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The Single-Layer Potential Associated with the Time-dependent Oseen System

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Abstract: We consider a boundary value problem for the time-dependent Oseen system in a 3D exterior domain. This problem is reduced to an integral equation for the single-layer potential related to the that system. This integral equation, in turn, is solved by applying a result by Shen [11] pertaining to the evolutionary Stokes system. Since it automatically leads to an integral representation of the solution of our boundary value problem, our approach may yield a new access for studying spatial decay of exterior time-dependent Oseen flows.

Key Words: Exterior domain, Navier-Stokes system, time-dependent Oseen system, decay.

1 Introduction

The steady motion of a rigid body in a viscous incompressible fluid is usually modeled by the incompressible Navier-Stokes system in an exterior domain $\mathbb{R}^3\backslash\overline{\Omega}$, with the open bounded set $\Omega \subset \mathbb{R}^3$ representing the rigid object. Its steady motion is then described by a boundary condition which states that the velocity of the fluid at infinity converges to a nonzero constant vector. By a normalization procedure, we may assume this constant vector to be equal to $(1,0,0)$. If in this situation, we add the vector $(-1,0,0)$ to the velocity, we obtain a homogeneous boundary condition at infinity. In this way, we arrive at the following initial-boundary value problem:

$$\begin{aligned}&\partial_t v(x,t) - \Delta_x v(x,t) + \tau \cdot \partial_{x_1} v(x,t) \qquad (1)\\ &\quad +\tau \cdot \big(v(x,t)\cdot \nabla_x\big) v(x,t) + \nabla_x p(x,t)\\ &\quad = f(x,t), \quad \mathrm{div}_x v(x,t) = 0\\ &\quad \text{for } x \in \mathbb{R}^3\backslash\overline{\Omega},\ t \in (0,T),\end{aligned}$$

$$v(x,t) = (-1,0,0) \text{ for } x \in \partial\Omega,\ t \in (0,T), \quad (2)$$

$$v(x,t) \to 0 \ (|x| \to \infty) \ \text{ for } t \in (0,T), \quad (3)$$

$$v(x,0) = b(x) \ \text{ for } x \in \mathbb{R}^3\backslash\overline{\Omega}, \quad (4)$$

for some $T \in (0,\infty)$, where $\tau \in (0,\infty)$ is the Reynolds number, f a volume force, v the velocity field of the fluid, p its pressure field, and b its velocity field at the instant $t = 0$. The velocity v and the pressure p are the unknowns; all the other quantities are given. The term $\tau \cdot \partial_{x_1} v$ (the "Oseen term") arises due to the translation of the velocity by the factor $(-1,0,0)$. For the same reason, the usual homogeneous boundary condition on $\partial\Omega \times (0,T)$ is replaced by a non-homogeneous one. Of course, the domain Ω introduced at the beginning is also modified by this translation, but we used the same notation for the modified domain as for the original one. We will follow the usual custom and call the differential equation in (1) "Navier-Stokes system", although it is different from this system because of the Oseen term.

A solution of (1) - (4) should reflect the main features of the of the physical flow in question. In particular, such a solution should exhibit a "wake". This means that in a paraboloidal downstream region, the velocity converges slower than elsewhere to its constant boundary value at infinity. Such an asymptotic behaviour is well established for solutions of the stationary Navier-Stokes system, which describes a flow whose streamlines do not change with time. We refer to [5, Section IX.8] and the references therein for results on this stationary case.

Less is known on solutions of the instationary problem (1) - (4). Knightly [6] seems to have been the first to consider it; his results are valid only under restrictive smallness assumptions. Mizumachi [9] exhibits spatial decay rates of the velocity which are the same as the ones found in the stationary case. In particular, he shows the existence of a wake. But he starts from the assumption that the velocity decays pointwise in time and in space. It is an open question whether this assumption may be eliminated. Moreover he does not consider how the gradient of the velocity or its second derivatives or the pressure decay in space or time. Shibata e.a. [12], [4] consider L^p-

norms of solutions of a problem somewhat more general than (1) - (4), with these L^p-norms taken with respect to the space variables. They study how these norms decay in time, under the assumption that the Reynolds number τ is small. But of course, L^p-norms with respect to the space variables do not carry much information on the spatial decay of the functions under consideration.

In the work at hand, we consider a suitable linearization of (1) - (4), where "suitable" means that the linear problem is well suited for studying the spatial decay of a solution to (1) - (4). This linearization is obtained by first dropping the nonlinear term $\tau \cdot (v \cdot \nabla_x)v$, and then subtracting a solution of the stationary problem

$$-\Delta w + \tau \cdot \partial_1 w + \nabla q = 0, \operatorname{div} w = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{\Omega},$$
$$w \,|\, \partial\Omega = (-1,0,0), \quad w(x) \to 0 \;\; (|x| \to \infty),$$

from the solution of equations (1) - (4) without a nonlinear term in (1). In this way we arrive at the following linear problem with a homogeneous boundary condition on $\partial\Omega \times (0,\infty)$:

$$\partial_t u(x,t) - \Delta_x u(x,t) + \tau \cdot \partial_{x_1} u(x,t) \qquad (5)$$
$$+\nabla_x \pi(x,t) = f(x,t), \;\; \operatorname{div}_x u(x,t) = 0$$
$$\text{for } x \in \mathbb{R}^3\backslash\overline{\Omega}, \;\; t \in (0,\infty),$$
$$u(x,t) = 0 \;\text{ for }\; x \in \partial\Omega, \;\; t \in (0,\infty), \qquad (6)$$
$$u(x,t) \to 0 \; (|x| \to \infty) \;\text{ for }\; t \in (0,\infty), \qquad (7)$$
$$u(x,0) = a(x) \;\text{ for }\; x \in \mathbb{R}^3\backslash\overline{\Omega}, \qquad (8)$$

where the velocity is now denoted by u, and the pressure by π. The partial differential equations in (5) are called "(time-dependent) Oseen system". Shibata e.a. [7], [12], [3], [4] consider time decay of spatial L^p-norms of solutions to (5) - (8). But as far as we know, no spatial decay rates of these solutions have been exhibited up to now. It seems that a crucial problem consisted in the lack of a suitable representation formula for these solutions.

With such a formula in mind, we will solve (5) - (8) by applying the method of integral equations. We will use a variant of this method whose main idea consists in reducing the given initial-boundary value problem to an integral equation on $\partial\Omega\times(0,\infty)$, which involves the single-layer potential related to the time-dependent Oseen system. This approach automatically yields an integral representation of our solutions – a representation which should be well adapted to the study of the asymptotic behaviour of these solutions. A first result in this respect is given in Lemma 18 below.

We remark that the present article should be considered as an overview; some of the proofs will be presented elsewhere.

2 Notations. Volume potentials.

We fix $\tau \in (0,\infty)$, an open bounded set $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary, as well as functions $f \in C_0^\infty\big(\mathbb{R}^3 \times (0,\infty)\big)^3$, $a \in C_0^\infty(\mathbb{R}^3\backslash\overline{\Omega})^3$ with $\operatorname{div} a = 0$. Of course, the smoothness assumptions on f and a may be reduced, but this is a point we do not want to pursue here. Let N denote the outward unit normal to Ω. Moreover, let $m^{(\Omega)} \in C_0^\infty(\mathbb{R}^3)^3$ be a non-tangential vector field to Ω, that is, there are constants $\epsilon(\Omega)$, $\widetilde{\epsilon}(\Omega) \in (0,\infty)$ such that

$$|x + \delta \cdot m^{(\Omega)}(x) - x' - \delta' \cdot m^{(\Omega)}(x')| \qquad (9)$$
$$\geq \; \epsilon(\Omega) \cdot \big(|x - x'| + |\delta - \delta'|\big)$$

for $x, x' \in \partial\Omega$, $\delta, \delta' \in [-\widetilde{\epsilon}(\Omega), \widetilde{\epsilon}(\Omega)]$, and

$$x + \delta \cdot m^{(\Omega)}(x) \in \mathbb{R}^3\backslash\overline{\Omega}, \;\; x - \delta \cdot m^{(\Omega)}(x) \in \Omega \qquad (10)$$

for $x \in \partial\Omega$, $\delta \in (0, \widetilde{\epsilon}(\Omega)]$. Some indications on how to construct such a field are given in [10, p. 246]. Since Ω is only Lipschitz bounded, the relations in (9) and (10) do not hold in general when $m^{(\Omega)}$ is replaced by the outward unit normal N to Ω.

