WALKS ARE JUST CONNECTED HIKES

T. Espinasse (Lyon I), P.L. Giscard (York) and P. Rochet (Nantes)

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OUTLINE

WALKS ON A GRAPH AND LINEAR ALGEBRA

HIKES INCIDENCE ALGEBRA

SPECTRAL ANALYSIS

OPEN HIKES

Outline

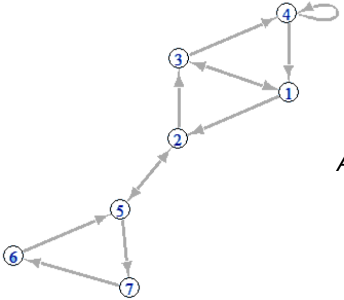
Walks on a graph and linear algebra

Hikes incidence algebra

Spectral analysis

Open hikes

- $G = (V, E)$ be a directed graph
- $A(x)$ its variable adjacency matrix.



$$A(x) = \begin{bmatrix} 0 & x_{12} & x_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{23} & 0 & x_{25} & 0 & 0 \\ x_{31} & 0 & 0 & x_{34} & 0 & 0 & 0 \\ x_{41} & 0 & 0 & x_{44} & 0 & 0 & 0 \\ 0 & x_{52} & 0 & 0 & 0 & 0 & x_{57} \\ 0 & 0 & 0 & 0 & x_{65} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{76} & 0 \end{bmatrix}$$

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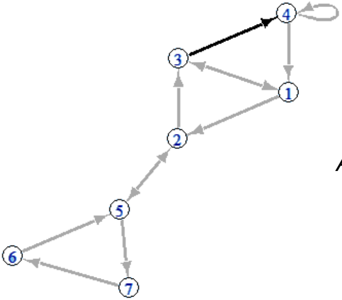

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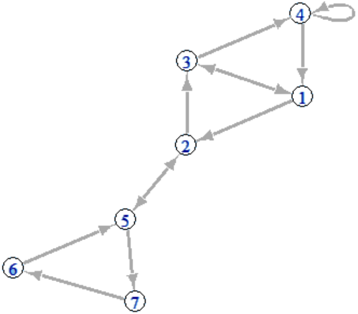

DEFINITIONS

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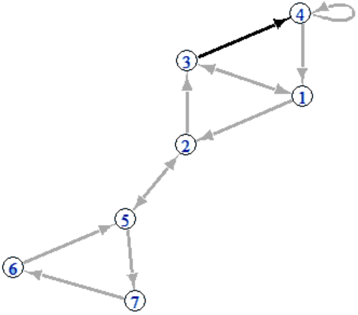

$w = x_{34}$

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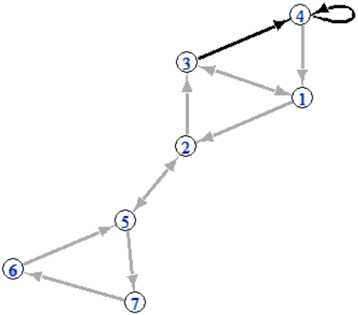

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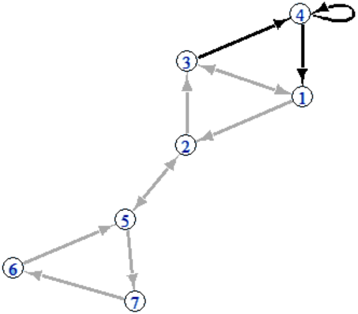

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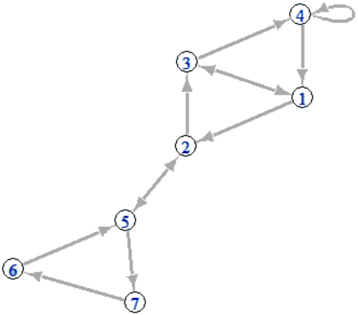

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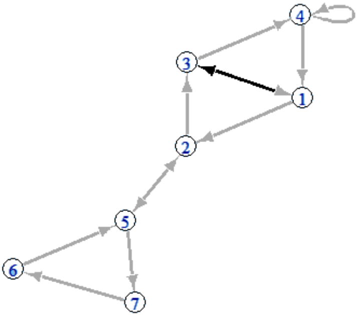

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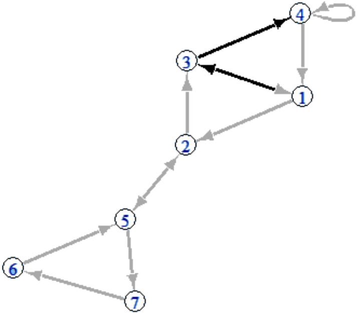

