The Method Generating Theorem

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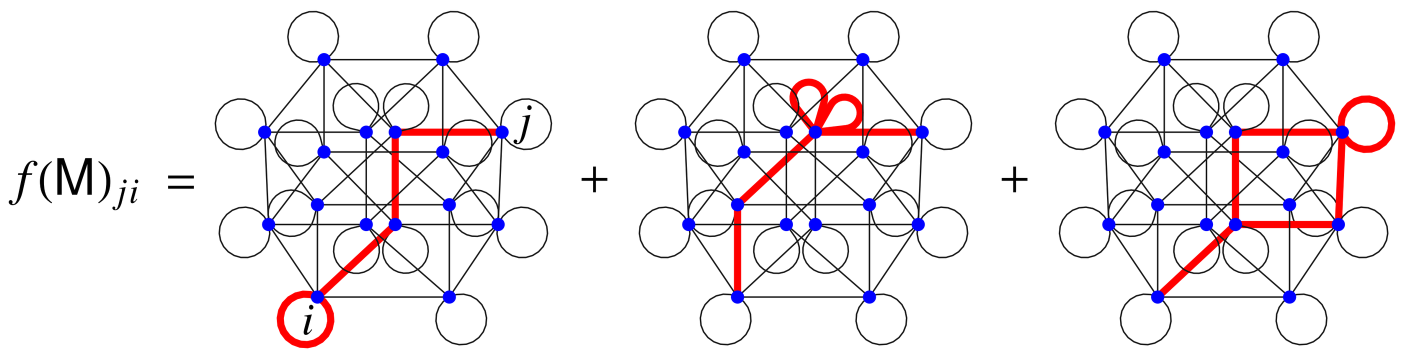
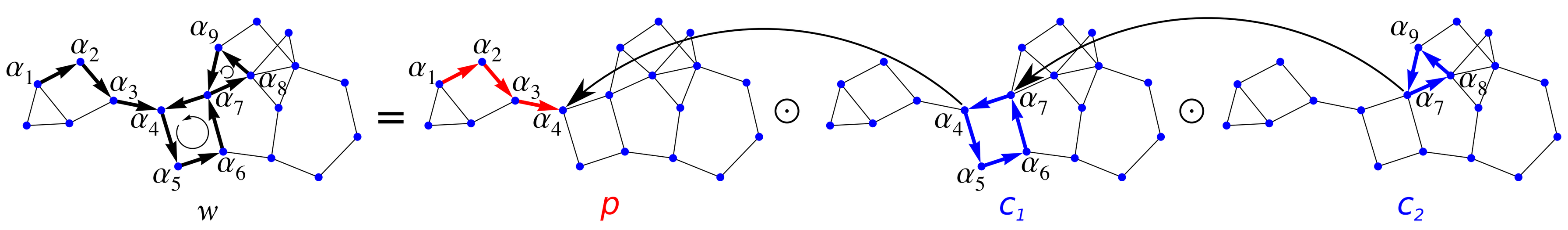
Introduction

Matrix functions = walks on graphs



Matrix multiplications are in one to one correspondence with walks on weighted directed graphs. Regarding a matrix $\mathsf{M} \in \mathbb{C}^{n\times n}$ or any partition of M as the adjacency matrix of a graph $\mathcal{G}$, the power-series expression of a function $f(\mathsf{M})$ is seen to be a sum of weighted walks on $\mathcal{G}$. Regardless of $\mathcal{G}$ and M, these walks are fundamentally structured and can be systematically grouped into families amenable to exact resummations, thereby improving the computation of $f(\mathsf{M})$.

Walks factorize into primes



Let $\mathbb{K}$ be a field. The ensemble of *all* walks on graph $\mathcal{G}$, denoted $W_{\mathcal{G}}$, together with an addition $+$ and a walk product $\bullet$ forms a (near) $\mathbb{K}$-algebra $\mathbb{K}\mathcal{G}_\bullet = (W_{\mathcal{G}}, +, \bullet)$. Depending on $\bullet$, walks may uniquely factorize into products of shorter, irreducible walks that are also prime

$$p \,|\, w \bullet w' \Rightarrow p \,|\, w \quad \text{or} \quad p \,|\, w'$$

Walks then fall into families F *generated by the primes.* For example the family F_{p,c_1c_2}

$$F_{p,c_1c_2} = \{w\}_{k,\ell} = p \bullet {c_1}^k \bullet {c_2}^\ell, \quad k,\ell \in \mathbb{N}$$

is generated by a simple path p and two simple cycles c_1 and c_2. Summing over the walks of this family is now easy !

