Approximating exterior flows by flows on truncated exterior domains: piecewise polygonal artificial boundaries

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Abstract

The article deals with a mixed finite element method for discretizing the Stokes system in truncated exterior domains. This method yields approximate solutions of the Stokes equations on exterior domains, as is shown by error estimates. An implementation of the method is discussed, and the results of test computations are presented. In particular, it is indicated how a truncated exterior domain with spherical outer boundary may be triangulated in a suitable way. Similar mixed finite element methods work for other exterior problems which may be written as mixed variational problems.

1. Introduction

In this article, we discuss how exterior flows may be approximated by solutions of mixed variational problems in truncated exterior domains. In particular we will be concerned with truncated exterior domains which are polyhedral and thus may easily be decomposed into grids suitable for finite element methods.

The model problem we consider is the Stokes system in the exterior domain $\overline{\Omega}^c := \mathbb{R}^3\backslash\overline{\Omega}$, with a Dirichlet boundary condition on $\partial\Omega$ and a decay condition near infinity:

$$-\Delta u + \nabla\pi = f, \quad \operatorname{div} u = 0 \quad \text{in } \overline{\Omega}^c, \quad u|\partial\Omega = b, \qquad (1.1)$$
$$\int_{\partial B_1} |u(x)|^2\, do_x = O(R^{-1}) \quad \text{for } R \to \infty,$$

where $\Omega \subset \mathbb{R}^3$ is either the empty set, or a bounded open polyhedron with Lipschitz boundary and with $0 \in \Omega$. We further suppose $b \in H^{1/2}(\partial\Omega)^3, f \in L^2(\mathbb{R}^3)^3$, and we assume there is $\sigma \in (4,\infty)$, $S, \gamma \in (0,\infty)$ with $\overline{\Omega} \subset B_S$, $|\, f(x)\,| \le \gamma \cdot |x|^{-\sigma}$ for $x \in \mathbb{R}^3\backslash B_S$. The quantities σ, S, γ and the functions b and f will be kept fixed throughout. Here and in the following, for $\epsilon \in (0,\infty)$, we denote by B_ϵ the open ball in $\mathbb{R}^3$ with radius $\epsilon > 0$ and center in the origin.

Under the preceding assumptions, problem (1.1) admits a unique solution (u,π) with the properties $u \in H^2_{loc}(\overline{\Omega}^c)^3$, $\pi \in H^1_{loc}(\overline{\Omega}^c) \cap L^2(\overline{\Omega}^c)$, $\nabla u \in L^2(\overline{\Omega}^c)^9$ ("exterior Stokes flow"); see [6, Theorem V.2.1, V.1.1].

Now take $R \in [S,\infty)$ with $\overline{\Omega} \subset B_R$, and denote by Ω_R that part of the exterior domain $\overline{\Omega}^c$ which lies inside the ball B_R, that is, $\Omega_R := B_R\backslash\overline{\Omega}$. Then we consider the boundary value problem

$$-\Delta v + \nabla\varrho = f\,|\,\Omega_R, \quad \operatorname{div} v = 0 \quad \text{in } \Omega_R, \quad v\,|\,\partial\Omega = b, \qquad (1.2)$$
$$\mathcal{L}_R(B_R)(v,\varrho)(x) = 0 \quad \text{for } x \in \partial B_R,$$

where $\mathcal{L}_R$ is defined as follows:

$$\mathcal{L}_R(A)(w,\sigma)_j(x) := 3\cdot w_j(x)/(2\cdot R) + \sum_{k=1}^{3}\Big(D_k w_j - D_j w_k/2 - \delta_{jk}\cdot\sigma\Big)(x)\cdot n_k^{(A)}(x)$$

for $j \in \{1, 2, 3\}$, $R \in (0,\infty)$, $x \in \partial A$, $A \subset \mathbb{R}^3$ a Lipschitz domain, $n^{(A)}$ its outward unit normal, w a vector-valued function and σ a scalar function with the property that w, ∇w and σ are defined on ∂A. The symbols D_k and D_j denote partial derivatives.

This choice of $\mathcal{L}_R$ is based on two considerations: first, it allows to write problem (1.2) in mixed variational form, and second, the difference between the exterior Stokes flow and the solution of (1.2) ("truncation error"), measured in some suitable norm, decays with an optimal rate. More precisely, setting for $v, w \in H^1(\Omega_R)^3$, $\varrho \in L^2(\Omega_R)$,

$$\alpha_R(v,w) := 3/(2\cdot R_i)\cdot\int_{\partial\Omega} v\cdot w\, do_x + \int_{\Omega_R}\sum_{j,k=1}^{3}\Big(D_j v_k\cdot D_j w_k \qquad (1.3)$$
$$-(1/2)\cdot D_j v_k(x)\cdot D_k w_j(x)\Big)\,dx, \qquad \beta_R(v,\varrho) := -\int_{\Omega_R}\operatorname{div} v\cdot\varrho\,dx,$$

we could show (see [1, Theorem 4.2]) there is a uniquely determined pair of functions $(v_R, \varrho_R) \in H^1(\Omega_R)^3 \times L^2(\Omega_R)$ with

$$\alpha_R(v_R,w) + \beta_R(w,\varrho_R) = \int_{\Omega_R} f\cdot w\,dx \quad \text{for } w \in H^1(\Omega_R)^3 \text{ with } w\,|\,\partial\Omega = 0, \qquad (1.4)$$
$$\operatorname{div} v_R = 0, \quad v_R\,|\,\partial\Omega = b.$$

Note that problem (1.4) is a weak form of (1.2); see [1, Lemma 4.2]. The equation $\operatorname{div} v_R = 0$ may be replaced by the relation $\beta_R(v_R,\sigma) = 0$ for $\sigma \in L^2(\Omega_R)$, so problem (1.4) is a mixed variational problem. It is further shown in [1] that

$$\|\,(u-v_R)\,|\,\Omega_S\,\|_2 + \|\,\nabla(u\,|\,\Omega_R - v_R)\,\|_2 + R^{1/2}\cdot\|\,(u-v_R)\,|\,\partial B_R\,\|_2 \qquad (1.5)$$
$$+\,\|\,\pi\,|\,\Omega_R - \varrho_R\,\|_2 \;\le\; C\cdot(\,\|\,f\,|\,\Omega_S\,\|_2 + \gamma + \|b\|_{1/2,2}\,) \text{ for any } R \in [S,\infty),$$

with a constant $C > 0$ which only depends on Ω, S and σ ([1, Theorem 5.1, Lemma 5.1]). Actually, only the case $b = 0$ is considered in [1], but a technique for extending this result to the case $b \neq 0$ may be found in [5]. The important point of this extension is that the constant C in (1.5) remains independent of R. We further remark that the results in [1] were inspired by a similar theory in [8], [9] pertaining to the Poisson equation.

An artificial boundary condition for the Stokes system different from the one in (1.2) is presented in [10]. Similar estimates as in (1.5) are proved in [5] for Oseen flows.

Due to its mixed variational form, problem (1.4) may be discretized by finite element methods. A finite element solution to (1.4) may then be considered an approximation of the exterior Stokes flow (u, π). The corresponding error consists of the sum of the truncation and the discretization error, the latter one arising due to the transition from (1.4) to a finite-dimensional problem. In [1, Theorem 6.3, Corollary 6.2], we estimated this sum, under the assumption that the Babushka-Brezzi condition [1, (6.9)] is valid with a constant independent of R. This assumption is satisfied by the Mini element ([3, Theorem 4.1]).