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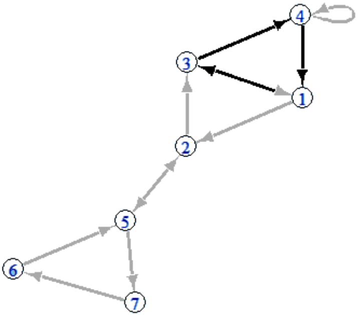

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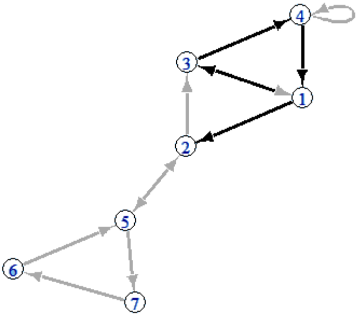

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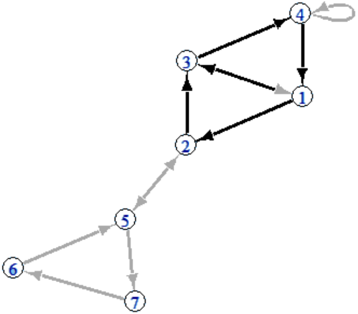

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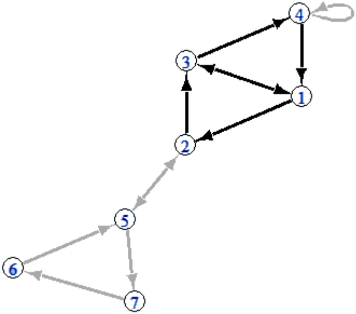

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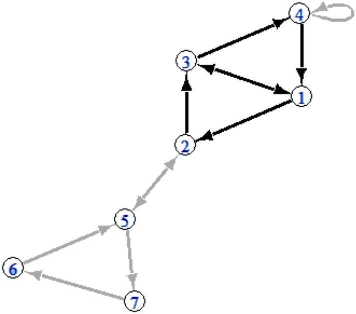

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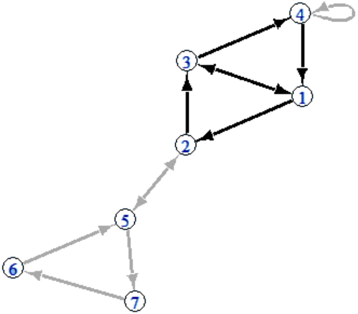

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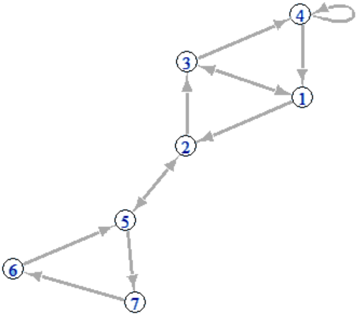

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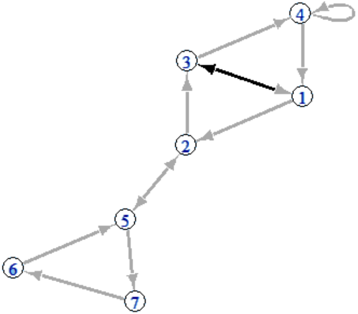

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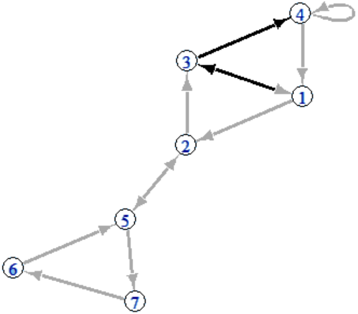

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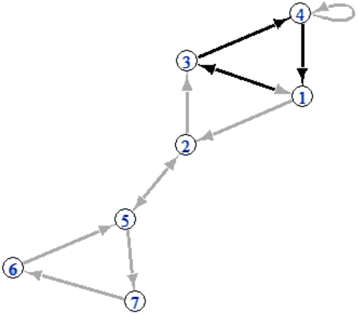

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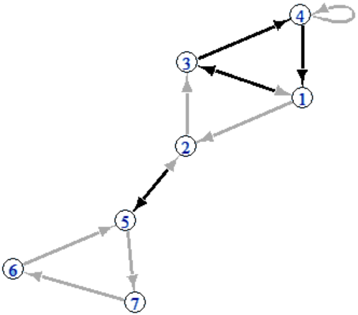

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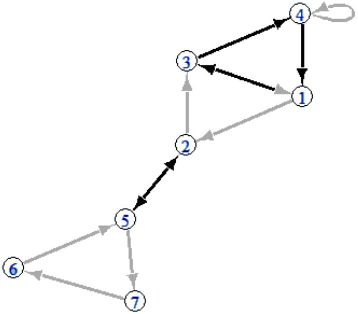

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This representation is unique, up to permutations of vertex-disjoint cycles

