Of Walks and Graphs

An Introduction to Walk Theory II

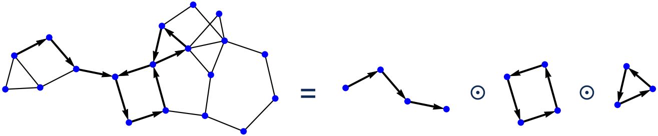
P.-L. Giscard, S. Thwaite, D. Jaksch

Seminar
September 2014

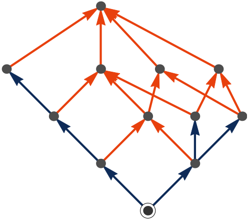
Outline

Summary

- Unique factorisation of walks into primes



- Walk-posets: a 'graph-free' representation of walk sets



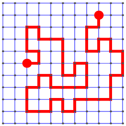
- Prime representation of walk series: path-sums

How many primes of length ℓ?

Prime counting

- Tempting analogies

$$\text{Walks} \iff \text{Integers}$$
$$\text{Prime walks} \iff \text{Prime integers}$$



2, 3, 5...

- Rigorous?

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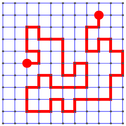

2, 3, 5...

- Rigorous?

$$\text{Prime walk theorem} \iff \text{Prime number theorem}$$

- Can we follow the same route?

$$?? \iff \zeta(s)$$

- *Can we construct a 'number theory of walks'?*

Elements of Walk Theory

Walk Zeta Functions

Let's start from the end...

Riemann's original proof

1) Zeta function $\zeta(s)$

2) Euler product

3) Analysis connecting the zeros to primes

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$\hookrightarrow$ Functions on poset intervals (G. C. Rota)...

$$f_\alpha : \text{Intervals}[P_\alpha] \longrightarrow \mathbb{N}$$

$$\big(f_\alpha \,.\, g_\alpha\big)(w, w') = \sum_{w \le w_1 \le w'} f_\alpha(w, w_1)\, g_\alpha(w_1, w')$$

$\hookrightarrow$ ...form the incidence algebra $I_\alpha := (\{f_\alpha\}, .)$

$$Z_\alpha \in I_\alpha : \text{Intervals}[P_\alpha] \longrightarrow 1$$

Similarly for P_α^Ω and P_α^ω

Walk Zeta Functions

Why?

- Probability loop-erased random walk: $\vec{1}_p^{\mathsf{T}} \,.\, Z_\alpha \,.\, \vec{\mathrm{Pr}}[w]$
- Möbius inversion: $u(w') = \sum_{w|w'} v(w) \Rightarrow \vec{v} = {Z_\alpha}^{-1} \,.\, \vec{u}$

- # Divisors of all walks: $\vec{d}^{\,\mathsf{T}} = \vec{1}^{\mathsf{T}} \,.\, Z_\alpha$
- # Number of common divisors: $1_w^{\mathsf{T}} \,.\, (Z_\alpha)^{\mathsf{T}} \,.\, Z_\alpha \,.\, \vec{1}_{w'}$

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Length poset

$$Z_\alpha^\ell(w, w') = 1 \iff \ell(w) < \ell(w') \text{ or } w = w'$$

- # Primes shorter than w: $\pi(w) = \vec{\omega}^{\,\mathsf{T}} \, . \, {Z_\alpha}^{-1} \, . \, Z_\alpha^\ell \, . \, \vec{1}_w$
- # Shorter non-divisors of w: $\left(\vec{1} - Z_\alpha \, . \, \vec{1}_w\right)^{\mathsf{T}} \, . \, Z_\alpha^\ell \, . \, \vec{1}_w$

Relations in parallel to number theory?