When we performed test computations ([2], [4]), we did not actually decompose Ω_R. Instead we used tetrahedral grids of such a type that the union of the tetrahedrons of each grid yields a polyhedron P, depending on the respective grid, with $P \subset \Omega_R$ and $\Omega_R \backslash P$ a slender set near the sphere ∂B_R. The artificial boundary condition was then imposed on the outer part $\partial P \backslash \partial\Omega$ of the boundary ∂P of P. It is the purpose of this paper to give a precise description of the transition from Ω_R to these polyhedra, and to quantify the error related to this transition.

To this end, we fix some index set $\mathcal{I}$. For $i \in \mathcal{I}$, let $n_i \in \mathbb{N}_0$ with $n_i \geq 3$, and put $R_i := 2^{n_i} \cdot S$. (The parameter S was introduced at the beginning of this section and refers to the size of Ω and the decay behaviour of f.) Assume that $\mathcal{T}_i = (K_l^{(i)})_{1 \leq l \leq \tau_i}$, with $\tau_i \in \mathbb{N}$, is a family of open tetrahedrons in $B_{R_i} \backslash \overline{\Omega}$. We will write K_l instead of $K_l^{(i)}$ since the dependence of K_l on i should be obvious. Denote by P_i the interior of the set $\cup\{\overline{K_l} \; : \; 1 \leq l \leq \tau_i\}$. We suppose that $P_i \subset \Omega_{R_i}$ and $\Omega_R \backslash P_i$ is a remainder set near ∂B_{R_i}. Let h_i denote an upper bound of $diam\, K_l$ for tetrahedrons K_l near Ω. Following Goldstein [8], [9], we suppose the mesh size of $\mathcal{T}_i$ to augment from this value h_i near Ω to $h_i \cdot R_i/S$ near ∂B_{R_i}. These and other properties of $\mathcal{T}_i$ will be made precise in Section 2.

For each $i \in \mathcal{I}$, the polyhedron P_i is used for setting up a mixed variational problem. In fact, define the bilinear forms $\alpha_i : H^1(P_i)^3 \times H^1(P_i)^3 \mapsto \mathbb{R}$, $\beta_i : H^1(P_i)^3 \times L^2(P_i) \mapsto \mathbb{R}$ in the same way as respectively α_R and β_R in (1.3), but with $R = R_i$, and with the domains of integration Ω_R and ∂B_R replaced by P_i and $\partial P_i \backslash \partial\Omega$, respectively. Then we consider the boundary value problem

$$-\Delta v_i + \nabla \varrho_i = f \,|\, P_i, \quad \operatorname{div} v_i = 0 \quad \text{in } P_i, \quad v_i \,|\, \partial\Omega = b,$$
$$\mathcal{L}_{R_i}(P_i \cup \overline{\Omega})(v_i, \varrho_i)(x) = 0 \quad \text{for } x \in \partial P_i \backslash \partial\Omega,$$

which admits a weak solution for any $i \in \mathcal{I}$ with h_i small enough:

Theorem 1.1 *There is $S_0 \in (0, S)$ such that for $i \in \mathcal{I}$ with $h_i \in (0, S_0]$, a uniquely determined pair of functions (v_i, ϱ_i) exists in $H^1(P_i)^3 \times L^2(P_i)$ with*

$$\alpha_i(v_i, w) + \beta_i(w, \varrho_i) = \int_{P_i} f \cdot w \, dx \quad \textit{for } w \in W_i, \qquad div\, v_i = 0, \qquad v_i \,|\, \partial\Omega = b, \tag{1.6}$$

where $W_i := \{ w \in H^1(P_i)^3 \; : \; w \,|\, \partial\Omega = 0 \}$. Moreover, for i as before, we have

$$\| (u - v_i) \,|\, \Omega_S \|_2 + \| \nabla(u \,|\, P_i - v_i) \|_2 + R_i^{-1/2} \cdot \| (u - v_i) \,|\, \partial P_i \backslash \partial\Omega \|_2 + \| \pi \,|\, P_i - \varrho_i \|_2 \tag{1.7}$$

$$\leq C \cdot R_i^{3/2} \cdot \big((\overline{\mathcal{D}_i} \cdot \widetilde{\mathcal{D}_i})^{1/2} + (\overline{\mathcal{D}_i} + \widetilde{\mathcal{D}_i})^{1/2} \cdot (\mathcal{D}_i \cdot h_i)^{1/2} + \widetilde{\mathcal{D}_i} + (\overline{\mathcal{D}_i} + \mathcal{D}_i) \cdot h_i + \overline{\mathcal{D}_i} \big),$$

with a constant $C > 0$ independent of i, and with the terms $\overline{\mathcal{D}_i}$, $\mathcal{D}_i$ and $\widetilde{\mathcal{D}_i}$ defined by

$$\overline{\mathcal{D}_i} := \sup\{ \,|\mathcal{L}_{R_i}(B_{R_i})(u, \pi)(x)| \; : \; x \in \partial B_{R_i} \},$$
$$\mathcal{D}_i := \sup\{ \,| F_i(t \cdot x) | \; : \; x \in \partial B_{R_i}, \; t \in [(1 - h_i^2/S^2)^{1/2}, 1] \},$$
$$\widetilde{\mathcal{D}_i} := \sup\{ \,| F_i(x) - F_i(t \cdot x) | \; : \; x \in \partial B_{R_i}, \; t \in [(1 - h_i^2/S^2)^{1/2}, 1] \},$$

where $F_i(x) := (R_i^{-1} \cdot u(x), \nabla u(x), \pi(x))$ for $x \in \overline{\Omega}^c$.

This theorem, which will be proved in Section 2, yields an analogue of inequality (1.5) once we are able to find upper bounds for $\overline{\mathcal{D}_i}$, $\mathcal{D}_i$ and $\widetilde{\mathcal{D}_i}$ in terms of $\|f \,|\, \Omega_S\|_2$, γ and $\|b\|_{1/2,2}$. In other words, we still have to establish pointwise estimates of the exterior Stokes flow (u, π) in the annular domain $B_{R_i} \backslash B_{R_i \cdot (1-h_i^2/S^2)^{1/2}}$. Such estimates follow from our assumptions on f. For lack of space, we can only give the result here, which says there is a constant $\mathcal{C} > 0$ only depending on Ω, S and σ such that

$$\overline{\mathcal{D}_i} \leq \mathcal{C} \cdot R_i^{-3} \cdot \Gamma, \quad \mathcal{D}_i \leq \mathcal{C} \cdot R_i^{-2} \cdot \Gamma, \quad \widetilde{\mathcal{D}_i} \leq \mathcal{C} \cdot (h_i/R_i)^2 \cdot (1 + \ln(S/h_i)) \cdot \Gamma \tag{1.8}$$

for $i \in \mathcal{I}$, where $\Gamma := \|f \,|\, B_S\|_2 + \gamma + \|b\|_{1/2,2}$. A proof of (1.8) may be based on an integral representation of u and π similar to the one in [1, (3.7), 3.8)]. In fact, the term $\overline{\mathcal{D}_i}$ is evaluated in this way in [1, p. 259], under the assumption $b = 0$. Combining (1.7) and (1.8) finally yields an estimate of the truncation error which corresponds to one given in (1.5) for the case of spherical truncating surfaces, and which states that the left-hand side of (1.7) is bounded by a constant times a sum of terms of the form $h_i^a \cdot R_i^{-b} \cdot (1 + \ln(S/h_i))^c$, with $a + b \geq 3/2$, $c \geq 0$.

