H. Amann, W. Arendt, M. Hieber, F. Neubrander, S. Nicaise, J. von Below (eds):
Functional Analysis and Evolution Equations. The Günter Lumer Volume. 171–186

Resolvent Estimates for a Perturbed Oseen Problem

Paul Deuring

Abstract. We consider a resolvent equation arising from a stability problem for exterior Navier-Stokes flows with nonzero velocity at infinity.

Mathematics Subject Classification (2000). Primary 35Q30, 35B35; Secondary 76D05, 76E09.

Keywords. Stability, Oseen system, resolvent estimate.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be an exterior domain in $\mathbb{R}^3$. Consider the Navier-Stokes system

$$\partial_t u - \Delta u + \tau \cdot (u \cdot \nabla)u + \nabla p = h, \;\; \operatorname{div} u = 0 \quad \text{in} \;\; \Omega \times (0, \infty), \tag{1.1}$$

with the boundary conditions

$$u \,|\, \partial\Omega \times (0, \infty) = 0, \quad u(x,t) \to e_1 \;\; (|x| \to \infty) \;\; \text{for} \;\; t \in (0, \infty), \tag{1.2}$$

where $e_1 := (1, 0, 0)$, and where the data h do not depend on the time variable t. Let (u, p) be a solution to problem (1.1), (1.2), and let (U, P) be a solution of the corresponding stationary boundary value problem

$$-\Delta U + \tau \cdot (U \cdot \nabla)U + \nabla P = h, \;\; \operatorname{div} U = 0 \quad \text{in} \;\; \Omega, \tag{1.3}$$

$$U \,|\, \partial\Omega = 0, \quad U(x) \to e_1 \;\; (|x| \to \infty). \tag{1.4}$$

In this situation, the question arises as to whether $u(t) - U$ tends to zero in some sense for t tending to infinity, provided $u(0) - U$ is small in a suitable way. This "stability problem" attracted much attention for some time now; see [2], [8], [9], [11], [12], [17], for example. Most of the results in these references are based on smallness assumptions on U. However, as explained in [13], [14], one would also like to find a criterion related to the spectrum of a suitable linear operator, similar to the situation with ODE. Recently Neustupa [15] came rather close to such a criterion. His result may be stated as follows:

Write P_2 for the usual Helmholtz operator on $L^2(\Omega)^3$. Define the operatc L by

$$L(v) := P_2\big(\,\Delta v - \tau\cdot(U\cdot\nabla)v - \tau\cdot(v\cdot\nabla)U\,\big),$$

with v from a suitable function space. Let $\mathfrak{B}_{\rm sym}$ denote the symmetric part c an operator $\mathfrak{B}$ given by $\mathfrak{B}(v) := -(U\cdot\nabla)v - (v\cdot\nabla)U$, and let H_0' be the finite dimensional subspace of $L^2(\Omega)^3$ consisting of the eigenfunctions associated to th positive eigenvalues of the operator $P_2\big(\,\Delta + \widetilde{a}\cdot\tau\cdot\mathfrak{B}_{\rm sym}\,\big)$, where $\widetilde{a}$ is some fixed rea number. (For rigorous definitions see Section 2.) Suppose there is some $R>0$ an some non-increasing, integrable and square-integrable function $\varphi : [0,\infty) \mapsto [0,\infty$ such that

$$\|\nabla e^{Lt}(f)\,|\,B_R\|_2 \le \varphi(t)\cdot\|f\|_2 \quad \text{for } t\in(0,\infty),\ f\in H_0'. \qquad (1.5$$

Then Neustupa [15] could show that for a strong solution (u,p) of (1.1), (1.3), th relation $\|\nabla(u(t)-U)\|_2 \to 0$ holds for $t\to\infty$ if $\|u(0)-U\|_{1,2}$ is small. Neustup considers (1.5) as a substitute of the assumption that all eigenvalues of L hav negative real part.

In the work at hand, we show that this point of view is justified at least i the case $\Omega = \mathbb{R}^3$ (the case of the whole space). It turned out that for such Ω inequality (1.5) is valid provided all the eigenvalues of L have negative real par and the point 0 is almost in the resolvent of L, in the same sense as the point 0 i almost in the resolvent of respectively the Stokes and the Oseen operator. A precis statement of these conditions may be found in assertion (C1) and (C2) in Sectio 2; our results are stated in Theorem 2.3. These results are by no means obviou since the spectrum of L touches the imaginary axis from the left, independentl of the concrete form of the function U.

In this article, we are only able to indicate the way we proceed and elaborat some selected points. More detailed proofs will be given in [5].

2. Notations, definitions and main result

For $\epsilon>0$, we put $B_\epsilon(x) := \{\, y\in\mathbb{R}^3 \,:\, |y-x|<\epsilon \,\}$. Set $B_\epsilon := B_\epsilon(0)$. For $A\subset\mathbb{R}^3$ we abbreviate $A^c := \mathbb{R}^3\backslash A$. The length $\alpha_1+\alpha_2+\alpha_3$ of a multi-index $\alpha\in\mathbb{N}_0^3$ i denoted by $|\alpha|_1$.

If $\sigma\in\mathbb{N}$, and if $f:\mathbb{R}^3\mapsto\mathbb{R}$, $g:\mathbb{R}^3\mapsto\mathbb{R}^\sigma$ are measurable functions, with

$$\int_{\mathbb{R}^3} |f(x-y)|\cdot|g(y)|\,dy < \infty \quad \text{for a.e. } x\in\mathbb{R}^3,$$

then we set

$$(f*g)(x) := \Big(\int_{\mathbb{R}^3} f(x-y)\cdot g_j(y)\,dy\Big)_{1\le j\le 3} \quad \text{for a.e. } x\in\mathbb{R}^3.$$

We define $\mathfrak{D}_0^{1,2}(\mathbb{R}^3)$ as the space of all functions $v\in W^{1,1}_{\rm loc}(\mathbb{R}^3)\cap L^6(\mathbb{R}^3)$ suc that $\nabla v\in L^2(\mathbb{R}^3)^3$; see [6, Remark II.5.2, Theorem II.5.1, II.6.1]. This space i equipped with the gradient norm. Furthermore, put $\mathfrak{D} := [\mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3]'$. We define the norm $\| \ \|_{-1,2}$ on $\mathfrak{D}$ by setting

$$\|F\|_{-1,2} := \sup\{\, |F(v)|/\|\nabla v\|_2 \ :\ v \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3,\ \nabla v \neq 0 \,\}. \tag{2.1}$$

We note that in (2.1), it is sufficient to take the sup with respect to all functions $v \in C_0^\infty(\mathbb{R}^3)^3$ with $\nabla v \neq 0$. Any function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^3)^3$ with

$$\gamma_f := \sup\Big\{ \Big| \int_{\mathbb{R}^3} f \cdot v \, dx \Big| / \|\nabla v\|_2 \ :\ v \in C_0^\infty(\mathbb{R}^3)^3,\ \nabla v \neq 0 \Big\} < \infty$$

defines an element of $\mathfrak{D}$, which we also denote by f, and which verifies the relation $\|f\|_{-1,2} = \gamma_f$. This is true in particular for $f \in L^{6/5}(\mathbb{R}^3)^3$, due to the standard Sobolev estimate $\|v\|_6 \leq C \cdot \|\nabla v\|_2$ for $v \in C_0^\infty(\mathbb{R}^3)^3$.

For $p \in (1,\infty)$, let $H_p(\mathbb{R}^3)$ denote the closure of the set $\{\, \varphi \in C_0^\infty(\mathbb{R}^3)^3 \ :\ \operatorname{div}\varphi = 0 \,\}$ with respect to the norm $\| \ \|_p$. Then, for any element $f \in L^p(\mathbb{R}^3)^3$, there is a unique function $P_p f \in H_p(\mathbb{R}^3)$ and some $g \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^3)$ with $P_p f + \nabla g = f$. The mapping $P_p : H_p(\mathbb{R}^3) \mapsto L^p(\mathbb{R}^3)^3$ is linear and bounded. We refer to [6, Section III.1] for these results. Since for any $p, q \in (1,\infty)$ and any $f \in L^p(\mathbb{R}^3)^3 \cap L^q(\mathbb{R}^3)^3$, we have $P_p f = P_q f$, we will only write P instead of P_q in the following.