Put $B_r := \{y \in \mathbb{R}^3 \;:\; |y| < r\}$, $B_r^c := \mathbb{R}^3\backslash B_r$ for $r \in (0,\infty)$. We fix some $R_0 > 0$ with $\overline{\Omega} \subset B_{R_0/2}$. The letter C will denote constants which only depend on Ω, R_0 or τ. If such a constant depends on τ, it is an increasing function of this parameter.

We further define $S_T := \partial\Omega \times (0,T)$ for $T \in (0,\infty]$, $|\alpha|_1 := \alpha_1 + \alpha_2 + \alpha_3$ (length of α) for multi-indices $\alpha \in \mathbb{N}_0^3$, $e_1 := (1,0,0)$, $s(x) := |x| - x_1$ for $x \in \mathbb{R}^3$. We write $< , >$ for the usual scalar product in $\mathbb{R}^3$. Let H denote the usual fundamental solution of the heat equation in $\mathbb{R}^3$, that is,

$$H(z,t) := (4 \cdot \pi \cdot t)^{-3/2} \cdot e^{-|z|^2/(4\cdot t)}$$

for $t \in (0,\infty)$, $z \in \mathbb{R}^3$. We further introduce a fundamental solution of the time-dependent Stokes system by setting as in [11]:

$$\Gamma_{jk}(z,t) := \delta_{jk} \cdot H(z,t) + \int_t^\infty \partial_j \partial_k H(z,s)\, ds,$$

$$E_k(x) := (4 \cdot \pi)^{-1} \cdot x_k \cdot |x|^{-3}$$

for $z \in \mathbb{R}^3$, $t \in (0,\infty)$, $x \in \mathbb{R}^3\backslash\{0\}$, $1 \leq j,k \leq 3$. Finally we define a fundamental solution of the time-dependent Oseen system by putting

$$\Lambda_{jk}(z,t,\kappa) := \Gamma_{jk}(z - \kappa \cdot t \cdot e_1, t)$$

for $z \in \mathbb{R}^3$, $t, \kappa \in (0,\infty)$, $j,k \in \{1, 2, 3\}$. The following results are well known:

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Lemma 1 *The functions H, Γ_{jk} belong to $C^\infty\big(\mathbb{R}^3 \times (0,\infty)\big)$, and E_k to $C^\infty(\mathbb{R}^3\backslash\{0\})$, for $1 \le j,k \le 3$. Moreover,*

$$\big|\,\partial_t^l\partial_z^\alpha H(z,t)\,\big| + \big|\,\partial_t^l\partial_z^\alpha \Gamma_{jk}(z,t)\,\big| \\ \le\ \mathcal{C}\cdot(|z|^2+t)^{-3/2-|\alpha|_1/2-l},$$

$$|\partial_x^\alpha E_k(x)| \le \mathcal{C}\cdot|x|^{-2-|\alpha|_1}$$

for $z \in \mathbb{R}^3,\ t \in (0,\infty),\ x \in \mathbb{R}^3\backslash\{0\},\ 1 \le j,k \le 3,\ \alpha \in \mathbb{N}_0^3,\ l \in \mathbb{N}_0$ with $|\alpha|_1 + 2\cdot l \le 4$.

Of course, the estimates in the previous lemma also hold if $|\alpha|_1 + 2\cdot l \le m$, for any given $m \in \mathbb{N}$, with the constant $\mathcal{C}$ additionally depending on m. However, we will not need this fact here. In order to obtain a similar estimate for Λ_{jk}, we observe

Lemma 2 *Let $K \in (0,\infty)$. Then there is $C(K) > 0$ such that for $x \in \mathbb{R}^3,\ \kappa, t \in (0,\infty),\ r \in [0,t]$,*

$$\big(|x-\kappa\cdot r\cdot e_1|^2 + t\big)^{-1} \\ \le\ C(K)\cdot\max\{1,\kappa\}\cdot\gamma(x,t)^{-1},$$

where

$$\gamma(x,t) := |x|^2 + t, \quad \textit{if } |x| \le K,$$

$$\gamma(x,t) := |x|\cdot\big(1+\kappa\cdot s(x)\big) + t, \quad \textit{if } |x| > K.$$

Proof: In this proof, the symbol $\mathcal{D}$ denotes constants only depending on K. Let x, κ, t, r be given as in the lemma. Instead of $\max\{1,\kappa\}$, we write $1 \vee \kappa$. Abbreviate $\mathcal{G} := |x-\kappa\cdot r\cdot e_1|^2 + t$. We distinguish several cases.

1st case: $\kappa\cdot s(x) \le K$ and $|x| > K$. Then

$$\begin{aligned} |x| + t\ &\le\ |x-\kappa\cdot r\cdot e_1| + \kappa\cdot r + t \\ &\le\ |x-\kappa\cdot r\cdot e_1| + 2\cdot(\kappa\vee 1)\cdot t \\ &\le\ (2\cdot K)^{-1}\cdot|x-\kappa\cdot r\cdot e_1|^2 + K/2 \\ &\qquad +(\kappa\vee 1)\cdot 2\cdot t \\ &\le\ \mathcal{D}\cdot(\kappa\vee 1)\cdot\mathcal{G} + K/2, \end{aligned}$$

hence $|x| - K/2 + t\ \le\ \mathcal{D}\cdot(\kappa\vee 1)\cdot\mathcal{G}$, so that by the assumption $|x| \ge K$:

$$(1/2)\cdot|x| + t\ \le\ \mathcal{D}\cdot(\kappa\vee 1)\cdot\mathcal{G}.$$

Now it follows that $|x|+t \le \mathcal{D}\cdot(\kappa\vee 1)\cdot\mathcal{G}$. Exploiting this inequality and the assumption $\kappa\cdot s(x) \le K$, we find

$$\begin{aligned} |x|\cdot\big(1+\kappa\cdot s(x)\big) + t\ &\le\ |x|\cdot(1+K) + t \\ &\le\ (1+K)\cdot(|x|+t)\ \le\ \mathcal{D}\cdot(\kappa\vee 1)\cdot\mathcal{G}. \end{aligned}$$

2nd case: $\kappa\cdot s(x) > K$ and $x_1 > 0$. Then

$$\begin{aligned} |(x_2,x_3)|^2 &= |x|^2 - x_1^2 = (|x|+x_1)\cdot s(x) \\ &=\ \kappa^{-1}\cdot\kappa\cdot(|x|+x_1)\cdot s(x) \\ &\ge\ \kappa^{-1}\cdot(|x|+x_1)\cdot\big(K+\kappa\cdot s(x)\big)/2, \end{aligned}$$

where the last inequality follows from the assumption $\kappa\cdot s(x) > K$. Thus we obtain

$$\begin{aligned} &|(x_2,x_3)|^2 \\ &\quad\ge\ \mathcal{D}\cdot\kappa^{-1}\cdot(|x|+x_1)\cdot\big(1+\kappa\cdot s(x)\big) \\ &\quad\ge\ \mathcal{D}\cdot\kappa^{-1}\cdot|x|\cdot\big(1+\kappa\cdot s(x)\big). \end{aligned}$$

with the last inequality holding because of the assumption $x_1 > 0$. Now we find

$$\begin{aligned} \mathcal{G}\ &\ge\ |(x_2,x_3)|^2 + t \\ &\ge\ \mathcal{D}\cdot\kappa^{-1}\cdot|x|\cdot\big(1+\kappa\cdot s(x)\big)\ +\ t \\ &\ge\ \mathcal{D}\cdot(1\vee\kappa)^{-1}\cdot\big(|x|\cdot\big(1+\kappa\cdot s(x)\big) + t\big). \end{aligned}$$

3rd case: $\kappa\cdot s(x) > K,\ x_1 \le 0$.