Also for lack of space, we cannot establish error estimates associated with finite element discretisations of the mixed variational problem (1.6). We only mention that the error estimates in [1, Theorem 6.3, Corollary 6.2], which pertain to finite element discretisations of (1.4), must be modified by adding to their right-hand side some additional terms which may be traced back to (1.8) and (1.7). The proof of this result does not differ much from that of [1, loc. cit.]. The main changes concern [3, Theorem 4.1], which yields the Babushka-Brezzi condition [1, (6.9)] for the Mini element, with a constant independent of R_i. In fact, [3, Theorem 4.1] pertains to domains Ω_{R_i}; if we consider P_i instead of Ω_{R_i}, we have to replace [3, Corollary 3.1] by Theorem 3.1 below, and [3, Theorem 2.2] by Theorem 3.5 below. But otherwise, the transition from Ω_{R_i} to P_i simplifies the proof of [3, Theorem 4.1]. In particular, we may drop the smallness condition on h_i required in [3, loc.cit].

2. Partitionings of truncated exterior domains

We begin by describing our grids $\mathcal{T}_i$ in a more precise way. First we characterize the set $\Omega_{R_i} \backslash P_i$:

(A1) *Let $i \in \mathcal{I}$. Then $K_l \subset B_{R_i} \backslash \overline{\Omega}$ for $l \in \{1, ..., \tau_i\}$; the set P_i is a connected polyhedron; any vertex of P_i is either a vertex of Ω, or it is located on ∂B_{R_i}.*

We do not admit hanging nodes, and we require that our grids do not degenerate:

(A2) *If $i \in \mathcal{I}$, $k, l \in \{1, ..., \tau_i\}$ with $k \neq l$, and if $F := \overline{K_k} \cap \overline{K_l} \neq \emptyset$, then F is either a common face or side or vertex of K_k and K_l.*

(A3) *Put $B_r(x) := \{ y \in \mathbb{R}^3 : |y - x| < r \}$ for $x \in \mathbb{R}^3$, $r > 0$; $B_r := B_r(0)$ for $r > 0$;*

$$U_0 := B_S \backslash \overline{\Omega}; \quad U_j := B_{2^j \cdot S} \backslash \overline{B_{2^{j-1} \cdot S}} \quad \text{for } j \in \mathbb{N}.$$

Then there is a constant $\sigma_1 > 0$ and a family $(h_i)_{i \in \mathcal{I}}$ in $(0, S/4)$ such that

$$diam\, K_l \leq 2^j \cdot h_i, \ \sup\{ r \in (0, \infty) : B_r(x) \subset K_l \text{ for some } x \in K_l \} \geq \sigma_1 \cdot 2^j \cdot h_i$$

for $i \in \mathcal{I}$, $l \in \{1, ..., \tau_i\}$, $j \in \{0, ..., n_i\}$ with $U_j \cap K_l \neq \emptyset$.

Note that although the mesh size of $\mathcal{T}_i$ increases with growing distance from Ω, as already indicated in Section 1, our assumptions in (A3) imply that the family $(\mathcal{T}_i)_{i \in \mathcal{I}}$ is non-degenerate in the sense that the ratio of the smallest circumscribing sphere to the largest inscribing sphere of any element $K_l^{(i)}$ is bounded independently of $i \in \mathcal{I}$ and $l \in \{1, ..., \tau_i\}$.

In the three-dimensional situation we are considering, it might happen that all four vertices of a tetrahedron are situated on the sphere ∂B_{R_i}. We want to exclude this case, so we require

(A4) *For $i \in \mathcal{I}$, $l \in \{1, ..., \tau_i\}$, at least one vertex of K_l is located in the open set P_i.*

Finally we prescribe a condition which allows us to project the points of ∂B_R onto $\partial P_i \backslash \partial \Omega$ and vice versa.

(A5) *For $i \in \mathcal{I}$, the ray $\{ r \cdot (\cos \varrho \cdot \cos \vartheta, \sin \varrho \cdot \cos \vartheta, \sin \vartheta) : r \in (0, \infty) \}$ has precisely one point of intersection with $\partial P_i \backslash \partial \Omega$, for any $\varrho, \vartheta \in \mathbb{R}$.*

Let $i \in \mathcal{I}$. By (A4) and (A1), we may assume there is some $\kappa_i \in \{1, ..., \tau_i - 1\}$ with the properties to follow: For $l \in \{1, ..., \kappa_i\}$, at most two vertices of K_l are located on ∂B_{R_i}. If $l \in \{\kappa_i + 1, ..., \tau_i\}$, three vertices of K_l are situated on ∂B_{R_i}.

For $l \in \{1, ..., \tau_i\}$, let $a^{(1)} = a^{(1)}(i, l)$, ..., $a^{(4)} = a^{(4)}(i, l)$ denote the four vertices of K_l. If $l > \kappa_i$, then, by the choice of κ_i, we may suppose that $a^{(1)}, a^{(2)}, a^{(3)} \in \partial B_{R_i}$ and $a^{(4)} \in P_i$.

We note a well known consequence of assumption (A3):

Lemma 2.1 *There is a constant $\sigma_2 \in (0, 1)$ such that*

$$|a^{(1)} - a^{(3)}|^{-1} \cdot |a^{(2)} - a^{(3)}|^{-1} \cdot (a^{(1)} - a^{(3)}) \cdot (a^{(2)} - a^{(3)}) \leq \sigma_2$$

for $i \in \mathcal{I}$, $l \in \{1, ..., \tau_i\}$, where $a^{(k)} := a^{(k)}(i, l)$ for $k \in \{1, 2, 3\}$.

If $i \in \mathcal{I}$, $l \in \{\kappa_i + 1, ..., \tau_i\}$, we write $T_l^{(i)}$, or more briefly T_l, for the plane open triangle in $\mathbb{R}^3$ with vertices $a^{(1)}(i, l)$, $a^{(2)}(i, l)$, $a^{(3)}(i, l)$. Due to (A1), (A2) and (A4), we have

$$\partial P_i \backslash \partial \Omega = \cup \{ \overline{T_l} : \kappa_i + 1 \leq l \leq \tau_i \} \quad \text{for } i \in \mathcal{I}. \tag{2.1}$$

We introduce a two- and a three-dimensional standard tetrahedron $\widehat{K}^{(2)}$ and $\widehat{K}^{(3)}$, respectively, by setting

$$\widehat{K}^{(2)} := \{ x \in (0, 1)^2 : x_2 < 1 - x_1 \},$$
$$\widehat{K}^{(3)} := \{ x \in (0, 1)^3 : x_2 < 1 - x_1, \ x_3 < 1 - x_1 - x_2 \}.$$

