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Discrete quotients of Stein manifolds, the geometry of holomorphic Lie group actions, and domains in complex homogeneous spaces

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Introduction

The main focus of my research lies within the theory of Lie group actions in complex analysis and geometry. As Akhiezer wrote in his book [Akh95], the role of Lie groups in complex analysis is twofold. On the one hand they appear as transformation groups of complex manifolds, while on the other hand complex Lie groups and their homogeneous spaces provide interesting examples of complex manifolds where many questions that are rather difficult in the general case can be answered explicitly. In the first case one would like to understand the interplay between the symmetries of a complex manifold and its analytic properties, while in the second case one asks for a Lie theoretic characterization of analytic and geometric properties of the homogeneous space G/H in terms of the pair $H \subset G$.

In this monograph I will present the results that I have obtained since my doctoral thesis [Mie07]. They concern proper holomorphic actions of discrete groups on hyperbolic Stein manifolds, a special case of the Levi problem for domains in homogeneous spaces and Hopf surfaces, questions about the global geometry of complex homogeneous spaces, geometric invariant theory for actions of complex reductive groups on Kähler manifolds, and the theory of gradient maps developed by Peter Heinzner and Gerald Schwarz for actions of real reductive Lie groups, with an emphasis on real spherical gradient manifolds.

In the first chapter I discuss the papers [MO09], [Mie10a] and [Mie10b] where quotients of hyperbolic Stein manifolds by proper holomorphic actions of the group $\mathbb{Z}$ of integers are studied. Motivated by a question of Peter Heinzner, Alan T. Huckleberry conjectured that the quotient of a contractible bounded domain of holomorphy $D \subset \mathbb{C}^n$ by any $\mathbb{Z}$-action that extends to a proper $\mathbb{R}$-action on D is a Stein manifold. In [MO09] we considered more generally hyperbolic Stein surfaces admitting proper $\mathbb{R}$-actions and were able to verify Huckleberry's conjecture for simply-connected bounded domains of holomorphy $D \subset \mathbb{C}^2$. In [Mie10a] it is shown that the conjecture holds for every homogeneous bounded domain in $\mathbb{C}^n$. This result is used in [Mie10b] in order to show that quotients of compression semigroups associated with a bounded symmetric domain by proper $\mathbb{Z}$-actions are Stein as well.

The second chapter discusses the papers that are concerned with complex analytic and geometric properties of complex homogeneous spaces and their subdomains. In [Mie09] the joint action by left and right multiplication of a pair of real forms of a given complex semisimple group is studied and the Cauchy-Riemann structure of closed orbits is determined. The paper [GMO11] deals with the action of a complex reductive group $G = K^{\mathbb{C}}$ on a Kähler manifold X such that the K-action on X admits an equivariant moment map. It is shown that in this case the G-action on X shares many features of algebraic actions: The topological closure of any G-orbit is analytic, every G-orbit is analytically Zariski open in its closure, and the isotropy groups are algebraic subgroups of G. As an application, we obtained a characterization of homogeneous spaces G/H that possess a Kähler form. The idea that in many cases holomorphic actions of complex reductive groups behave like algebraic ones also motivated the research leading to [GM12] where a Rosenlicht-type quotient is constructed for holomorphic actions of complex reductive groups on Stein spaces. In [GMO13] we consider pseudoconvex domains spread over a complex homogeneous manifold $X = G/H$ where G is an arbitrary connected complex Lie group and H is a closed complex subgroup of G. Relying on ideas of Hirschowitz we formulate an obstruction to their being Stein in Lie theoretic terms. This result is then applied to the study of the holomorphic reduction of a pseudoconvex homogeneous space $X = G/H$ for G solvable or reductive. In [Mie13] pseudoconvex non-Stein domains in primary Hopf surfaces are characterized by using similar techniques.

In the last chapter I describe two results obtained for actions of real reductive groups by the method of gradient maps. In [Mie08] it is shown how the theory of gradient map can be used to yield a better understanding of Matsuki's double coset decomposition of a real reductive Lie group with respect to a pair of symmetric subgroups. In [MS10] the characterization of spherical compact Kähler manifolds by their moment maps (see [Bri87] and [HW90]) is carried over to actions of real reductive Lie groups where one has a gradient map, which leads to the notion of (real) spherical gradient manifolds.

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Many of the results presented here are the outcome of joint work. I would like to thank my colleagues for the stimulating and enlightening collaboration.

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Chapter 1

Quotients of Stein manifolds by discrete group actions

1.1 Background

Given a complex space $(X, \mathcal{O}_X)$ endowed with a Lie group G of holomorphic transformations, it is natural to ask under which conditions it is possible to construct a holomorphic quotient of X by the action of G. More precisely, one would like to know under which conditions there exist a complex space $(X/\!\!/G, \mathcal{O}_{X/\!\!/G})$ and a G-invariant holomorphic map $\pi\colon X \to X/\!\!/G$ such that, locally, holomorphic functions on $X/\!\!/G$ are G-invariant holomorphic functions on X, i.e., such that $\mathcal{O}_{X/\!\!/G} = (\pi_*\mathcal{O}_X)^G$. Given the existence of such a holomorphic quotient $\pi\colon X \to X/\!\!/G$, the main question is which analytic properties of X are inherited by $X/\!\!/G$.

If X is a Stein space, e.g. a closed complex subspace of some $\mathbb{C}^n$, and if $G = U^{\mathbb{C}}$ is complex reductive with compact real form U, then the existence of the categorical quotient $\pi\colon X \to X/\!\!/G$ onto a Stein space $X/\!\!/G$ has been shown in [Sno82]. This result has been generalized in [Hei91] where the quotient $\pi\colon X \to X/\!\!/U$ onto a Stein space $X/\!\!/U$ is constructed for the action of a compact Lie group U by holomorphic transformations on a Stein space X. If G is a complex Lie group acting holomorphically and properly on X, the orbit space X/G carries the structure of a complex space such that the natural map $\pi\colon X \to X/G$ is a geometric quotient, see [Hol63]. However, if G is non-compact, in general one cannot expect that X/G inherits the property of being Stein from X unless G is reductive. Hence, we are lead to the following general problem.

Problem A. *Let G be a complex Lie group acting holomorphically and properly on a Stein space X. Under which conditions is the quotient space X/G again Stein?*

Let us study this problem for infinite discrete groups $G = \Gamma$. Examples of co-compact discrete subgroups of complex semisimple Lie groups indicate that Γ should be "small" compared to X. In the papers [MO09], [Mie10a] and [Mie10b] I therefore concentrated on the simplest and already non-trivial case $\Gamma = \mathbb{Z}$. In order to exclude counter-examples such as elliptic curves $\mathbb{C}^*/\mathbb{Z}$, we shall assume that X is a bounded domain of holomorphy or, more generally, a Kobayashi-hyperbolic Stein manifold.

In [CL85] Cœuré and Loeb described a bounded Reinhardt domain D in $\mathbb{C}^2$ endowed with a proper holomorphic action of $\mathbb{Z}$ such that $D/\mathbb{Z}$ is not Stein. Peter Heinzner asked to what extent the rather complicated topology of D and the fact that the group $\mathbb{Z}$ is not contained in the identity component of $\mathrm{Aut}(D)$ are responsible for $D/\mathbb{Z}$ not being Stein. Alan T. Huckleberry conjectured that, if D is a contractible bounded domain of holomorphy in $\mathbb{C}^n$ and if the generator of the $\mathbb{Z}$-action on D is contained in a closed one parameter group of automorphisms, then the quotient manifold $D/\mathbb{Z}$ is Stein. This conjecture was first shown to be true in [dFI01] for the unit ball of $\mathbb{C}^n$.

In [MO09] we proved the conjecture for arbitrary bounded simply-connected domains of holomorphy in $\mathbb{C}^2$, and in [Mie10a] I extended the result of de Fabritiis and Iannuzzi to any bounded homogeneous domain in $\mathbb{C}^n$. In [Mie10b] actions of general discrete groups on certain semigroups in complex semisimple Lie groups were studied. In the following I will describe the content of these papers in greater detail.

Building on the ideas developed in [Mie10a], Vitali showed that the quotient of the Akhiezer-Gindikin domain in the complexification of a non-compact symmetric space G/K by any holomorphic action of $\mathbb{Z}$ is Stein, see [Vi14]. Her paper also contains results about actions of more general nilpotent discrete groups. In [BHH03] the authors established Steinness of quotients of the Akhiezer-Gindikin domain by torsion free co-compact discrete subgroups of G. In [Che13] the author studied holomorphic actions of discrete groups on Kähler-Hadamard manifolds by differential-geometric methods.

1.2 Proper $\mathbb{R}$-actions on hyperbolic Stein surfaces

[MO09] Christian Miebach and Karl Oeljeklaus, *On proper $\mathbb{R}$-actions on hyperbolic Stein surfaces*, Doc. Math. **14** (2009), 673–689.

The main result of [MO09] is the following theorem that proves Huckleberry's conjecture in dimension 2.