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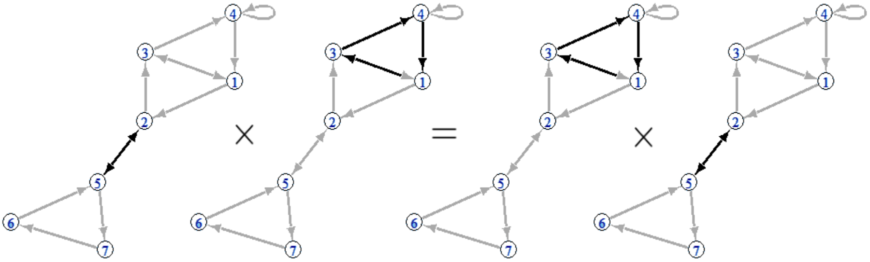

$$x_{25}x_{52} \times x_{13}x_{34}x_{41} = x_{13}x_{34}x_{41} \times x_{25}x_{52}$$

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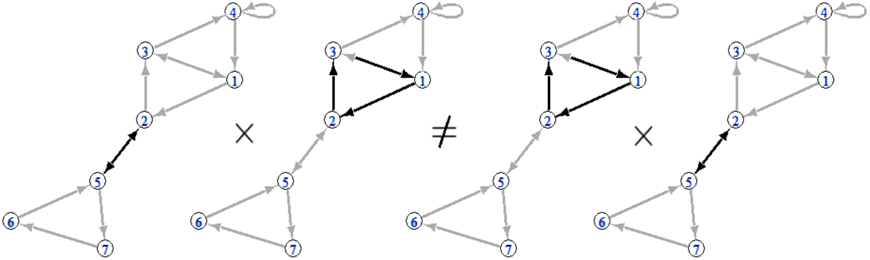

$$x_{25}x_{52} \times x_{12}x_{23}x_{31} \neq x_{12}x_{23}x_{31} \times x_{25}x_{52}$$

Let

- $\ell(h)$ the length of h (its degree)
- $\Omega(h)$ the number of primes composing it

Proposition

$$\det(-A(x)) = \sum_{\substack{h \in \mathcal{H} \\ h \text{ self-avoiding} \\ \ell(h)=N}} (-1)^{\Omega(h)} h$$

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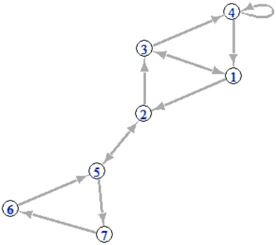

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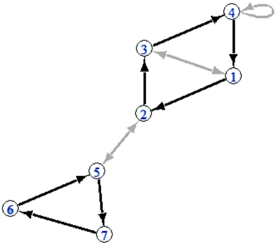

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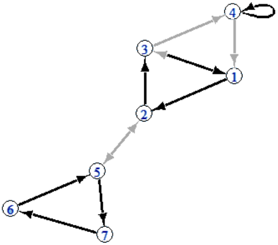

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$$-(x_{12}x_{23}x_{31}) \times x_{44} \times (x_{57}x_{76}x_{65})$$

COROLLARY

$$\det(I - zA(x)) = \sum_{\substack{h \in \mathcal{H} \\ h \text{ self-avoiding}}} (-1)^{\Omega(h)} h \, z^{\ell(h)}$$

Define $\mu : \mathcal{H} \to \{-1, 0, 1\}$

$$\mu(h) = \begin{cases} (-1)^{\Omega(h)} & \text{if } h \text{ is self-avoiding} \\ 0 & \text{otherwise} \end{cases}$$

Looks like a Mobius function...

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Division in $\mathcal{H}$

$$h = d \times h' \iff h' = \frac{h}{d} \quad , \quad h, d, h' \in \mathcal{H}.$$

Incidence algebra

Set of functions $f : \mathcal{H} \to \mathbb{R}$, endowed with the Dirichlet convolution

$$f * g(h) = \sum_{d|h} f(d) g\left(\frac{h}{d}\right).$$

Why the Dirichlet convolution?

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Multiplication of formal series

$$\left(\sum_{h\in\mathcal{H}} f(h) h\, z^{\ell(h)}\right) \times \left(\sum_{h\in\mathcal{H}} g(h) h\, z^{\ell(h)}\right) = \sum_{h\in\mathcal{H}} f * g(h)\, h\, z^{\ell(h)}$$

The Dirichlet convolution is

1. associative
2. distributive over addition
3. **not** commutative

Important functions

- The identity: $\delta(1) = 1$ and $\forall h \neq 1$, $\delta(h) = 0$
- The caracteristic function: $\zeta(h) = 1$
- The Mobius function: ζ^{-1}

THEOREM (MOBIUS INVERSION)