Reduced Incidence Algebras

Matrices, vectors are unwieldy

$\hookrightarrow$ Reduced incidence algebra

$$R(P,\sim) := \{f \in I[P],\ a \sim b \Rightarrow f(a) = f(b)\}$$

Proposition (G.C. Rota)

$R(P,\sim) \simeq$ *some algebra of generating functions*

Example (G.C. Rota)

S countable set, $P := (2^S, \subseteq)$

$$[s_1, s_2] \sim [s_3, s_4] \iff |s_2 \backslash s_1| = |s_4 \backslash s_3|$$

$$R(2^S,\sim) \simeq \big(\{f = \sum_n f_n/n!\}, \times\big)$$

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There must be functions corresponding to Z_α, Z_α^Ω, Z_α^ω

Ω and ω Path-Sums

Riemann's original proof

1) Zeta function $\zeta(s)$

➡ 2) Euler product

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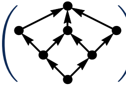
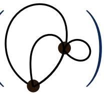
Prime representation of Ω- and ω-walk series

$$\Sigma^{\Omega}_{\mathcal{G};\,\alpha\beta} := \sum_{w\,\in\, W^{\Omega}_{\mathcal{G};\alpha\beta}} k^{\Omega}(w)$$

Theorem

If $k^{\Omega}(w \odot w') = k^{\Omega}(w)k^{\Omega}(w')$ and P^{Ω}_{α} is distributive then $\Sigma^{\Omega}_{\mathcal{G};\,\alpha\beta}$ admits a form involving only prime walks.

P^{Ω}_{α} distributive = Forbidden() = Forbidden()

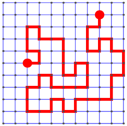
$\longleftrightarrow$ Unique factorisation into join-irreducibles in lattice theory

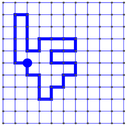
Ω and ω Path-Sums

- Explicit Ω path-sum

$$\Sigma^{\Omega}_{\mathcal{G};\,\alpha\beta} = \sum_{p\in\Pi_{\mathcal{G};\,\alpha\beta}} k^{\Omega}(p) \prod_{j=0}^{\ell_p-1} \Sigma^{\Omega}_{\mathcal{G}\backslash\{\alpha,\nu_1,\cdots,\nu_{j-1}\};\,\nu_j\nu_j}$$

$$\Sigma^{\Omega}_{\mathcal{G};\,\alpha\alpha} = \prod_{\gamma\in\Gamma_{\mathcal{G};\,\alpha\alpha}} \left\{1 + \frac{k^{\Omega}(\gamma)}{1-k^{\Omega}(\gamma)} \prod_{j=1}^{\ell_\gamma} \Sigma^{\Omega}_{\mathcal{G}\backslash\{\alpha,\mu_1,\mu_{j-1}\};\,\mu_j\mu_j}\right\}$$





Prime counting using Ω-walks

- Explicit Ω path-sum

$$\Sigma^{\Omega}_{\mathcal{G};\,\alpha\beta} = \sum_{p\in\Pi_{\mathcal{G};\,\alpha\beta}} k^{\Omega}(p) \prod_{j=0}^{\ell_p-1} \Sigma^{\Omega}_{\mathcal{G}\backslash\{\alpha,\nu_1,\cdots,\nu_{j-1}\};\,\nu_j\nu_j}$$

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- Explicit ω path-sum

$$\Sigma^{\omega}_{\mathcal{G};\,\alpha\beta} = \sum_{p\in\Pi_{\mathcal{G};\,\alpha\beta}} k^{\omega}(p) \prod_{j=0}^{\ell_p-1} \Sigma^{\omega}_{\mathcal{G}\backslash\{\alpha,\nu_1,\cdots,\nu_{j-1}\};\,\nu_j\nu_j}$$

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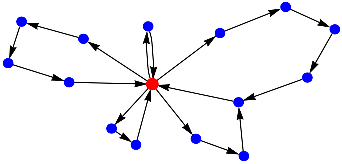
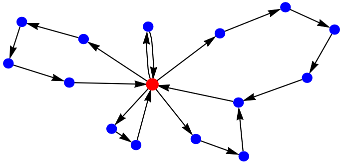
Algebraic Walk Theory