Theorem. *Let $D \subset \mathbb{C}^2$ be a simply-connected bounded domain of holomorphy that admits a proper $\mathbb{R}$-action by holomorphic transformations. Then the quotient $D/\mathbb{Z}$ of D by the induced holomorphic $\mathbb{Z}$-action is a Stein manifold. Moreover, $D/\mathbb{Z}$ can be realized as a domain in $\mathbb{C}^2$.*

As an application of this result we obtain a normal form of simply-connected bounded domains of holomorphy in $\mathbb{C}^2$ with positive-dimensional automorphism group, without any assumption on the regularity of the boundary of D.

The proof relies on Palais' theory (cf. [Pal57]) of globalizations of local Lie group actions, see [HI97] for a formulation in the holomorphic category. Considering the holomorphic local flow of the holomorphic vector field on D given by the $\mathbb{R}$-action, we see that the $\mathbb{R}$-action extends to a local holomorphic $\mathbb{C}$-action on D. Since D is simply-connected, we can deduce from [CTIT00] that the Palais globalization of this local action is a simply-connected Stein manifold. More precisely, there exists a simply-connected Stein manifold D^* endowed with a holomorphic $\mathbb{C}$-action that contains D as an open submanifold such that the local $\mathbb{C}$-action on D is given as the restriction of the global $\mathbb{C}$-action on D^*. The key step in the proof of [MO09, Theorem 1.1] consists in showing that the $\mathbb{C}$-action on D^* is proper. Then D^* is equivariantly isomorphic to the product $\mathbb{C} \times S$ where S is either the unit disk or the complex plane. Since $D/\mathbb{Z}$ is a locally Stein domain in the Stein manifold $D^*/\mathbb{Z} \cong \mathbb{C}^* \times S$, the theorem of Docquier-Grauert ([DG60]) implies that $D/\mathbb{Z}$ is Stein. Note that this proof of the two-dimensional case relies on the solution of Problem A for the proper $\mathbb{C}$-action on D^*. The main advantage is that the Riemann surface $D^*/\mathbb{C}$ is Stein if and only if it is non-compact.

Properness of the $\mathbb{C}$-action on D^* is a consequence of the following more general theorem [MO09, Theorem 1.2].

Theorem. *Let X be a Kobayashi-hyperbolic Stein surface on which $\mathbb{R}$ acts properly by holomorphic transformations. Suppose that X is taut, or that X*

admits the Bergman metric and $H^1(X, \mathbb{R}) = 0$*. Then the universal globalization* X^* *of the induced local* $\mathbb{C}$*-action on* X *is Hausdorff and* $\mathbb{C}$ *acts properly on it. If* X *is simply-connected, then the* $\mathbb{C}$*-principal bundle* $X^* \to X^*/\mathbb{C}$ *is holomorphically trivial and* $X^*/\mathbb{C}$ *is a simply-connected Riemann surface.*

In [MO09, Section 5] we give examples of complete Kobayashi-hyperbolic Stein surfaces X endowed with proper $\mathbb{R}$-actions such that $X/\mathbb{Z}$ is isomorphic to the product of the punctured unit disk and a compact Riemann surface of genus g, for any $g \geq 2$. Moreover, we construct a Kobayashi-hyperbolic domain of holomorphy D in a three-dimensional Stein manifold homogeneous under the three-dimensional complex Heisenberg group that admits a proper $\mathbb{R}$-action by holomorphic transformations such that $D/\mathbb{Z}$ is holomorphically separable but not Stein. Note that none of these examples are simply-connected, which underlines the important role of topology in studying Problem A.

1.3 On quotients of bounded homogeneous domains by cyclic groups

[Mie10a] Christian Miebach, *Quotients of bounded homogeneous domains by cyclic groups*, Osaka J. Math. **47** (2010), no. 2, 331–352.

The main result obtained in [Mie10a] is the following theorem.

Theorem. *Let* $D \subset \mathbb{C}^n$ *be a bounded homogeneous domain. Let* φ *be an automorphism of* D *such that the group* $\Gamma = \langle \varphi \rangle$ *is a discrete subgroup of* $\mathrm{Aut}(D)$*. Then the quotient* D/Γ *is a Stein space.*

The main steps of the proof are as follows. Since the group $\mathrm{Aut}(D)$ has only finitely many connected components, we may assume that φ is contained in its identity component $G = \mathrm{Aut}(D)^0$. By Kaneyuki's theorem ([Ka67]) the group G is isomorphic to the identity component of a real algebraic group. Hence, every element $\varphi \in G$ has a Jordan decomposition $\varphi = \varphi_{\mathsf{e}}\varphi_{\mathsf{h}}\varphi_{\mathsf{u}}$ where φ_{e} is elliptic, φ_{h} is hyperbolic, φ_{u} is unipotent and where these elements commute. It can be shown that the group Γ' generated by $\varphi_{\mathsf{h}}\varphi_{\mathsf{u}}$ is again discrete in G. Since the groups Γ and Γ' differ by the compact torus generated by φ_{e}, the quotient D/Γ' is Stein if and only if X is Stein, cf. [Mie10a, Proposition 3.6].

Consequently, we may work with the group Γ' which has the advantage of being contained in a maximal split solvable subgroup S of G that acts simply transitively on D. Since the exponential map $\exp\colon \mathfrak{s} \to S$ is a diffeomorphism, we may transfer the complex structure and the Bergman metric from D to $\mathfrak{s}$. In this way we obtain the normal j-algebra associated with D. Exploiting its general structure theory (see [PS69, Chapters 2.3, 2.4]) we see that there is an S-equivariant holomorphic submersion $\pi\colon D \to D'$ onto a bounded homogeneous domain D' having all fibers biholomorphically equivalent to the unit ball $\mathbb{B}_m$. In order to overcome the problem that $\pi\colon D \to D'$ is in general not holomorphically locally trivial, we consider the universal globalizations D^* and $(D')^*$ of the induced local actions of $S^{\mathbb{C}}$ and prove that we obtain a holomorphic fiber bundle $\pi^*\colon D^* \to (D')^*$. This allows us to use an inductive argument to prove Steinness of D/Γ' if Γ' acts properly on D'. On the other hand, if Γ' stabilizes every π–fiber we can use the fact that the quotients $\mathbb{B}_m/\Gamma'$ are already known to be Stein, for which a new and more conceptual proof is given in [Mie10a, Section 4].

1.4 Discrete quotients of complex semigroups

[Mie10b] Christian Miebach, *Sur les quotients discrets de semigroupes complexes*, Ann. Fac. Sci. Toulouse Math. (6) **19** (2010), no. 2, 269–276.

In order to state the main result of [Mie10b] we have to introduce some notation. Let G be a connected real semisimple Lie group of hermitian type with maximal compact subgroup K and complexification $G^{\mathbb{C}}$. We suppose that G embeds as a closed subgroup into $G^{\mathbb{C}}$. Since G is hermitian, the symmetric space G/K can be realized as a bounded symmetric domain D in some $\mathbb{C}^n$. We embed D as an open G-orbit into its compact dual $G^{\mathbb{C}}/P$ where P is a parabolic subgroup of $G^{\mathbb{C}}$. Consequently, we may define the compression semigroup S in $G^{\mathbb{C}}$ associated with D as

$$S := \{g \in G^{\mathbb{C}};\ g \cdot \overline{D} \subset D\},$$

where $\overline{D}$ denotes the topological closure of D in $G^{\mathbb{C}}/P$. One can show that S is a hyperbolic domain of holomorphy in $G^{\mathbb{C}}$ that is invariant under the action of $G \times G$ by left and right multiplication, see [Nee00]. Consequently, every discrete subgroup $\Gamma \subset G$ acts properly on S by left multiplication. We write $X := \Gamma\backslash S$ for the quotient manifold.

In [ABK04] the authors showed that $X = \Gamma\backslash S$ is holomorphically separable for any $\Gamma \subset G$. Moreover, they conjectured that X is Stein and verified this conjecture for $G = \mathrm{SL}(2, \mathbb{R})$ and Γ any subgroup of $\mathrm{SL}(2, \mathbb{Z})$.

The following theorem [Mie10b, Théorème 2.4] gives a sufficient condition for X to be Stein. In particular, it yields a new proof of the fact that X is Stein for $G = \mathrm{SL}(2,\mathbb{R})$ and $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$.