Then $\kappa\cdot r - x_1 \ge -x_1 \ge 0$, hence $(\kappa\cdot r - x_1)^2 \ge x_1^2$, and we may conclude that

$$\begin{aligned} |x|^2 &= |(x_2,x_3)|^2 + x_1^2 \\ &\le\ |(x_2,x_3)|^2 + (x_1-\kappa\cdot r)^2 \\ &=\ |x-\kappa\cdot r\cdot e_1|^2. \end{aligned} \tag{11}$$

Moreover, $s(x) \le 2\cdot|x|$ so that

$$\begin{aligned} &\kappa\cdot s(x)\cdot|x| + t \le 2\cdot\kappa\cdot|x|^2 + t \\ &\quad\le\ 2\cdot(\kappa\vee 1)\cdot(|x|^2+t) \le 2\cdot(\kappa\vee 1)\cdot\mathcal{G}, \end{aligned} \tag{12}$$

with the last inequality following from (11). On the other hand, we assumed $\kappa\cdot s(x) > K$, so that

$$\begin{aligned} \kappa\cdot s(x)\ &\ge\ \big(K+\kappa\cdot s(x)\big)/2 \\ &\ge\ (1/2)\cdot\min\{1,K\}\cdot\big(1+\kappa\cdot s(x)\big). \end{aligned}$$

We conclude that

$$\begin{aligned} &\kappa\cdot s(x)\cdot|x| + t \\ &\quad\ge\ \mathcal{D}\cdot\big(\big(1+\kappa\cdot s(x)\big)\cdot|x| + t\big), \end{aligned}$$

hence by (12):

$$\big(1+\kappa\cdot s(x)\big)\cdot|x| + t\ \le\ \mathcal{D}\cdot(1\vee\kappa)\cdot\mathcal{G}.$$

4th case: $|x| \le K,\ \kappa\cdot t \le K$. Then

$$\begin{aligned} |x|^2 + t\ &\le\ 2\cdot|x-\kappa\cdot r\cdot e_1|^2 + 2\cdot\kappa^2\cdot r^2 + t \\ &\le\ 2\cdot|x-\kappa\cdot r\cdot e_1|^2 + 2\cdot K\cdot\kappa\cdot t + t, \end{aligned}$$

where we used that $r \le t$ and $\kappa\cdot t \le K$. We may conclude that $|x|^2 + t\ \le\ \mathcal{D}\cdot(1\vee\kappa)\cdot\mathcal{G}$.

5th case: $|x| \le K$ and $\kappa\cdot t > K$. Then

$$\begin{aligned} |x|^2 + t\ &\le\ K^2 + t\ \le\ K\cdot\kappa\cdot t + t \\ &\le\ \mathcal{D}\cdot(\kappa\vee 1)\cdot t\ \le\ \mathcal{D}\cdot(1\vee\kappa)\cdot\mathcal{G}. \end{aligned}$$

Thus we have found in any case that

$$\gamma(x,t) \le \mathcal{D}\cdot(1\vee\kappa)\cdot\mathcal{G}.$$

This proves the lemma. □

Combining Lemma 1 and Lemma 2 yields

Lemma 3 *The function $\Lambda_{jk}(\,\cdot\,,\,\cdot\,,\kappa)$ belongs to the space $C^\infty(\mathbb{R}^3 \times (0,\infty))$, for $\kappa > 0$, $1 \le j,k \le 3$.*

For any $K > 0$, there is some $C(K) > 0$ with

$$\left|\,\partial_t^l \partial_z^\alpha \Lambda_{jk}(z,t,\kappa)\,\right| \le C(K) \cdot \max\{1,\kappa\}^{3/2+|\alpha|_1/2+l}$$

LC [$+\,\gamma(x,t)^{-3/2-|\alpha|_1/2-l/2}\big)$

$$\cdot\, \gamma(z,t)^{-3/2-|\alpha|_1/2-l},$$

for z, t, j, k, α, l as in Lemma 1, $\gamma(z,t)$ as in Lemma 2, and for $\kappa \in (0,\infty)$.

Next we introduce some volume potentials related to the time-dependent Oseen system. We put

$$\mathcal{R}_j(f)(x,t) := \int_0^t \int_{\mathbb{R}^3} \sum_{k=1}^3 \Lambda_{jk}(x-y,t-\sigma,\tau) \cdot f_k(y,\sigma)\,dy\,d\sigma,$$

$$\mathcal{P}(f)(x,t) := \int_{\mathbb{R}^3} \sum_{k=1}^3 E_k(x-y) \cdot f_k(y,t)\,dy,$$

$$\mathcal{I}(a)(x,0) := a(x), \quad \mathcal{I}_j(a)(x,r) := \int_{\mathbb{R}^3} H(x-\tau\cdot r\cdot e_1 - y, r) \cdot a_j(y)\,dy,$$

for $x \in \mathbb{R}^3$, $t \in [0,\infty)$, $r \in (0,\infty)$, $1 \le j \le 3$. Then we have

Theorem 4 *The functions $\mathcal{R}_j(f)$, $\mathcal{P}(f)$ and $\mathcal{I}_j(a)$ belong to $C^\infty(\mathbb{R}^3 \times [0,\infty))$, for $1 \le j \le 3$. Moreover, for $x \in \mathbb{R}^3$, $t \in (0,\infty)$,*

$$\partial_t \mathcal{R}(f)(x,t) - \Delta_x \mathcal{R}(f)(x,t) + \tau\cdot\partial_{x_1}\mathcal{R}(f)(x,t) + \nabla_x \mathcal{P}(f)(x,t) = f(x,t),$$

$$\operatorname{div}_x \mathcal{R}(f)(x,t) = 0, \quad \mathcal{R}(f)(x,0) = 0,$$

$$\partial_t \mathcal{I}(a)(x,t) - \Delta_x \mathcal{I}(a)(x,t) + \tau\cdot\partial_{x_1}\mathcal{I}(a)(x,t) = 0,$$

$$\operatorname{div}_x \mathcal{I}(a)(x,t) = 0, \quad \mathcal{I}(a)(x,0) = a(x).$$

3 The single-layer potential related to the evolutionary Oseen system.

We want to apply a result from the proof of [11, Theorem 5.2.1]. To this end, we introduce some function spaces used in that reference. Put

$$L_n^2(\partial\Omega) := \{v \in L^2(\partial\Omega)^3 : \int_{\partial\Omega} < v, N > d\Omega = 0\}.$$

Let $T \in (0,\infty]$. (Note that the case $T = \infty$ is admitted.) Then we put

$$L_n^2(S_T) := \{v \in L^2(S_T)^3 : v(\,\cdot\,,t) \in L_n^2(\partial\Omega) \text{ for almost every } t \in (0,T)\},$$

$$\mathcal{H}_T := \{v\,|\,S_T : v \in C_0^\infty(\mathbb{R}^4)^3,\ v\,|\,\mathbb{R}^3 \times (-\infty,0] = 0\}.$$

For $v \in C^1((-\infty,T))$ with $v\,|\,(-\infty,0] = 0$, and for $t \in (0,T)$, we put

$$\partial_t^{1/2} v(t) := \Gamma(1/2)^{-1} \cdot \partial_t\Big(\int_0^t (t-r)^{-1/2} \cdot v(r)\,dr\Big)$$