We fix $\tau \in (0,\infty)$. Put $s(x) := \tau \cdot (|x| - x_1)$ for $x \in \mathbb{R}^3$, and define

$$E^{(0)}(z) := (4\cdot\pi)^{-1} \cdot |z|^{-1} \cdot e^{-s(z)/2},$$
$$E^{(\lambda)}(z) := (4\cdot\pi)^{-1} \cdot |z|^{-1} \cdot e^{-\sqrt{\lambda+(\tau/2)^2}\cdot|z| + \tau\cdot z_1/2}$$

for $z \in \mathbb{R}^3\backslash\{0\}$, $\lambda \in \mathbb{C}\backslash\{0\}$. Then $E^{(\varrho)}$, for $\varrho \in \mathbb{C}$ with $\Re\varrho \geq 0$, is a fundamental solution of the equation $-\Delta v + \tau \cdot \partial_1 v + \varrho \cdot v = g$.

Concerning the function h in (1.1) and (1.3), we suppose that $h \in L^s(\mathbb{R}^3)^3$ for $s \in (1, 3+\epsilon]$, with some $\epsilon > 0$. Then there is a pair $(U,P) \in H^2_{\mathrm{loc}}(\mathbb{R}^3)^3 \times H^1_{\mathrm{loc}}(\mathbb{R}^3)$ which solves (1.3) with $\Omega = \mathbb{R}^3$, and which verifies the relations

$$U - e_1 \in L^s(\mathbb{R}^3)^3 \ \text{ for } \ s \in (2, 3+\epsilon], \tag{2.2}$$
$$\nabla U \in L^s(\mathbb{R}^3)^9 \ \text{ for } \ s \in [4/3,\, 3+\epsilon], \quad \text{with some } \ \epsilon > 0.$$

For this result, see [7, Section IX.7], [4, Theorem 4.9]. For the rest of this article, we fix such a solution (U,P).

For $v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^3)^3$, we put

$$\mathfrak{B}(v) := \Big(-\sum_{k=1}^{3} \big(\partial_k U_j \cdot v_k + (U - e_1)_k \cdot \partial_k v_j \big) \Big)_{1\leq j\leq 3},$$

$$\mathfrak{B}_{\mathrm{sym}}(v) := \Big((-1/2) \cdot \sum_{k=1}^{3} v_k \cdot (\partial_k U_j + \partial_j U_k) \Big)_{1\leq j\leq 3}.$$

Further put $\mathfrak{D}(L) := H_2(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)^3$, and define an operator $L : \mathfrak{D}(L) \mapsto H_2(\mathbb{R}^3)$ by setting

$$Lv := P\big(\Delta v - \tau \cdot \partial_1 v + \tau \cdot \mathfrak{B}(v) \big) \quad \text{for } \ v \in \mathfrak{D}(L),$$

where we used implicitly that $\mathfrak{B}(v) \in L^p(\mathbb{R}^3)^3$ for some $p \in (1, \infty)$. In fact, the relation $\mathfrak{B}(v) \in L^2(\mathbb{R}^3)^3$ holds for $v \in \mathfrak{D}$. It should further be noted that

$$P\big(\Delta v - \tau \cdot \partial_1 v\big) = \Delta P v - \tau \cdot \partial_1 P v = \Delta v - \tau \cdot \partial_1 v$$

for $v \in \mathfrak{D}(L)$, a relation which is not valid for functions in a corresponding space on Ω with $\Omega \neq \mathbb{R}^3$. The ensuing theorem holds according to [1], [13], [14].

Theorem 2.1. *The set $\mathfrak{D}(L)$ is dense in $H_2(\mathbb{R}^3)$. The operator L is closed. Let $\varrho(L)$ denote the resolvent set of L, and $\sigma(L)$ the spectrum of L. Then there is a countable set $\mathfrak{K}$ of isolated eigenvalues of L such that*

$$\sigma(L) \backslash \mathfrak{K} \subset \big\{\, \lambda \in \mathbb{C} \;:\; \Re\lambda \leq -(\Im\lambda)^2/\tau^2 \,\big\}. \tag{2.3}$$

Moreover, there are $a \in (0, \infty)$, $\vartheta \in (\pi/2, \pi)$ such that

$$S_{\vartheta,a} := \big\{\, \lambda \in \mathbb{C}\backslash\{a\} \;:\; |\arg(\lambda - a)| \leq \vartheta \,\big\} \subset \varrho(L),$$

and there is $C_1 > 0$ with

$$\|(\lambda \cdot I - L)^{-1}(\Phi)\|_2 \leq C_1 \cdot |\lambda - a|^{-1} \cdot \|\Phi\|_2 \tag{2.4}$$

for $\Phi \in H_2(\mathbb{R}^3)$, $\lambda \in S_{\vartheta,a}$.

We require that the spectrum of L satisfies the following two conditions:

(C1) $\Re\lambda < 0$ for $\lambda \in \mathfrak{K}$.

(C2) For any $G \in \mathfrak{D}$, there is one and only one function $u \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$ with $\operatorname{div} u = 0$ and

$$\int_{\mathbb{R}^3} \big(\nabla u \cdot \nabla v + \tau \cdot \partial_1 u \cdot v - \tau \cdot P\mathfrak{B}(u) \cdot v\big)\, dx = G(v)$$

for $v \in C_0^\infty(\mathbb{R}^3)^3$ with $\operatorname{div} v = 0$.

Note that an existence and uniqueness result as in (C2) is valid for the Oseen system ([7, Theorem IX.4.1]). Thus condition (C2) may be interpreted in the sense that the term $\tau \cdot P\mathfrak{B}(u)$ should not destroy this existence and uniqueness result for the Oseen system.

We fix some $\widetilde{a} \in \mathbb{R}$. For the ensuing theorem, we refer to [15].

Theorem 2.2. *The set of all numbers $\lambda \in (0, \infty)$ with $\Delta f + \widetilde{a} \cdot \tau \cdot P\mathfrak{B}_{\text{sym}}(f) = \lambda \cdot f$ for some $f \in \mathfrak{D}(L)$ with $f \neq 0$ is finite.*

Let H_0' be the set consisting of these functions f and of the zero function. Then H_0' is a vector space of finite dimension.

We remark that according to Lemma 5.1, the term $P\mathfrak{B}_{\text{sym}}(f)$ is well defined for functions f as in Theorem 2.2. By Theorem 2.1, the operator $-L$ is sectorial ([10, Definition 1.3.1]), and thus generates an analytic semigroup $(e^{Lt})_{t \geq 0}$ of linear operators from $H_2(\mathbb{R}^3)$ into $\mathfrak{D}(L)$; see [10, Theorem 1.3.4]. Our aim is to show the following

Theorem 2.3. *Let $R \in (0,\infty)$. Then there is some $C > 0$ depending on τ, U, a, ϑ, C_1, $\widetilde{a}$ and R such that*

$$\|\nabla e^{Lt}(f) \,|\, B_R\|_2 \le C \cdot (1+t)^{-9/8} \cdot \|f\|_2 \quad \textit{for } f \in H_0',\ \ t \in (0,\infty).$$

Theorem 2.3 will be proved via resolvent estimates related to the operator L. These estimates are stated in Theorem 5.3, 5.4 and Lemma 5.5 below. Since L may be considered as a perturbed Oseen operator, we thus establish resolvent estimates for a perturbed Oseen system.

Concerning the constants appearing in the following, the symbol $\mathfrak{C}$ will denote constants only depending on τ, U, the parameters a, ϑ and C_1 from Theorem 2.1, and on the constant $\widetilde{a}$ appearing in Theorem 2.2. We will write $\mathfrak{C}(\gamma_1, \dots, \gamma_n)$ for constants depending on the preceding quantities as well as on the parameters $\gamma_1, \dots, \gamma_n$. A constant which only depends on $\gamma_1, \dots, \gamma_n$ and on no other quantity will be denoted by $C(\gamma_1, \dots, \gamma_n)$.

3. Convolutions of $E^{(\varrho)}$; estimates of $P\mathfrak{B}(E^{(\varrho)} * \Phi)$

We begin by stating some estimates of the fundamental solution $E^{(\varrho)}$ of the equation $-\Delta v + \tau \cdot \partial_1 v + \varrho \cdot v = g$. We remark that in this section and in the following, we will use the letter ϱ to denote complex numbers including 0, whereas the variable λ stands for non-vanishing complex numbers.