Theorem. *Let $D = G/K$ be a bounded symmetric domain in $\mathbb{C}^n$ and let $S \subset G^{\mathbb{C}}$ be the associated compression semigroup. Let $\Gamma \subset G$ be a discrete subgroup that acts freely on D. If D/Γ is Stein, the same holds for $X = \Gamma\backslash S$.*

In [Mie10b, Section 3] the general conjecture in [ABK04] is disproved as a consequence of the following

Theorem. *Let $S \subset \mathrm{SL}(2,\mathbb{C})$ be the compression semigroup associated with the upper half plane in $\mathbb{C}$. If $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ is a co-compact discrete subgroup, then $X = \Gamma\backslash S$ is not Stein.*

Chapter 2

Analytic properties of complex homogeneous spaces and their subdomains

2.1 Background

Complex homogeneous spaces are complex manifolds of the form $X = G/H$ where G is a complex Lie group and H is a closed complex subgroup of G. Their importance stems from the fact that many problems that are rather difficult for arbitrary complex manifolds have particularly nice answers when dealing with complex homogeneous spaces. A typical question is how one can characterize analytic or geometric properties of $X = G/H$ by algebraic properties of the pair $H \subset G$. For instance, if G is complex reductive, it was shown in [Mat60] and [Oni60] that X is a Stein manifold if and only if H is complex reductive. Due to [BO73], $X = G/H$ with G complex reductive is holomorphically separable if and only if H is an algebraic subgroup of G and X is a quasi-affine variety. If G is semisimple, it was proven in [BO88] and [Ber87] that G/H admits a Kähler form if and only if H is an algebraic subgroup of G. In the spirit of these results, the paper [GMO13] discusses global properties of pseudoconvex complex homogeneous spaces.

Closely related to the study of complex homogeneous spaces is the investigation of (holomorphic) G-actions. It is a recurrent theme in some of the papers that I will discuss in this chapter that under certain assumptions of analytic nature the holomorphic actions of complex reductive groups share many properties of algebraic ones. For example, a key result that allowed to carry over the characterization of existence of Kähler metrics on $X = G/H$ for reductive G in [GMO11] states that the orbits of Hamiltonian G-actions are locally closed in the analytic Zariski topology and that stabilizers are *algebraic* subgroups of G. In [GM12] the construction of Rosenlicht quotients for algebraic group actions was extended to the case of Stein spaces endowed with holomorphic actions of reductive groups.

Another typical problem is to investigate complex-analytic properties of domains in a complex homogeneous space G/H that are assumed to be invariant with respect to certain subgroups of G. In my thesis I studied the existence of domains of holomorphy in a complex semisimple group G that are invariant with respect to a pair of real forms of G. In [GMO13] resp. [Mie13] the obstruction of pseudoconvex domains in complex homogeneous spaces resp. Hopf surfaces to be Stein is described in group theoretical terms.

2.2 Geometry of invariant domains in complex semisimple Lie groups

In [Las78] Lassalle characterized the domains of holomorphy in a complex reductive group $G = U^{\mathbb{C}}$ that are invariant under left and right multiplication by elements of the compact real form U.

In [FG98] Fels and Geatti study the case of domains in connected semisimple complex Lie groups G that are invariant under left and right multiplication by elements of a *non-compact* real form G_0 of G. Their approach is based on the determination of the intrinsic CR geometry of generic $(G_0 \times G_0)$-orbits in G. More precisely, since $G_0 \times G_0$ acts by holomorphic transformations on G, every orbit is a generic CR submanifold of G, and hence its intrinsic Levi form and Levi cone can be defined. According to [BP82] the Levi cone of a generic CR submanifold of G governs the local holomorphic extension of CR functions from the CR submanifold to some open subset of G. In [FG98, Theorem 5.3] the authors give a Lie theoretic description of the Levi cone of $(G_0 \times G_0)$-orbits that are generic, i.e., closed and of maximal dimension. Combining this result with the Boggess-Polking extension theorem, Fels and Geatti obtain a necessary criterion for a bi-invariant domain in G containing a generic $(G_0 \times G_0)$-orbit in its boundary to be a domain of holomorphy.

In the paper [Mie09] (based on my thesis [Mie07]) I investigated complex analytic properties of domains in a connected semisimple complex Lie group G that are invariant with respect to a *pair* of real forms of G. To be more precise, let G_1 and G_2 be two real forms of G defined by two anti-holomorphic involutive automorphisms σ_1 and σ_2 of G. The group $G_1 \times G_2$ acts on G by $(g_1, g_2) \cdot z = g_1 z g_2^{-1}$. I determined the Levi cone of any generic $(G_1 \times G_2)$-orbit in G and thus obtained a necessary criterion for a $(G_1 \times G_2)$-invariant domain in G containing a closed orbit in its boundary to be a domain of holomorphy, see [Mie09, Theorems 3.23 and 4.2].

In order to state these results, let $z \in G$ be such that the orbit $G_1 z G_2$ is generic, i.e., closed and of maximal dimension. Due to [Mat97] we may assume $z = n\exp(i\eta) \in n\exp(i\mathfrak{c})$ where n lies in a compact real form of G and where $\mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{g}_2 \cap \mathrm{Ad}(n^{-1})\mathfrak{g}_1$. Moreover, we may identify $\mathfrak{c}$ with the normal space to $T_z(G_1 z G_2)$ in $T_z G$. Without loss of generality and in order to simplify the notation, we assume that the triple (G, G_1, G_2) is irreducible.

Theorem. *Let $z = n\exp(i\eta) \in G$ such that $G_1 z G_2$ is generic. Then the Levi cone $\mathcal{C}_z$ of $G_1 z G_2$ coincides with $\mathfrak{c}$ unless $\mathfrak{g}_1 = \mathfrak{g}_2 =: \mathfrak{g}_0$ is of Hermitian type and $i\eta$ lies in the maximal elliptic cone $C_{\max}$, in which case the Levi cone is pointed.*

As a first application, this result allows us generalize the criterion of Fels and Geatti to the case of two different real forms. Consequently, we see that $(G_1 \times G_2)$-invariant domains of holomorphy in G that contain a generic orbit in their boundary are rather rare. In particular, under the assumptions of the previous theorem such domains of holomorphy do not exist unless G_1 and G_2 are conjugate in G.

On the other hand, if G_1 and G_2 are not conjugate, it is more reasonable to look for invariant q-complete domains, a natural generalization of domains of holomorphy. In [Mie09, Theorem 4.6] those triples (G, G_1, G_2) are determined for which the generic $G_1 \times G_2$-orbit is a hypersurface in G. In this case every generic orbit coincides with the boundary of a $(G_1 \times G_2)$-invariant domain in G. In [Mie09, Theorem 4.9] it is shown that there are always q-complete $(G_1 \times G_2)$-invariant domains with boundary $G_1 z G_2$ in G and that the number q can be computed in Lie theoretic terms.

In arbitrary rank, [Mie09, Theorem 4.16] says that G contains a $(G_1 \times G_2)$-invariant q-complete domain whenever $\mathfrak{g}_1 \cap \mathfrak{g}_2$ is of hermitian type.

2.3 On homogeneous Kähler and Hamiltonian manifolds

[GMO11] Bruce Gilligan, Christian Miebach, and Karl Oeljeklaus, *Homogeneous Kähler and Hamiltonian manifolds*, Math. Ann. **349**

(2011), no. 4, 889–901.

In [BO88] and [Ber87] the authors characterize complex homogeneous Kähler manifolds $X = G/H$ where G is a semisimple complex Lie group and H is a closed complex subgroup of G. More precisely, they show that such $X = G/H$ admits a Kähler form if and only if H is an algebraic subgroup of G. The simple example of an elliptic curve $\mathbb{C}^*/\mathbb{Z}$ shows that this result is no longer true when G is merely supposed to be reductive.

In [GMO11, Theorem 5.1] existence of a Kähler form on the complex homogeneous manifold $X = G/H$ with G reductive is characterized as follows. Recall that any complex reductive group G can be written as $G = S \cdot Z$ where the derived group $S := G'$ is semisimple and Z is the center of G.

Theorem. *Let $G = S \cdot Z$ be complex reductive and let H be a closed complex subgroup of G. Then $X = G/H$ admits a Kähler form if and only if $S \cap H$ is algebraic and $S \cdot H$ is closed in G.*

The proof that these two conditions allow for the construction of a Kähler form on X relies on Blanchard's Théorème Principal II in [Bla56]. In order to show that they are also necessary we make use of the theory of moment maps as follows.

Let $G = U^{\mathbb{C}}$ be a complex reductive group acting holomorphically on a Kähler manifold (X, ω). We may assume that the compact real form U of G leaves the Kähler form ω invariant. A *moment map* for the U-action on X is a smooth map $\mu\colon X \to \mathfrak{u}^*$, equivariant with respect to the co-adjoint U-representation on $\mathfrak{u}^*$, such that for every $\xi \in \mathfrak{u}$ the smooth function μ^ξ given by $\mu^\xi(x) := \mu(x)\xi$ verifies $d\mu^\xi = \omega(\xi_X, \cdot)$. If a moment map $\mu\colon X \to \mathfrak{u}^*$ exists, we say that X is a *Hamiltonian G-manifold*. Basic examples of Hamiltonian G-manifolds arise as G-invariant closed complex submanifolds of V or $\mathbb{P}(V)$ where V is a G-representation space. If G is semisimple, then any Kähler manifold on which G acts holomorphically is Hamiltonian.

It is not hard to give examples of projective manifolds endowed with free holomorphic $\mathbb{C}^*$-actions such that the topological closure of every $\mathbb{C}^*$-orbit is a smooth real hypersurface. In particular the orbits are then not even locally closed. It is shown in [GMO11] that the presence of a moment map prevents such irregular behavior. More precisely, the following theorem shows that Hamiltonian G-manifolds share many of the good properties of algebraic G-varieties, cf. [GMO11, Theorem 3.6].