For $i \in \mathcal{I}$, $l \in \{1, ..., \tau_i\}$, we put

$$A_l := A_l^{(i)} := (a^{(1)} - a^{(4)}, \ a^{(2)} - a^{(4)}, \ a^{(3)} - a^{(4)}) \in \mathbb{R}^{3 \times 3},$$

where the vectors $a^{(v)} - a^{(4)} = a^{(v)}(i, l) - a^{(4)}(i, l)$ are to be understood as columns ($v \in \{1, 2, 3\}$). We further set

$$\mathcal{A}_l(x) := \mathcal{A}_l^{(i)}(x) := A_l^{(i)} \cdot x + a^{(4)}(i, l) \quad \text{for } i \in \mathcal{I}, \ 1 \leq l \leq \tau_i, \ x \in \mathbb{R}^3.$$

Denote by $n^{(P_i)}$, $n^{(B_{R_i})}$ the outward unit normal to P_i and B_{R_i}, respectively. Define the functions $\varphi_l := \varphi_l^{(i)} : \widehat{K}^{(2)} \mapsto \partial P_i \backslash \partial \Omega$, $\quad \psi_l := \psi_l^{(i)} : \widehat{K}^{(2)} \mapsto \partial B_{R_i}$, $\quad g_l := g_l^{(i)} : \widehat{K}^{(2)} \mapsto \mathbb{R}$, $\quad \nu_l := \nu_l^{(i)} : \widehat{K}^{(2)} \mapsto \mathbb{R}$ by

$$\varphi_l(\eta) := \mathcal{A}_l(\eta, 1 - \eta_1 - \eta_2) = a^{(3)} + (a^{(1)} - a^{(3)}) \cdot \eta_1 + (a^{(2)} - a^{(3)}) \cdot \eta_2,$$
$$\psi_l(\eta) := |\varphi_l(\eta)|^{-1} \cdot R_i \cdot \varphi_l(\eta),$$
$$g_l(\eta) := \Big(|a^{(1)} - a^{(3)}|^2 \cdot |a^{(2)} - a^{(3)}|^2 - ((a^{(1)} - a^{(3)}) \cdot (a^{(2)} - a^{(3)}))^2 \Big)^{1/2},$$
$$\nu_l(\eta) := \Big(\det (D_j \psi_l(\eta) \cdot D_k \psi_l(\eta))_{1 \leq j,k \leq 2} \Big)^{1/2}$$

for $\eta \in \widehat{K}^{(2)}$, where $a^{(j)} := a^{(j)}(i, l)$, $j \in \{1, 2, 3\}$, $\kappa_i + 1 \leq l \leq \tau_i$, $i \in \mathcal{I}$. Put

$$F_l := F_l^{(i)} := \{ \psi_l^{(i)}(\eta) : \eta \in \widehat{K}^{(2)} \} \quad \text{for } i \in \mathcal{I}, \ \kappa_i + 1 \leq l \leq \tau_i.$$

We note that φ_l is a local parameter of ∂P_i, with range$(\varphi_l) = T_l$, and g_l is the corresponding element of surface area. The function $\mathcal{A}_l \,|\, \widehat{K}^{(3)}$ is the usual affine mapping from $\widehat{K}^{(3)}$ onto K_l, and F_l is defined by a radial projection of the triangle T_l on ∂B_{R_i}. Due to (A5), ψ_l is one-to-one. We shall show below (Corollary 2.3) that if h_i is small enough, ψ_l is regular and hence a local parameter of ∂B_{R_i}. The related element of surface area is given by the function ν_l. As a consequence of (A5) and (2.1), we note

$$\partial B_{R_i} = \cup \{ \overline{F_l} : \kappa_i + 1 \leq l \leq \tau_i \}, \tag{2.2}$$

$$\int_{\partial P_i \backslash \partial \Omega} f \, do_y = \sum_{l = \kappa_i + 1}^{\tau_i} \int_{T_l} f \, do_y = \sum_{l = \kappa_i + 1}^{\tau_i} \int_{\widehat{K}^{(2)}} (f \circ \varphi_l)(\eta) \cdot g_l(\eta) \, d\eta \tag{2.3}$$

for $i \in \mathcal{I}$, $f \in L^1(\partial P_i \backslash \partial \Omega)$. In order to show that ψ_l is a regular mapping, and to estimate the difference between integrals over $\partial P_i \backslash \Omega$ and ∂B_R, we begin by recalling a result from [3]:

Lemma 2.2 ([3, Lemma 4.2]) *We have $B_{R_i \cdot (1 - h_i^2 \cdot S^{-2})^{1/2}} \backslash \overline{B_S} \subset P_i$ for $i \in \mathcal{I}$.*

We note that the quantity $h/4$ in [3] corresponds to h_i in the present context. This explains why a factor $h_i^2 \cdot S^{-2}$ arises here instead of $h^2 \cdot (4 \cdot S)^{-2}$ as in [3, Lemma 4.2]. By Lemma 2.2 and (A3) we obtain

Corollary 2.1 *For $i \in \mathcal{I}$, $x \in \partial P_i \backslash \partial\Omega$, we have* $|x| \geq R_i \cdot (1 - h_i^2 \cdot S^{-2})^{1/2}$.

Another technical result which will be useful is

Lemma 2.3 *Let $i \in \mathcal{I}$, $l \in \{\kappa_i + 1, ..., \tau_i\}$, $j \in \{1, 2\}$, $\eta \in \widehat{K}^{(2)}$. Then*

$$\big|\, |\varphi_l(\eta)|^{-1} \cdot (\varphi_l(\eta) \cdot D_j\varphi_l(\eta))\,\big| \leq \mathit{diam}\, K_l \cdot (8/\sqrt{15}) \cdot R_i^{-1} \cdot |D_j\varphi_l(\eta)|.$$

Proof: Abbreviate $a^{(k)} := a^{(k)}(i,l)$ for $k \in \{1, ..., 4\}$. Then $D_j\varphi_l(\eta) = a^{(j)} - a^{(3)}$. Since $a^{(1)}, a^{(2)}, a^{(3)} \in \partial B_{R_i}$, it follows $(a^{(j)} + a^{(3)}) \cdot D_j\varphi_l(\eta) = 0$. Thus

$$\begin{aligned}
&\big|\, |\varphi_l(\eta)|^{-1} \cdot (\varphi_l(\eta) \cdot D_j\varphi_l(\eta))\,\big| \\
&= \Big|\big(|\varphi_l(\eta)|^{-1} \cdot \varphi_l(\eta) - |a^{(j)} + a^{(3)}|^{-1} \cdot (a^{(j)} + a^{(3)})\big) \cdot D_j\varphi_l(\eta)\Big| \\
&\leq 4 \cdot |(1/2) \cdot (a^{(j)} + a^{(3)}) - \varphi_l(\eta)| \cdot |a^{(j)} + a^{(3)}|^{-1} \cdot |D_j\varphi_l(\eta)|.
\end{aligned}$$

But $|(1/2) \cdot (a^{(j)} + a^{(3)}) - \varphi_l(\eta)| \leq \operatorname{diam} K_l$, and by Corollary 2.1

$$(1/2) \cdot |a^{(j)} + a^{(3)}| \geq R_i \cdot (1 - h_i^2 \cdot S^{-2})^{1/2}.$$