Theorem 3.1. *Let $\kappa, \gamma \in [0,\infty)$. Then*

$$|\partial_z^\alpha E^{(\lambda)}(z)| \le C(\tau,\kappa,\gamma) \cdot |\lambda|^{-2\cdot\gamma} \cdot \big(|z|^{-\gamma-1-|\alpha|_1/2} + |z|^{-\gamma-1-|\alpha|_1}\big) \cdot \big(1+s(z)\big)^{-\kappa} \cdot e^{-\mu\cdot|\lambda|^2\cdot|z|} \tag{3.1}$$

for $z \in \mathbb{R}^3 \backslash \{0\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha|_1 \le 1$, $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \ge 0$ and $|\lambda| \le (\tau/2)^2$, where μ is a constant only depending on τ. Moreover,

$$|\partial_z^\alpha E^{(\varrho)}(z)| \le C(\tau) \cdot \big(|z|^{-1-|\alpha|_1/2} + |z|^{-1-|\alpha|_1}\big) \cdot \big(1+s(z)\big)^{-1-|\alpha|_1/2} \tag{3.2}$$

for z, α as in (3.1), and for $\varrho \in \mathbb{C}$ with $\Re\varrho \ge 0$, $|\varrho| \le (\tau/2)^2$.

Theorem 3.1, Young's and Minkowski's inequality and the Hardy-Littlewood-Sobolev inequality yield

Theorem 3.2. *Let $p \in (1,2]$, $q \in [1,p]$ with $p < 2$ or $q > 1$. Then*

$$\|\,|E^{(\lambda)}| * |f|\,\|_p \le C(\tau,p,q) \cdot |\lambda|^{2-4\cdot(1-1/q+1/p)} \cdot \|f\|_q \tag{3.3}$$

for $f \in L^q(\mathbb{R}^3)$, $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \ge 0$, $|\lambda| \le (\tau/2)^2$.

Let $q \in [1,2)$ and

$$p \in \big((1/q-1/2)^{-1},\, \infty\big] \ \textit{ if } q \ge 3/2,$$
$$p \in \big((1/q-1/2)^{-1},\, (1/q-2/3)^{-1}\big) \ \textit{ if } q < 3/2.$$

Then, for $f \in L^q(\mathbb{R}^3)$, $\varrho \in \mathbb{C}$ with $\Re\varrho \ge 0$ and $|\varrho| \le (\tau/2)^2$,

$$\|\,|E^{(\varrho)}| * |f|\,\|_p \le C(\tau,p,q) \cdot \|f\|_q. \tag{3.4}$$

Let $p, q \in [1, \infty]$ *with* $1/q - 1/3 < 1/p < 1/q - 1/4$. *Then*

$$\| \, |\partial_l E^{(\varrho)}| * |f| \, \|_p \le C(\tau, p, q) \cdot \|f\|_q \tag{3.5}$$

for $1 \le l \le 3$ *and for* f *and* ϱ *as in* (3.4). *Finally*

$$\| \, |E^{(\varrho)}| * |f| \, \|_6 + \| \, |\partial_l E^{(\varrho)}| * |f| \, \|_2 \le C(\tau) \cdot \|f\|_{6/5}$$

for $1 \le l \le 3$, $f \in L^{6/5}(\mathbb{R}^3)$, $\varrho \in \mathbb{C}$ *with* $\Re\varrho \ge 0$, $|\varrho| \le (\tau/2)^2$.

Among other results, the next theorem gives a precise form of the assertion that $E^{(\varrho)}$ is a fundamental solution of the equation $-\Delta v + \tau \cdot \partial_1 v + \varrho \cdot v = g$.

Theorem 3.3. *Let* $q \in (1, 2)$, $f \in L^q(\mathbb{R}^3)$, $\varrho \in \mathbb{C}$ *with* $\Re\varrho \ge 0$, $|\varrho| \le (\tau/2)^2$. *Then*

$$E^{(\varrho)} * f \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^3), \quad \partial_l(E^{(\varrho)} * f) = (\partial_l E^{(\varrho)}) * f \ \ (1 \le l \le 3),$$
$$-\Delta(E^{(\varrho)} * f) + \tau \cdot \partial_1(E^{(\varrho)} * f) + \varrho \cdot (E^{(\varrho)} * f) = f,$$
$$\|\partial_l \partial_m (E^{(\varrho)} * f) \,|\, B_R\|_q \le C(\tau, q, R) \cdot \|f\|_q \quad \big(1 \le l, m \le 3, \ R > 0\big). \tag{3.6}$$

We note a consequence of Theorem 3.2 and a remark in [6, p. 391/392]:

Lemma 3.4. $\|\nabla E^{(\varrho)} * w\|_2 \le C(\tau) \cdot \|w\|_{-1,2}$ *for* $\varrho \in \mathbb{C}$ *with* $\Re\varrho \ge 0$, $|\varrho| \le (\tau/2)^2$ $w \in C_0^\infty(\mathbb{R}^3)^3$.

Due to this lemma, we may define convolutions of $E^{(\varrho)}$ with elements of $\mathfrak{D}$:

Corollary 3.5. *Let* $\varrho \in \mathbb{C}$ *with* $\Re\varrho \ge 0$, $|\varrho| \le (\tau/2)^2$. *Then there is a linear mapping* $\Gamma := \Gamma_\varrho : \mathfrak{D} \mapsto L^6(\mathbb{R}^3)^3$ *with*

$$\Gamma(\Phi) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^3)^3, \quad \nabla\Gamma(\Phi) \in L^2(\mathbb{R}^3)^9, \quad \partial_1\Gamma(\Phi) \in \mathfrak{D},$$
$$\|\nabla\Gamma(\Phi)\|_2 \le C(\tau) \cdot \|\Phi\|_{-1,2} \quad \text{for } \Phi \in \mathfrak{D},$$
$$\Gamma(\Phi) = E^{(\varrho)} * \Phi \quad \text{for } \Phi \in C_0^\infty(\mathbb{R}^3)^3.$$

Moreover, $\Gamma(w) = E^{(\varrho)} * w$ *if* $w \in \mathfrak{D} \cap L^q(\mathbb{R}^3)^3$ *for some* $q \in (1, 2)$, *or if* $\varrho \ne 0$ *and* $w \in \mathfrak{D} \cap L^2(\mathbb{R}^3)^3$.

If $w \in \mathfrak{D} \cap L^2(\mathbb{R}^3)^3$, *then*

$$\partial_l\Gamma(w) = (\partial_l E^{(\varrho)}) * w \ \ (1 \le l \le 3), \quad \Gamma(w) \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^3)^3,$$
$$\partial_l\partial_m\Gamma(w) \in L^2(\mathbb{R}^3)^3 \ \ (1 \le l, m \le 3), \quad -\Delta\Gamma(w) + \tau \cdot \partial_1\Gamma(w) + \varrho \cdot \Gamma(w) = w$$

Finally, if $w \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$, *the equation* $\operatorname{div}\Gamma(w) = 0$ *holds.*

The last statement of Corollary 3.5 means that for $w \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$, the pair (v, π) with $v = \Gamma(w)$, $\pi = 0$ is a solution in $\mathbb{R}^3$ of the resolvent problem

$$-\Delta v + \tau \cdot \partial_1 v + \varrho \cdot v + \nabla\pi = w, \quad \operatorname{div} v = 0$$

associated to the Oseen operator. This observation explains why convolutions of $E^{(\varrho)}$ are studied here.

In the ensuing theorem, which is a consequence of (2.2), we evaluate the operator $P\mathfrak{B}$ applied to convolutions of $E^{(\varrho)}$.

Theorem 3.6. *Let $q \in (1,2)$, $p \in (1,2]$. Then $\mathfrak{B}(E^{(\varrho)} * \Phi) \in L^q(\mathbb{R}^3)^3$ for $\Phi \in L^q(\mathbb{R}^3)^3$, and $\mathfrak{B}\big(\Gamma_\varrho(w)\big) \in L^p(\mathbb{R}^3)^3$ for $w \in \mathfrak{D}$, where $\varrho \in \mathbb{C}$ with $\Re\varrho \geq 0$ and $|\varrho| \leq (\tau/2)^2$. Moreover,*

$$\|P\mathfrak{B}(E^{(\varrho)} * \Phi)\|_q \leq \mathfrak{C}(q) \cdot \|\Phi\|_q, \quad \|P\big(\Gamma_\varrho(w)\big)\|_p \leq \mathfrak{C}(p) \cdot \|w\|_{-1,2} \tag{3.7}$$

for Φ, w, ϱ as above. In addition, there are non-increasing functions $\mathfrak{D}_1^{(q)}, \mathfrak{D}_2^{(p)} : [0,\infty) \mapsto (0,\infty)$ such that $\mathfrak{D}_1^{(q)}(R) \to 0$, $\mathfrak{D}_2^{(p)}(R) \to 0$ for $R \to \infty$, and

$$\begin{aligned} &\|P\big(\chi_{B_R^c} \cdot \mathfrak{B}(E^{(\varrho)} * \Phi)\big)\|_q \leq \mathfrak{D}_1^{(q)}(R) \cdot \|\Phi\|_q, \\ &\|P\big(\chi_{B_R^c} \cdot \mathfrak{B}(\Gamma_\varrho(w))\big)\|_p \leq \mathfrak{D}_2^{(p)}(R) \cdot \|w\|_{-1,2} \end{aligned} \tag{3.8}$$

for Φ, w, ϱ as above, and for $R \in [R_0, \infty)$.