$$\sum_{w\in F_{p,c_1c_2}} w = \sum_{k,\ell\in\mathbb{N}} p \bullet {c_1}^k \bullet {c_2}^\ell = p \bullet (\mathbb{1}-c_1)^{-1} \bullet (\mathbb{1}-c_2)^{-1}$$

- If *all* walks factorize *uniquely* into $\bullet$ products of primes, summing over all walk families sums over all walks

Method Generating Theorem

Let $\mathcal{G}$ be the graph associated to M and $\mathrm{W}[.]$ a function weighting the edges of the graph with entries or submatrices of M. Let $W_{\mathcal{G}}$ and $W_{\mathcal{G};\alpha\omega}$ be the ensembles of all walks on $\mathcal{G}$ and of all walks on $\mathcal{G}$ from vertex α to vertex ω, respectively.

If $\mathbb{K}\mathcal{G}_\bullet = (W_{\mathcal{G}}, +, \bullet)$ is a unique factorization (near-) $\mathbb{K}$-algebra with $\mathcal{F}_\bullet(w)$ the unique prime factorization of a walk w, then

$$\sum_{w\in W_{\mathcal{G};\alpha\omega}} w = \sum_{w\in W_{\mathcal{G};\alpha\omega}} \mathcal{F}_\bullet(w)$$

$$\equiv \sum_{F_{\alpha\omega}} (\mathbb{1}-p_i)^{-1} \bullet (\mathbb{1}-p_{i-1})^{-1} \cdots \bullet (\mathbb{1}-p_1)^{-1}$$

with $F_{\alpha\omega}$ walks families and p_i their prime generators.

This transforms the power-series for $f(\mathsf{M})_{\omega\alpha} = \mathrm{W}\big[\sum_{w\in W_{\mathcal{G};\alpha\omega}} w\big]$ into a sum over walks' factorized forms. For example, for the resolvent function $\mathsf{R}_\mathsf{M} = (s\mathsf{I}-\mathsf{M})^{-1}$, we have

$$\big(\mathsf{R}_\mathsf{M}(s)\big)_{\omega\alpha} = s^{-1}\mathrm{W}\Big[\sum_{F_{\alpha\omega}} (\mathbb{1}-p_i)^{-1} \bullet (\mathbb{1}-p_{i-1})^{-1} \cdots \bullet (\mathbb{1}-p_1)^{-1}\Big]$$

- Did we say anything specific about $\bullet$?
  $\hookrightarrow$ Only requirements: existence and uniqueness of $\mathcal{F}_\bullet(w)$

Method	Walk product	Primes	Form of $\mathsf{R}_\mathsf{M}(s)$
Power-series	Concatenation	Edges	Infinite sum
Primitive series	Self-conc.	Primitive orbits	Infinite sum of fractions
Path-sums	Nesting	Simple paths & cycles	Finite continued fraction
Language eq.	Inc. nesting	Simple paths & cycles	System of equations

P.-L. Giscard, S. J. Thwaite and D. Jaksch, *Continued Fractions and Unique Factorization on Quivers*, under review, arXiv:1202.5523

Path-sum

Nesting $\odot$: inserts a cycle c into a walk w, as late as possible and only if c does not visit any vertex previously visited by w

- $\odot$ *non commutative & non associative* !
- Yet factorization $\mathcal{F}_\odot(w)$ into primes always exists and is unique
  $\hookrightarrow$ Efficient algorithm producing $\mathcal{F}_\odot(w)$
- Primes are simple paths $\pi \in \Pi_{\mathcal{G}}$ and simple cycles $\gamma \in \Gamma_{\mathcal{G}}$

Let $\mathsf{M}_{\omega\alpha} = \mathrm{W}[(\alpha\omega)]$ be edge weights, $\pi = (\alpha\eta_1\cdots\eta_n\omega)$

$$\big(\mathsf{R}_\mathsf{M}(s)\big)_{\omega\alpha} = \sum_{\Pi_{\mathcal{G};\alpha\omega}} (\omega) \quad \mathsf{M}_{\omega\eta_n} \quad (\eta_n) \quad \cdots \quad (\eta_1) \quad \mathsf{M}_{\eta_1\alpha} \quad (\alpha)$$

$$(\eta_1) \leftarrow \odot\mathrm{W}\Big[\big(\mathbb{1} - \sum_{\Gamma_{\mathcal{G}\backslash\{\alpha\};\eta_1}} \gamma_{\eta_1}\big)^{-1}\Big] \qquad (\alpha) \leftarrow \odot\,\mathrm{W}\Big[\big(\mathbb{1} - \sum_{\Gamma_{\mathcal{G};\alpha}} \gamma_\alpha\big)^{-1}\Big]$$