The function

$$\mu(h) = \begin{cases} (-1)^{\Omega(h)} & \text{if } h \text{ is self-avoiding} \\ 0 & \text{otherwise} \end{cases}$$

is the Mobius function on $(\mathcal{H}, \times)$*:*

1. $\forall h \neq 1,\ \mu * \zeta(h) = \sum_{d|h} \mu(d) = 0$
2. $\left(\sum_{h\in\mathcal{H}} \mu(h)h\, z^{\ell(h)}\right) \times \sum_{h\in\mathcal{H}} \zeta(h)h\, z^{\ell(h)} = \det(I - zA(x)) \times \sum_{h\in\mathcal{H}} h\, z^{\ell(h)} = 1$

Analogy with $\mathbb{N}$

Natural integer $n = p_1 ... p_k$

- $\mu(n) = \begin{cases} (-1)^k & \text{if } p_i \neq p_j \\ 0 & \text{otherwise} \end{cases}$
- n_1, n_2 co-prime

$$\mu(n_1)\mu(n_2) = \mu(n_1 n_2)$$

Hike $h = c_1 \times ... \times c_k$

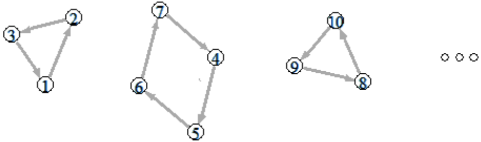
- $\mu(h) = \begin{cases} (-1)^k & \text{if } c_i \cap c_j = \emptyset \\ 0 & \text{otherwise} \end{cases}$
- h_1, h_2 disjoint

$$\mu(h_1)\mu(h_2) = \mu(h_1 \times h_2)$$

Co-prime = vertex-disjoint $\neq$ different

$$p_1 \neq p_2 \nRightarrow p_1, p_2 \text{ co-prime.}$$

GRAPH REPRESENTATION OF $\mathbb{N}$



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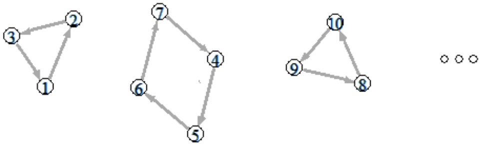
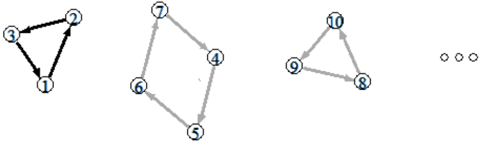


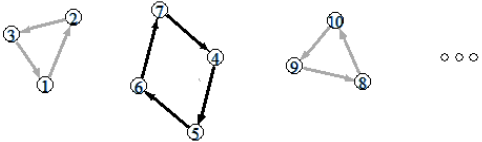
- The primes:

Graph representation of $\mathbb{N}$



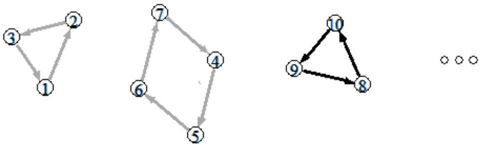
- The primes: c_1

GRAPH REPRESENTATION OF $\mathbb{N}$



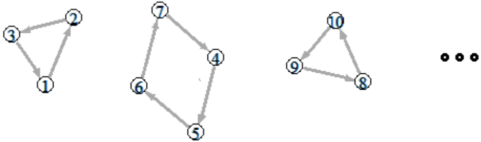
- The primes: c_1 , c_2

Graph representation of $\mathbb{N}$



- The primes: c_1 , c_2 , c_3

Graph representation of $\mathbb{N}$



- The primes: c_1 , c_2 , c_3 , ...

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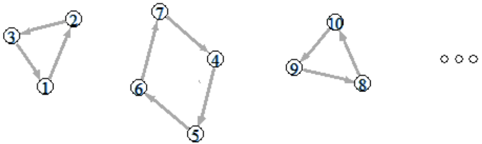
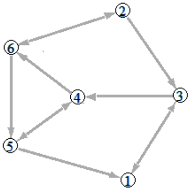


- The primes: c_1 , c_2 , c_3 , ...
- Different primes are disjoint: all closed hikes commute

General case



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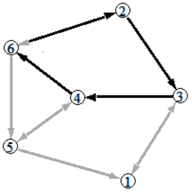

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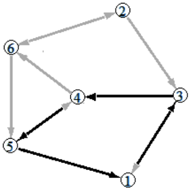

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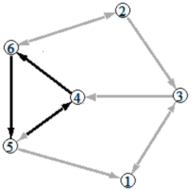

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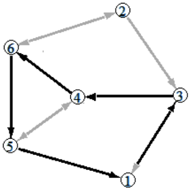
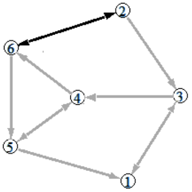