Since $h_i \leq S/4$ (see (A3)), we obtain Lemma 2.3 by combining the preceding estimates. ◇

Lemma 2.3 allows us to estimate $n^{(P_i)} \circ \varphi_l - n^{(B_{R_i})} \circ \psi_l$ and $g_l^{(i)} - \nu_l^{(i)}$:

Lemma 2.4 *There is $C > 0$ such that for i, l, η as in Lemma 2.3, we have*

$$|n^{(P_i)}(\varphi_l(\eta)) - n^{(B_{R_i})}(\psi_l(\eta))| \leq C \cdot h_i.$$

Proof: Take i, l, η as in Lemma 2.3. Put $v := |\varphi_l(\eta)|^{-1} \cdot \varphi_l(\eta)$, and for $j \in \{1, 2\}$,

$$\alpha^{(j)} := |a^{(j)}(i,l) - a^{(3)}(i,l)|^{-1} \cdot (a^{(j)}(i,l) - a^{(3)}(i,l)) = |D_j\varphi_l(\eta)|^{-1} \cdot D_j\varphi_l(\eta).$$

Let n denote the unit vector in $\mathbb{R}^3$ satisfying two properties: first, $n \cdot \alpha^{(j)} = 0$ for $j \in \{1, 2\}$, and second, there is some $\epsilon > 0$ with $t \cdot n + \varphi_l(\eta) \notin K_l$ for $t \in (0, \epsilon)$. Then $v = n^{(B_{R_i})}(\psi_l(\eta))$ and $n = n^{(P_i)}(\varphi_l(\eta))$, and there are $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$ with $v = \delta_1 \cdot \alpha^{(1)} + \delta_2 \cdot \alpha^{(2)} + \delta_3 \cdot n$. We have $\delta_3 = n \cdot v > 0$, $|v - n| = \sqrt{2} \cdot (1 - \delta_3)^{1/2}$, and

$$\begin{aligned}
1 - \delta_3^2 &= |\delta_1 \cdot \alpha^{(1)} + \delta_2 \cdot \alpha^{(2)}|^2 = (\delta_1 \cdot \alpha^{(1)} + \delta_2 \cdot \alpha^{(2)}) \cdot v \\
&\leq (|\delta_1| + |\delta_2|) \cdot 8 \cdot (\sqrt{15} \cdot S)^{-1} \cdot h_i,
\end{aligned}$$

where the last inequality follows with Lemma 2.3 and (A3). We further find with Lemma 2.1:

$$1 = |v|^2 \geq \delta_1^2 + \delta_2^2 - 2 \cdot |\delta_1 \cdot \delta_2| \cdot \sigma_2 + \delta_3^2 \geq (1 - \sigma_2) \cdot (\delta_1 + \delta_2)^2/2 + \delta_3^2.$$

Lemma 2.4 follows from the preceding estimates. ◇

Lemma 2.5 *Let $i \in \mathcal{I}$, $l \in \{\kappa_i + 1, ..., \tau_i\}$, $\eta \in \widehat{K}^{(2)}$. Assume in addition that*

$$h_i < (\sqrt{15}/16) \cdot S \cdot (1 - \sigma_2^2)^{1/2}, \tag{2.4}$$

with σ_2 from Lemma 2.1. Then

$$|\nu_l(\eta) - g_l(\eta)| \leq (16^3/15^2) \cdot g_l(\eta) \cdot h_i^2 \cdot S^{-2} \cdot (1 - \sigma_2^2)^{-1}.$$

Proof: Abbreviate $\gamma := |\varphi_l(\eta)|^{-1} \cdot \varphi_l(\eta)$, $a^{(j)} := a^{(j)}(i,l)$ for $j \in \{1, 2, 3\}$. Put $\alpha^{(j)} := a^{(j)} - a^{(3)} = D_j\varphi_l(\eta)$ for $j \in \{1, 2\}$. Then

$$D_j\psi_l(\eta) = |\varphi_l(\eta)|^{-1} \cdot R_i \cdot (\alpha^{(j)} - (\gamma \cdot \alpha^{(j)}) \cdot \gamma) \quad \text{for } j \in \{1, 2\}.$$

Using this equation, we get after some computations

$$\nu_l(\eta)^2 = |\varphi_l(\eta)|^{-4} \cdot R_i^4 \cdot (g_l(\eta)^2 - \sigma(\eta)), \tag{2.5}$$

where $\sigma(\eta)$ is defined by $\sigma(\eta) := |(\gamma \cdot \alpha^{(1)}) \cdot \alpha^{(2)} - (\gamma \cdot \alpha^{(2)}) \cdot \alpha^{(1)}|^2$. We conclude with (2.5):

$$\begin{aligned}
|g_l(\eta) - \nu_l(\eta)| &= g_l(\eta) \cdot \Big| 1 - (R_i/|\varphi_l(\eta)|)^2 \\
&\qquad + (R_i/|\varphi_l(\eta)|)^2 \cdot \big(1 - (1 - \sigma(\eta) \cdot g_l(\eta)^{-2})^{1/2}\big)\Big|.
\end{aligned} \tag{2.6}$$

On the other hand, we have by Lemma 2.1

$$\begin{aligned}
g_l(\eta)^2 &= |\alpha^{(1)}|^2 \cdot |\alpha^{(2)}|^2 \cdot \Big[1 - \big(|\alpha^{(1)}|^{-1} \cdot |\alpha^{(2)}|^{-1} \cdot (\alpha^{(1)} \cdot \alpha^{(2)})\big)^2\Big] \\
&\geq |\alpha^{(1)}|^2 \cdot |\alpha^{(2)}|^2 \cdot (1 - \sigma_2^2),
\end{aligned}$$

and we may deduce from Lemma 2.3 and (A3)

$$\sigma(\eta) \leq (16^2/15) \cdot R_i^{-2} \cdot (\operatorname{diam} K_l)^2 \cdot |\alpha^{(1)}|^2 \cdot |\alpha^{(2)}|^2 \leq (16^2/15) \cdot (h_i/S)^2 \cdot |\alpha^{(1)}|^2 \cdot |\alpha^{(2)}|^2.$$

Therefore assumption (2.4) on h_i implies

$$\begin{aligned}
&(1 - \sigma(\eta) \cdot g_l(\eta)^{-2})^{1/2} \geq 1 - \sigma(\eta) \cdot g_l(\eta)^{-2} \\
&\geq 1 - (1 - \sigma_2^2)^{-1} \cdot (16^2/15) \cdot h_i^2 \cdot S^{-2} > 0.
\end{aligned}$$

Using this estimate, recalling that $|\varphi_l(\eta)|^2 / R_i^2 \geq 1 - h_i^2 \cdot S^{-2} \geq 15/16$ (see Corollary 2.1), and noting that $R_i / |\varphi(\eta)| > 1$, we may now deduce from (2.6):

$$\begin{aligned}
|g_l(\eta) - \nu_l(\eta)| &\leq g_l(\eta) \cdot \max\Big\{(1 - h_i^2 \cdot S^{-2})^{-1} - 1, \\
&\qquad (1 - h_i^2 \cdot S^{-2})^{-1} \cdot (1 - \sigma_2^2)^{-1} \cdot (16^2/15) \cdot h_i^2 \cdot S^{-2}\Big\} \\
&\leq g_l(\eta) \cdot h_i^2 \cdot S^{-2} \cdot \max\{16/15, (16^3/15^2) \cdot (1 - \sigma_2^2)^{-1}\} \\
&\leq g_l(\eta) \cdot (16^3/15^2) \cdot h_i^2 \cdot S^{-2} \cdot (1 - \sigma_2^2)^{-1}.
\end{aligned}$$

◇

Let us state some consequences of the preceding lemma.