The following theorem, although technical, is a crucial part of our theory. Its significance will become apparent in the next section.

Theorem 3.7. *Let $q \in (1,2)$, $p \in (1,2]$. Then there are constants $\delta_1^{(q)} = \delta_1(\tau, U, q)$, $\delta_2^{(p)} = \delta_2(\tau, U, p) \in (0,1)$ and non-decreasing functions $\gamma_1^{(q)}, \gamma_2^{(p)} : (0,\infty) \mapsto (0,\infty)$ such that the following holds:*

Let $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$ and $|\lambda| \leq (\tau/2)^2$. Let $R \in [R_0, \infty)$, $\widetilde{R} \in [2 \cdot R + 1, \infty)$. Then

$$\begin{aligned} &\|P\mathfrak{B}(E^{(\lambda)} * w) - P\mathfrak{B}(E^{(0)} * w)\|_q \\ &\leq \Big(2 \cdot \mathfrak{D}_1^{(q)}(R) + \mathfrak{C}(q) \cdot \gamma_1^{(q)}(R) \cdot \widetilde{R}^{-\delta_1^{(q)}} + \gamma_1^{(q)}(\widetilde{R}) \cdot |\lambda|^{1/3}\Big) \cdot \|w\|_q \\ &\text{for } w \in L^q(\mathbb{R}^3)^3, \end{aligned}$$

$$\begin{aligned} &\|P\mathfrak{B}(E^{(\lambda)} * w) - P\mathfrak{B}\big(\Gamma_0(w)\big)\|_p \\ &\leq \Big(2 \cdot \mathfrak{D}_2^{(p)}(R) + \mathfrak{C}(p) \cdot \gamma_2^{(p)}(R) \cdot \Big(\widetilde{R}^{-\delta_2^{(p)}} + \big(\ln(\widetilde{R}/(\widetilde{R}-1))\big)^{1/2}\Big) \\ &\qquad + \gamma_2^{(p)}(\widetilde{R}) \cdot |\lambda|^{1/3}\Big) \cdot (\|w\|_2 + \|w\|_{1,2}) \quad \text{for } w \in L^2(\mathbb{R}^3)^3 \cap \mathfrak{D}. \end{aligned}$$

4. Solving a perturbed Oseen system

We are now looking for solutions in $H_2(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)^3$ of the perturbed Oseen system

$$-\Delta v + \tau \cdot \partial_1 v + \lambda v - \tau \cdot P\mathfrak{B}(v) = f, \quad \operatorname{div} v = 0. \tag{4.1}$$

To this end, we intend to use the formula

$$\begin{aligned} &(-\Delta + \tau \cdot \partial_1 + \lambda \cdot I - \tau \cdot P\mathfrak{B})^{-1} \\ = \ &(-\Delta + \tau \cdot \partial_1 + \lambda \cdot I)^{-1} \circ \big(I - \tau \cdot P\mathfrak{B} \circ (-\Delta + \tau \cdot \partial_1 + \lambda \cdot I)^{-1}\big)^{-1}. \end{aligned} \tag{4.2}$$

Of course, this equation is only formal, and the problem consists in giving it a sense. Our idea is to start with $\lambda = 0$. In that case, equation (4.2) may be transformed into something rigorous due to our assumption (C2) in Section 2. Then we use perturbation argument and Theorem 3.7 in order to deal with the case $\lambda \in \mathbb{C}\backslash\{0$ with $\Re\lambda \geq 0$ and $|\lambda|$ small. The results for this case will be used in Section 5 i order to derive estimates of solutions of (4.1) (that is, resolvent estimates of th operator L), under the assumptions that $|\lambda|$ small, $\Re\lambda \geq 0$, and the right-han side of f in (4.1) belongs to the space H_0' introduced in Theorem 2.2.

We begin by looking for a rigorous form of (4.2) in the case $\lambda = 0$. In a firs step, we deduce from Corollary 3.5 and a standard uniqueness result (see [6, p 397 and p. 391]):

Theorem 4.1. *Define $D(A)$ as the space of all functions $v \in L^6(\mathbb{R}^3)^3 \cap W^{2,1}_{loc}(\mathbb{R}^3)$ such that $\partial_l v$, $\partial_m \partial_l v \in L^2(\mathbb{R}^3)^3$ for $1 \leq l, m \leq 3$, $\partial_1 v \in \mathfrak{D}$ and* $\operatorname{div} v = 0$.

Put $Av := -\Delta v + \tau \cdot \partial_1 v$ for $v \in D(A)$. Then the operator $A : D(A) \mapsto$ $\mathfrak{D} \cap H_2(\mathbb{R}^3)$ is linear and bijective, with $A^{-1} = \Gamma_0$, where Γ_0 was introduced i Corollary 3.5.

Theorem 4.1 and assumption (C2) imply

Corollary 4.2. *The operator $\widetilde{A} : D(A) \mapsto \mathfrak{D} \cap H_2(\mathbb{R}^3)$, with $\widetilde{A}v := Av - \tau \cdot P\mathfrak{B}(v$ for $v \in D(A)$, is linear and bijective.*

In view of Corollary 4.2, we may use the simple operator calculus indicate by (4.2). It follows with Corollary 3.5 and Theorem 3.6:

Corollary 4.3. *The mapping*

$$\widetilde{Z}_0 : \mathfrak{D} \cap H_2(\mathbb{R}^3) \ni w \mapsto w - \tau \cdot P\mathfrak{B}\big(\Gamma_0(w)\big) \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$$

is linear, bounded, and bijective, and

$$\|w\|_{-1,2} + \|w\|_2 \leq \mathfrak{C} \cdot \big(\|\widetilde{Z}_0(w)\|_{-1,2} + \|\widetilde{Z}_0(w)\|_2\big) \quad \textit{for } w \in \mathfrak{D} \cap H_2(\mathbb{R}^3).$$

But the invertibility $\widetilde{Z}_0$ implies that the operator $Z_0^{(q)}$ defined below an acting on L^q-spaces is also invertible:

Theorem 4.4. *Let $q \in (1, 2)$. Then the operator*

$$Z_0^{(q)} : L^q(\mathbb{R}^3)^3 \ni w \mapsto w - \tau \cdot P\mathfrak{B}(E^{(0)} * w) \in L^q(\mathbb{R}^3)^3$$

is linear, bounded and bijective, with $\|w\|_q \leq \mathfrak{C}(q) \cdot \|Z_0^{(q)}(w)\|_q$ for $w \in L^q(\mathbb{R}^3)^3$

The idea of the proof of Theorem 4.4 consists in showing that Z_q is Fredholn with index zero. This may be done by a compactness argument involving the space $W^{2,q}(B_R)$ and $L^q(B_R)$ for $R \in (0, \infty)$ as well as inequality (3.6), and by referrin to a contraction principle and inequality (3.8) with large R. On the other hand, i may be shown that $Z_0^{(q)}$ is one-to-one because $\widetilde{Z}_0$ has the same property. Theoren 4.4 then follows.

Now we use a perturbation argument together with Theorem 3.7 in orde to show that corresponding operators $\widetilde{Z}_\lambda$ and $Z_\lambda^{(q)}$, with $\lambda \neq 0$ but $|\lambda|$ small, ar bijective, too. We get

Theorem 4.5. *There is $\epsilon_1 \in (0, (\tau/2)^2]$ depending on τ and U such that for $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$ and $|\lambda| \leq \epsilon_1$, the operator*

$$\widetilde{Z}_\lambda : \mathfrak{D} \cap H_2(\mathbb{R}^3) \ni w \mapsto w - \tau \cdot P\mathfrak{B}(E^{(\lambda)} * w) \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$$

is linear, bounded, and bijective, with

$$\|w\|_{-1,2} + \|w\|_2 \leq \mathfrak{C} \cdot \big(\|\widetilde{Z}_\lambda(w)\|_{-1,2} + \|\widetilde{Z}_\lambda(w)\|_2 \big)$$

for $w \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$ and for λ as before.