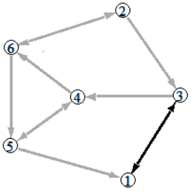
- The primes: c_1 , c_2 , c_3 , c_4

GENERAL CASE



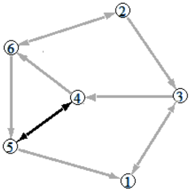
- The primes: c_1 , c_2 , c_3 , c_4 , c_5

General case



- The primes: c_1 , c_2 , c_3 , c_4 , c_5 , c_6

General case



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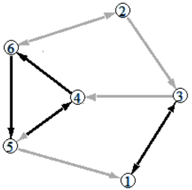

- The primes: c_1 , c_2 , c_3 , c_4 , c_5 , c_6 , c_7
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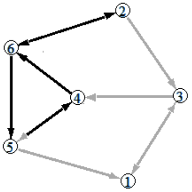

- The primes: c_1 , c_2 , c_3 , c_4 , c_5 , c_6 , c_7
- c_3 and c_6 are co-prime (disjoint), they commute: $c_3 \times c_5 = c_5 \times c_3$
- c_3 and c_5 are not co-prime: $c_3 \times c_4 \neq c_4 \times c_3$

What about connected hikes (cycles)?

Let $\Lambda(h)$ denote the number of connected representations of h

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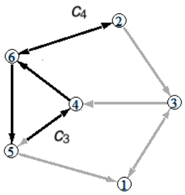
Example



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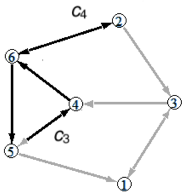

$$c_3 \times c_4 \quad = (x_{54}x_{46}x_{65})\,(x_{62}x_{26})$$

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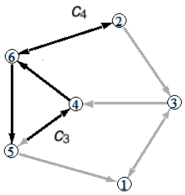

$$\begin{aligned} c_3 \times c_4 &= (x_{54}x_{46}x_{65})\,(x_{62}x_{26}) \\ &= x_{65}x_{54}x_{46}x_{62}x_{26} \end{aligned}$$

WHAT ABOUT CONNECTED HIKES (CYCLES)?

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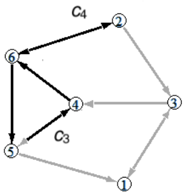

$$
\begin{aligned}
c_3 \times c_4 &= (x_{54}x_{46}x_{65})\,(x_{62}x_{26}) \\
&= x_{65}x_{54}x_{46}x_{62}x_{26} \\
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\end{aligned}
$$

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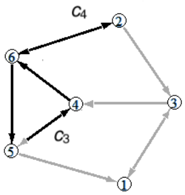

$$\left.\begin{aligned} c_3 \times c_4 &= (x_{54}x_{46}x_{65})\,(x_{62}x_{26}) \\ &= x_{65}x_{54}x_{46}x_{62}x_{26} \\ &= x_{26}x_{65}x_{54}x_{46}x_{62} \end{aligned}\right\} \Lambda(c_3 \times c_4) = 2$$

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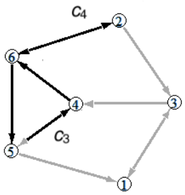

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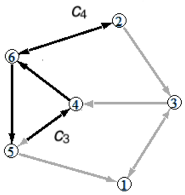

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$$\begin{aligned} c_4 \times c_3 &= (x_{62}x_{26})\,(x_{54}x_{46}x_{65}) \\ &= x_{62}x_{26}x_{65}x_{54}x_{46} \end{aligned}$$

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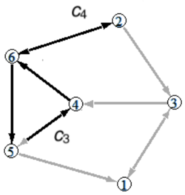
Example



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$$\begin{aligned} c_4 \times c_3 &= (x_{62}x_{26})\,(x_{54}x_{46}x_{65}) \\ &= x_{62}x_{26}x_{65}x_{54}x_{46} \\ &= x_{46}x_{62}x_{26}x_{65}x_{54} \end{aligned}$$

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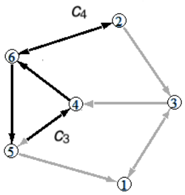

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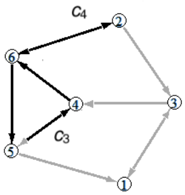


$$\left.\begin{aligned} c_3 \times c_4 &= (x_{54}x_{46}x_{65})\,(x_{62}x_{26}) \\ &= x_{65}x_{54}x_{46}x_{62}x_{26} \\ &= x_{26}x_{65}x_{54}x_{46}x_{62} \end{aligned}\right\}\ \Lambda(c_3 \times c_4) = 2$$

$$\left.\begin{aligned} c_4 \times c_3 &= (x_{62}x_{26})\,(x_{54}x_{46}x_{65}) \\ &= x_{62}x_{26}x_{65}x_{54}x_{46} \\ &= x_{46}x_{62}x_{26}x_{65}x_{54} \\ &= x_{54}x_{46}x_{62}x_{26}x_{65} \end{aligned}\right\}\ \Lambda(c_4 \times c_3) = 3$$