Realization of Number Theory



A bouquet graph $\mathcal{B}$

Realization of Number Theory



A bouquet graph $\mathcal{B}$

Theorem

On ∞-bouquet graphs $P^{\Omega}_{\bullet} \simeq (\mathbb{N}, \leq_{|})$, $(W^{\Omega}_{\mathcal{B};\bullet\bullet}, \odot) \simeq (\mathbb{N}, \times)$

$\hookrightarrow$ *Ω-walks $\equiv$ integers*

Furthermore $R(P^{\Omega}_{\bullet}, \sim) \simeq$ (Dirichlet series, $\times$)

$\hookrightarrow Z^{\Omega}_{\alpha} \equiv \zeta(s)$

$\hookrightarrow$ *Path-sum representation* $Z^{\Omega}_{\alpha} \equiv \prod_p \frac{1}{1-p^{-s}}$

$\zeta(s)/\zeta(2s)$

ω-walks $\equiv$ square-free integers

$\left(1-\zeta_p(s)\right)^{-1}$

walks $\equiv$ ordered prime factorisations

Realization of Number Theory

Relations between Dirichlet series follow

$\hookrightarrow$ Via path-sums

$$D_{2^\Omega} = \prod_p \frac{1}{1-2p^{-s}}, \quad D_\phi = \prod_p \frac{p^s - 1}{p^s - p}, \quad L(s,\chi) = \prod_p \frac{1}{1-\chi_n(p)p^{-s}}$$

$\hookrightarrow$ From the incidence algebra

Prime zeta function

$$\vec{d} = \vec{1} \,.\, Z_\alpha^\Omega \quad \Rightarrow \quad D_d = \zeta(s)^2$$

$$\vec{\omega} = \sum_p \vec{1}_p \,.\, Z_\alpha^\Omega \quad \Rightarrow \quad D_\omega = \zeta_p(s)\zeta(s)$$

$$\vec{\Omega} = \sum_{p,k>0} \vec{1}_{p^k} \,.\, Z_\alpha^\Omega \quad \Rightarrow \quad D_\Omega = \sum_p \frac{p^{-s}}{1-p^{-s}}\zeta(s)$$

and *many more*!
Also Lambert series, residue classes, almost Euler representations modular arithmetic (arXiv:1409.3555)...

General Reduced Incidence Algebras

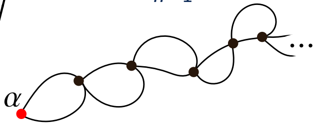
If P_α^Ω is distributive

$$R(P_\alpha^\Omega, \sim) \simeq \text{ some algebra of Dirichlet series}$$

$$Z_\mathcal{B}^\Omega \equiv \zeta_\mathcal{B}^\Omega(s) = \zeta(s), \qquad Z_\mathcal{P}^\Omega \equiv \zeta_\mathcal{P}^\Omega(s) = 1 + \sum_{n=1}^{\infty} \frac{p_n\#^{-s}}{\prod_{k=1}^{n}\left(1 - p_n^{-s}\right)}$$

Bouquet graph

$\mathcal{P} \equiv$



If P_α^ω is distributive

$$R(P_\alpha^\omega, \sim) \simeq \text{ some algebra of power series}$$

$$Z_\mathcal{B}^\omega \equiv \zeta_\mathcal{B}^\omega(x) = e^x, \qquad Z_\mathcal{P}^\omega \equiv \zeta_\mathcal{P}^\omega(x) = \frac{1}{1-x}$$

$\hookrightarrow$ All of these have path-sums, e.g. $(1-x)^{-1} = 1 + x(1 + x(\cdots$

General Reduced Incidence Algebras

General case

Homomorphic surjection

$$s: \quad \text{Intervals}[P] \longrightarrow P$$
$$i_1 \cup i_2 = i_3 \implies s(i_1)\,.\,s(i_2) = s(i_3)$$

