Why walks lead us astray in the study of graphs

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Joint work with Pierre-Louis Giscard and Jean Fromentin

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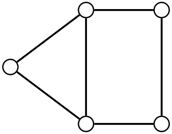
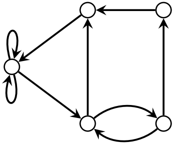
December 16, 2021

Definitions



Graph



Digraph

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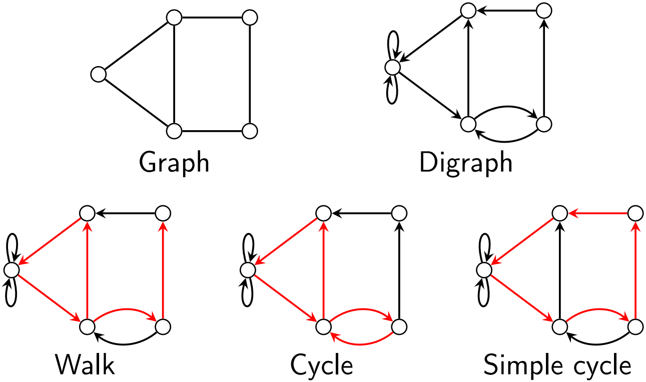

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Motivation

(Di)Graphs have many uses:

- Social networks,
- Population interaction,
- Protein protein interaction.

But too big for graph theory.

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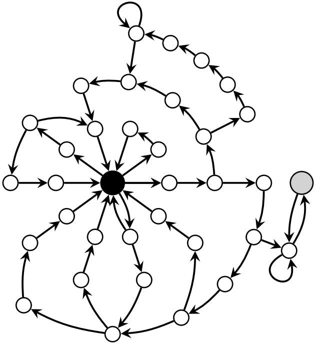
$\rightarrow$ Solution: (closed) walks

- Centrality measure,
- Similarity measure.

Motivation

Walks are not foolproof



Node ∘ as central as node •.

Motivations

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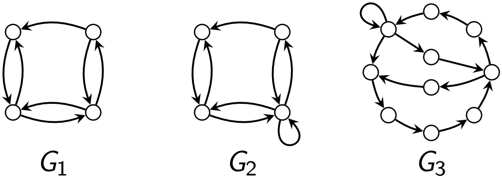

Graph G_1 more similar to G_3.

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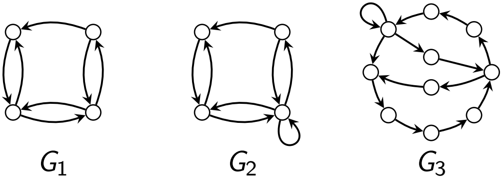

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Do closed walks determine directed graphs ?

Hike graph

Need to consider closed walks without graphs → Hikes.

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$$\phi : G \mapsto H$$

Where:

- vertices of H = simple cycles of G,
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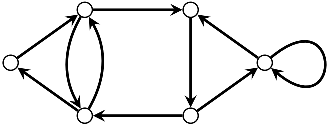
Example:

$G =$ 

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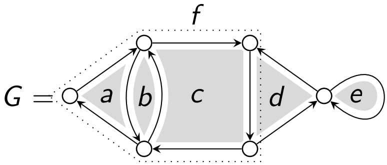

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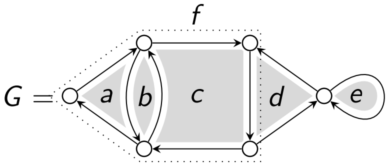

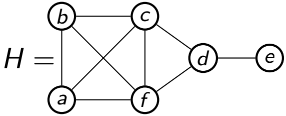

Problem formulation

Two questions:

Question (ϕ-Surjectivity.)

Given a graph H, is there a digraph G such that $\phi(G) = H$?

If yes, we say that H is *realizable*.

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Question (ϕ-Injectivity.)

What are all digraph transformations that conserve hike graphs ?

If $\phi(G_1) = \phi(G_2)$ we say that G_1 and G_2 are ϕ-*equivalent*.

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Surjectivity: first approach

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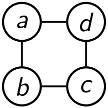
Surjectivity: first approach

ϕ is not surjective: square not realizable.



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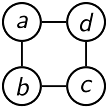

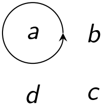

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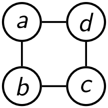

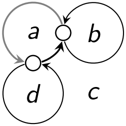

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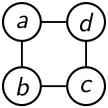

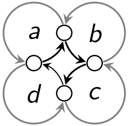

Surjectivity: first approach

Proposition

Let H be a realizable graph. Then for any induced simple cycle $(c_1, \ldots, c_n)$ of length at least 4 in H, there exist two vertices w_1, w_2 such that edges $\{w_1, c_i\}$ and $\{w_2, c_i\}$ are in H for $i \in [n]$.

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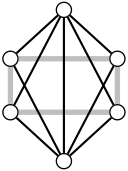
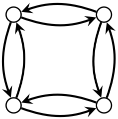
$H =$ 

 $\in \phi^{-1}(H)$

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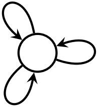
- What about triangles ?