Theorem

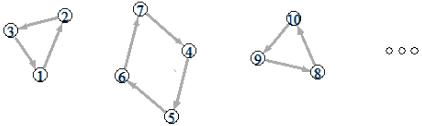
For all $h \in \mathcal{H}$,

$$\sum_{d|h} \Lambda(d) = \ell(h) \quad \textit{and} \quad \sum_{d|h} \ell(d)\mu\left(\frac{h}{d}\right) = \Lambda(h)$$

1. $\Lambda * \zeta = \ell \iff \ell * \mu = \Lambda$
2. Λ is the von Mangoldt function
3. $\ell(h)$ is the “logarithm” of a hike

EXPLANATION

On $\mathbb{N}$:



1. Connected hikes are powers of primes
2. $\Lambda(h) = \begin{cases} \ell(p) & \text{if } h = p^k \\ 0 & \text{otherwise} \end{cases} \Longrightarrow$ this is the von Mangoldt function on $\mathbb{N}$!
3. Additive property of ℓ: $\ell(h \times h') = \ell(h) + \ell(h') \Longrightarrow \ell \equiv \log$

The von Mangoldt function is a characteristic function over connected hikes

PERSPECTIVES

I) Generalization of number theory: $\zeta(z) = \det(I - zA(x))^{-1}$

- $\zeta^2(z) = \sum_{h \in \mathcal{H}} d(h)h\, z^{\ell(h)}$ with $d(.)$ the number of divisors

PERSPECTIVES

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- $\zeta^2(z) = \sum_{h \in \mathcal{H}} d(h) h\, z^{\ell(h)}$ with $d(.)$ the number of divisors
- $\frac{\zeta'(z)}{\zeta(z)} = -\sum_{h \in \mathcal{H}} \Lambda(h) h\, z^{\ell(h)}$

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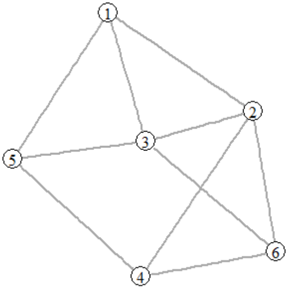
- ...

These hold for any graph, for $\mathbb{N}$ in particular

II) Understanding oriented cycle covers

- Covering hike:

$$h = x_{15}x_{35}x_{21}x_{54}x_{51}x_{63}x_{13}x_{23}x_{42}x_{12}x_{26}x_{32}x_{31}x_{45}x_{64}x_{53}x_{24}x_{36}x_{62}x_{46}$$



1	2	3	4	5	6

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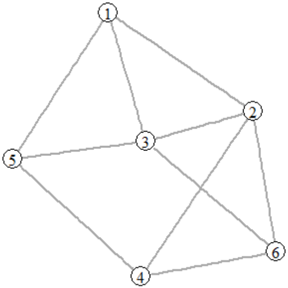

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(15)					

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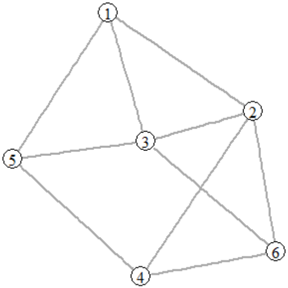

1	2	3	4	5	6
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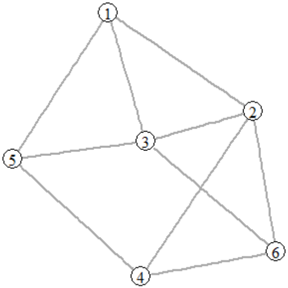

1	2	3	4	5	6
(15)	(21)	(35)			

Perspectives

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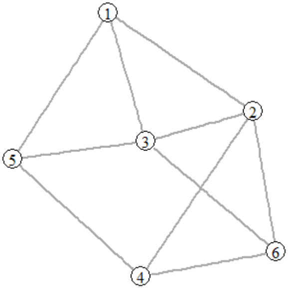

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Perspectives

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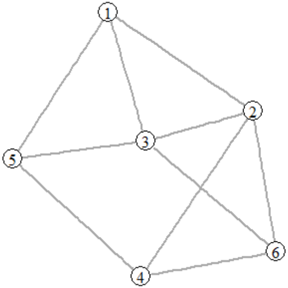

1	2	3	4	5	6
				(51)	
(15)	(21)	(35)		(54)	

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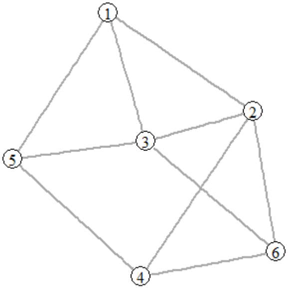

