The Theory of Operator-Lattices
A New Analytical Tool for Quantum Many-Body Physics

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**GOAL**

*Efficiently find the time dynamics of a many-body system*

- System with $N$ interacting 2-level atoms

  $$\mathcal{H} \Rightarrow 2^N \times 2^N \gg 1 \Rightarrow \text{Hard}$$

- Same system, all atoms frozen except 1

  $$\mathcal{H} \Rightarrow 2 \times 2 \Rightarrow \text{Easy}$$

Can we express the true evolution as $\mathcal{F}$ (frozen situations)?
Freezing atoms = put + keep in some state $|\psi_\mu\rangle$

- Operator-lattice $\Rightarrow$ lattice of projectors

\[ \varepsilon_\mu = P_\mu \otimes I_{\text{atom}} \]

\[ P_\mu = |\psi_\mu\rangle \langle \psi_\mu| \]

\[ \varepsilon_\mu = E_1 E_2 G_3 G_4 E_5 G_6 \otimes I_{\text{red}} \]
Freezing the System

- Introduce closure relation \( U(t) = \lim_{\delta t \to 0} \prod_{n}^{N} \left( \sum_{\mu} \varepsilon_{\mu} \right) \delta U \)

and expand

\[ U = \lim_{\delta t \to 0} \varepsilon_{\mu} \delta U \varepsilon_{\mu} \delta U \ldots \varepsilon_{\mu} \delta U \varepsilon_{\mu} + \]

\[ \lim_{\delta t \to 0} \varepsilon_{\mu} \delta U \varepsilon_{\mu} \delta U \ldots \varepsilon_{\mu} \delta U \varepsilon_{\nu} \ldots \varepsilon_{\nu} \delta U \varepsilon_{\nu} + \ldots \]

- Zeno measurement of \( \varepsilon_{\mu} \)
- Zeno measurement of \( \varepsilon_{\nu} \)
  + jump +
- Zeno measurement of \( \varepsilon_{\mu} \)

- Full expansion = all jumps + all jumping times
Conditional-Evolution Operators

$$U_{\nu \rightarrow \mu}(t) = \sum_n (i\hbar)^{-n} \sum_{P_n(G)} \int_0^t \int_0^{t_n} \ldots \int_0^{t_2} e^{-iH_\mu(t-t_n)/\hbar} H_{\eta \rightarrow \mu}$$

Number of jumps \( n \) \quad Strings of \( n \) \quad Times of the jumps

- \( H_\mu \) effective Hamiltonian when system in \( \varepsilon_\mu \), 2 \( \times \) 2 matrix
  \[ \varepsilon_\mu \mathcal{H} \varepsilon_\mu = H_\mu \otimes \varepsilon_\mu \]

- \( H_{\eta \rightarrow \mu} \) how a jump from \( \varepsilon_\eta \) to \( \varepsilon_\mu \) affects the atom, 2 \( \times \) 2 matrix
  \[ \varepsilon_\mu \mathcal{H} \varepsilon_\eta = H_{\eta \rightarrow \mu} \otimes |\varepsilon_\mu\rangle\langle\varepsilon_\eta| \]
Conditional-Evolution Operators

Conditional-evolution operators have nice properties!

- $U_{\nu \rightarrow \mu}(t) = \epsilon_\mu U(t) \epsilon_\nu$

- They evolve quantum conditional expectation values

\[
\langle A/\epsilon_\mu \rangle = \langle \psi_\nu | U_{\nu \rightarrow \mu}^\dagger A U_{\nu \rightarrow \mu} | \psi_\nu \rangle
\]

- Fourier domain $\tilde{U}_{\nu \rightarrow \mu}(\omega)$ only $\times$ and $+$ of universal $2 \times 2$ matrices

- Can be interpreted as interference of information TSVF

Future $\rightarrow$ Past & Past $\rightarrow$ Future
Is it **practical** for calculations?

$$U_{\nu \to \mu}(t) \leftarrow \sum_n \sum P_n(G)$$

- Huge number of jumps?
- Find all the possible strings of jumps? some $H_{\eta \to \chi} = 0$

**Simple graph** $G$

- For each operator-lattice $\varepsilon_\mu$ place a vertex
- If $H_{\eta \to \mu} \neq 0$ draw an edge between $\varepsilon_\mu$ and $\varepsilon_\eta$
The Simple Graph

Graphs $G$ for 1D, 2D and 3D XYZ-Hamiltonians

- String of jumps = Path on $G$
- Number of strings = Number of paths on $G$
- N.n Hamiltonians reduce to small graphs

Path-integrals on $G$
Are calculations efficient?

- Number of operations
  \[ U(t) \propto 2^{3N}K \]
  \[ U_{\nu \rightarrow \mu}(t) \propto N^K \]

  \( U_{\nu \rightarrow \mu} \) generates a small piece of the full \( U(t) \)

- Operations / elements of \( U(t) \) generated
  \[ U(t) \propto 2^N \]
  \[ U_{\nu \rightarrow \mu}(t) \propto N^K \]

Exponential speed-up
The Underlying Mathematical Theory

Function of matrices = Paths on graphs

- Extension to any matrix-function
- Derivation all the results independently of physics
- Exponential speed-up on \( \text{Exp}(\mathcal{M}) \) and \( \mathcal{M}^{-1} \)
Rydberg Atoms in Deep Optical Lattices

\[ \langle P \rangle \text{ ? Distribution of Rydberg excitations ?} \]

\[ \mathcal{H} = \sum_i \mathcal{H}_{\text{Laser}} + \sum_{i,j} \frac{A}{R^3} P_i P_j \]

Strong anisotropic dipole-dipole interaction
Blockade, antiblockade \( \Rightarrow \) Mean Field

- MF-based \( \frac{A}{R^3} P_i P_j \approx \frac{A}{R^3} P_i \langle P \rangle g_2 \)
- Using operator-lattices \( \langle P \rangle = \mathcal{F}(\langle P_i/P_j \rangle, \langle P_i/G_j \rangle) \)
Results

- Lowest order
- $n$-atom correlation functions
- Rydberg fraction

- Lattice coverings
- Quantum phase-transition
Results

- 2D and 3D
- Selective population of lattice-coverings
"Path integrals suffer most grievously from a serious defect. They do not permit a discussion of spin operators or other such operators in a simple and lucid way."

R. Feynman

Main results

- Path-integrals (sums) on discrete spaces
- System of $N$ particles with $n_\ell$ levels.
- Time-dynamics $\Rightarrow \times$ and $+$ of $n_\ell \times n_\ell$ matrices
- Large speed-up in computations
- General theory of matrix functions, independent of physics
Perspectives

- Systematic application to physical systems
- Integration of damping
- Link with TSVF
- Recovering usual path-integrals (continuous $G$)
- Efficient computation of the matrix-log