Corollary 2.2 *Put $S_0 := (15/128) \cdot S \cdot (1 - \sigma_2^2)^{1/2}$.*
Then, if $i \in \mathcal{I}$ with $h_i \leq S_0$, $l \in \{\kappa_i + 1, ..., \tau_i\}$, $\eta \in \widehat{K}^{(2)}$, we have $\nu_l(\eta)/2 \leq g_l(\eta) \leq 2 \cdot \nu_l(\eta)$.

Corollary 2.2 yields $\nu_l(\eta) \geq g_l(\eta)/2 > 0$. This, in turn, implies

Corollary 2.3 *In the situation of Corollary 2.2, the vectors $D_1\psi_l(\eta)$, $D_2\psi_l(\eta)$, with $\psi_l := \psi_l^{(i)}$, are linearly independent.*

Thus $\psi_l^{(i)}$ is a local parameter of ∂B_{R_i}, for $l \in \{\kappa_i+1, ..., \tau_i\}$, $i \in \mathcal{I}$, hence by (2.2),

$$\int_{\partial B_{R_i}} h\, do_y \;=\; \sum_{l=\kappa_i+1}^{\tau_i} \int_{F_l} h\, do_y \;=\; \sum_{l=\kappa_i+1}^{\tau_i} \int_{\widehat{K}^{(2)}} (h \circ \psi_l)(\eta) \cdot \nu_l(\eta)\, d\eta \tag{2.7}$$

for $i \in \mathcal{I}$ with $h_i \leq S_0$, $h \in L^1(\partial B_{R_i})$.

Corollary 2.4 *There is a constant $C > 0$ with the ensuing properties:*
Let $G \in H^1_{loc}(\overline{\Omega}^c)$, $H \in H^1_{loc}(\overline{\Omega}^c)^3$, $\mathcal{E}, \widetilde{\mathcal{E}}, \overline{\mathcal{E}} \in (0,\infty)$, $i \in \mathcal{I}$ with

$$h_i \leq S_0, \quad |G(t\cdot x)| + |H(t\cdot x)| \leq \mathcal{E}, \quad |G(x) - G(t\cdot x)| \;+\; |H(x) - H(t\cdot x)| \;\leq\; \widetilde{\mathcal{E}},$$
$$|G(x) + H(x)\cdot n^{(B_{R_i})}| \leq \overline{\mathcal{E}} \qquad \text{for } x \in \partial B_{R_i},\; t \in [(1 - h_i^2 \cdot S^{-2})^{1/2}, 1].$$

Abbreviate $a_i := G\,|\,\partial B_{R_i} + (H\,|\,\partial B_{R_i})\cdot n^{(B_{R_i})}$, $\quad b_i := G\,|\,\partial P_i\backslash\partial\Omega + (H\,|\,\partial P_i\backslash\partial\Omega)\cdot n^{(P_i)}$. *Then*

$$|\,\|a_i\|_2 - \|b_i\|_2\,| \;\leq\; C\cdot\big((\overline{\mathcal{E}}\cdot\widetilde{\mathcal{E}})^{1/2} \;+\; (\overline{\mathcal{E}}+\widetilde{\mathcal{E}})^{1/2}\cdot(\mathcal{E}\cdot h_i)^{1/2} \;+\; \widetilde{\mathcal{E}} \;+\; (\mathcal{E}+\overline{\mathcal{E}})\cdot h_i\big)\cdot R_i.$$

Proof: Let $l \in \{\kappa_i+1, ..., \tau_i\}$, $\eta \in \widehat{K}^{(2)}$, and put $t := |\varphi_l(\eta)|/R_i$. Then $|\psi_l(\eta)| = R_i$, $\varphi_l(\eta) = t\cdot\psi(\eta)$. Since $\varphi_l(\eta) \in \partial P_i\backslash\partial\Omega$, Corollary 2.1 yields $t \in [(1 - h_i^2/S^2)^{1/2}, 1]$. Moreover, for $g, \widetilde{g} \in \mathbb{R}$, $h, \widetilde{h}, n, \widetilde{n} \in \mathbb{R}^3$, we have

$$\begin{aligned} &|(g \;+\; h\cdot n)^2 - (\widetilde{g} \;+\; \widetilde{h}\cdot\widetilde{n})^2| \\ &\leq\; 2\cdot|g \;+\; h\cdot n|\cdot|g \;+\; h\cdot n \;-\; \widetilde{g} \;-\; \widetilde{h}\cdot\widetilde{n}| \;+\; (g \;+\; h\cdot n \;-\; \widetilde{g} \;-\; \widetilde{h}\cdot\widetilde{n})^2. \end{aligned}$$

Starting from these observations, we may deduce Corollary 2.4 from (2.3), (2.7), Lemma 2.4, Lemma 2.5 and Corollary 2.2. $\diamond$

3. Proof of Theorem 1.1

Let $i \in \mathcal{I}$. The space W_i defined in Theorem 1.1 is a Hilbert space with respect to the inner product

$$(u,v)^{(i)} := \int_{P_i} \sum_{j,k=1}^{3} D_j u_k \cdot D_j v_k\, dx \;+\; R_i^{-1}\cdot\int_{\partial P_i\backslash\partial\Omega} u\cdot v\, do_x \quad (u, v \in W_i).$$

We denote the corresponding norm by $|\;\;|^{(i)}$, that is,

$$|\,v\,|^{(i)} := \Big(\int_{P_i}\sum_{j,k=1}^{3}(D_j v_k)^2\, dx \;+\; R_i^{-1}\cdot\|\,v\,|\,\partial P_i\backslash\partial\Omega\,\|_2^2\Big)^{1/2} \quad \text{for } v \in W_i.$$

Theorem 3.1 *For $A \subset \mathbb{R}^3$, let $\mathcal{P}_1(A)$ denote the set of polynomials over A with degree less than or equal to 1, and put $|w|_{0,2} := \|w\|_2$ for $w \in L^2(A)^3$, $|w|_{1,2} := (\|D_1 w\|_2^2 + \|D_2 w\|_2^2 + \|D_3 w\|_2^2)^{1/2}$ for $w \in H^1(A)^3$. Set*

$$V_i := \{\, v \in C^0(\overline{P_i})^3 \;:\; v\,|\,K_l \in \mathcal{P}_1(K_l) \text{ for } 1 \leq l \leq \tau_i \,\} \quad \text{for } i \in \mathcal{I}.$$