Theorem. *Let G be complex reductive and let X be a Hamiltonian G-manifold. Then the topological closure of every G-orbit is complex-analytic in X. Moreover, every G-orbit has only orbits of strictly smaller dimension in its closure.*

If a Hamiltonian G-manifold X is semistable, i.e., if the topological closure of any G-orbit in X intersects the zero fiber of the moment map, then this theorem follows from the quotient theory developed in [HL94]. The difficulties in the proof of our result stem from the fact that the zero fiber of the moment map might even be empty. To circumvent these problems we use the fact that every Hamiltonian G-manifold is covered by open subsets that are semistable with respect to a maximal torus T of G. Since $G = UTU$, we then can deduce a weak regularity property of G-orbits from the corresponding statement for T-orbits. More precisely, we show that every G-orbit is a locally subanalytic subset of X, which turns out to be enough in order to apply Bishop's theorem [Bis64] about extensions of analytic sets to the closure of the G-orbits.

In the homogeneous case these methods yield the following characterization of homogeneous Hamiltonian G-manifolds, see [GMO11, Theorem 4.11].

Theorem. *Let G be a complex reductive group and let H be a closed complex subgroup. Then $X = G/H$ is a Hamiltonian G-manifold if and only if H is an algebraic subgroup of G.*

2.4 Invariant meromorphic functions on Stein spaces

> [GM12] Daniel Greb and Christian Miebach, *Invariant meromorphic functions on Stein spaces*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 5, 1983–2011.

One of the fundamental results relating invariant theory and the geometry of algebraic group actions is Rosenlicht's Theorem [Ros56, Thm. 2]: for any action of a linear algebraic group on an algebraic variety there exists a finite set of invariant rational functions that separate orbits in general position. Moreover, there exists a rational quotient, i.e., a Zariski-open invariant subset on which the action admits a geometric quotient. It was the purpose of the paper [GM12] to study meromorphic functions invariant under holomorphic group actions and to construct quotients of Rosenlicht-type in the analytic category.

Examples of non-algebraic holomorphic actions of $\mathbb{C}^*$ on projective surfaces having an orbit space that is not locally Hausdorff show that even in the compact analytic case an analogue of Rosenlicht's Theorem does not hold without further assumptions. If a complex reductive group acts *meromorphically* on a compact Kähler space (and more generally a compact complex space of class $\mathcal{C}$), existence of meromorphic quotients was shown by Lieberman [Lie78] and Fujiki [Fuj78]. One should note that the notions of meromorphic and of Hamiltonian actions are closely related.

As a natural starting point in the non-compact case we consider group actions on spaces with rich function theory such as Stein spaces. Actions of reductive groups and their subgroups on these spaces are known to possess many features of algebraic group actions. However, while the holomorphic invariant theory in this setup is well understood, cf. [Hei91], invariant meromorphic functions had been less studied.

In [GM12] we developed fundamental tools to study meromorphic functions in an equivariant setup. These tools were then used to prove the following result, which provides a natural generalization of Rosenlicht's Theorem to Stein spaces with actions of complex reductive groups.

Main Theorem. *Let $H \subset G$ be an algebraic subgroup of a complex reductive Lie group G, and let X be a Stein G-space. Then, there exists an H-invariant Zariski-open dense subset Ω in X and a holomorphic map $p\colon \Omega \to Q$ to a Stein space Q such that*

(1) the map p is a geometric quotient for the H-action on Ω,

(2) the map p is universal with respect to H-stable analytic subsets of Ω,

(3) the map p is a submersion and realizes Ω as a topological fiber bundle over Q,

(4) the map p extends to a weakly meromorphic map (in the sense of Stoll) from X to Q,

(5) for every H-invariant meromorphic function $f \in \mathcal{M}_X(X)^H$, there exists a unique meromorphic function $\overline{f} \in \mathcal{M}_Q(Q)$ such that $f|_\Omega = \overline{f} \circ p$, i.e., the map $p\colon \Omega \to Q$ induces an isomorphism between $\mathcal{M}_X(X)^H$ and $\mathcal{M}_Q(Q)$, and

(6) the H-invariant meromorphic functions on X separate the H-orbits in Ω.

It is essential that the H-action on X extends to a holomorphic action of the complex reductive group G. In fact, there exists a domain of holomorphy in $\mathbb{C}^2$ endowed with a free holomorphic $\mathbb{C}$-action such that the topological closure of every $\mathbb{C}$-orbit is a real hypersurface, see [GM12, Example 3.17] and [HOV94, Sections 7-8].

The idea of proof is to first establish a weak equivariant analogue of Remmert's and Narasimhan's embedding theorem for Stein spaces [Nar60]. More precisely, given a G-irreducible Stein G-space we prove the existence of a G-equivariant holomorphic map into a finite-dimensional G-representation space V that is a proper embedding when restricted to a big Zariski-open G-invariant subset, see [GM12, Proposition 5.2]. Since the G-action on V is algebraic, we may then apply Rosenlicht's Theorem to this linear action. Subsequently, a careful comparison of algebraic and holomorphic geometric quotients allows us to carry over the existence of a Rosenlicht-type quotient from V to X.

The geometric quotient constructed in this paper provides us with a new and effective tool to investigate invariant meromorphic functions on Stein spaces. In the following let me describe two typical applications of the main result.

Given a Stein G-space we show that every invariant meromorphic function is a quotient of two invariant holomorphic functions precisely if the generic fiber of the natural invariant-theoretic quotient $\pi\colon X \to X/\!\!/G$ contains a dense orbit, see [GM12, Theorem 3.5]. An important class of examples for this situation consists of representation spaces of semisimple groups G.

An important fundamental result of Richardson [Ric74] states that in every connected Stein G-manifold there exists an open and dense subset on which all isotropy groups are conjugate in G. Under further assumptions on the group action we use the Main Theorem to sharpen Richardson's result by showing that there exists a Zariski-open subset on which the conjugacy class of stabilizer groups is constant, see [GM12, Proposition 3.11]. In particular, for every effective torus action on a Stein manifold we find a Zariski-open subset that is a principal fiber bundle over the meromorphic quotient, cf. [GM12, Remark 3.14].

2.5 Pseudoconvex domains spread over complex homogeneous manifolds

[GMO13] Bruce Gilligan, Christian Miebach, and Karl Oeljeklaus, *Pseudoconvex domains spread over complex homogeneous manifolds*, Manuscripta Math. **142** (2013), no. 1-2, 35–59.

Let G be a connected complex Lie group and H a closed complex subgroup of G. The complex homogeneous manifold $X = G/H$ admits a Lie theoretic holomorphic reduction $\pi\colon G/H \to G/J$ where G/J is holomorphically separable and $\mathcal{O}(G/H) \simeq \pi^*\mathcal{O}(G/J)$. If $X = G/H$ is holomorphically convex, then $\pi\colon G/H \to G/J$ coincides with the Remmert reduction of X, but in general the base G/J is not Stein nor do we have $\mathcal{O}(J/H) = \mathbb{C}$. In some cases one can say more. If G is solvable, then G/J is always a Stein manifold, although $\mathcal{O}(J/H) = \mathbb{C}$ is not true in general (see [HO86]). If G is complex reductive, then due to [BO73] the base G/J is a quasi-affine variety and in particular J contains the Zariski closure of H in G. However, the fiber J/H can even be Stein as it is the case for $G/H = \mathrm{SL}(2,\mathbb{C})/\left(\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right) \to \mathrm{SL}(2,\mathbb{C})/\left(\begin{smallmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{smallmatrix}\right) = G/J$.

From a complex-analytic viewpoint the Lie theoretic holomorphic reduction is optimal for a complex homogeneous manifold G/H with G nilpotent: its base G/J is Stein and its fiber J/H does not admit any non-constant holomorphic function. It was observed in [Huc10] that for nilpotent G the homogeneous manifold G/H is *pseudoconvex*, i.e., that it possesses a continuous plurisubharmonic exhaustion function. The main results of [GMO13] describe the holomorphic reduction of a pseudoconvex homogeneous space G/H under the assumption that G is complex reductive or solvable. It turns out that it is precisely the pseudoconvexity of G/H that is responsible for the good properties of its holomorphic reduction. The following theorem summarizes [GMO13, Theorems 7.4, 7.5, 8.5].

Theorem. *Suppose that the complex homogeneous manifold $X = G/H$ is pseudoconvex and let $X = G/H \to G/J$ be its holomorphic reduction.*

(1) If G is a complex reductive Lie group, then the base G/J is Stein and the fiber J/H satisfies $\mathcal{O}(J/H) = \mathbb{C}$. If, in addition, X is Kähler, then J/H is a product of the Cousin group $\overline{H}/H$ and the homogeneous rational manifold $J/\overline{H}$ where $\overline{H}$ denotes the Zariski closure of H in G.

(2) If G is solvable, then the fiber J/H is a Cousin group tower and thus $\mathcal{O}(J/H) = \mathbb{C}$.