("fractional derivative of v"), where Γ denotes the usual gamma function. We further define for $v \in \mathcal{H}_T$:

$$\|v\|_{H^{1,1/2}(S_T)} := \Big(\int_0^T \big(\,\|v(\,\cdot\,,t)\|_{1,2}^2 + \int_{\partial\Omega} |\,\partial_t^{1/2} v(x,t)\,|^2\,d\Omega(x)\,\big)\,dt\Big)^{1/2},$$

where $\|\ \ \|_{1,2}$ denotes the usual norm of $H^1(\partial\Omega)^3$. Obviously the mapping $\|\ \ \|_{H^{1,1/2}(S_T)}$ is a norm on $\mathcal{H}_T$. For any $v \in L^2(S_T)$, we may define $F_v \in L^2(0,T,H^1(\partial\Omega)')$ by setting for $\sigma \in H^1(\partial\Omega)$ and for almost every $t \in (0,T)$:

$$F_v(t)(\sigma) := \int_{\partial\Omega} v(x,t)\cdot\sigma(x)\,dx.$$

We will write v instead of F_v. For $v \in \mathcal{H}_T$, set

$$< \partial_t v, N > (x,t) := < \partial_t v(x,t), N(x) > \quad \text{for } (x,t) \in S_T,$$

$$\|v\|_{H_T} := \|v\|_{H^{1,1/2}(S_T)} + \|< \partial_t v, N >\|_{L^2(0,T,H^1(\partial\Omega)')}.$$

The mapping $\|\ \ \|_{H_T}$ is a norm on $\mathcal{H}_T$. Let the space H_T consist of all functions $v \in L_n^2(S_T)$ such that there exists a sequence (w_n) in $C_0^\infty(\mathbb{R}^4)^3$ with the properties that

$$w_n\,|\,S_T \in \mathcal{H}_T \text{ for } n \in \mathbb{N}, \quad \|v - w_n\,|\,S_T\|_2 \to 0,$$

and $(w_n\,|\,S_T)$ is a Cauchy sequence with respect to the norm $\|\ \ \|_{H_T}$. This means in particular that the sequences $\big(\|< \partial_t w_n, N >\|_{L^2(0,T,H^1(\partial\Omega)')}\big)$ and $\big(\|w_n\,|\,S_T\|_{H^{1,1/2}(S_T)}\big)$ are convergent. Their limit value does not depend on the choice of the sequence (w_n) with the above properties, for a given $v \in H_T$. Thus, for $v \in H_T$, we may define the quantity $\|v\|_{H_T}$ in an obvious way. The mapping $\|\ \ \|_{H_T}$ is a norm on H_T, and the pair $(H_T, \|\ \ \|_{H_T})$ is a Banach space.

Next we introduce our single-layer potentials. For $\phi \in L^2(S_T)^3$, $\kappa \in (0,\infty)$, $x \in \mathbb{R}^3$, $t \in [0,T] \cap \mathbb{R}$,

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we put

$$\mathcal{V}_T^{(0)}(\phi)(x,t) := \Big(\int_0^t \int_{\partial\Omega} \sum_{k=1}^{3} \Gamma_{jk}(x-y, t-\sigma) \cdot \phi_k(y,\sigma)\, d\Omega(y)\, d\sigma \Big)_{1\le j\le 3},$$

$$\mathcal{V}_T^{(\kappa)}(\phi)(x,t) := \Big(\int_0^t \int_{\partial\Omega} \sum_{k=1}^{3} \Lambda_{jk}(x-y, t-\sigma, \kappa) \cdot \phi_k(y,\sigma)\, d\Omega(y)\, d\sigma \Big)_{1\le j\le 3}.$$

We call $\mathcal{V}^{(0)}(\phi)$ and $\mathcal{V}^{(\kappa)}(\phi)$ the "single-layer potential related to the time-dependent Stokes- and Oseen-system", respectively. We further set

$$Q_T(\phi)(x,t) := \int_{\partial\Omega} \sum_{k=1}^{3} E_k(x-y)\cdot \phi(y,t)\, d\Omega(y)$$

for $\phi \in L^2(S_T)^3$, $x \in \mathbb{R}^3\backslash\partial\Omega$, $t \in [0,T] \cap \mathbb{R}$. The following lemma holds:

Lemma 5 *Let* $T \in (0,\infty]$, $\kappa \in [0,\infty)$, $\phi \in L^2(S_T)^3$. *Then the function*

$$v := \mathcal{V}_T^{(\kappa)}(\phi) \,|\, (\mathbb{R}^3\backslash\partial\Omega) \times ([0,T]\cap\mathbb{R})$$

belongs to $C^\infty\big((\mathbb{R}^3\backslash\partial\Omega) \times ([0,T]\cap\mathbb{R}) \big)^3$.

Set $q := Q_T(\phi)$. *Then* $q(\cdot, t) \in C^\infty(\mathbb{R}^3\backslash\partial\Omega)$ *and*

$$\partial_t v(x,t) - \Delta_x v(x,t) + \tau\cdot \partial_{x_1} v(x,t) + \nabla_x q(x,t) = 0,$$
$$\mathrm{div}_x v(x,t) = 0, \quad v(x,0) = 0$$
$$\text{for } x \in \mathbb{R}^3\backslash\partial\Omega,\ t \in (0,T).$$

The restriction $\mathcal{V}_T^{(\kappa)}(\phi) \,|\, S_T$ may be considered as the boundary value of $\mathcal{V}_T^{(\kappa)}(\phi) \,|\, (\mathbb{R}^3\backslash\overline{\Omega}) \times (0,T)$ on S_T in the following sense:

Lemma 6 *Take* T, κ, ϕ *as in Lemma 5. Then*

$$\mathcal{V}_T^{(\kappa)}(\phi) \,|\, S_T \in L^2(S_T)^3,$$

$$\int_0^T \int_{\partial\Omega} \Big| \mathcal{V}_T^{(\kappa)}(\phi)(x+\epsilon\cdot m^{(\Omega)}(x),\, t) - \mathcal{V}_T^{(\kappa)}(\phi)(x,t) \Big|^2 d\Omega(x)\, dt \longrightarrow 0 \ \ (\epsilon \downarrow 0).$$

Stronger convergence results are probably valid; for example, see [11, Theorem 2.1.7] in the case $T < \infty$ and $\kappa = 0$. But this is also a point we do not want to pursue here.

Now we can state how the problem of finding a solution to (5) - (8) reduces to an integral equation on S_∞:

Lemma 7 *Let* $\phi \in L^2(S_\infty)^3$ *with*

$$\mathcal{V}_\infty^{(\tau)}(\phi) \,|\, S_\infty = \big(-\mathcal{R}(f) - \mathcal{I}(a) \big) \,|\, S_\infty. \qquad (13)$$

Put

$$u := \big(\mathcal{R}(f) + \mathcal{I}(a) + \mathcal{V}_\infty^{(\tau)}(\phi) \big) \,|\, (\mathbb{R}^3\backslash\overline{\Omega}) \times [0,\infty), \qquad (14)$$

$$\pi := \big(\mathcal{P}(f) + \mathcal{Q}(\phi) \big) \,|\, (\mathbb{R}^3\backslash\overline{\Omega}) \times (0,\infty). \qquad (15)$$

Then $u \in C^\infty\big((\mathbb{R}^3\backslash\overline{\Omega}) \times [0,\infty) \big)^3$, *and* $\pi(\cdot, t)$ *is a* C^∞*-function on* $\mathbb{R}^3\backslash\overline{\Omega}$ *for almost every* $t \in (0,T)$. *Moreover, the pair* (u,π) *is a solution of (5) - (8), with the boundary condition on* S_∞ *verified in the sense that*

$$\int_0^\infty \int_{\partial\Omega} \big| u(x+\epsilon\cdot m^{(\Omega)}(x),\, t) \big|^2 d\Omega(x)\, dt \longrightarrow 0 \ \ \text{for } \epsilon \downarrow 0. \qquad (16)$$

Note that the equation in (13) is in fact an integral equation on S_∞, with ϕ as unknown. In order to solve (13), we introduce operators $\mathcal{F}_T^{(\kappa)} : L_n^2(S_T) \mapsto H_T$, for $\kappa \in [0,\infty)$, $T \in (0,\infty]$, by setting

$$\mathcal{F}_T^{(\kappa)}(\phi) := \mathcal{V}_T^{(\kappa)}(\phi) \,|\, S_T \quad \text{for } \phi \in L_n^2(S_T).$$

We want to show that $\mathcal{F}_\infty^{(\tau)}$ is bijective. Our starting point is a result by Shen [11], which we state as

Theorem 8 *For* $T \in (0,\infty]$, *the operator* $\mathcal{F}_T^{(0)} : L_n^2(S_T) \mapsto H_T$ *is well defined, linear, bounded and bijective. In particular,* $\mathcal{F}_T^{(0)}$ *is Fredholm with index 0.*

Proof: See [11, p. 365-367]. Note that the arguments given there also hold for $T = \infty$, although only the case $T < \infty$ is considered. □

We further need the ensuing continuity result.