Suppose there is $C_1 > 0$ and for each $i \in \mathcal{I}$, a linear operator $\Pi_i : H^1(P_i)^3 \mapsto V_i$ such that

$$\Big(\sum_{l=1}^{\tau_i}\big(\operatorname{diam} K_l\big)^{2\cdot k-2}\cdot|\,(w - \Pi_i(w))\,|\,K_l\,|_{k,2}^2\Big)^{1/2} \;\leq\; C_1\cdot|w|_{1,2} \tag{3.1}$$

for $w \in H^1(P_i)^3$, $k \in \{0,1\}$. Then there is a constant $C_2 > 0$ with

$$R_i^{-1/2}\cdot\|\,(w - \Pi_i(w))\,|\,\partial P_i\backslash\partial\Omega\,\|_2 \;\leq\; C\cdot|\,w\,|_{1,2} \quad \text{for } i \in \mathcal{I},\; w \in H^1(P_i)^3.$$

Proof: Let $i \in \mathcal{I}$, $w \in H^1(P_i)^3$, $l \in \{\kappa_i+1, ..., \tau_i\}$. We shall write $\mathcal{C}$ for constants which do not depend on i, w or l. For brevity, we set $v := w - \Pi_i(w)$. Recalling that $\mathcal{A}_l(\eta, 1-\eta_1-\eta_2) = \varphi_l(\eta)$ for $\eta \in \widehat{K}^{(2)}$ (see Section 2), we get

$$\begin{aligned} \int_{\widehat{K}^2}|\,v(\varphi_l(\eta))\,|^2\cdot g_l(\eta)\, d\eta \;&\leq\; \mathcal{C}\cdot(\operatorname{diam} K_l)^2\cdot\int_{\widehat{K}^{(2)}}|\,v(\varphi_l(\eta))\,|^2\, d\eta \\ &\leq\; \mathcal{C}\cdot(\operatorname{diam} K_l)^2\cdot\|\,v\circ\mathcal{A}_l\,|\,\widehat{K}^{(3)}\,\|_{1,2}^2, \end{aligned}$$

where the last inequality follows by a standard trace theorem on $\widehat{K}^{(3)}$. Using the transformation rule and recalling the standard estimate $\det A_l^{-1} \leq \mathcal{C}\cdot(\operatorname{diam} K_l)^{-3}$, we conclude

$$\begin{aligned} &\int_{\widehat{K}^{(2)}}|\,v(\varphi_l(\eta))\,|^2\cdot g_l(\eta)\, d\eta \\ &\leq\; \mathcal{C}\cdot\operatorname{diam} K_l\cdot\int_{K_l}\Big((\operatorname{diam} K_l)^{-2}\cdot|\,v(y)\,|^2 \;+\; |\,\nabla v(y)\,|^2\Big)\, dy. \end{aligned}$$

But $\operatorname{diam} K_l \leq h_i\cdot R_i/S \leq R_i$ by (A3), so Theorem 3.1 is implied by (2.3) and (3.1). $\diamond$

We need some further technical results related to the outer part $\partial P_i\backslash\partial\Omega$ of the boundary of P_i.

Theorem 3.2 *There is a constant $C > 0$ with*

$$\|\,w\,|\,\partial B_S\,\|_2 \;\leq\; C\cdot|\,w\,|^{(i)} \qquad \text{for } i \in \mathcal{I} \text{ with } h_i \leq S_0,\;\; w \in H^1(P_i)^3,$$

where S_0 was introduced in Corollary 2.2.

Proof: Take $i \in \mathcal{K}$ with $h_i \leq S_0$, $w \in C^1(\overline{P_i})$. Recall the definition of the patches $F_l = F_l^{(i)}$ and $T_l = T_l^{(i)}$ on ∂B_{R_i} and $\partial P_i\backslash\partial\Omega$, respectively, and of their local parameters $\varphi_l = \varphi_l^{(i)}$ and $\psi_l = \psi_l^{(i)}$, respectively ($\kappa_i + 1 \leq l \leq \tau_i$); see Section 2. Then $R_i^{-1}\cdot F_l$ is a subset of ∂B_1, and

$$(\varphi_l\circ\psi_l^{-1})(R_i\cdot y) \in T_l \subset \partial P_i\backslash\partial\Omega \qquad \text{for } y \in R_i^{-1}\cdot F_l \text{ and for } l \in \{\kappa_i+1, ..., \tau_i\}.$$

For such l, we set

$$\Gamma_l := \{\, r\cdot y \;:\; y \in R_i^{-1}\cdot F_l,\;\; r \in \mathbb{R} \text{ with } S \;<\; r \;<\; |\,(\varphi_l\circ\psi_l^{-1})(R_i\cdot y)\,|\,\}.$$

Then we have by (A5)

$$\cup\{\,\overline{\Gamma_l} \;:\; \kappa_i+1 \leq i \leq \tau_i\,\} \;=\; \overline{P_i}\backslash B_S, \qquad \Gamma_l\cap\Gamma_k = \emptyset \text{ for } k, l \in \{\kappa_i+1, ..., \tau_i\},\; k \neq l.$$

Thus, in view of (2.2) and (2.3), it suffices to show for $l \in \{\kappa_i + 1, ..., \tau_i\}$:

$$\int_{(S/R_i)\cdot F_l} |w|^2 \, do_x \quad \leq \quad C \cdot \Big(\int_{\Gamma_l} |\nabla w|^2 \, dx \; + \; R_i^{-1} \cdot \int_{T_l} |w|^2 \, do_x \Big). \tag{3.2}$$

Here and in the rest of this proof, the letter C denotes constants only depending on S. In order to prove (3.2), we fix $l \in \{\kappa_i + 1, ..., \tau_i\}$. Then we find for $y \in R_i^{-1} \cdot F_l$, with the abbreviation $\gamma := |(\varphi_l \circ \psi_l^{-1})(R_i \cdot y)|$:

$$\begin{aligned} |w(S \cdot y)| &= |w(S \cdot y) - w(\gamma \cdot y) + w(\gamma \cdot y)| = \Big| \int_S^\gamma \nabla w(r \cdot y) \cdot y \, dr + w(\gamma \cdot y) \Big| \\ &\leq \Big(\int_S^\gamma r^{-2} \, dr \Big)^{1/2} \cdot \Big(\int_S^\gamma r^2 \cdot |\nabla w(r \cdot y)|^2 \, dr \Big)^{1/2} + |w(\gamma \cdot y)|, \end{aligned}$$

hence

$$\begin{aligned} &\int_{R_i^{-1}\cdot F_l} S^2 \cdot |w(S \cdot y)|^2 \, do_y \\ &\leq \Big(S \cdot \int_S^\gamma \int_{R_i^{-1}\cdot F_l} r^2 \cdot |\nabla w(r \cdot y)|^2 \, do_y \, dr \; + \; (S/R_i)^2 \cdot A \Big), \end{aligned}$$

with

$$A := \int_{R_i^{-1}\cdot F_l} R_i^2 \cdot \Big| w(|(\varphi_l \circ \psi_l^{-1})(R_i \cdot y)| \cdot y) \Big|^2 \, do_y.$$

It follows

$$\int_{(S/R_i)\cdot F_l} |w|^2 \, do_x \quad \leq \quad C \cdot \Big(\int_{\Gamma_l} |\nabla w|^2 \, dx \; + \; R_i^{-1} \cdot A \Big).$$