Let $q \in (1,2)$. Then there is $\epsilon_2 = \epsilon_2(q) = \epsilon_2(\tau, U, q) \in (0, \epsilon_1]$ such that for $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$ and $|\lambda| \leq \epsilon_2$, the operator

$$Z_\lambda^{(q)} : L^q(\mathbb{R}^3)^3 \ni w \mapsto w - \tau \cdot P\mathfrak{B}(E^{(\lambda)} * w) \in L^q(\mathbb{R}^3)^3$$

is linear, bounded and bijective, with $\|w\|_q \leq \mathfrak{C}(q) \cdot \|Z_\lambda^{(q)}(w)\|_q$ for $w \in L^q(\mathbb{R}^3)^3$ and for $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$, $|\lambda| \leq \epsilon_2$.

Now, for the second time, we apply the straightforward operator calculus implicit in (4.2). This argument combined with Theorem 4.5, Corollary 3.5 and Theorem 3.2 (which yields that $E^{(\lambda)} * w \in L^2(\mathbb{R}^3)^3$ for $w \in L^2(\mathbb{R}^3)^3$, $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$ and $|\lambda| < (\tau/2)^2$) implies the ensuing corollary.

Corollary 4.6. *Let $g \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$, $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$, $|\lambda| \leq \epsilon_1$. Put $\psi := \widetilde{Z}_\lambda^{-1}(g)$, $u := E^{(\lambda)} * \psi$. Then $u \in H^2(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$,*

$$-\Delta u + \tau \cdot \partial_1 u + \lambda \cdot u - \tau \cdot P\mathfrak{B}(u) = g, \quad \operatorname{div} u = 0.$$

If in addition $q \in (1,2)$, $|\lambda| \leq \epsilon_2(q)$ and $g \in L^q(\mathbb{R}^3)^3$, we have $\psi \in L^q(\mathbb{R}^3)^3$ and $Z_\lambda^{(q)}(\psi) = g$.

In view of (C1) and Theorem 2.1, the preceding corollary implies

Corollary 4.7. *Let $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$ and $|\lambda| \leq \epsilon_1$, with ϵ_1 from Theorem 4.5. Then $\lambda \in \varrho(L)$, the operator $\widetilde{Z}_\lambda$ (Theorem 4.5) is bijective, and $(\lambda \cdot I - L)^{-1}(g) = E^{(\lambda)} * \widetilde{Z}_\lambda^{-1}(g)$.*

5. Some resolvent estimates for a perturbed Oseen system

The aim of this section is to present some estimates of solutions of equation (4.1). We begin by an observation with respect to the operator $\mathfrak{B}_{\mathrm{sym}}$. Hölder's inequality, the Sobolev imbedding $\|w\|_\infty \leq C \cdot \|w\|_{2,2}$ for $w \in H^2(\mathbb{R}^3)$, and the assumptions in (2.2), imply

Lemma 5.1. *Let $q \in [1, 6/5]$. Then $\|\mathfrak{B}_{\mathrm{sym}}(w)\|_q \leq \mathfrak{C}(q) \cdot \|w\|_2$ for $w \in L^2(\mathbb{R}^3)^3$.*
Let $q \in (6/5, 2]$. Then $\|\mathfrak{B}_{\mathrm{sym}}(w)\|_q \leq \mathfrak{C}(q) \cdot \|w\|_{2,2}$ for $w \in H^2(\mathbb{R}^3)^3$.

The special role of the exponent $6/5$ in Lemma 5.1 is due to the inequality $(1 - q/2)^{-1} \cdot q \leq 3$, which is valid for $q \in [1, 6/5]$. This inequality and (2.2) yield $\|\nabla U\|_{(1-q/2)^{-1} \cdot q} < \infty$.

Next we observe that for $f \in \mathfrak{D}(L)$, $\sigma \in (0, \infty)$ with $\Delta f + \widetilde{a} \cdot \tau \cdot P\mathfrak{B}_{\mathrm{sym}}(f) = \sigma \cdot f$, this function f verifies the Stokes resolvent system $-\Delta f + \sigma \cdot f = g$, $\operatorname{div} f = 0$ with a right-hand side g given by $g = \widetilde{a} \cdot \tau \cdot P\mathfrak{B}_{\mathrm{sym}}(f)$. (Note that $f \in \mathfrak{D}(L) \subset H_2(\mathbb{R}^3)$, so that it is in fact the Stokes resolvent system which appears here.) Thus we may combine Lemma 5.1 and the regularity theory for the Stokes resolvent problem. The latter theory yields the inequality $\|\nabla f\|_2 \leq C \cdot \|\widetilde{a} \cdot \tau \cdot P\mathfrak{B}(f)\|_{6/5}$ and much more deep-lying $W^{2,q}$-estimates; see [3], for example. As a consequence we get

Theorem 5.2. *Let $f \in \mathfrak{D}(L)$, and suppose that $\Delta f + \widetilde{a} \cdot \tau \cdot P\mathfrak{B}_{\mathrm{sym}}(f) = \sigma \cdot f$ for some $\sigma \in (0, \infty)$. (These assumptions are verified by the functions f from the space H_0' introduced in Theorem 2.2.) Let $s \in (1, 2]$.*

Then $f \in W^{2,s}(\mathbb{R}^3)^3$ and $\|f\|_{2,s} \leq \mathfrak{C}(s, \sigma) \cdot \|f\|_{2,2}$. In the case $s \leq 6/5$, the estimate $\|f\|_{2,s} \leq \mathfrak{C}(s, \sigma) \cdot \|f\|_2$ holds. Moreover, $\|\nabla f\|_2 \leq \mathfrak{C} \cdot \|f\|_2$.

The next theorem, which exploits Theorem 4.5, Corollary 4.6 and 4.7, as well as Theorem 5.2, is the principal tool in the proof of Theorem 2.3. It yields resolvent estimates for the operator L under the assumptions that the resolvent parameter λ is small and the right-hand side in the resolvent equation (4.1) belongs to the space H_0' from Theorem 2.2.

Theorem 5.3. *Let f and σ be given as in Theorem 5.2. Let $R \in (0, \infty)$, $\delta \in (0, 1)$.*

Then there is $\epsilon_3 = \epsilon_3(\delta) = \epsilon_3(\tau, U, \sigma, \widetilde{a}, \delta) \in (0, \epsilon_1]$ (the constant ϵ_1 was introduced in Theorem 4.5) such that for $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$, $|\lambda| \leq \epsilon_3$, the ensuing inequalities hold:

$$\|\nabla(\lambda \cdot I - L)^{-1}(f)\|_2 \leq \mathfrak{C}(\sigma) \cdot \|f\|_2, \tag{5.1}$$

$$\|\nabla(\lambda \cdot I - L)^{-2}(f) \,|\, B_R\|_2 \leq \mathfrak{C}(\sigma, \delta, R) \cdot |\lambda|^{-\delta} \cdot \|f\|_2, \tag{5.2}$$

$$\begin{aligned} &\|\nabla(\overline{\lambda} \cdot I - L)^{-1} \circ (\lambda \cdot I - L)^{-1}(f) \,|\, B_R\|_2 \\ &\qquad \leq \ \mathfrak{C}(\sigma, \delta, R) \cdot |\lambda|^{-\delta} \cdot \|f\|_2, \end{aligned} \tag{5.3}$$

$$\|\nabla(\lambda \cdot I - L)^{-3}(f) \,|\, B_R\|_2 \leq \mathfrak{C}(\sigma, \delta, R) \cdot |\lambda|^{-2-\delta} \cdot \|f\|_2. \tag{5.4}$$

Next we state a resolvent estimate valid for large values of $|\lambda|$.

Theorem 5.4. *There is $\widetilde{C} = \widetilde{C}(\tau, U, a, \vartheta, C_1) > 0$ such that for $\lambda \in S_{\vartheta,a}$ with $|\lambda| \geq \widetilde{C}$, and for $g \in H_2(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)^3$, the inequality*

$$|\lambda| \cdot \|\nabla(\lambda \cdot I - L)^{-1}(g)\|_2 \leq \mathfrak{C} \cdot \|\nabla g\|_2$$

is valid.

This theorem may be established by multiplying equation (4.1) by $-\Delta\overline{v}$, with v an abbreviation for $(\lambda \cdot I - L)^{-1}(g)$, and then integrating by parts.

For values of $|\lambda|$ which may be considered as neither large nor small, we exploit the continuity of the resolvent, to obtain

Lemma 5.5. *Let $\kappa_1, \kappa_2 \in (0, \infty)$ with $\kappa_1 < \kappa_2$. Put*

$$\mathfrak{M} := \{\, \lambda \in \mathbb{C} \,:\, \Re\lambda \geq 0,\ \kappa_1 \leq |\lambda| \leq \kappa_2 \,\}.$$

Then $\mathfrak{M} \subset \varrho(L)$ and $\|(\lambda \cdot I - L)^{-1}(w)\|_{1,2} \leq \mathfrak{C}(\kappa_1, \kappa_2) \cdot \|w\|_2$ for $w \in H_2(\mathbb{R}^3)$.