Lemma 9 *For* $\kappa \in [0,\infty)$, $T \in (0,\infty]$, *the operator* $\mathcal{F}_T^{(\kappa)} : L_n^2(S_T) \mapsto H_T$ *is well defined, linear and bounded. There is* $\delta \in (0,1)$ *such that*

$$\|\mathcal{F}_T^{(\kappa)}(\phi) - \mathcal{F}_T^{(\varrho)}(\phi)\|_{H_T} \le \mathcal{C}\cdot |\kappa - \varrho|^\delta \cdot \|\phi\|_2$$

for $T \in (0,\infty]$, $\kappa, \varrho \in [0,\tau]$, $\phi \in L_n^2(S_T)$.

Indication of a proof of the first part of the lemma: Let $\kappa \in (0,\infty)$, $T \in (0,\infty]$, $\phi \in L_n^2(S_T)$. We are going to show that $\mathcal{F}_T^{(\kappa)}(\phi) - \mathcal{F}_T^{(0)}(\phi) \in H_T$ and $\|\mathcal{F}_T^{(\kappa)}(\phi) - \mathcal{F}_T^{(0)}(\phi)\|_{H_T} \le \mathcal{C}\cdot\|\phi\|_2$. In view of Theorem 8, this proves that $\mathcal{F}_T^{(\kappa)} : L_n^2(S_T) \mapsto H_T$ is well defined and bounded.

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For $j, k \in \{1, 2, 3\}$, $t \in (0, \infty)$, $z \in \mathbb{R}^3$, we put $K_{jk}(z, t) := \Lambda_{jk}(z, t, \kappa) - \Gamma_{jk}(z, t)$.

Then we have for $\alpha \in \mathbb{N}_0^3$, $l \in \mathbb{N}_0$ with $|\alpha|_1 + l \leq 1$ (use Lemma 1 as well as Lemma 2 with $K = R_0$):

$$|\partial_t^l \partial_z^\alpha K_{jk}(z, t)| = \tag{17}$$
$$\Big| \int_0^1 \Big[\partial_z^\alpha \partial_1^{l+1} \Gamma_{jk}(z - \vartheta \cdot \kappa \cdot t \cdot e_1, t) \cdot (-\kappa \cdot \vartheta)^l$$
$$+ \delta_{l1} \cdot \partial_z^\alpha \partial_1 \partial_4^l \Gamma_{jk}(z - \vartheta \cdot \kappa \cdot t \cdot e_1, t) \Big] \, d\vartheta \Big| \cdot \kappa$$
$$\leq \; C \cdot \big((|z|^2 + t)^{-1-|\alpha|_1/2-l/2}$$
$$+ (|z|^2 + t)^{-1-|\alpha|_1/2-l} \big),$$

and by a direct application of Lemma 1 and 2:

$$|\partial_t^l \partial_z^\alpha K_{jk}(z, t)| \;\leq\; C \cdot (|z|^2 + t)^{-3/2-|\alpha|_1/2-l}. \tag{18}$$

Moreover, for $1 \leq j, k \leq 3$, $l \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^3$, $z \in \mathbb{R}^3 \backslash \{0\}$:

$$\partial_t^l \partial_z^\alpha K_{jk}(z, t) \to 0 \;\; (t \downarrow 0). \tag{19}$$

This is obvious in the case $l \geq 1$. If $l = 0$, this relation follows from the theorem on dominated convergence. Consider the non-tangential vector field $m^{(\Omega)}$ introduced in Section 2, and the constants $\epsilon(\Omega)$ and $\widetilde{\epsilon}(\Omega)$ associated with it. Let $\delta \in (0, \widetilde{\epsilon}(\Omega)]$. According to (9), the relation $|x - y - \delta \cdot m^{(\Omega)}(y)| \geq \epsilon(\Omega) \cdot \delta$ holds for $x, y \in \partial\Omega$, hence with (10),

$$|x - y - \delta \cdot m^{(\Omega)}(y)| \geq \epsilon(\Omega) \cdot \delta/2 \tag{20}$$

for $y \in \partial\Omega$, $x \in \mathbb{R}^3$ with $dist(x, \overline{\Omega}) \leq \epsilon(\Omega) \cdot \delta/2$.

Let $\varphi^{(\delta)} \in C_0^\infty(\mathbb{R}^3)$ be such that $\varphi^{(\delta)}(x) = 1$ for $x \in \mathbb{R}^3$ with $dist(x, \overline{\Omega}) \leq \epsilon(\Omega) \cdot \delta/4$, and $\varphi^{(\delta)}(x) = 0$ if $dist(x, \overline{\Omega}) \geq \epsilon(\Omega) \cdot \delta/2$.

Let $\zeta \in C^\infty(\mathbb{R})$ with $\zeta \,|\, (-\infty, 1] = 1$ and $\zeta \,|\, [2, \infty) = 0$. Put $\zeta^{(\delta)}(t) := \zeta(\delta \cdot t)$ for $t \in \mathbb{R}$, $\delta \in (0, \widetilde{\epsilon}(\Omega)]$. Then $\zeta^{(\delta)}(t)$ equals 1 if $t \leq 1/\delta$, and vanishes in the case $t \geq 2/\delta$. In addition,

$$|\zeta^{(\delta)\prime}(t)| \leq C \cdot \chi_{(1/\delta, 2/\delta)}(t) \cdot 1/t \tag{21}$$

for any $t \in (0, \infty)$. For $n \in \mathbb{N}$, put $\epsilon_n := \widetilde{\epsilon}(\Omega)/n$, and define $w_n : \mathbb{R}^4 \mapsto \mathbb{R}^3$ by

$$w_{n,j}(x, t) := \varphi^{(\epsilon_n)}(x) \cdot \zeta^{(\epsilon_n)}(t) \tag{22}$$
$$\cdot \int_0^t \int_{\partial\Omega} \sum_{k=1}^3 K_{jk}(x - y - \epsilon_n \cdot m^{(\Omega)}(y), t - \sigma)$$
$$\cdot \phi_k(y, \sigma) \, d\Omega(y) \, d\sigma$$

for $x \in \mathbb{R}^3$ with $dist(\overline{\Omega}, x) \leq \epsilon(\Omega) \cdot \epsilon_n/2$, $t \in (0, \infty)$, $1 \leq j \leq 3$. Set $w_n(x, t) := 0$ for any other $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$. Then, in view of (19), we have $w_n \,|\, S_T \in \mathcal{H}_T$. Note that $\varphi^{(\epsilon_n)}(x) = 1$ in (22) if $x \in \overline{\Omega}$. Without loss of generality, we may suppose that $\epsilon_n \leq 1$ for $n \in \mathbb{N}$.