As a first step towards the proof of this theorem we discuss the Levi problem for pseudoconvex domains spread over complex homogeneous spaces and present a Lie theoretic description of the obstruction to their being Stein. A characterization of relatively compact, smoothly bounded, pseudoconvex domains D in complex homogeneous manifolds such that D is not Stein is given in [KLY11]. The incorporation of methods of Hirschowitz [Hir75] allows us to simplify their proof and to show that the assumptions on the smoothness of the boundary and relative compactness of D are not needed. This is of course necessary in order to apply our result to the case that $D = G/H$ is pseudoconvex.

One of the essential tools that Hirschowitz uses is the concept of an *inner integral curve*, i.e., a non-constant holomorphic image of $\mathbb{C}$ in D that is relatively compact and is the integral curve of a vector field. Indeed, Hirschowitz proves that if a non-compact pseudoconvex domain D is spread over an infinitesimally homogeneous complex manifold and D has no inner integral curves, then D is Stein.

Our generalization of the main result of [KLY11] reads then as follows.

Theorem. *Let $p\colon D \to X$ be a pseudoconvex domain spread over the complex homogeneous manifold $X = G/H$ such that $p(D)$ contains the base point $eH \in X$. If D is not Stein, then there exist a connected complex Lie subgroup $\widehat{H}$ of G with $H^0 \subset \widehat{H}$ and $\dim H < \dim \widehat{H}$ and a foliation $\mathcal{F} = \{F_x\}_{x\in D}$ of D such that*

(1) every leaf of $\mathcal{F}$ is a relatively compact immersed complex submanifold of D,

(2) every inner integral curve in D passing through $x \in D$ lies in the leaf F_x containing x, and

(3) the leaves of $\mathcal{F}$ are homogeneous under a covering group of $\widehat{H}$.

Moreover, we have the following strengthening of this theorem in the projective setting.

Theorem. *Suppose that G is a connected complex Lie group acting holomorphically on the complex projective space $\mathbb{P}_n$ and that $X = G/H$ is an orbit in $\mathbb{P}_n$. Then, every pseudoconvex domain spread over X is holomorphically convex and the fibers of its Remmert reduction are rational homogeneous manifolds.*

2.6 Pseudoconvex non-Stein domains in primary Hopf surfaces

[Mie13] Christian Miebach, *Pseudoconvex non-Stein domains in primary Hopf surfaces*, arXiv:1306.5595v2 (2013), to appear in Izvestiya: Mathematics **78** (2014), no. 5.

Let us fix two complex numbers $a_1, a_2 \in \mathbb{C}$ with $0 < |a_1| \leq |a_2| < 1$ and let H_a denote the quotient of $\mathbb{C}^2 \setminus \{0\}$ with respect to the $\mathbb{Z}$-action given by $m \cdot (z_1, z_2) := (a_1^m z_1, a_2^m z_2)$. It is well known that H_a is a compact complex surface diffeomorphic to $S^1 \times S^3$ and that the quotient map $p\colon \mathbb{C}^2 \setminus \{0\} \to H_a$, $p(z_1, z_2) =: [z_1, z_2]$, is the universal covering of H_a, hence that H_a is a primary Hopf surface. Moreover, H_a is almost homogeneous with respect to the holomorphic $(\mathbb{C}^*)^2$-action given by $(t_1, t_2) \cdot [z_1, z_2] := [t_1 z_1, t_2 z_2]$. We write H_a^* for the open $(\mathbb{C}^*)^2$-orbit. Its complement in H_a is the union of two elliptic curves.

For a generic choice of a_1 and a_2, the open orbit H_a^* is a Cousin group. The maximal complex direction in the maximal compact subgroup of H_a^* induces a holomorphic foliation $\mathcal{F}$ of H_a such that H_a^* contains with every point the topological closure of the leaf of $\mathcal{F}$ passing through that point, and such that the leaves of $\mathcal{F}|_{H_a^*}$ are relatively compact in H_a^*.

On the other hand, if there exist integers k_1, k_2 such that $a_1^{k_1} = a_2^{k_2}$, then H_a is fibered by elliptic curves over $\mathbb{P}_1$, which gives rise to a holomorphic foliation by compact curves that we denote again by $\mathcal{F}$. As before, H_a^* contains with every point the whole leaf of $\mathcal{F}$ passing through that point.

In [LY13] Levenberg and Yamaguchi characterize locally pseudoconvex domains $D \subset H_a$ having smooth real-analytic boundary that are not Stein. Extending Hirschowitz' techniques developed in [Hir74] and [Hir75] for infinitesimally homogeneous complex manifolds to the almost homogeneous case, the main result of [LY13] is generalized to arbitrary pseudoconvex domains in H_a, which yields the following

Theorem. *Let $D \subset H_a$ be a pseudoconvex domain. If D is not Stein, then D contains with every point $x \in D$ the topological closure $\overline{F}_x$ of the leaf $F_x \in \mathcal{F}$ passing through x.*

Let us outline the main steps of the proof. In order to remove the hypotheses on the boundary regularity of D and to be able to apply Hirschowitz' ideas, we define pseudoconvexity of D by the existence of a continuous plurisubharmonic exhaustion function on D. This is justified by [Mie13, Proposition 4.1] which says that any smoothly bounded locally Stein domain $D \subset H_a$ admits a continuous plurisubharmonic exhaustion function.

Let us assume that $D \subset H_a$ is a pseudoconvex non-Stein domain. If $D^* := D \cap H_a^*$ is not Stein, then the theorem follows from the homogeneous case using [GMO13, Theorem 3.1]. Therefore suppose that D^* is Stein. The main work in the proof of the above theorem consists in the construction of a plurisubharmonic exhaustion function on D that is strictly plurisubharmonic on D^*, and in the proof that in this situation D must already be Stein, from which we obtain the desired contradiction.

Chapter 3

Actions of real reductive groups and gradient maps

3.1 Background

As we have seen in Section 2.3, moment maps are useful tools in order to study holomorphic actions of complex reductive groups on Kähler manifolds and, what is more, prevent such actions from behaving "too wildly". In [HS07a] Heinzner and Schwarz introduced an analogue of a moment map, suitable for the investigation of actions of real reductive Lie groups, as follows.

Let $U^{\mathbb{C}}$ be a complex reductive group with compact real form U and let Z be a Hamiltonian $U^{\mathbb{C}}$-manifold. We consider closed subgroups G of $U^{\mathbb{C}}$ that are compatible with the Cartan decomposition of $U^{\mathbb{C}}$, i.e., that verify $G = K\exp(\mathfrak{p})$ for $K := G \cap U$ and $\mathfrak{p} := \mathfrak{g} \cap i\mathfrak{u}$, and call such groups real reductive. We will always assume that the real reductive group $G = K\exp(\mathfrak{p})$ is non-compact, i.e., that $\mathfrak{p} \neq \{0\}$. Note that being real reductive is not an intrinsic property of G but depends on a fixed embedding into some $U^{\mathbb{C}}$. The fact that the G-action on Z extends to a holomorphic action of $U^{\mathbb{C}}$ controls in particular the action of the center of G. For instance, if Z is a $U^{\mathbb{C}}$-representation space, then the center of G acts by *semisimple* transformations on Z.

Let $\mu\colon Z \to \mathfrak{u}^*$ be a U-equivariant moment map. Identifying $\mathfrak{u}^*$ with $i\mathfrak{u}$ via a U-invariant inner product $\langle\cdot,\cdot\rangle$ on $i\mathfrak{u}$ and composing $\mu\colon Z \to i\mathfrak{u}$ with the orthogonal projection onto $\mathfrak{p} \subset i\mathfrak{u}$, we obtain a K-equivariant map $\mu_{\mathfrak{p}}\colon Z \to \mathfrak{p}$. For any $\xi \in \mathfrak{p}$ we define $\mu_{\mathfrak{p}}^{\xi} \in \mathcal{C}^\infty(Z)$ by $\mu_{\mathfrak{p}}^{\xi}(z) := \big\langle \mu_{\mathfrak{p}}(z), \xi \big\rangle$. One verifies easily that the defining property of the moment map translates into $\operatorname{grad} \mu_{\mathfrak{p}}^{\xi} = \xi_Z$ where the gradient of a smooth map is defined with respect to the Kähler metric on Z. For this reason $\mu_{\mathfrak{p}}$ is called a *gradient map* for the G-action on Z. We refer the reader to [HS07a] and [HSS08] for a more detailed exposition and proofs of the basic properties of gradient maps.

We want to study the action of G on locally closed G-stable submanifolds $X \subset Z$, referred to as *G-gradient manifolds* in the following. Let us cite three fundamental results from [HS07a] that underline the importance of the gradient map $\mu_{\mathfrak{p}} \colon X \to \mathfrak{p}$. The first result shows the relevance of the zero fiber $\mu_{\mathfrak{p}}^{-1}(0)$.