- $\mathsf{R}_\mathsf{M}(s)$, M^q, $\mathrm{Exp}(\mathsf{M})$, M^{-1}, $\mathrm{Log}(\mathsf{M})$
- Generalized matrix powers $\mathrm{im}(\mathsf{M})^q \rightarrow$ Drazin inverse $q = -1$
- Exponential convergence for *sufficiently random matrices*
  $\hookrightarrow$ Based on an approach to Anderson localization

P.-L. Giscard, S. J. Thwaite and D. Jaksch, *Evaluating matrix functions by resummations on graphs: the method of path-sums*, to appear in SIAM Matrix Analysis and Applications, arXiv:1112.1588

M. Aizenman and S. Molchanov, *Localization at large disorder and at extreme energies: an elementary derivation*, Commun. Math. Phys. 157, 245-278 (1993)

Primitive series

Self-concatenation $\tilde{\circ}$: allows only the concatenation of a cycle with itself

- Factorization of cycles $\mathcal{F}_{\tilde{\circ}}(w)$ into primes always exists and is unique
- Primes are primitive orbits, $p \in P_{\mathcal{G}}$ a cycle not multiple of any other cycle $p \neq q^k$ and $\mathcal{F}_{\tilde{\circ}}(c) = p^\ell \Rightarrow F_p = \{p^\ell\}_{\ell\in\mathbb{N}}$

$$\big(\mathsf{R}_\mathsf{M}(s)\big)_{ii} = s^{-1}\Big[1 + \sum_{p\,\in\, P_{\mathcal{G};ii}} \mathrm{W}[p]\,\big(s^\ell\mathsf{I} - \mathrm{W}[p]\big)^{-1}\Big]$$

- Primitive orbits counting theorem, Ihara Zeta function, $\det(\mathsf{M})$

Language equations

Incomplete nesting $\bigcirc$: inserts a cycle c into a walk w, as late as possible and only if c does not visit any vertex previously visited by w except its 1st vertex

- Factorization $\mathcal{F}_\bigcirc(w)$ exists and is unique, primes are $\Pi_{\mathcal{G}} \cup \Gamma_{\mathcal{G}}$
- Give $W_{\mathcal{G};\alpha\alpha}$ in terms of itself and of other $W_{\mathcal{G};\omega\omega}$
- Weighted version gives equations for the resolvent

$$\big(\mathsf{R}_\mathsf{M}(s)\big)_\alpha = \sum_{\Gamma_{\mathcal{G};\alpha}} \big(\mathsf{R}_{\mathsf{M};\mathcal{G}\backslash\{\eta_2,\cdots,\eta_\ell\}}(s)\big)_\alpha \mathsf{M}_{\alpha\eta_\ell}\cdots\mathsf{M}_{\eta_2\eta_3}\big(\mathsf{R}_\mathsf{M}(s)\big)_{\eta_2}\mathsf{M}_{\eta_2\alpha}$$

$$= \mathbb{1} + \underbrace{\big(\mathsf{R}_\mathsf{M}(s)\big)_\alpha \mathsf{M}_{\alpha\alpha}}_{\textit{loop}} + \underbrace{\big(\mathsf{R}_\mathsf{M}(s)\big)_{\mathcal{G};\backslash\{\eta\};\alpha}\mathsf{M}_{\alpha\eta}\big(\mathsf{R}_\mathsf{M}(s)\big)_\eta\mathsf{M}_{\eta\alpha}}_{\textit{backtrack}} + \cdots$$

$\Theta(.)$ a function assigning labels to edges. Walks on $\mathcal{G}$ are the words of a formal language $L = \Theta(W_{\mathcal{G};\alpha\omega})$. The language equations

- Provide the formal grammar of L
- Characterize the graph uniquely, up to isomorphism

Summary and ongoing research

The Method Generating Theorem allows for complete freedom in the design of walk products so long as the existence and uniqueness of $\mathcal{F}_\bullet(w)$ hold. Then, novel methods to evaluate sums of walks and matrix functions can be designed at will.

Numerical analysis, matrix computations :

- Development of a MATLAB package
- Solution to matrix quadratic equations
- Exponential convergence for non-random matrices ?
  $\hookrightarrow$ research of Prof. Benzi

Theory :

- Factorization into primes not crossing any edge more than once
- Characterization of graphs through $W_{\mathcal{G}}$ and prime counting

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