Note that the relation $\mathfrak{M} \subset \varrho(L)$ holds by (2.3) and assumption (C1).

6. Estimate of the semigroup e^{Lt}

Put $\overline{C} := \max\{\, \widetilde{C},\ \epsilon_3(1/16),\ 2^{-1/2},\ a,\ 2 \cdot a \cdot \tan(\pi - \vartheta) \,\}$, where $\widetilde{C}$ was introduced in Theorem 5.4, $\epsilon_3(1/16)$ in Theorem 5.3 (with $\delta = 1/16$), and a and ϑ in Theorem 2.1.

Since $\overline{C} \geq 2 \cdot a \cdot \tan(\pi - \vartheta)$ and $\overline{C} \geq a$, we may choose $\vartheta_0 \in (\pi/2, \vartheta)$ so close to $\pi/2$ that for any $s \in [\overline{C}, \infty)$, the relation

$$\{\, s \cdot e^{i \cdot \varphi} \,:\, \varphi \in [-\vartheta_0, \vartheta_0] \,\} \cup \{\, r \cdot e^{i \cdot \vartheta_0} \,:\, r \in [s, \infty) \,\} \subset S_{\vartheta, a} \tag{6.1}$$

holds. Let $\alpha, \beta \in (0, \infty)$ with $\alpha < \beta$, $\beta \geq \overline{C}$. Then we define the curves $\Gamma_1^{(\alpha,\beta)}, \dots, \Gamma_5^{(\alpha,\beta)} \subset \mathbb{C}$ by

$$\Gamma_1^{(\alpha,\beta)} := \{\, \alpha \cdot e^{i \cdot \varphi} \,:\, \varphi \in [-\pi/2, \pi/2] \,\}, \quad \Gamma_2^{(\alpha,\beta)} := \{\, i \cdot r \,:\, r \in [\alpha, \beta] \,\},$$
$$\Gamma_3^{(\alpha,\beta)} := \{\, i \cdot \beta + r \cdot e^{i \cdot \vartheta} \,:\, r \in [0, \infty) \,\},$$
$$\Gamma_i^{(\alpha,\beta)} := \{\, \overline{y} \,:\, y \in \Gamma_{i-2}^{(\alpha,\beta)} \,\} \ \text{ for } \ i \in \{4, 5\}.$$

Let $s \in [\overline{C}, \infty)$ and define

$$\Lambda_1^{(s)} := \{\, s \cdot e^{i \cdot \varphi} \,:\, \varphi \in [-\vartheta_0, \vartheta_0] \,\}, \quad \Lambda_2^{(s)} := \{\, r \cdot e^{i \cdot \vartheta_0} \,:\, r \in [s, \infty) \,\},$$
$$\Lambda_3^{(s)} := \{\, \overline{y} \,:\, y \in \Lambda_2^{(s)} \,\}.$$

Then, in view of (6.1), (2.3), (C1) and the choice of $\overline{C}$, the curves $\Gamma_\nu^{(\alpha,\beta)}$ and $\Lambda_\mu^{(s)}$ are contained in $\varrho(L)$ ($1 \leq \nu \leq 5$, $1 \leq \mu \leq 3$), and we have by [10, Theorem 1.3.4], for $t \in (0, \infty)$, $w \in H_2(\mathbb{R}^3)$:

$$\begin{aligned} e^{Lt}(w) &= (2 \cdot \pi \cdot i)^{-1} \cdot \sum_{\nu=1}^{5} \int_{\Gamma_\nu^{(\alpha,\beta)}} e^{\lambda \cdot t} \cdot (\lambda \cdot I - L)^{-1}(w)\, d\lambda \qquad (6.2) \\ &= (2 \cdot \pi \cdot i)^{-1} \cdot \sum_{\mu=1}^{3} \int_{\Lambda_\mu^{(s)}} e^{\lambda \cdot t} \cdot (\lambda \cdot I - L)^{-1}(w)\, d\lambda. \end{aligned}$$

A remark is perhaps in order with respect to the difficulties we have to face in this section. In Theorem 2.3, it is claimed that for large t, the term $\|\nabla e^{Lt}(f) \,|\, B_R\|_2$ is bounded by $t^{-1-\epsilon} \cdot \|f\|_2$, for some $\epsilon > 0$, times a factor independent of t and f. (Incidentally we chose $\epsilon = 1/8$, but this is only for definiteness.) We will obtain such an estimate by considering the first sum on the right-hand side of (6.2). Thi means in particular that we have to show that

$$\Big\| \int_{\Gamma_1^{(\alpha,\beta)}} e^{\lambda \cdot t} \cdot \nabla(\lambda \cdot I - L)^{-1}(f) \,|\, B_R \, d\lambda \Big\|_2 \le \mathfrak{C}(R, \vartheta, \sigma) \cdot \|f\|_2 \cdot t^{-1-\epsilon}.$$

In view of (5.1), this should require $\alpha \le t^{-1-\epsilon}$. On other hand, in order to produc a factor $t^{-\mu}$ for some $\mu > 0$ in the estimate of $\int_{\Gamma_\nu^{(\alpha,\beta)}} e^{\lambda \cdot t} \cdot \nabla(\lambda \cdot I - L)^{-1}(f) \,|\, B_R \, d\lambda$ for $\nu = 2$ and $\nu = 4$, we integrate by parts after introducing the local paramete $\varphi(r) := i \cdot r$ $(r \in [\alpha, \beta])$, so that the factor $e^{i \cdot r \cdot t}$ is transformed into $e^{i \cdot r \cdot t} \cdot (i \cdot t)^{-1}$ But this means that a single partial integration does not suffice to generate a factor $t^{-1-\epsilon}$. On the other hand, after two such integrations, we obtain a tern $\nabla(i \cdot r \cdot I - L)^{-3}(f) \,|\, B_R$, which gives rise to a factor $r^{-2-\delta}$ for some $\delta > 0$ (see (5.4)) Integrating this term on the interval $[\alpha, \beta]$ leads to a factor $\alpha^{-1-\delta} = t^{(1+\epsilon)(1+\delta}$ which cancels the effect of the second partial integration. Therefore, in view of th fact that the term $\nabla(i \cdot r \cdot I - L)^{-2}(f) \,|\, B_R$ only produces a factor $r^{-\delta}$ (see (5.2)) we perform some kind of interpolation between one and two partial integrations To this end, we use fractional derivatives, as introduced in the next lemma.

Lemma 6.1. *Let $\kappa, b \in \mathbb{R}$ with $\kappa < b$, $\mu \in (0, 1)$, $h \in C^1([\kappa, b])$ with $h(b) = 0$. Define $\overline{h} : [\kappa, b] \mapsto \mathbb{C}$ by*

$$\overline{h}(r) := \Gamma(1-\mu)^{-1} \cdot \int_r^b (s-r)^{-1+\mu} \cdot h(s) \, ds \quad \text{for } r \in [\kappa, b].$$

Then $\overline{h} \in C^1([\kappa, b])$ with

$$\overline{h}'(r) = \Gamma(1-\mu)^{-1} \cdot \int_r^b (\alpha - r)^{-1+\mu} \cdot h'(\alpha) \, d\alpha \quad \text{for } r \in [\kappa, b]. \qquad (6.3$$

Define $\gamma : [\kappa, b] \ni r \mapsto \Gamma(\mu)^{-1} \cdot \int_r^b (s-r)^{-\mu} \cdot \overline{h}'(s) \, ds \in \mathbb{C}$. Then $h = -\gamma$.

Now we prove an inequality which will be the key element in the estimate o the integrals over $\Gamma_2^{(\alpha,\beta)}$ and $\Gamma_4^{(\alpha,\beta)}$ in (6.2).