Theorem. *Let $X \subset Z$ be a G-gradient manifold with gradient map $\mu_{\mathfrak{p}} \colon X \to \mathfrak{p}$, let $x \in \mu_{\mathfrak{p}}^{-1}(0)$. Then*

(1) $(G \cdot x) \cap \mu_{\mathfrak{p}}^{-1}(0) = K \cdot x$,

(2) the isotropy subgroup of x in G is real reductive, i.e., we have $G_x = K_x \exp(\mathfrak{p}_x)$ where $\mathfrak{p}_x := \big\{\xi \in \mathfrak{p};\ \xi_X(x) = 0\big\}$, and

(3) the isotropy representation of G_x on T_xX is completely reducible.

Suppose that $x \in \mu_{\mathfrak{p}}^{-1}(0)$. Due to the last property of the above theorem, there exists a G_x-invariant decomposition $T_xX = (\mathfrak{g} \cdot x) \oplus W$ where $\mathfrak{g} \cdot x = \big\{\xi_X(x);\ \xi \in \mathfrak{g}\big\} = T_x(G \cdot x)$. The Slice Theorem says that in an invariant neighborhood of x the G-action is determined by the G_x-representation on W.

Slice Theorem. *Let $X \subset Z$ be a G-gradient manifold and let $x \in \mu_{\mathfrak{p}}^{-1}(0)$. Then there exist a G_x-invariant open neighborhood S of $0 \in W$, a G-stable open neighborhood Ω of $x \in X$, and a G-equivariant diffeomorphism $G \times_{G_x} S \to \Omega$*[1] *with $[e, 0] \mapsto x$.*

The twisted product $G \times_{G_x} S$ is called a *slice model* for the G-action near x. The representation of G_x on W is called the *slice representation.*

In closing we discuss the notion of a topological Hilbert quotient. For $x, y \in X$ we define

$$x \sim y \quad :\Longleftrightarrow \overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset.$$

If this relation is an equivalence relation, we write $\pi \colon X \to X /\!\!/ G$ for the corresponding quotient and call it the topological Hilbert quotient of X with respect to the G-action. However, in general it cannot be expected that the

[1] For any closed subgroup H of G that acts on some set Y, the twisted product $G \times_H Y$ is defined as the quotient of $G \times Y$ by the diagonal H-action $h \cdot (g, y) := (gh^{-1}, h \cdot y)$. The orbit $H \cdot (g, y)$ is denoted by $[g, y] \in G \times_H Y$.

topological Hilbert quotient exists. Instead, we must replace X by its subset of semistable points that is defined with the help of $\mu_{\mathfrak{p}}$ by

$$\mathcal{S}_G\big(\mu_{\mathfrak{p}}^{-1}(0)\big) := \big\{x \in X;\ \overline{G \cdot x} \cap \mu_{\mathfrak{p}}^{-1}(0) \neq \emptyset\big\}.$$

As shown in [HS07b], the semistable points form an open G-invariant subset of X. By the following Quotient Theorem the topological Hilbert quotient exists on the set of semistable points.

Quotient Theorem. *Suppose that* $X = \mathcal{S}_G\big(\mu_{\mathfrak{p}}^{-1}(0)\big)$. *Then the topological Hilbert quotient* $\pi\colon X \to X/\!\!/G$ *exists and has the following properties.*

(1) Every fiber $\pi^{-1}\big(\pi(x)\big)$ *contains a unique closed* G*-orbit, which lies in the closure of every other* G*-orbit in* $\pi^{-1}\big(\pi(x)\big)$.

(2) Every non-closed G*-orbit in a fiber of* π *has strictly bigger dimension than the closed* G*-orbit in that fiber.*

(3) The inclusion $\mu_{\mathfrak{p}}^{-1}(0) \hookrightarrow X$ *induces a homeomorphism* $\mu_{\mathfrak{p}}^{-1}(0)/K \to X/\!\!/G$.

In other words, the topological Hilbert quotient parametrizes the set of closed G*-orbits in* X.

In the rest of this chapter I will discuss the content of the papers [Mie08] and [MS10] where this theory is applied to two particular problems of real reductive group actions.

3.2 Matsuki's double coset decomposition via gradient maps

[Mie08] Christian Miebach, *Matsuki's double coset decomposition via gradient maps*, J. Lie Theory **18** (2008), no. 3, 555–580.

Let G be a real reductive Lie group equipped with two involutive automorphisms σ_1 and σ_2. Let G^{σ_j} denote the subgroup of σ_j-fixed points in G and let G_j be an open subgroup of G^{σ_j} for $j = 1, 2$. The group $G_1 \times G_2$ acts on G by $(g_1, g_2) \cdot g := g_1 g g_2^{-1}$ and one would like to describe the orbit space $G/(G_1 \times G_2) = G_1\backslash G/G_2$, i.e., the set of double cosets.

This problem arises naturally when one is dealing with symmetric spaces, and has applications in the representation theory of real or complex reductive groups. In the case that G is complex semisimple and that $\sigma_1 = \sigma_2 =: \sigma$ is anti-holomorphic, $G_1 = G_2 =: G_{\mathbb{R}}$ is a real form of G. The structure of the closed $(G_{\mathbb{R}} \times G_{\mathbb{R}})$-orbits in G was investigated in [Sta86] and [Bre96] by methods coming from the theory of real-algebraic transformation groups. Their results were extended to a description of the non-closed orbits in [BF00]. In [HS01] the problem is solved for G a connected reductive algebraic group defined over an algebraically closed field of characteristic not equal to 2 and for σ_1, σ_2 two commuting regular involutions. The authors also consider the case that G is complex reductive defined over $\mathbb{R}$ and that the involutions are likewise defined over $\mathbb{R}$. In [Mat97] Matsuki solves the general problem described above under a mild technical assumption and parametrizes the closed double cosets by certain cross sections. His proof relies essentially on the existence of the Jordan decomposition in the algebraic group $\mathrm{Aut}(\mathfrak{g})$.

In [Mie08] the orbit structure of the $(G_1 \times G_2)$-action on G is described using the analytic technique of gradient maps. In doing so, one obtains from the outset finer geometric information than Matsuki. For instance, it turns out that Matsuki's cross section actually give local slices at closed $(G_1 \times G_2)$-orbits. Hence, by the Slice Theorem, this yields also a description of the non-closed orbits that contain a given closed orbit in their closure.

In order to see how the $(G_1 \times G_2)$-action on G fits into the general framework of gradient maps, we note first that there exists a Cartan involution on G that commutes with σ_1 and σ_2. Consequently, the group $G_j = K_j \exp(\mathfrak{p}_j)$ is real reductive where K_j is an open subgroup of K^{σ_j} and $\mathfrak{p}_j := \mathfrak{p}^{\sigma_j}$[2] for $j = 1, 2$. Embedding G as a compatible subgroup into a complex reductive group $U^{\mathbb{C}}$, we see that $G_1 \times G_2$ is a real reductive subgroup of $U^{\mathbb{C}} \times U^{\mathbb{C}}$ and that the $(G_1 \times G_2)$-action on G is given as the restriction of the $(U^{\mathbb{C}} \times U^{\mathbb{C}})$-action on $U^{\mathbb{C}}$ by left and right multiplication. On $U^{\mathbb{C}} = U \exp(i\mathfrak{u})$ we have a $(U \times U)$-invariant strictly plurisubharmonic exhaustion function given by $\rho\big(u \exp(i\xi)\big) = \frac{1}{2} \operatorname{Tr}(\xi \overline{\xi}^t)$ and the induced Kähler form $\omega = \frac{i}{2} \partial \overline{\partial} \rho$. In this setting a corresponding moment map was explicitly determined in [HS07a, Lemma 9.1]. The induced gradient map $\mu_{\mathfrak{p}} \colon G \to \mathfrak{p}_1 \oplus \mathfrak{p}_2$ is of the form

$$k \exp(\xi) \mapsto \Big(\mathrm{Ad}(k)\xi + \sigma_1\big(\mathrm{Ad}(k)\xi\big), -\big(\xi + \sigma_2(\xi)\big)\Big).$$

Note that we have $\mathcal{S}_G\big(\mu_{\mathfrak{p}}^{-1}(0)\big) = G$ since the Kähler form on $U^{\mathbb{C}}$ is given by a plurisubharmonic exhaustion.

For $x \in G$ we set $H^x := G_2 \cap (x^{-1} G_1 x)$ and $\mathfrak{q}^x := \mathfrak{g}^{-\sigma_2} \cap \mathrm{Ad}(x^{-1}) \mathfrak{g}^{-\sigma_1}$. The key result for the description of the orbit space $G_1 \backslash G / G_2$ is the following theorem,

[2] By abuse of notation we write σ_j also for the differentiated involution of $\mathfrak{g}$.

cf. [Mie08, Theorem 2.11].