Let $n, p \in \mathbb{N}$. Abbreviate

$$\mathcal{K}_{jk}^{(n,p)}(x, y, t, r) :=$$
$$\zeta^{(\epsilon_n)}(t) \cdot K_{jk}(x - y - \epsilon_n \cdot m^{(\Omega)}(y), r)$$
$$-\zeta^{(\epsilon_p)}(t) \cdot K_{jk}(x - y - \epsilon_p \cdot m^{(\Omega)}(y), r),$$

$$\mathcal{L}_{jk}^{(n,p)}(x, y, t, r) :=$$
$$\zeta^{(\epsilon_n)\prime}(t) \cdot K_{jk}(x - y - \epsilon_n \cdot m^{(\Omega)}(y), r)$$
$$-\zeta^{(\epsilon_p)\prime}(t) \cdot K_{jk}(x - y - \epsilon_p \cdot m^{(\Omega)}(y), r)$$

for $j, k \in \{1, 2, 3\}$, $x \in \overline{\Omega}$, $y \in \partial\Omega$, $t, r \in (0, \infty)$. Then we find for $x, y \in \partial\Omega$ with $x \neq y$, $t \in (0, \infty)$, $j, k \in \{1, 2, 3\}$, using (19):

$$\partial_t \Big(\int_0^t (t - r)^{-1/2} \tag{23}$$
$$\cdot \int_0^r \mathcal{K}_{jk}^{(n,p)}(x, y, r, r - \sigma) \cdot \phi_k(y, \sigma) \, d\sigma \, dr \Big)$$
$$= \int_0^t \phi_k(y, \sigma) \cdot \int_0^{t-\sigma} r^{-1/2}$$
$$\cdot \partial_t \big[\mathcal{K}_{jk}^{(n,p)}(x, y, t - r, t - \sigma - r) \big] \, dr \, d\sigma$$
$$= \int_0^t \phi_k(y, \sigma) \cdot \Big(\int_0^{(t-\sigma)/2} r^{-1/2}$$
$$\cdot \Big[\mathcal{L}_{jk}^{(n,p)}(x, y, t - r, t - \sigma - r)$$
$$- \partial_\sigma \mathcal{K}_{jk}^{(n,p)}(x, y, t - r, t - \sigma - r) \Big] \, dr$$
$$+ 2^{1/2} \cdot (t - \sigma)^{-1/2}$$
$$\cdot \mathcal{K}_{jk}^{(n,p)}\big(x, y, (t + \sigma)/2, (t - \sigma)/2\big)$$
$$- (1/2) \cdot \int_{(t-\sigma)/2}^{t-\sigma} r^{-3/2}$$
$$\cdot \mathcal{K}_{jk}^{(n,p)}(x, y, t - r, t - \sigma - r) \, dr \Big) \, d\sigma.$$

Let $E : H^{1/2}(\partial\Omega) \mapsto H^1(\Omega)$ be a linear and bounded extension operator. Take $v \in H^1(\partial\Omega)$. Since the equation $\sum_{j=1}^3 \partial_{x_j} \mathcal{K}_{jk}^{(n,p)}(x, y, t, r) = 0$ holds for $x \in \Omega$, $y \in \partial\Omega$, $t, r \in (0, \infty)$, $1 \leq k \leq 3$, we find for $t \in (0, T)$:

$$\Big| \int_{\partial\Omega} < \partial_t (w_n - w_p)(x, t), \, N(x) > \cdot v(x) \, d\Omega(x) \Big|$$
$$= \; \Big| \sum_{j=1}^3 \int_\Omega \partial_t (w_n - w_p)_j(x, t) \cdot \partial_j E(v)(x) \, dx \Big|$$
$$\leq \; C \cdot \big\| \, \partial_t (w_n - w_p)(\,\cdot\,, t) \,|\, \Omega \, \big\|_2 \cdot \|\nabla E(v)\|_2,$$

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with $\|\nabla E(v)\|_2 \leq C \cdot \|v\|_{1/2,2} \leq C \cdot \|v\|_{1,2}$. Therefore

$$\| < \partial_t(w_n - w_p) \,|\, S_T,\, N > \|_{L^2(0,T,H^1(\partial\Omega)')} \qquad (24)$$
$$\leq \; C \cdot \Big(\int_0^T \int_\Omega |\, \partial_t(w_n - w_p)(x,t)\,|^2 \, dx \, dt \Big)^{1/2}.$$

We further note that for $t \in (0,T)$,

$$\| \, (w_n - w_p)(\,\cdot\,, t) \,|\, \partial\Omega \, \|_{1,2}^2$$
$$\leq \; C \cdot \sum_{j,k=1}^{3} \sum_{\alpha \in \mathbb{N}_0^3, |\alpha|_1 \leq 1} \int_{\partial\Omega} \Big(\int_0^t \int_{\partial\Omega}$$
$$\Big| \partial_x^\alpha \mathcal{K}_{jk}^{(n,p)}(x,y,t,t-\sigma) \cdot \phi_k(y,\sigma) \Big| \, d\Omega(y) \, d\sigma \Big)^2 \, d\Omega(x).$$

We may thus conclude that

$$\|(w_n - w_p) \,|\, S_T\|_{H_T} \qquad (25)$$
$$\leq \; C \cdot \sum_{j,k=1}^{3} \Big(\sum_{\alpha \in \mathbb{N}_0^3, |\alpha|_1 \leq 1} \|A_{j,k,\alpha}^{(n,p)}(\phi_k)\|_2$$
$$+ \|B_{j,k}^{(n,p)}(\phi_k)\|_2 + \|C_{j,k}^{(n,p)}(\phi_k)\|_2 \Big).$$

Here we used the definitions

$$A_{j,k,\alpha}^{(n,p)}(\psi)(x,t) := \int_0^t \int_{\partial\Omega}$$
$$\cdot \Big| \partial_x^\alpha \mathcal{K}_{jk}^{(n,p)}(x,y,t,t-\sigma) \cdot \psi(y,\sigma) \Big| \, d\Omega(y) \, d\sigma,$$

$$B_{j,k}^{(n,p)}(\psi)(x,t) := \int_0^t \int_{\partial\Omega}$$
$$\cdot \Big| \mathcal{Z}_{jk}^{(n,p)}(x,y,t,\sigma) \cdot \psi(y,\sigma) \Big| \, d\Omega(y) \, d\sigma$$

for $\psi \in L^2(S_T)$, $x \in \partial\Omega$, $t \in (0,T)$, $1 \leq j,k \leq 3$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha|_1 \leq 1$, where the term $\mathcal{Z}_{jk}^{(n,p)}(x,y,t,\sigma)$ denotes the factor of $\phi_k(y,\sigma)$ in the integral with respect to σ on the right-hand side of (23). In (25), we further used the notation

$$C_{j,k}^{(n,p)}(\psi)(x,t) := \int_0^t \int_{\partial\Omega}$$
$$\cdot \Big| \partial_t \mathcal{K}_{jk}^{(n,p)}(x,y,t,t-\sigma) \cdot \psi(y,\sigma) \Big| \, d\Omega(y) \, d\sigma$$

for $x \in \Omega$ and for ψ, t, j, k as above. Now put

$$H_1(x,y,t,\sigma) := (|x-y|^2 + t - \sigma)^{-3/2},$$

$$H_2(x,y,t,\sigma) := \int_0^{(t-\sigma)/2} r^{-1/2}$$
$$\cdot \Big[\chi_{(1,\infty)}(t) \cdot t^{-1} \cdot (|x-y|^2 + t - \sigma - r)^{-3/2}$$
$$+ \chi_{(0,1)}(t-\sigma) \cdot (|x-y|^2 + t - \sigma - r)^{-2}$$
$$+ \chi_{(1,\infty)}(t-\sigma) \cdot (|x-y|^2 + t - \sigma - r)^{-5/4} \Big] \, dr$$
$$+ (t-\sigma)^{-1/2} \cdot (|x-y|^2 + t - \sigma)^{-1}$$
$$+ \int_{(t-\sigma)/2}^{t-\sigma} r^{-3/2} \cdot (|x-y|^2 + t - \sigma - r)^{-1} \, dr,$$