Lemma 6.2. *Let f and σ be given as in Theorem 5.2. Let $R \in (0, \infty)$, $\delta \in (0, 1/4)$. Abbreviate $b := \min\{\, 2^{-1/2}, \epsilon_3(\delta) \,\}$, with $\epsilon_3(\delta)$ from Theorem 5.3. Le $\kappa \in (0, b)$, $t \in (0, \infty)$. Then*

$$\Big\| \int_\kappa^b e^{i \cdot r \cdot t} \cdot \nabla(i \cdot r \cdot I - L)^{-2}(f) \,|\, B_R \, dr \Big\|_2 \le \mathfrak{C}(\sigma, \delta, R) \cdot t^{-1/4} \cdot \kappa^{-\delta} \cdot \|f\|_2.$$

To give some indications on the proof of this lemma, we first observe that by (2.3) and (C1), we have $\{\, i \cdot r \;:\; r \in [\kappa, b] \,\} \subset \varrho(L)$. Therefore the mapping

$$g : [\kappa, b] \ni r \mapsto \nabla(i \cdot r \cdot I - L)^{-1}(f) \,|\, B_R \in L^2(B_R)^9$$

is in particular twice continuously differentiable, with

$$g^{(\nu)}(r) = (-i)^\nu \cdot \nu \cdot \nabla(i \cdot r \cdot I - L)^{-(\nu+1)}(f) \,|\, B_R \quad \text{for } \nu \in \{1, 2\},\ r \in [\kappa, b].$$

Thus, due to the assumption $b \le \epsilon_3(\delta)$, inequalities (5.2) and (5.4) yield

$$\|g'(r)\|_2 \le \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot r^{-\delta}, \quad \|g''(r)\|_2 \le \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot r^{-2-\delta} \tag{6.4}$$

for $r \in [\kappa, b]$. Put $h(r) := (i\cdot t)^{-1}\cdot(e^{i\cdot r\cdot t} - e^{i\cdot b\cdot t})$ for $r \in [\kappa, b]$. Define $\overline{h}$ and γ as in Lemma 6.1, with $\mu = 1/4$. Then we get by partial integration

$$\begin{aligned}\int_\kappa^b e^{i\cdot r\cdot t}\cdot\nabla(i\cdot r\cdot I - L)^{-2}(f)\,|\,B_R\,dr &= -i\cdot\int_\kappa^b \gamma'(r)\cdot g'(r)\,dr \\ &= i\cdot\Gamma(1/4)^{-1}\cdot\int_\kappa^b \overline{h}'(s)\cdot\Big(\int_\kappa^s (s-r)^{-1/4}\cdot g''(r)\,dr\Big)\,ds \\ &\quad + i\cdot\Gamma(1/4)^{-1}\cdot\int_\kappa^b (s-\kappa)^{-1/4}\cdot\overline{h}'(s)\,ds\cdot g'(\kappa).\end{aligned} \tag{6.5}$$

Take $s \in (\kappa, b)$. If $s - s^3 > \kappa$, we have

$$\begin{aligned}&\int_\kappa^s (s-r)^{-1/4}\cdot g''(r)\,dr \\ &= \int_{s-s^3}^s (s-r)^{-1/4}\cdot g''(r)\,dr - (1/4)\cdot\int_\kappa^{s-s^3} (s-r)^{-5/4}\cdot g'(r)\,dr \\ &\quad + s^{-3/4}\cdot g'(s-s^3) - (s-\kappa)^{-1/4}\cdot g'(\kappa).\end{aligned}$$

Further observe that $s \le b \le 2^{-1/2}$, hence $s^3 \le s/2$. Now we find with (6.4), in the case $s - s^3 > \kappa$,

$$\begin{aligned}&\Big\|\int_\kappa^s (s-r)^{-1/4}\cdot g''(r)\,dr\Big\|_2 \\ &\le \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot\Big(\int_{s-s^3}^s (s-r)^{-1/4}\cdot r^{-2-\delta}\,dr \\ &\quad + \int_\kappa^{s-s^3} (s-r)^{-5/4}\cdot r^{-\delta}\,dr + s^{-3/4}\cdot(s-s^3)^{-\delta} + (s-\kappa)^{-1/4}\cdot\kappa^{-\delta}\Big) \\ &\le \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot\Big(s^{-2-\delta}\cdot\int_{s-s^3}^s (s-r)^{-1/4}\,dr + \kappa^{-\delta}\cdot\int_\kappa^{s-s^3} (s-r)^{-5/4}\,dr \\ &\quad + s^{-3/4}\cdot\kappa^{-\delta} + (s-\kappa)^{-1/4}\cdot\kappa^{-\delta}\Big) \\ &\le \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot(s^{1/4-\delta} + \kappa^{-\delta}\cdot s^{-3/4} + (s-\kappa)^{-1/4}\cdot\kappa^{-\delta}),\end{aligned} \tag{6.6}$$

where we used the inequality $s^3 \le s/2$ and the assumption $s - s^3 > \kappa$ in the last but one inequality. If $s - s^3 \le \kappa$, we argue as follows, again using (6.4),

$$\begin{aligned}\Big\|\int_\kappa^s (s-r)^{-1/4}\cdot g''(r)\,dr\Big\|_2 &\le \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot\int_\kappa^s (s-r)^{-1/4}\cdot r^{-2-\delta}\,dr \\ &\le \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot\kappa^{-2-\delta}\cdot\int_{s-s^3}^s (s-r)^{-1/4}\,ds \\ &\le \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot\kappa^{-2-\delta}\cdot s^{9/4} \\ &\le \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot\kappa^{1/4-\delta} \le \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot s^{1/4-\delta},\end{aligned}$$

where the last but one inequality holds because $s^3 \leq s/2$ (see above), so that $s \leq s^3 + \kappa \leq s/2 + \kappa$, hence $s \leq 2 \cdot \kappa$. Therefore we see that inequality (6.6) holds in any case. Starting from (6.5), and applying (6.6), we now find

$$\begin{aligned}
&\Big\| \int_\kappa^b e^{i\cdot r\cdot t} \cdot \nabla(i \cdot r \cdot I - L)^{-2}(f) \,|\, B_R \, dr \Big\|_2 \qquad (6.7)\\
\leq\ & \mathfrak{C}(\sigma, \delta, R) \cdot \max\{|\overline{h}'(s)| \,:\, s \in [\kappa, b]\}\\
&\cdot\Big(\int_\kappa^b \Big\| \int_\kappa^s (s-r)^{-1/4} \cdot g''(r)\, dr \Big\|_2 \, ds \;+\; \|f\|_2 \cdot \kappa^{-\delta} \cdot \int_\kappa^b (s-\kappa)^{-1/4} \, ds \Big)\\
\leq\ & \mathfrak{C}(\sigma, \delta, R) \cdot \max\{|\overline{h}'(s)| \,:\, s \in [\kappa, b]\} \cdot \|f\|_2\\
&\cdot\Big(\int_\kappa^b \big(s^{1/4-\delta} + \kappa^{-\delta} \cdot s^{-3/4} + \kappa^{-\delta} \cdot (s-\kappa)^{-1/4} \big) \, ds + \kappa^{-\delta} \Big)\\
\leq\ & \mathfrak{C}(\sigma, \delta, R) \cdot \max\{|\overline{h}'(s)| \,:\, s \in [\kappa, b]\} \cdot \|f\|_2 \cdot \kappa^{-\delta}.
\end{aligned}$$

This leaves us to consider $|\overline{h}'(s)|$, for $s \in [\kappa, b]$. In this respect, we observe that $|h(s)| \leq 2 \cdot t^{-1}$, $|h'(s)| \leq 2$ for $s \in [\kappa, b]$. Thus, in the case $s + 1/t < b$, by (6.3) and a partial integration,

$$\begin{aligned}
|\overline{h}'(s)| \;=\; & \Gamma(3/4)^{-1} \cdot \Big| \int_s^{s+1/t} (\alpha - s)^{-3/4} \cdot h'(\alpha) \, d\alpha\\
& + (3/4) \cdot \int_{s+1/t}^b (\alpha - s)^{-7/4} \cdot h(\alpha) \, d\alpha \;-\; t^{3/4} \cdot h(s + 1/t) \Big|\\
\leq\ & \mathfrak{C} \cdot \Big(\int_s^{s+1/t} (\alpha - s)^{-3/4} \, d\alpha + \int_{s+1/t}^b (\alpha - s)^{-7/4} \, d\alpha \cdot t^{-1} + t^{-1/4} \Big)\\
\leq\ & \mathfrak{C} \cdot t^{-1/4}.
\end{aligned}$$

If $s \in [\kappa, b]$ with $s + 1/t \geq b$, we deduce from (6.3) and the inequality $|h'(s)| \leq 2$ for $s \in [\kappa, b]$ that $|\overline{h}'(s)| \leq \mathfrak{C} \cdot (b - s)^{1/4} \leq \mathfrak{C} \cdot t^{-1/4}$. The estimate $|\overline{h}'(s)| \leq \mathfrak{C} \cdot t^{-1/4}$ thus holds in any case. When we insert this estimate into (6.7), we obtain the inequality stated in the lemma. ◇

Lemma 6.2 enters into the proof of

Theorem 6.3. *Let* $R \in (0, \infty)$, $t \in [\max\{\epsilon_3(1/16)^{-1}, 2^{1/2}\}, \infty)$, *with* $\epsilon_3(1/16)$ *from Theorem* 5.3 *with* $\delta = 1/16$. *Let* f, σ *be given as in Theorem* 5.2. *Then*

$$\|\nabla e^{Lt}(f) \,|\, B_R\|_2 \leq \mathfrak{C}(\sigma, R, \vartheta) \cdot \|f\|_2 \cdot t^{-9/8}.$$

Let us give some indications on the proof of this theorem. We consider the first sum in (6.2), with $\alpha = t^{-2}$ (hence $\alpha = t^{-2} \leq t^{-1} \leq \min\{2^{-1/2}, \epsilon_3(1/16)\}$) and $\beta \in [\overline{C}, \infty)$, where $\overline{C}$ was introduced at the beginning of this section. Then the gradient of the integral over $\Gamma_1^{(\alpha,\beta)}$ in (6.2) may be estimated in the L^2-norm on B_R by a constant times $t^{-2} \cdot \|f\|_2$, as follows from (5.1).