Theorem. *Suppose that $x \in \mu_{\mathfrak{p}}^{-1}(0)$. Then $\mathfrak{h}^x \oplus \mathfrak{q}^x$ is a reductive symmetric Lie algebra, and the slice representation of $(G_1 \times G_2)_x$ is isomorphic to the adjoint representation of H^x on $\mathfrak{q}^x$.*

Consequently, in a neighborhood of a closed orbit the $(G_1 \times G_2)$-action on G looks like the isotropy representation of a reductive symmetric space. According to their well-known structure theory, every closed orbit of H^x in $\mathfrak{q}^x$ intersects a so-called Cartan subspace of $\mathfrak{q}^x$, i.e., a maximal Abelian subspace consisting of semisimple elements, in finitely many points. In particular we see that $\mathfrak{q}^x$ is itself a Cartan subspace if G_1xG_2 is closed and of maximal dimension.

Transferring the Cartan subspaces of the various $\mathfrak{q}^x$ to the group level, we obtain the notion of *Cartan subsets* of G. The following theorem summarizes the main results Theorem 4.8, Proposition 4.20, Theorem 4.24 and Proposition 4.25 of [Mie08]. For its statement set $G_r := \{x \in G;\ G_1xG_2 \text{ is closed}\}$ and $G_{sr} := \{x \in G_r;\ \dim G_1xG_2 \text{ is maximal}\}$.

Theorem. *The set G_{sr} is open and dense in G, and there are finitely many Cartan subsets $C_1, \dots, C_k$ of G such that*

$$G_r = G_1\mu_{\mathfrak{p}}^{-1}(0)G_2 = \bigcup_{1\le j\le k} G_1C_jG_2 \quad \text{and} \quad G_{sr} = \dot{\bigcup_{1\le j\le k}} G_1(C_j \cap G_{sr})G_2.$$

Moreover, the set $C_j \cap G_{sr}$ defines a local slice at every $x \in C_j \cap G_{sr}$ and locally the open subset $G_1(C_j \cap G_{sr})G_2$ of G has the structure of a trivial fiber bundle over $C_j \cap G_{sr}$ with fiber G_1xG_2 for some $x \in C_j \cap G_{sr}$.

3.3 Spherical gradient manifolds

> [MS10] Christian Miebach and Henrik Stötzel, *Spherical gradient manifolds*, Ann. Inst. Fourier (Grenoble) **60** (2010), no. 6, 2235–2260.

Let $U^{\mathbb{C}}$ be a complex-reductive Lie group with compact real form U and let Z be a Hamiltonian $U^{\mathbb{C}}$-manifold with moment map μ.

In the special case that Z is a closed connected subvariety of $\mathbb{P}(V)$ for some finite dimensional $U^{\mathbb{C}}$-representation space V, Brion showed in [Bri87] that μ separates the U-orbits in Z if and only if Z is spherical, i.e., if and only if a Borel subgroup of $U^{\mathbb{C}}$ has an open orbit in Z. Here we say that μ separates the U-orbits if and only if it induces an injective map $Z/U \hookrightarrow \mathfrak{u}/U$. Moreover, this is equivalent to the property that the U-action on Z is coisotropic. In [HW90] the authors extended Brion's observation to any compact connected Hamiltonian $U^{\mathbb{C}}$-manifold Z and showed that spherical such Z are necessarily projective algebraic.

In the paper [MS10] we generalized these results to actions of real reductive groups on real-analytic gradient manifolds which moreover are not assumed to be compact. More precisely, we consider a closed real reductive subgroup G of $U^{\mathbb{C}}$ and a G-invariant real-analytic submanifold X of Z. By restriction, the moment map μ induces a K-equivariant gradient map $\mu_{\mathfrak{p}} \colon X \to \mathfrak{p}$.

There are two main differences between the complex compact and the real situation: Even if X is connected, an open G-orbit in X does not have to be dense and in general the fibers of $\mu_{\mathfrak{p}}$ are not connected. Therefore one cannot expect $\mu_{\mathfrak{p}}$ to separate the K-orbits globally in X. We say that $\mu_{\mathfrak{p}}$ *locally almost separates* the K-orbits if there exists a K-invariant open subset Ω of X such that $K \cdot x$ is open in $\mu_{\mathfrak{p}}^{-1}\big(K \cdot \mu_{\mathfrak{p}}(x)\big)$ for all $x \in \Omega$. Geometrically this means that the induced map $\Omega/K \to \mathfrak{p}/K$ has discrete fibers. If $\Omega = X$, we say that $\mu_{\mathfrak{p}}$ *almost separates* the K-orbits in X.

We suppose throughout that X/G is connected. Now we can state the main result of [MS10].

Theorem. *For a real analytic G-gradient manifold X the following are equivalent.*

(1) The gradient map $\mu_{\mathfrak{p}}$ locally almost separates the K-orbits in X.

(2) The gradient map $\mu_{\mathfrak{p}}$ almost separates the K–orbits in X.

(3) A minimal parabolic subgroup Q_0 of G has an open orbit in X.

Hence, this theorem gives a sufficient condition on the G-action for $\mu_{\mathfrak{p}}$ to induce a map $X/K \to \mathfrak{p}/K$ whose fibers are discrete, while on the other hand the gradient map yields a criterion for X to be spherical. Moreover, we see that sphericity is independent of the particular choice of $\mu_{\mathfrak{p}}$, i.e., if one gradient map for the G-action on X locally almost separates the K-orbits in X, then this is true for every gradient map. Note that this result is new also in the complex case, since X is not assumed to be compact.

Let us outline the main ideas of the proof. First we observe that X contains an open Q_0-orbit if and only if $(G/Q_0) \times X$ contains an open G-orbit with respect to the diagonal action of G. The gradient map $\mu_\mathfrak{p}$ on X induces a gradient map $\widetilde{\mu}_\mathfrak{p}$ on $(G/Q_0) \times X$. Now we are in a situation where we can apply the methods introduced in [HS07a]. These allow us to show that open G-orbits correspond to isolated minimal K-orbits of the norm squared of $\widetilde{\mu}_\mathfrak{p}$. In order to relate the property that $\mu_\mathfrak{p}$ locally almost separates the K-orbits to the existence of an isolated minimal K–orbit, we need the following result. We consider the restriction $\mu_\mathfrak{p}|_{K\cdot x}\colon K \cdot x \to K \cdot \mu_\mathfrak{p}(x)$ which is a smooth fiber bundle with fiber $K_{\mu_\mathfrak{p}(x)}/K_x$. In the special case $G = K^\mathbb{C}$ it is proven in [GS84] that for generic x the fiber $K_{\mu_\mathfrak{p}(x)}/K_x$ is a torus. As a generalization we prove the following proposition, which moreover allows us to extend the notion of "K-spherical" defined in [HW90] to actions of real-reductive groups.

Proposition. *Let $x \in X$ be generic and choose a maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ containing $\mu_\mathfrak{p}(x)$. Then the orbits of the centralizer $\mathcal{Z}_K(\mathfrak{a})$ of $\mathfrak{a}$ in K are open in $K_{\mu_\mathfrak{p}(x)}/K_x$.*

These arguments yield the existence of an open Q_0-orbit under the assumption that $\mu_\mathfrak{p}$ locally almost separates the K-orbits. For the other direction we apply the shifting technique for gradient maps.

Note that our proof of Brion's theorem is different from the ones in [Bri87] and [HW90]. In particular, for every generic element $x \in X$ we construct a minimal parabolic subgroup Q_0 of G such that $Q_0 \cdot x$ is open in X.

Bibliography

[ABK04] Dehbia Achab, Frank Betten, and Bernhard Krötz, *Discrete group actions on Stein domains in complex Lie groups*, Forum Math. **16** (2004), no. 1, 37–68.

[Akh95] Dmitri N. Akhiezer, *Lie group actions in complex analysis*, Aspects of Mathematics, E27, Friedr. Vieweg & Sohn, Braunschweig, 1995.

[Ber87] François Berteloot, *Existence d'une structure kählérienne sur les variétés homogènes semi-simples*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), no. 19, 809–812.

[BF00] Ralph Bremigan and Gregor Fels, *Bi-invariant domains in semisimple groups: nonprincipal orbits and boundaries*, Geom. Dedicata **82** (2000), no. 1-3, 225–283.

[BHH03] D. Burns, S. Halverscheid, and R. Hind, *The geometry of Grauert tubes and complexification of symmetric spaces*, Duke Math. J. **118** (2003), no. 3, 465–491.

[Bis64] Errett Bishop, *Conditions for the analyticity of certain sets*, Michigan Math. J. **11** (1964), 289–304.

[Bla56] André Blanchard, *Sur les variétés analytiques complexes*, Ann. Sci. Ecole Norm. Sup. (3) **73** (1956), 157–202.

[BO73] Wolf Barth and Michael Otte, *Invariante holomorphe Funktionen auf reduktiven Liegruppen*, Math. Ann. **201** (1973), 97–112.

[BO88] F. Berteloot and K. Oeljeklaus, *Invariant plurisubharmonic functions and hypersurfaces on semisimple complex Lie groups*, Math. Ann. **281** (1988), no. 3, 513–530.

[BP82] Al Boggess and John C. Polking, *Holomorphic extension of CR functions*, Duke Math. J. **49** (1982), no. 4, 757–784.

[Bre96] R. J. Bremigan, *Invariant analytic domains in complex semisimple groups*, Transform. Groups **1** (1996), no. 4, 279–305.

