The Theory of Operator-Lattices

A New Analytical Tool for Quantum Many-Body Physics

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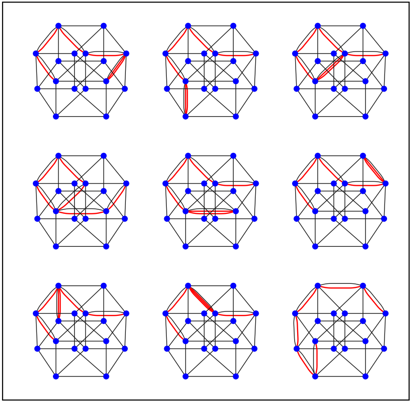
September 17, 2010



Original Motivation

GOAL

Efficiently find the time dynamics of a many-body system

- System with N interacting 2-level atoms

$$\mathcal{H} \Rightarrow 2^N \times 2^N \gg 1 \Rightarrow \textbf{Hard}$$

- Same system, all atoms frozen except 1

$$\mathcal{H} \Rightarrow 2 \times 2 \Rightarrow \textbf{Easy}$$

Can we express the true evolution as $\mathcal{F}$(frozen situations) ?

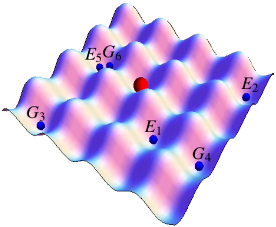
Operator-Lattices

Freezing atoms = put + keep in some state $|\psi_\mu\rangle$

- Operator-lattice $\Rightarrow$ lattice of projectors

$$\varepsilon_\mu = P_\mu \otimes \mathcal{I}_{atom}$$

$$P_\mu = |\psi_\mu\rangle\langle\psi_\mu|$$



$$\varepsilon_\mu = E_1 E_2 G_3 G_4 E_5 G_6 \otimes \mathcal{I}_{red}$$

Freezing the System

- Introduce closure relation $U(t) = \lim_{\delta t \to 0} \prod_n^N \left(\sum_\mu \varepsilon_\mu\right) \delta U$ and *expand*

$$U = \lim_{\delta t \to 0} \varepsilon_\mu \delta U \varepsilon_\mu \delta U \ldots \varepsilon_\mu \delta U \varepsilon_\mu + \lim_{\delta t \to 0} \varepsilon_\mu \delta U \varepsilon_\mu \delta U \ldots \varepsilon_\mu \delta U \varepsilon_\nu \ldots \varepsilon_\nu \delta U \varepsilon_\nu + \ldots$$

Zeno measurement of ε_μ

Zeno measurement of ε_ν
+ jump +
Zeno measurement of ε_μ

- Full expansion = *all* jumps + *all* jumping times

Conditional-Evolution Operators

$$U_{\nu\to\mu}(t) = \sum_n (i\hbar)^{-n} \sum_{P_n(G)} \int_0^t \int_0^{t_n} \dots \int_0^{t_2} e^{-iH_\mu(t-t_n)/\hbar} H_{\eta\to\mu} \dots H_{\nu\to\chi} e^{-iH_\nu t_1/\hbar} dt_1 \dots dt_n$$

Number of jumps

Strings of n consecutive jumps

Times of the jumps

- H_μ effective Hamiltonian when system in ε_μ, 2×2 matrix

$$\varepsilon_\mu \mathcal{H} \varepsilon_\mu = H_\mu \otimes \varepsilon_\mu$$

- $H_{\eta\to\mu}$ how a jump from ε_η to ε_μ affects the atom, 2×2 matrix

$$\varepsilon_\mu \mathcal{H} \varepsilon_\eta = H_{\eta\to\mu} \otimes |\varepsilon_\mu\rangle\langle\varepsilon_\eta|$$

Conditional-Evolution Operators

Conditional-evolution operators have nice properties !

- $U_{\nu\to\mu}(t) = \varepsilon_\mu U(t)\varepsilon_\nu$
- They evolve quantum conditional expectation values

$$\langle A/\varepsilon_\mu\rangle = \langle\psi_\nu|U^\dagger_{\nu\to\mu}\, A\, U_{\nu\to\mu}|\psi_\nu\rangle$$

- Fourier domain $\tilde{U}_{\nu\to\mu}(\omega)$ only $\times$ and $+$ of universal 2×2 matrices
- Can be interpreted as interference of information TSVF

$$Future \longrightarrow Past \qquad \& \qquad Past \longrightarrow Future$$

The Simple Graph

Is it **practical** *for calculations ?*

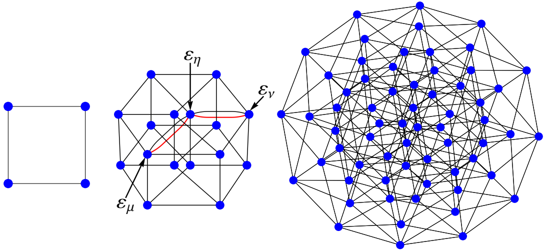
$$U_{\nu\to\mu}(t) \leftarrow \sum_n \sum_{P_n(G)}$$