The same norm of the gradient of the integrals over $\Gamma_3^{(\alpha,\beta)}$ and $\Gamma_5^{(\alpha,\beta)}$ is evaluated by referring to Theorem 5.4 and to the last inequality in Theorem 5.2. (Recall that $\beta \geq \overline{C} \geq \widetilde{C}$.) We obtain the upper bound $\mathfrak{C}(\sigma) \cdot (\beta \cdot t)^{-1} \cdot \|f\|_2$. This leaves us to consider the integrals over $\Gamma_2^{(\alpha,\beta)}$ and $\Gamma_4^{(\alpha,\beta)}$. In this respect, we observe that after a partial integration,

$$\int_{\Gamma_2^{(\alpha,\beta)}} e^{\lambda \cdot t} \cdot \nabla(\lambda \cdot I - L)^{-1}(f) \,|\, B_R \, d\lambda = \sum_{j=1}^{5} N_j,$$

where

$$\begin{aligned}
N_1 &:= t^{-1} \cdot e^{i \cdot t \cdot \beta} \cdot \nabla(i \cdot \beta \cdot I - L)^{-1}(f) \,|\, B_R, \\
N_2 &:= -t^{-1} \cdot e^{i \cdot t \cdot \alpha} \cdot \nabla(i \cdot \alpha \cdot I - L)^{-1}(f) \,|\, B_R, \\
N_3 &:= (i/t) \cdot \int_{\alpha}^{b} e^{i \cdot t \cdot r} \cdot \nabla(i \cdot r \cdot I - L)^{-2}(f) \,|\, B_R \, dr, \\
N_4 &:= (i/t) \cdot \int_{b}^{\beta} e^{i \cdot t \cdot r} \cdot \nabla(i \cdot r \cdot I - L)^{-2}(f) \,|\, B_R \, dr,
\end{aligned}$$

with $b := \min\{\epsilon_3(1/16), 2^{-1/2}\}$. Note that $b \leq \overline{C} \leq \beta$. The integral over $\Gamma_4^{(\alpha,\beta)}$ is split into a sum $\sum_{j=1}^{5} \overline{N}_j$, where $\overline{N}_j$ is defined in an analogous way as N_j ($1 \leq j \leq 5$). Now the terms $\|N_1\|_2$ and $\|\overline{N}_1\|_2$ are estimated by Theorem 5.4 and 5.2; we obtain the upper bound $\mathfrak{C}(\sigma) \cdot \beta^{-1} \cdot \|f\|_2$. Moreover, the resolvent formula and (5.3) yield

$$\|N_2 + \overline{N}_2\|_2 \leq \mathfrak{C} \cdot (\alpha + \alpha^{15/16} \cdot t^{-1}) \cdot \|f\|_2.$$

Concerning N_3 and $\overline{N}_3$, we get by Lemma 6.2:

$$\|N_3\|_2 + \|\overline{N}_3\|_2 \leq \mathfrak{C} \cdot t^{-5/4} \cdot \alpha^{-1/16} \cdot \|f\|_2.$$

Finally, in the integrals defining N_4 and $\overline{N_4}$, we perform another partial integration in order to generate an additional term t^{-1}. The term $\|\nabla(i \cdot r \cdot I - L)^{-3}(f) \,|\, B_R\|_2$ arising in this way is evaluated for $r \in [b, \overline{C}]$ by referring to Lemma 5.5, and for $r \in [\overline{C}, \beta]$ by applying Theorem 5.4 and 5.2. Combining all these estimates and letting β tend to infinity, we arrive at Theorem 6.3

Theorem 6.3 dealt with the case of large t. This leaves us to consider small and intermediate values of t. To this end, we use the representation of $e^{Lt}(\Phi)$ by the second sum in (6.2), with

$$s = \overline{C} \ \text{ if } \ t \in [\overline{C}^{-1}, \max\{\epsilon_3(1/16)^{-1}, 2^{1/2}\}], \quad s = 1/t \ \text{ if } \ t \in (0, \overline{C}^{-1}].$$

By referring to Theorem 5.4 and 5.2, we then obtain

Theorem 6.4. *Let $t \in (0, \max\{\epsilon_3(1/16)^{-1}, 2^{1/2}\}]$, and let f, σ be given as in Theorem 5.2. Then $\|\nabla e^{Lt}(f)\|_2 \leq \mathfrak{C}(\sigma) \cdot \|f\|_2$.*

Combining Theorem 6.3 and 6.4 yields Theorem 2.3.

References

[1] K.I. Babenko, *Spectrum of the linearized problem of flow of a viscous incompressible liquid round a body.* Sov. Phys. Dokl. **27** (1982), 25–27.

[2] W. Borchers, T. Miyakawa, *L^2-decay for Navier-Stokes flows in unbounded domains, with application to exterior stationary flows.* Arch. Rat. Mech. Anal. **118** (1992), 273–295.

[3] P. Deuring, *The resolvent problem for the Stokes system in exterior domains: an elementary approach.* Math. Methods Appl. Sci. **13** (1990), 335–349.

[4] P. Deuring, S. Kračmar, *Exterior stationary Navier-Stokes flows in 3D with non-zero velocity at infinity: approximation by flows in bounded domains.* Math. Nachr. **269-270** (2004), 86–115.

[5] P. Deuring, J. Neustupa, *An eigenvalue criterion for stability of Navier-Stokes flows in $\mathbb{R}^3$.* Submitted.

[6] G.P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I. Linearized steady problems* (rev. ed.). Springer, 1998.

[7] G.P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. II. Nonlinear steady problems.* Springer, 1994.

[8] G.P. Galdi, M. Padula, *A new approach to energy theory in the stability of fluid motion.* Arch. Rat. Mech. Anal. **110** (1990) 187–286.

[9] G.P. Galdi, S. Rionero, *Weighted energy methods in fluid dynamics and elasticity.* Lecture Notes in Mathematics **1134**, Springer, 1985.

[10] D. Henry, *Geometric theory of semilinear parabolic equations.* Lecture Notes in Mathematics **840**, Springer, 1981.

[11] P. Maremonti, *Asymptotic stability theorems for viscous fluid motions in exterior domains.* Rend. Sem. Mat. Univ. Padova **71** (1984), 35–72.

[12] K. Masuda, *On the stability of incompressible viscous fluid motions past bodies.* J. Math. Soc. Japan **27** (1975), 294–327.

[13] J. Neustupa, *Stabilizing influence of a skew-symmetric operator in semilinear parabolic equations.* Rend. Mat. Sem. Univ. Padova **102** (1999), 1–18.

[14] J. Neustupa, *Stability of a steady solution of a semilinear parabolic system in an exterior domain.* Far East J. Appl. Math. **15** (2004), 309–331.

[15] J. Neustupa, *Stability of a steady viscous incompressible flow past an obstacle.* To appear in J. Math. Fluid Mech.

[16] A. Pazy, *Semigroups of linear operators and applications to partial differential equations.* Springer, 1983.

[17] Y. Shibata, *On an exterior initial boundary value problem for Navier-Stokes equations.* Quarterly Appl. Math. **57** (1999), 117–155.

Paul Deuring
Université du Littoral "Côte d'Opale"
Laboratoire de Mathématiques Pures et Appliquées
F-62228 Calais cédex, France
e-mail: `Paul.Deuring@lmpa.univ-littoral.fr`