[Bri87] Michel Brion, *Sur l'image de l'application moment*, Séminaire d'algèbre Paul Dubreil et Marie-Paule Malliavin (Paris, 1986), Lecture Notes in Math., vol. 1296, Springer, Berlin, 1987, pp. 177–192.

[Che13] Bo-Yong Chen, *Discrete groups and holomorphic functions*, Math. Ann. **355** (2013), no. 3, 1025–1047.

[CL85] G. Cœuré and J.-J. Loeb, *A counterexample to the Serre problem with a bounded domain of* $\mathbf{C}^2$ *as fiber*, Ann. of Math. (2) **122** (1985), no. 2, 329–334.

[CTIT00] Enrico Casadio Tarabusi, Andrea Iannuzzi, and Stefano Trapani, *Globalizations, fiber bundles, and envelopes of holomorphy*, Math. Z. **233** (2000), no. 3, 535–551.

[dFI01] Chiara de Fabritiis and Andrea Iannuzzi, *Quotients of the unit ball of* $\mathbb{C}^n$ *for a free action of* $\mathbb{Z}$, J. Anal. Math. **85** (2001), 213–224.

[DG60] Ferdinand Docquier and Hans Grauert, *Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten*, Math. Ann. **140** (1960), 94–123.

[FG98] Gregor Fels and Laura Geatti, *Geometry of biinvariant subsets of complex semisimple Lie groups*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26** (1998), no. 2, 329–356.

[Fuj78] Akira Fujiki, *On automorphism groups of compact Kähler manifolds*, Invent. Math. **44** (1978), no. 3, 225–258.

[GM12] Daniel Greb and Christian Miebach, *Invariant meromorphic functions on Stein spaces*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 5, 1983–2011.

[GMO11] Bruce Gilligan, Christian Miebach, and Karl Oeljeklaus, *Homogeneous Kähler and Hamiltonian manifolds*, Math. Ann. **349** (2011), no. 4, 889–901.

[GMO13] ———, *Pseudoconvex domains spread over complex homogeneous manifolds*, Manuscripta Math. **142** (2013), no. 1-2, 35–59.

[GS84] Victor Guillemin and Shlomo Sternberg, *Symplectic techniques in physics*, Cambridge University Press, Cambridge, 1984.

[Hei91] Peter Heinzner, *Geometric invariant theory on Stein spaces*, Math. Ann. **289** (1991), no. 4, 631–662.

[HI97] Peter Heinzner and Andrea Iannuzzi, *Integration of local actions on holomorphic fiber spaces*, Nagoya Math. J. **146** (1997), 31–53.

[Hir74] André Hirschowitz, *Pseudoconvexité au-dessus d'espaces plus ou moins homogènes*, Invent. Math. **26** (1974), 303–322.

[Hir75] ______, *Le problème de Lévi pour les espaces homogènes*, Bull. Soc. Math. France **103** (1975), no. 2, 191–201.

[HL94] P. Heinzner and F. Loose, *Reduction of complex Hamiltonian G-spaces*, Geom. Funct. Anal. **4** (1994), no. 3, 288–297.

[HO86] A. T. Huckleberry and E. Oeljeklaus, *On holomorphically separable complex solv-manifolds*, Ann. Inst. Fourier (Grenoble) **36** (1986), no. 3, 57–65.

[Hol63] Harald Holmann, *Komplexe Räume mit komplexen Transformationsgruppen*, Math. Ann. **150** (1963), 327–360.

[HOV94] John H. Hubbard and Ralph W. Oberste-Vorth, *Hénon mappings in the complex domain. I. The global topology of dynamical space*, Inst. Hautes Études Sci. Publ. Math. (1994), no. 79, 5–46.

[HS01] Aloysius G. Helminck and Gerald W. Schwarz, *Orbits and invariants associated with a pair of commuting involutions*, Duke Math. J. **106** (2001), no. 2, 237–279.

[HS07a] Peter Heinzner and Gerald W. Schwarz, *Cartan decomposition of the moment map*, Math. Ann. **337** (2007), no. 1, 197–232.

[HS07b] Peter Heinzner and Henrik Stötzel, *Semistable points with respect to real forms*, Math. Ann. **338** (2007), no. 1, 1–9.

[HSS08] Peter Heinzner, Gerald W. Schwarz, and Henrik Stötzel, *Stratifications with respect to actions of real reductive groups*, Compos. Math. **144** (2008), no. 1, 163–185.

[Huc10] Alan Huckleberry, *Remarks on homogeneous complex manifolds satisfying Levi conditions*, Boll. Unione Mat. Ital. (9) **3** (2010), no. 1, 1–23.

[HW90] A. T. Huckleberry and T. Wurzbacher, *Multiplicity-free complex manifolds*, Math. Ann. **286** (1990), no. 1-3, 261–280.

[Ka67] Soji Kaneyuki, *On the automorphism groups of homogeneous bounded domains*, J. Fac. Sci. Univ. Tokyo Sect. I **14** (1967), 89–130.

[KLY11] Kang-Tae Kim, Norman Levenberg, and Hiroshi Yamaguchi, *Robin functions for complex manifolds and applications*, Mem. Amer. Math. Soc. **209** (2011), no. 984, viii+111.

[Las78] Michel Lassalle, *Séries de Laurent des fonctions holomorphes dans la complexification d'un espace symétrique compact*, Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 2, 167–210.

[Lie78] David I. Lieberman, *Compactness of the Chow scheme: applications to automorphisms and deformations of Kähler manifolds*, Fonctions de plusieurs variables complexes, III (Sém. François Norguet, 1975–1977), Lecture Notes in Math., vol. 670, Springer, Berlin, 1978, pp. 140–186.

[LY13] Norman Levenberg and Hiroshi Yamaguchi, *Pseudoconvex domains in the Hopf surface*, to appear in J. Math. Soc. Japan.

[Mat60] Yozô Matsushima, *Espaces homogènes de Stein des groupes de Lie complexes*, Nagoya Math. J **16** (1960), 205–218.

[Mat97] Toshihiko Matsuki, *Double coset decompositions of reductive Lie groups arising from two involutions*, J. Algebra **197** (1997), no. 1, 49–91.

[Mie07] Christian Miebach, *Geometry of invariant domains in complex semi-simple Lie groups*, Dissertation, Bochum, 2007.

[Mie08] ______, *Matsuki's double coset decomposition via gradient maps*, J. Lie Theory **18** (2008), no. 3, 555–580.

[Mie09] ______, *Geometry of invariant domains in complex semi-simple Lie groups*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **8** (2009), no. 3, 509–541.

[Mie10a] ______, *Quotients of bounded homogeneous domains by cyclic groups*, Osaka J. Math. **47** (2010), no. 2, 331–352.

[Mie10b] ______, *Sur les quotients discrets de semi-groupes complexes*, Ann. Fac. Sci. Toulouse Math. (6) **19** (2010), no. 2, 269–276.

[Mie13] ———, *Pseudoconvex non-Stein domains in primary Hopf surfaces*, to appear in Izvestiya: Mathematics **78** (2014), no. 5.

[MO09] Christian Miebach and Karl Oeljeklaus, *On proper* $\mathbb{R}$*-actions on hyperbolic Stein surfaces*, Doc. Math. **14** (2009), 673–689.

[MS10] Christian Miebach and Henrik Stötzel, *Spherical gradient manifolds*, Ann. Inst. Fourier **60** (2010), no. 6, 2235–2260.

[Nar60] Raghavan Narasimhan, *Imbedding of holomorphically complete complex spaces*, Amer. J. Math. **82** (1960), 917–934.

[Nee00] Karl-Hermann Neeb, *Holomorphy and convexity in Lie theory*, de Gruyter Expositions in Mathematics, vol. 28, Walter de Gruyter & Co., Berlin, 2000.

[Oni60] A. L. Oniščik, *Complex hulls of compact homogeneous spaces*, Soviet Math. Dokl. **1** (1960), 88–91.

[Pal57] Richard S. Palais, *A global formulation of the Lie theory of transformation groups*, Mem. Amer. Math. Soc. No. **22** (1957), iii+123.

[PS69] I. I. Pyateskii-Shapiro, *Automorphic functions and the geometry of classical domains*, Translated from the Russian. Mathematics and Its Applications, Vol. 8, Gordon and Breach Science Publishers, New York, 1969.

[Ric74] R. W. Richardson, Jr., *Principle orbit types for reductive groups acting on Stein manifolds*, Math. Ann. **208** (1974), 323–331.

[Ros56] Maxwell Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1956), 401–443.

[Sno82] Dennis M. Snow, *Reductive group actions on Stein spaces*, Math. Ann. **259** (1982), no. 1, 79–97.

[Sta86] Robert J. Stanton, *Analytic extension of the holomorphic discrete series*, Amer. J. Math. **108** (1986), no. 6, 1411–1424.

[Vi14] Sara Vitali, *Quotients of the crown domain by a proper action of a cyclic group*, Trans. Amer. Math. Soc. **366** (2014), no. 6, 3227–3239.