$$H_3(x,y,t,\sigma) := (|x-y|^2 + t - \sigma)^{-2}$$
$$+ \chi_{(1,\infty)}(t) \cdot t^{-1} \cdot (|x-y|^2 + t - \sigma)^{-3/2}$$

for $x \in \overline{\Omega}$, $y \in \partial\Omega$ with $x \neq y$, $t > 0$, $\sigma \in (0,t)$. Note that the preceding estimate of the first term in the definition of $\mathcal{Z}_{jk}^{(n,p)}$, as well as the estimate of $C_{jk}^{(n,p)}$ involve (21) and the assumption $\epsilon_n \leq 1$. Refering to (17), (18) and (20), we find

$$|\, \partial_x^\alpha \mathcal{K}_{jk}^{(n,p)}(x,y,t,t-\sigma) \,| \leq C \cdot H_1(x,y,t,\sigma), \quad (26)$$

$$|\, \mathcal{Z}_{jk}^{(n,p)}(x,y,t,\sigma) \,| \leq C \cdot H_2(x,y,t,\sigma), \quad (27)$$

$$|\, \partial_t \mathcal{K}_{jk}^{(n,p)}(\overline{x},y,t,t-\sigma) \,| \leq C \cdot H_3(\overline{x},y,t,\sigma) \quad (28)$$

for $j,k \in \{1,2,3\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha|_1 \leq 1$, $n,p \in \mathbb{N}$, $x,y \in \partial\Omega$ with $x \neq y$, $\overline{x} \in \Omega$, $t \in (0,T)$, $\sigma \in (0,T)$. Note that $H_2(x,y,t,\sigma)$ is bounded by

$$C \cdot |x-y|^{-15/8} \cdot \Big(\chi_{(1,\infty)}(t) \cdot t^{-1} \cdot r^{-1/16}$$
$$+ \chi_{(0,1)}(r) \cdot r^{-9/16} + \chi_{(1,\infty)}(r) \cdot r^{-23/16} \Big)$$

for x, y, t, σ as in the definition of $H_2(x,y,t,\sigma)$, and for $r := t - \sigma$. Using this observation, and applying Hölder's and Young's inequality, we get

$$\int_0^\infty \int_{B_i} \Big(\int_0^t \int_{\partial\Omega} H_i(x,y,t,\sigma) \qquad (29)$$
$$\cdot |\phi(y,\sigma)| \, d\Omega(y) \, d\sigma \Big)^2 \, d\Omega(x) \, dt \; \leq \; C \cdot \|\phi\|_2^2$$

for $i \in \{1,2,3\}$, with $B_1 = B_2 = \partial\Omega$, and $B_3 = \Omega$, and with the term $d\Omega(x)$ replaced by dx in the case $i = 3$. Lebesgue's theorem on dominated convergence now yields that for j, k, α as in (26) - (28), the terms

$$\|A_{j,k,a}^{(n,p)}(\phi_k)\|_2, \; \|B_{j,k}^{(n,p)}(\phi_k)\|_2, \; \|C_{j,k}^{(n,p)}(\phi_k)\|_2$$

tend to zero for n and p tending to infinity. In view of (25), we have thus shown that $(w_n \,|\, S_T)$ is a Cauchy sequence with respect to the norm $\| \; \|_{H_T}$.

A variant of estimate (29) yields that the difference $(\mathcal{F}_T^{(\kappa)} - \mathcal{F}_T^{(0)})(\phi)$ belongs to $L^2(S_T)$, and

$$\|(\mathcal{F}_T^{(\kappa)} - \mathcal{F}_T^{(0)})(\phi) - w_n \,|\, S_T\|_2 \to 0$$

for $n \to \infty$. Now we may conclude that $(\mathcal{F}_T^{(\kappa)} - \mathcal{F}_T^{(0)})(\phi) \in H_T$.

Similar but somewhat simpler arguments imply the inequality $\|(\mathcal{F}_T^{(\kappa)} - \mathcal{F}_T^{(0)})(\phi)\|_{H_T} \leq \|\phi\|_2$. □

The following lemma holds because the difference $\Lambda_{jk}(\cdot\,,\cdot\,,\kappa) - \Gamma_{jk}$, for $\kappa > 0$, is weakly singular in a suitable sense.

Lemma 10 *Let $\kappa, T \in (0,\infty)$. (The case $T = \infty$ is not admitted here.) Then the operator $\mathcal{F}_T^{(\kappa)} - \mathcal{F}_T^{(0)} : L_n^2(S_T) \mapsto H_T$ is compact.*

The next corollary is a consequence of Lemma 10, Theorem 8 and [8, Theorem I.3.4].

Corollary 11 *For $T \in (0,\infty)$, $\kappa \in [0,\infty)$, the operator $\mathcal{F}_T^{(\kappa)}$ is Fredholm.*

Now we may show

Lemma 12 *Let $\kappa \in [0,\infty)$. Then $\mathcal{F}_\infty^{(\kappa)} : L_n^2(S_\infty) \mapsto H_\infty$ has closed range.*

Proof: Let (ϕ_n) be a sequence in $L_n^2(S_\infty)$, $\phi \in L_n^2(S_\infty)$, $y \in H_\infty$ such that

$$\|\phi_n - \phi\|_2 \to 0, \quad \|y - \mathcal{F}_\infty^{(\kappa)}(\phi_n)\|_{H_\infty} \to 0.$$

Let $T \in (0,\infty)$. Then

$$\mathcal{F}_\infty^{(\kappa)}(\psi) \,|\, S_T = \mathcal{F}_T^{(\kappa)}(\psi \,|\, S_T) \quad \text{for } \psi \in L_n^2(S_\infty),$$

$$g|S_T \in H_T, \ \|g \,|\, S_T\|_{H_T} \leq \|g\|_{H_\infty} \ \text{for } g \in H_\infty,$$

hence $\|y \,|\, S_T - \mathcal{F}_T^{(\kappa)}(\phi_n \,|\, S_T)\|_{H_T} \to 0$ for $n \to \infty$. Moreover, we have $\|(\phi_n - \phi) \,|\, S_T\|_2 \to 0$. But $\mathcal{F}_T^{(\kappa)}$ has closed range (Corollary 11), so we may conclude that $\mathcal{F}_T^{(\kappa)}(\phi \,|\, S_T) = y \,|\, S_T$, that is,

$$\mathcal{F}_\infty^{(\kappa)}(\phi) \,|\, S_T = y \,|\, S_T.$$

Since this is true for any $T \in (0,\infty)$, it follows that $\mathcal{F}_\infty^{(\kappa)}(\phi) = y$. This proves the lemma. □

Theorem 13 *Let $\kappa \in [0,\infty)$, $\phi \in L_n^2(S_\infty)$ with $\mathcal{F}_\infty^{(\kappa)}(\phi) = 0$. Then $\phi = 0$.*

This theorem may be shown by standard arguments; compare [11, p. 365/366] in the case $\kappa = 0$.

Corollary 14 *The mapping $\mathcal{F}_\infty^{(\tau)} : L_n^2(S_\infty) \mapsto H_\infty$ is bijective.*

Proof: From Lemma 12, Lemma 9 and Theorem 13, we may conclude that for $\kappa \in [0,\infty)$, the operator $\mathcal{F}_\infty^{(\kappa)}$ is linear, bounded and semi-Fredholm. (Of course, in the case $\kappa = 0$, this is immediately clear from Theorem 8.) Moreover, the mapping which associates each $\kappa \in [0,\tau]$ with the operator $\mathcal{F}_\infty^{(\kappa)} : L_n^2(S_\infty) \mapsto H_\infty$ is continuous with respect to the operator norm (Lemma 9). Therefore, in view of the stability theorem for semi-Fredholm operators [8, Theorem I.3.9], and the arguments in the proof of the homotopy result [8, Theorem I.3.11], it follows that the operators $\mathcal{F}_\infty^{(\tau)}$ and $\mathcal{F}_\infty^{(0)}$ have the same index. But the index of the latter operator is zero according to Theorem 8, so the index of the first one is zero, too. Now Theorem 13 yields that $\mathcal{F}_\infty^{(\tau)}$ is bijective. □

In order to solve (13), we need two further theorems.