1	2	3	4	5	6
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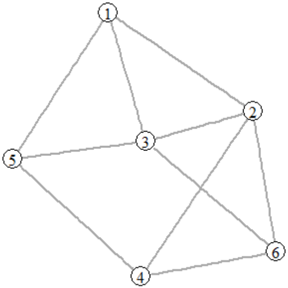


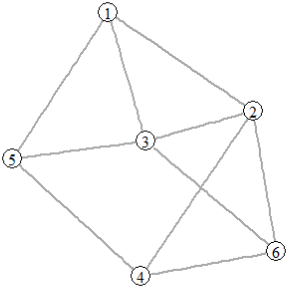
1	2	3	4	5	6
(13) (15)	(21)	(35)		(51) (54)	(63)

Perspectives

II) Understanding oriented cycle covers

- Covering hike:

$h = x_{42}x_{12}x_{26}x_{32}x_{31}x_{45}x_{64}x_{53}x_{24}x_{36}x_{62}x_{46}$



1	2	3	4	5	6
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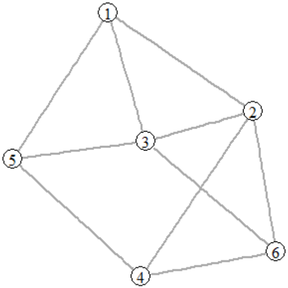

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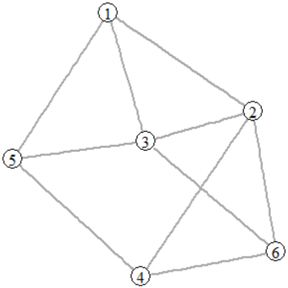

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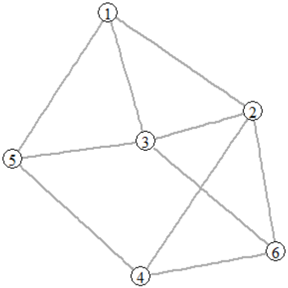

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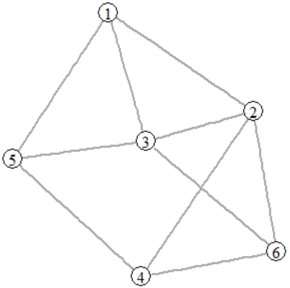

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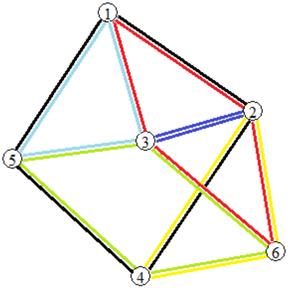

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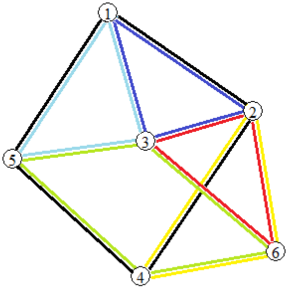

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PERSPECTIVES

III) Graph representation of monoids?

- If finitely generated and unique prime decomposition?
- If countable number of primes? (locally finite graphs)
- Minimal representation?
- Extensions to groups? Rings?

OUTLINE

WALKS ON A GRAPH AND LINEAR ALGEBRA

HIKES INCIDENCE ALGEBRA

SPECTRAL ANALYSIS

OPEN HIKES

$$\det(I - zA(x)) = \prod_{\lambda \in \mathrm{Sp}(A(x))} (1 - z\lambda)$$

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$$= 1 - \sum_{\lambda \in \mathrm{Sp}(A(x))} \lambda \, z + \sum_{\lambda_1, \lambda_2 \in \mathrm{Sp}(A(x))} \lambda_1 \lambda_2 \, z^2 - \ldots$$

$$
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&= \sum_{k=0}^{n} \sum_{\lambda_1, \ldots, \lambda_k \in \mathrm{Sp}(A(x))} (-1)^k \lambda_1 \ldots \lambda_k\, z^k
\end{aligned}
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Let $S = \{\lambda_1^{\alpha_1} \ldots \lambda_k^{\alpha_k} : \lambda_i \in \mathrm{Sp}(A(x))\}$

$$\det(I - zA(x)) = \sum_{s \in S} \mu_S(s) s\, z^{\ell(s)}$$

where μ_S is the “all-commutative” Mobius function

LEMMA

For all $k = 0, 1, ..., n$

$$\sum_{\substack{h \in \mathcal{H} \\ \ell(h)=k}} \mu(h)h = \sum_{\lambda_1,\dots,\lambda_k \in \Lambda} (-1)^k \lambda_1 \dots \lambda_k = \sum_{\substack{s \in S \\ \ell(s)=k}} \mu_S(s)s$$

Equality of the two Mobius functions over equal degree elements

Lemma

For all $k = 0, 1, ..., n$

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Equality of the two Mobius functions over equal degree elements

