\partial P^{(i)}_{N,J}$ does not coincide with $\partial\Omega = \partial B_1$.

We further remark that no error estimates are available in the case $j = 2$ (Dirichlet boundary conditions on $\partial P^{(o)}_{N,J}$). In fact, although it should be supposed by the results from [13], [14] that the truncation error is bounded by $C \cdot R^{-1/2}$ in that case, there remains the problem whether the family of spaces $(V^{(2)}_{N,J}, M^{(2)}_{N,J})_{(N,J) \in \mathbb{N}^2}$ satisfies the Babushka-Brezzi condition with a constant independent of N *and* J, that is, independent of the mesh-size of $\mathcal{C}_{N,J}$ and of the radius 2^{J+1} of the truncating sphere.

As test examples, we consider two exterior flows, $(u, \pi) = (\overline{u}, \overline{\pi})$ and $(u, \pi) = (\tilde{u}, \tilde{\pi})$, with

$$\overline{u}_k(x) := (3/4) \cdot x_1 \cdot x_k \cdot |x|^{-3} \cdot (1 - |x|^{-2}) + \delta_{1k} \cdot (1/4) \cdot |x|^{-1} \cdot (3 + |x|^{-2}),$$

$$\overline{\pi}(x) := (3/2) \cdot x_1 \cdot |x|^{-3} \quad \text{for } 1 \leq k \leq 3, \; x \in R^3 \setminus B_1,$$

and

$$\tilde{u}(x) = 10 \cdot \mathrm{rot}(g(x), g(x), g(x)), \qquad \tilde{\pi} = 10 \cdot (|x| - 1)^2 \cdot |x|^{-5}$$

for $x \in R^3 \setminus B_1$, where $g(x) := (|x| - 1)^4 \cdot |x|^{-4}$ for $x \in R^3 \setminus \{0\}$. Note that $(\overline{u}, \overline{\pi})$ is the well known Stokes flow around a sphere; it solves (1) with $b = (1, 0, 0)$, $f = 0$. The exterior flow $(\tilde{u}, \tilde{\pi})$ solves (1) with $b = 0$ and $f := -\Delta\tilde{u} + \nabla\tilde{\pi}$. This flow is an interesting test case because the right-hand side f does not decay too strongly. In fact, it holds $f(x) = |x|^{-4}$ for $|x| \to \infty$.

We computed the L^2-error $\mathrm{Err}(N, J) := \|u \,|\, \Omega_S - v^{(j)}_{N,J} \,|\, \Omega_S\|_2$ arising when our finite element method (Va2) is used to compute these flows. Our results are presented in Table 2 and 3 below. To interpret them, we remark that if for a constant value of R (that is, for J constant), the error remains almost unchanged when the grid is refined (that is, when N is augmented), then the dis-

TABLE 1. – *Number of nodes \ tetrahedrons arising with N refinement steps and with a truncating sphere of radius* 2^{J+1}.

	$J=0$	$J=1$	$J=2$	$J=3$
$N=0$	46\144	78\288	110\432	142\576
$N=1$	294\1 152	538\2 304	782\3 456	1026\4 608
$N=2$	2 122\9 216	4 050\18 432	5 978\27 648	7 906\38 864
$N=3$	16 146\73 728	31 522\147 456	46 898\221 184	62 274\294 912

TABLE 2. – L^2-*error* Err(N, J) *arising in the computation of* $\bar{u}$, *left in the case* $\mathcal{L}_R = \mathcal{L}_R^{(1)}$, *and right if* $\mathcal{L}_R = \mathcal{L}_R^{(2)}$.

	$J=0$	$J=1$	$J=2$	$J=3$
$N=0$	1.86	1.86	1.96	2.02
$N=1$	1.15	1.17	1.21	1.23
$N=2$	0.67	0.68	0.61	0.70
$N=3$	0.38	0.33	0.37	

	$J=0$	$J=1$	$J=2$	$J=3$
$N=0$	2.74	1.90	1.84	2.15
$N=1$	2.15	1.49	1.24	1.13
$N=2$	2.39	1.50	0.98	0.76
$N=3$	3.11	1.62	0.82	

TABLE 3. – L^2-*error* Err(N, J) *arising if* $(u, \pi) = (\tilde{u}, \tilde{\pi})$; *the left table refers to the case* $\mathcal{L}_R = \mathcal{L}_R^{(1)}$, *the right one to the case* $\mathcal{L}_R = \mathcal{L}_R^{(2)}$.

	$J=0$	$J=1$	$J=2$	$J=3$
$N=0$	5.71	1.36	1.49	1.47
$N=1$	6.46	0.85	0.87	0.90
$N=2$	6.58	0.80	0.25	0.28
$N=3$	6.42	0.88	0.07	

	$J=0$	$J=1$	$J=2$	$J=3$
$N=0$	5.83	1.59	1.29	1.26
$N=1$	7.45	1.38	0.68	0.67
$N=2$	6.98	1.89	0.34	0.21
$N=3$	6.94	2.29	0.42	

cretisation error is small with respect to the truncation error. Similarly, if at one and the same refinement level (that is, for constant N), the error does not diminish when J is increased, then the truncation error vanishes compared to the discretisation error. In fact, when $u = \bar{u}$, $\mathcal{L}_R = \mathcal{L}_R^{(1)}$, $N \in \{0, \ldots, 3\}$, the value $J = 0$ suffices to obtain a more or less minimal total error. In the case $u = \tilde{u}$, $\mathcal{L}_R = \mathcal{L}_R^{(1)}$, no significant decrease of the error may be expected for values of J beyond $J = 1$ if $N \in \{0, 1\}$, and beyond $J = 2$ if $N \in \{2, 3\}$. If $\mathcal{L}_R = \mathcal{L}_R^{(2)}$, these limit values of J are larger by at least 1.

The available computation time was not enough to deal with the case $N = J = 3$. In fact, the conditioning of problem (Va2) worsens with growing values of N and J, and the solver we used — an Uzawa conjugate gradient method ([9] p. 289-294) with a cg procedure in the inner loop — was not sophisticated, in particular as concerns the preconditioner (a diagonal matrix) applied in this inner loop. In our most expensive computation, we needed about 4 hours and 16 minutes of CPU time on an Ultra Sparc 450 workstation in order to calculate an approximation of $\bar{u}$ in the case $N = 3$, $J = 2$, $\mathcal{L}_R = \mathcal{L}_R^{(2)}$. Note that the case $N = 3$, $J = 2$ already involves about 850 000 unknowns before internal condensation of the bubble functions, and 200 000 afterwards; see Table 1. These numbers indicate how important it is to keep J small, and hence to find an optimal artificial boundary condition in (2).

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