Induces an equivalence $s(i_1) = s(i_2) \iff i_1 \sim_s i_2$

$\hookrightarrow$ Isomorphism $R(P, \sim_s) \simeq (P, \,.\,)$

$\hookrightarrow$ $f_\alpha \in R(P_\alpha, \sim_s)$ is a walk series with a path-sum

Also true on distributive P_α^Ω, P_α^ω □

Number theory on $\mathcal{B}$

$P_\bullet^\Omega \simeq (\mathbb{N}, \leq_|)$

$\hookrightarrow$ Immediate from prime-tree theorem

$R(P_\bullet^\Omega, \sim) \simeq$ (Dirichlet series, $\times$)

$\hookrightarrow$ Immediate from surjection *division* $[3, 30] \xrightarrow{\div} 10$ □

On Ω and ω zeta functions

We can construct a rigorous ‘number theory’ of walks!

- Comparing to Riemann’s approach

  1) Zeta function $\zeta(s)$ ✗

  2) Euler product $\iff$ Path-sum ✓

  3) Connecting the zeros to primes $\iff$ Feasible ✓

On Ω and ω zeta functions

We can construct a rigorous 'number theory' of walks!

- Comparing to Riemann's approach

  1) Zeta function $\zeta(s)$

  2) Euler product $\iff$ Path-sum

  3) Connecting the zeros to primes $\iff$ Feasible

- Zeta functions only available from path-sums

  1) *Surjection* Intervals$[P] \to P$

  $\hookrightarrow$ Division $[3, 30] \xrightarrow{\div} 10$ *only on $\mathcal{B}$*

  2) Any surjection $\Rightarrow Z_\alpha^\Omega$ a series of Ω-walks

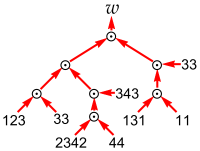
Basic objects: walks

Z_α walk series

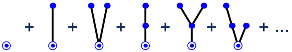
Non-commutative zeta functions

- Same construction for walk zeta functions Z_α
  1) Path-sums always exist
  2) Poset is non-commutative
  3) $R(P_\alpha, \sim)$ cannot be a commutative algebra

  $\hookrightarrow$ *Unique factorisation*: tree-series



$w = 133112343442333$

$\zeta_\alpha = \sum_{\text{Trees}}$ 

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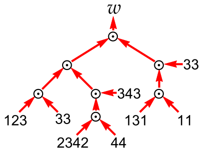

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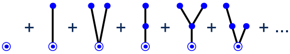
$\zeta_\alpha = \sum_{\text{Trees}}$ 

- Basic theory of non-commutative posets is largely lacking

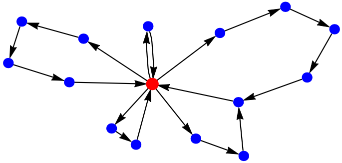
  Example: (P, Q) posets

  Poset formed by commutative product of P and Q

$$P \times Q \Rightarrow Z_{P\times Q} = Z_P \otimes Z_Q$$

  Non-commutative case ?

Non-commutative zeta functions



- On $\mathcal{B}$
  Z_α = zeta function of ordered prime factorizations

Proposition (Based on Björner)

Let $\mathcal{B}$ be a bouquet graph
Then $(Z_\alpha)^{-1}$ is the Möbius function of factor order

$\hookrightarrow$ Walks = words, primes = letters   *word* $\leq$ wor*word*rd

Prime number theorem must be workable from tree series

- Unlock access to the prime walk theorem

Anders Björner, *The Möbius function of factor order*, Theor. Comp. Sc. 117 (1993), p. 91-98

Walk Theory

Final Message

Main Message

A global approach to prime counting on multi-digraphs is possible. Algebraic walk theory is the 'number theory' of walks.

Results

- A 'number theory' of walks
- Number theory
- Aspects parallel to graph theory (not discussed)
- Fractal path-sums (not discussed)

Open problems

- Walk ideals anyone?
- Non-commutative zeta functions
- Analysis of tree series

Thank You!

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I am looking for a postdoctoral position!