Theorem 15 *The function $\mathcal{R}(f) \,|\, S_\infty$ belongs to H_∞. More precisely, let $q \in (1,2)$, $\alpha \in (1, q'/2)$, $\beta \in (4/3, 2)$. Then*

$$\begin{aligned}\|\mathcal{R}(f) \,|\, S_\infty\|_{H_T} \leq\ & C(q,\alpha,\beta) \cdot \big(\|f\|_2 \\ &+ \|f \,|\, (\mathbb{R}^3 \backslash B_{R_0}) \times (0,\infty)\|_{L^\alpha(0,\infty,L^q(B^c_{R_0})^3)} \\ &+ \|f \,|\, (\mathbb{R}^3 \backslash B_{R_0}) \times (0,\infty)\|_{L^\beta(0,\infty,L^q(B^c_{R_0})^3)} \big),\end{aligned}$$

where R_0 was introduced at the beginning of Section 2, and where the constant $C(q,\alpha,\beta) > 0$ depends on Ω, R_0, τ, q, α and β.

Theorem 16 *We have $\mathcal{I}(a) \,|\, S_\infty \in H_\infty$. More precisely, let $q \in (1,2)$. Then*

$$\|\mathcal{I}(a) \,|\, S_\infty\|_{H_\infty} \leq C(q) \cdot (\|a\|_2 + \|a\|_q),$$

with $C(q) > 0$ depending on Ω, R_0, τ and q.

As a consequence of Corollary 14 and Theorem 15 and 16, we arrive at the following result:

Corollary 17 *There is a solution (u,π) of (5) - (8) such that $u \in C^\infty\big((\mathbb{R}^3 \backslash \overline{\Omega}) \times [0,\infty)\big)^3$, $\pi(\cdot\,,t) \in C^\infty(\mathbb{R}^3 \backslash \overline{\Omega})$ for $t \in (0,\infty)$, and such that the boundary condition on S_∞ is verified as in (16). This solution is given by (14), (15), with $\phi \in L_n^2(S_\infty)$ fulfilling (13).*

Proof: On the one hand, the functions $\mathcal{R}(f) \,|\, S_\infty$ and $\mathcal{I}(a) \,|\, S_\infty$ belong to H_∞ (Theorem 15 and 16). On the other hand, the operator $\mathcal{F}_\infty^{(\tau)} : L_n^2(S_\infty) \mapsto H_\infty$ is bijective (Lemma 9, Corollary 14). Thus there is $\phi \in L_n^2(S_\infty)$ such that (13) holds. Now Corollary 17 follows from Lemma 7. □

4 Asymptotic behaviour of a solution of the time-dependent Oseen system.

The integral representation (14) may be used to study the asymptotic behaviour of a solution of (5) - (8). The ensuing lemma and the remarks on its proof indicate how to proceed in such a study.

Lemma 18 *Let the real $\overline{R} \in [R_0,\infty)$ be so large that $supp(f) \subset B_{\overline{R}/2} \times [0, \overline{R}/2]$ and $supp(a) \subset B_{\overline{R}/2}$, with R_0 introduced at the beginning of Section 2.*

Let $\phi \in L^2(S_\infty)^3$ be such that (13) holds, and let

(u,π) be the solution of (5) - (8) given by (14), (15).

Let $x \in B^c_{\overline{R}}$, $t \in [\overline{R},\infty)$, $\alpha \in \mathbb{N}_0^3$, $l \in \mathbb{N}_0$ with $2\cdot l + |\alpha|_1 \le 2$. Then

$$\begin{aligned} &|\,\partial_t^l \partial_x^\alpha u(x,t)\,| \\ &\le\; C(\overline{R}) \cdot \Big[\Big(|x| \cdot (1+\tau\cdot s(x)) + t \Big)^{-1-|\alpha|_1/2-l} \\ &\qquad \cdot (\, \|f\|_1 + \|a\|_1 + \|\phi\|_2 \,) \\ &\qquad + \Big(|x| \cdot (1+\tau\cdot s(x)) \Big)^{-1-|\alpha|_1/2-l} \\ &\qquad \cdot \|\phi \,|\, S_t \backslash S_{t/2}\|_2 \Big], \end{aligned}$$

where $C(\overline{R}) > 0$ depends on Ω, R_0, τ and $\overline{R}$.

Indication of a proof: Since $\overline{R} \ge R_0$, we have $\overline{\Omega} \subset B_{\overline{R}/2}$. In view of this relation and the assumptions on $\overline{R}$, we get

$$|x-y| \ge |x|/2 \ge R_0/2, \quad t-\sigma \ge t/2$$

for $(y,\sigma) \in supp(f)$, $y \in supp(a) \cup \partial\Omega$. Moreover, by [1, Lemma 4.8], we have

$$\begin{aligned} &(1+\tau\cdot s(z-y))^{-1} \\ &\le\; \mathcal{C}\cdot(1+|y|)\cdot(1+\tau\cdot s(z))^{-1} \;\text{ for } z,y \in \mathbb{R}^3, \end{aligned}$$

hence $(1+\tau\cdot s(x-y))^{-1} \le \mathcal{C}(\overline{R})\cdot(1+\tau\cdot s(x))^{-1}$ for $x \in B^c_{\overline{R}}$, $y \in supp(a)$ or $y \in \partial\Omega$ or $(y,\sigma) \in supp(f)$ for some $\sigma > 0$. Here and in the following, the symbol $\mathcal{C}(\overline{R}) > 0$ denotes constants depending on Ω, R_0, τ and $\overline{R}$. Now we conclude with Lemma 3:

$$\begin{aligned} &|\,\partial_t^l \partial_x^\alpha \mathcal{R}(f)(x,t)\,| + |\,\partial_t^l \partial_x^\alpha \mathcal{I}(a)(x,t)\,| \\ &\le\; \mathcal{C}(\overline{R}) \cdot [\,|x|\cdot(1+\tau\cdot s(x)) + t\,]^{-3/2-|\alpha|_1/2-l} \\ &\qquad \cdot (\,\|f\|_1 + \|a\|_1), \end{aligned}$$

$$\begin{aligned} &|\,\partial_t^l \partial_x^\alpha \mathcal{V}^{(\tau)}_\infty(\phi)(x,t)\,| \;\le\; \mathcal{C}(\overline{R}) \\ &\cdot\Big([\,|x|\cdot(1+\tau\cdot s(x)) + t\,]^{-1-|\alpha|_1/2-l} \cdot \|\phi \,|\, S_{t/2}\|_2 \\ &\qquad + [\,|x|\cdot(1+\tau\cdot s(x))\,]^{-1-|\alpha|_1/2-l} \\ &\qquad \cdot \|\phi \,|\, S_t \backslash S_{t/2}\|_2 \Big). \end{aligned}$$

Combining the preceding estimates yields the lemma. □

We note that the factor $1+\tau\cdot s(x)$ appearing in the preceding estimates is usually considered as a mathematical manifestation of the wake phenomenon.

5 Conclusion

We indicated how to solve the time-dependent Oseen system in a 3D exterior domain via the method of integral equations. This method automatically yields an integral representation of the solution, and thus is well suited for studying the asymptotic behaviour of that solution. As an example of how to exploit this representation, we presented a result on spatial decay of exterior Oseen flows.

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