The mapping $R_i^{-1} \cdot \psi_l$ is a local parameter of $R_i^{-1} \cdot F_l$, with $R_i^{-2} \cdot \nu_l$ as element of surface area. Thus, taking into account (2.3), Corollary 2.2 and the definition of ψ_l, we get

$$\begin{aligned} A &= \int_{\widehat{K}^{(2)}} |w(\varphi_l(\eta))|^2 \cdot \nu_l(\eta) \, d\eta \quad \leq \quad 2 \cdot \int_{\widehat{K}^{(2)}} |w(\varphi_l(\eta))|^2 \cdot g_l(\eta) \, d\eta \\ &= 2 \cdot \int_{T_l} |w(y)|^2 \, do_y. \end{aligned}$$

This estimate completes the proof of (3.2), and hence of Theorem 3.2. ◇

Theorem 3.3 *There is $C > 0$ such that for $i \in \mathcal{I}$ with $h_i \leq S_0$, $w \in H^1(\Omega_{R_i})^3$, we have*

$$R_i^{-1/2} \cdot \| w \,|\, \partial P_i \backslash \partial\Omega \|_2 \;\leq\; C \cdot (\| \nabla w \|_2 + R_i^{-1/2} \cdot \| w \,|\, \partial B_{R_i} \|_2).$$

Proof: Theorem 3.3 follows by arguments analogous to those used in the proof of Theorem 3.2. We omit the details. ◇

We note two consequences of the preceding theorems.

Theorem 3.4 *There is $C > 0$ with $\| w \,|\, B_S \backslash \overline{\Omega} \|_2 \leq C \cdot |w|^{(i)}$ for $i \in \mathcal{I}$ with $h_i \leq S_0$, $w \in W_i$.*

Proof: According to [1, Lemma 4.1], there is some $C > 0$ with

$$\|v\|_2 \leq C \cdot (\| \nabla v \|_2 + S^{-1/2} \cdot \| v \,|\, \partial B_S \|_2) \quad \text{for } v \in H^1(B_S \backslash \overline{\Omega})^3 \text{ with } v \,|\, \partial\Omega = 0.$$

Theorem 3.4 follows from this result and Theorem 3.2. ◇

Theorem 3.5 *There is a constant $C > 0$ and for any $i \in \mathcal{I}$ with $h_i \leq S_0$, an operator $\mathcal{D}_i : L^2(P_i) \mapsto W_i$ such that $div \mathcal{D}_i(\pi) = \pi$, $|\mathcal{D}_i(\pi)|^{(i)} \leq C \cdot \| \pi \|_2$ for $\pi \in L^2(P_i)$.*

Proof: According to [1, Theorem 4.1], there is a constant $C > 0$ and for any $i \in \mathcal{I}$, an operator $\widetilde{\mathcal{D}}_i : L^2(\Omega_{R_i}) \mapsto H^1(\Omega_{R_i})^3$ such that we have for $\varrho \in L^2(\Omega_{R_i})$:

$$\operatorname{div} \widetilde{\mathcal{D}}_i(\varrho) = \varrho, \quad \widetilde{\mathcal{D}}_i(\varrho) \,|\, \partial\Omega = 0, \quad \| \nabla \widetilde{\mathcal{D}}_i(\varrho) \|_2 + R_i^{-1/2} \cdot \| \widetilde{\mathcal{D}}_i(\varrho) \,|\, \partial B_{R_i} \|_2 \;\leq\; C \cdot \| \varrho \|_2.$$

Thus, if we set $\mathcal{D}_i(\varrho) := \widetilde{\mathcal{D}}_i(\widetilde{\varrho}) \,|\, P_i$ for $i \in \mathcal{I}$, $\varrho \in L^2(P_i)$, where $\widetilde{\varrho}$ denotes the trivial extension of ϱ to Ω_{R_i}, then Theorem 3.5 readily follows from Theorem 3.3. ◇

Now we are in a position to carry out the

Proof of Theorem 1.1: Take $i \in \mathcal{I}$ with $h_i \leq S_0$. Let $C > 0$ denote constants in $(0, \infty)$ which only depend on Ω, S and the parameter σ_2 from Lemma 2.1. We have

$$(|v|^{(i)})^2 \;\leq\; C \cdot \alpha_i(v, v), \quad |\alpha(v, w)| \leq C \cdot |v|^{(i)} \cdot |w|^{(i)} \quad \text{for } v, w \in H^1(P_i)^3. \tag{3.3}$$

Moreover, by Theorem 3.5, the pair of spaces $(W_i, L^2(P_i))$ satisfies the Babushka-Brezzi condition, uniformly in $i \in \mathcal{I}$. Due to these results, we may apply the theory of mixed variational problems, as presented in [7, p. 57-61]. It follows there is a pair $(v_i, \varrho_i) \in H^1(P_i)^3 \times L^2(P_i)$ satisfying (1.6). The first inequality in (3.3) and a partial integration which exploits (1.1) yield

$$\begin{aligned} &\| \nabla(u \,|\, P_i - v) \|_2 + R_i^{1/2} \cdot \| (u - v) \,|\, \partial P_i \backslash \partial\Omega \|_2 \\ &\leq C \cdot R_i^{1/2} \cdot \| \mathcal{L}_{R_i}(P_i \cup \overline{\Omega})(u, \pi) \|_2; \end{aligned} \tag{3.4}$$

compare the proof of [1, Theorem 5.1]. Using Theorem 3.5, we may deduce from (3.4) that the term $\| \pi \,|\, P_i - \varrho \|_2$ is also bounded by the right-hand side of (3.4). Details of this argument can again be found in the proof of [1, Theorem 5.1]. Due to (3.4) and Theorem 3.4, the term $\|(u - v) \,|\, \Omega_S \|_2$, too, may be seen to be bounded by the right-hand side of (3.4). Now inequality (1.7) follows with Corollary 2.4. ◇

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On the Stefan problem with surface tension

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Abstract

We present results on the uniqueness and the existence of smooth solutions to the one and two-phase Stefan problem with surface tension.

1 Introduction

The classical Stefan problem is a model for phase transitions in solid-liquid systems and accounts for heat diffusion and exchange of latent heat in a homogeneous medium. The strong formulation of this model corresponds to a moving boundary problem involving a parabolic diffusion equation for each phase and a transmission condition prescribed at the interface separating the phases. Molecular considerations attempting to explain supercooling and dendritic growth of crystals suggest to also include surface tension on the interface separating the solid from the liquid region.

In order to formulate the Stefan problem, we introduce the following notations. Let Ω be a smooth bounded domain in $\mathbb{R}^n$, whose boundary $\partial\Omega$ consists of two disjoint components, an 'interior' part J^1 and an 'exterior' part J^2. We think of Ω as a homogeneous medium which is occupied by a liquid and a solid phase, say water and ice, that initially occupy the regions Ω_0^1 and Ω_0^2, and that Ω_0^1 and Ω_0^2 are separated by a sharp interface Γ_0. More precisely, we assume that $\Gamma_0 \subset \Omega$ is a compact closed hypersurface, that Ω_0^1 and Ω_0^2 are disjoint open sets such that $\overline{\Omega} = \overline{\Omega_0^1} \cup \overline{\Omega_0^2}$ and such that $\partial\Omega_0^i = J^i \cup \Gamma_0$ for $i = 1, 2$. For the sake of definiteness, we consider the open set Ω_0^1 as the region occupied by the liquid phase. Consequently, the component J^1 is in contact with the liquid phase and J^2 is in contact with the solid phase. The boundaries J^1 and J^2, corresponding for