- Huge number of jumps ?
- Find all the possible strings of jumps ? some $H_{\eta\to\chi} = 0$

Simple graph G

- For each operator-lattice ε_μ place a vertex
- If $H_{\eta\to\mu} \neq 0$ draw an edge between ε_μ and ε_η

The Simple Graph



Graphs G for 1D, 2D and 3D XYZ-Hamiltonians

- String of jumps = Path on G
- Number of strings = Number of paths on G
- N.n Hamiltonians reduce to small graphs

Path-integrals on G

The Speed-Up

Are calculations **efficient** *?*

- Number of operations

$$U(t) \;\propto 2^{3N}K$$
$$U_{\nu\rightarrow\mu}(t) \;\propto N^K$$

$U_{\nu\rightarrow\mu}$ generates a small piece of the full $U(t)$

- Operations / elements of $U(t)$ generated

$$U(t) \;\propto 2^{N}$$
$$U_{\nu\rightarrow\mu}(t) \;\propto N^K$$

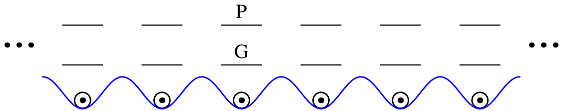
Exponential speed-up

The Underlying Mathematical Theory

Function of matrices = Paths on graphs

- Extension to any matrix-function
- Derivation all the results independently of physics
- Exponential speed-up on **Exp**$(\mathcal{M})$ and $\mathcal{M}^{-1}$

Rydberg Atoms in Deep Optical Lattices



$\langle P \rangle$ *? Distribution of Rydberg excitations ?*

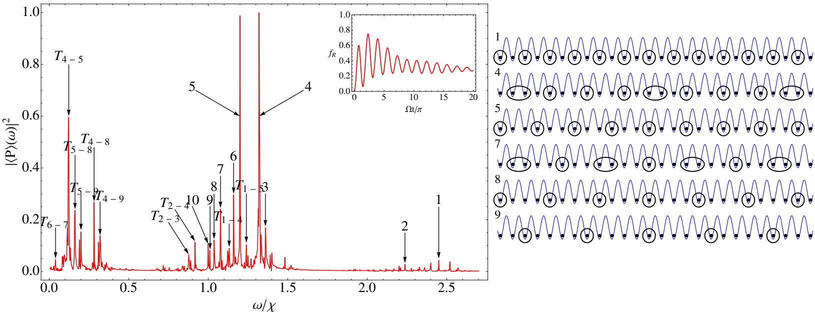
$$\mathcal{H} = \sum_i \mathcal{H}_{\text{Laser}} + \sum_{i,j} \frac{A}{R^3} P^i P^j$$

Strong anisotropic dipole-dipole interaction

Blockade, antiblockade $\Rightarrow$ ~~Mean-Field~~

- MF-based $\frac{A}{R^3} P^i P^j \simeq \frac{A}{R^3} P^i \langle P \rangle g_2$
- Using operator-lattices $\langle P \rangle = \mathcal{F}(\langle P^i / P^j \rangle, \langle P^i / G^j \rangle)$

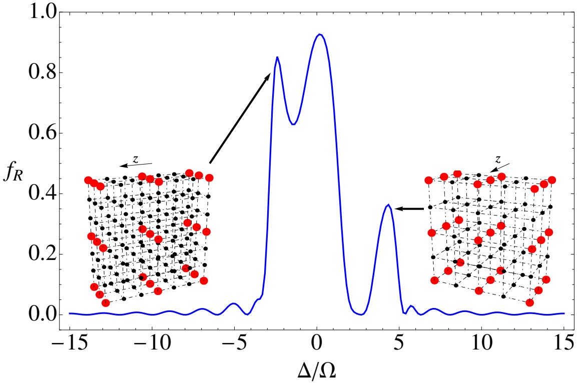
Results



- Lowest order
- *n*-atom correlation functions
- Rydberg fraction

- Lattice coverings
- Quantum phase-transition

Results



- 2D and 3D
- Selective population of lattice-coverings

Summary

"Path integrals suffer most grievously from a serious defect. They do not permit a discussion of spin operators or other such operators in a simple and lucid way."

R. Feynman

Main results

- Path-integrals (sums) on discrete spaces
- System of N particles with n_ℓ levels.

  Time-dynamics $\Rightarrow$ $\times$ and $+$ of $n_\ell \times n_\ell$ matrices

- Large speed-up in computations
- General theory of matrix functions, *independent of physics*

Future Research

Perspectives

- Systematic application to physical systems
- Integration of damping
- Link with TSVF
- Recovering usual path-integrals (continuous G)
- Efficient computation of the matrix-log