Theorem (Spectral Mobius inversion)

For all $k = 0, 1, ..., n$

$$\sum_{\substack{h \in \mathcal{H} \\ \ell(h)=k}} h = \sum_{\substack{s \in S \\ \ell(s)=k}} s$$

Equality of the formal series zeta functions over $\mathcal{H}$ *and* S

PERSPECTIVES

Clear connections between hikes and spectrum

- Hike interpretation of co-spectral graphs
- Bounds on spectrum using the hike poset
- Hike structure of special graphs (bipartite, stars, wheels etc...)
- Self-avoiding walks on a lattice

Spectral approach

1. $\zeta(z) = \det(I - zA(x))^{-1} = \prod_{\lambda \in \mathrm{Sp}(A(x))} (1 - z\lambda)^{-1}$

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OUTLINE

WALKS ON A GRAPH AND LINEAR ALGEBRA

HIKES INCIDENCE ALGEBRA

SPECTRAL ANALYSIS

OPEN HIKES

Definition

For $i \neq j$, h is an *open hike* from v_i to v_j if $h \times x_{ji}$ is a closed hike

- ▷ An open hike is an open walk without the connectedness condition

Notation

- ∘ $\mathcal{W}_{ij}$ the set of walks (connected hikes) from i to j
- ∘ $\mathcal{S}_{ij}$ the set of self-avoiding hikes from i to j

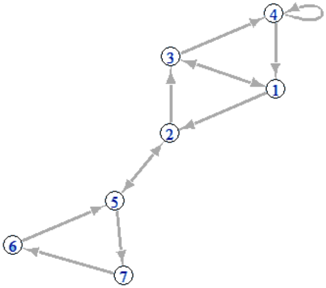
Proposition

$$(A(x)^k)_{ij} = \sum_{\substack{h \in \mathcal{W}_{ij} \\ \ell(h)=k}} h$$

PROPOSITION

$$(A(x)^k)_{ij} = \sum_{\substack{h \in \mathcal{W}_{ij} \\ \ell(h)=k}} h$$

EXAMPLE



$$(A(x)^3)_{14} =$$

Proposition

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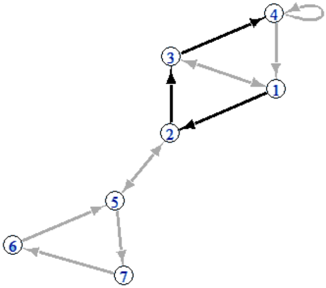
Example



$$(A(x)^3)_{14} = x_{12}x_{23}x_{34}$$

Proposition

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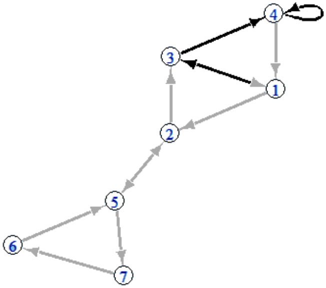

$$(A(x)^3)_{14} = x_{12}x_{23}x_{34} + x_{13}x_{34}x_{44}$$

Properties of open walks

Theorem

$$\sum_{\substack{d|h \\ d \in \mathcal{H}}} \mu(d)\, \mathbb{1}\left\{ \frac{h}{d} \in \mathcal{W}_{ij} \right\} = (-1)^{\Omega(h)} \mathbb{1}\{h \in \mathcal{S}_{ij}\}$$

- Mobius relation: $\mu * \mathbb{1}\{. \in \mathcal{W}_{ij}\} = (-1)^{\Omega(.)} \mathbb{1}\{. \in \mathcal{S}_{ij}\}$
- Walks are just connected hikes

PROOFS

1.

$$\mathrm{adj}(I - zA(x)) \,.\, (I - zA(x)) = \det(I - zA(x)) \,.\, I$$

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Proofs

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2. $$\mathrm{adj}(I - zA(x)) = \det(I - zA(x)) \,.\, (I + zA(x) + z^2 A(x)^2 + \ldots)$$

3. $$\sum_{\substack{h:i\to j \\ h \text{ self-avoiding}}} (-1)^{\Omega(h)}\, h\, z^{\ell(h)} = \left(\sum_{h\in\mathcal{H}} \mu(h) h\, z^{\ell(h)}\right)\left(\sum_{w:i\to j} w\, z^{\ell(w)}\right)$$

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4.

$$\sum_{h\in\mathcal{H}} (-1)^{\Omega(h)} \mathbb{1}\{h \in \mathcal{S}_{ij}\}\, h\, z^{\ell(h)} = \sum_{h\in\mathcal{H}} \mu * \mathbb{1}\{. \in \mathcal{W}_{ij}\}(h)\, h\, z^{\ell(h)}$$

GRACIAS POR SU ATENCION