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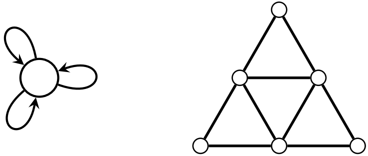

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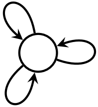
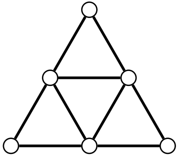
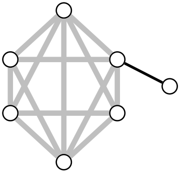
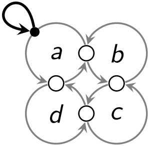




- Neighbourhood problems





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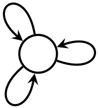

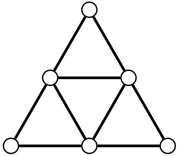

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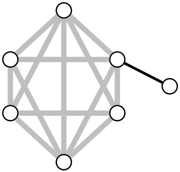
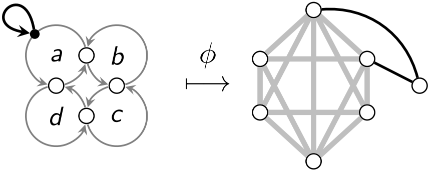
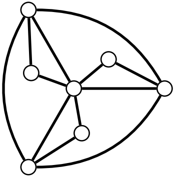




Surjectivity: first approach

Still not enough.



Surjectivity: equivalent condition

G digraph, $H = \phi(G)$.
Idea: vertices of H = cycles of G. Vertices of G = ?

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Idea: vertices of H = cycles of G. Vertices of G = cliques of H.

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$$v \in G, \quad \kappa_v = \{c \in H \,|\, v \in c\}$$

- The set $\{\kappa_v\}_{v \in G}$ is a *clique cover* of H, that is each clique κ_v is a subgraph of H and every edge and vertex of H appears in at last one κ_v.
- For $W \subseteq V$, the set $\bigcap_{v \in W} \kappa_v \setminus \bigcup_{v \in V \setminus W} \kappa_v$ corresponds to the simple cycles of G with exactly W as vertex set.

Surjectivity: equivalent condition

$\mathfrak{C}_W$: cyclic permutations of W.

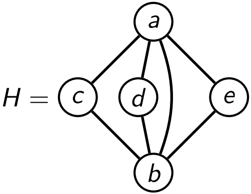
Theorem

Let H be a graph. Then H is realizable if and only if there exists a clique cover $\{\kappa_1, \ldots, \kappa_n\}$ of H such that the following polynomial system in variables $(m_{ij})_{i,j \in [n]}$ admits an integer solution:

$$\forall\, W \subseteq [n], \quad \sum_{\sigma \in \mathfrak{C}_W} \prod_{v \in W} m_{v,\sigma(v)} = \left|\mathcal{K}_W\right|,$$

where $\mathcal{K}_W := \bigcap_{v \in W} \kappa_v \setminus \bigcup_{v \in [n] \setminus W} \kappa_v$.
In this case, H is realized by the digraph G with vertex set $[n]$ and $m_{i,j}$ edges from i to j for $1 \leq i, j \leq n$.

Example



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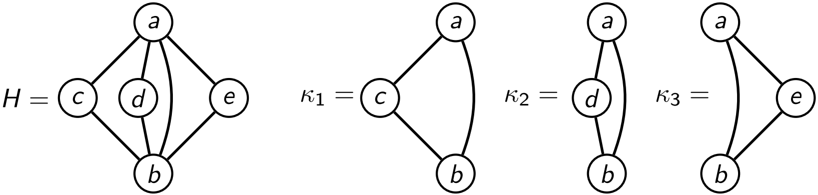

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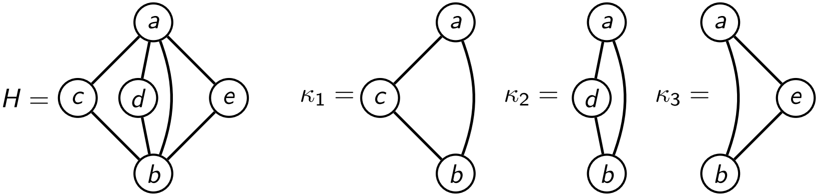

W	$\mathcal{K}_W$
$\{1\}$	$\kappa_1 \setminus (\kappa_2 \cup \kappa_3) = \{c\}$
$\{2\}$	$\kappa_2 \setminus (\kappa_1 \cup \kappa_3) = \{d\}$
$\{3\}$	$\kappa_3 \setminus (\kappa_1 \cup \kappa_2) = \{e\}$
$\{1,2\}$	$(\kappa_1 \cap \kappa_2) \setminus \kappa_3 = \emptyset$
$\{1,3\}$	$(\kappa_1 \cap \kappa_3) \setminus \kappa_2 = \emptyset$
$\{2,3\}$	$(\kappa_2 \cap \kappa_3) \setminus \kappa_1 = \emptyset$
$\{1,2,3\}$	$\kappa_1 \cap \kappa_2 \cap \kappa_3 = \{a, b\}$

Example

- $m_{1,1} = m_{2,2} = m_{3,3} = 1,$
- $m_{1,2}m_{2,1} = m_{2,3}m_{3,2} = m_{3,1}m_{1,3} = 0,$
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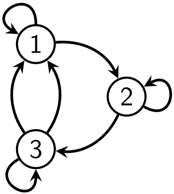
$G =$ 

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Corollary

All trees are realizable.

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First 7 terms: $1, 1, 2, 5, 15, 58, 265$. Compare to $1, 1, 2, 6, 21, 112, 853$

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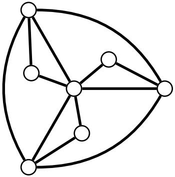
Conjecture

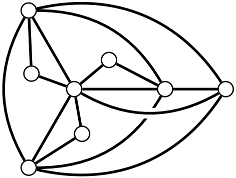
The preceding theorem continues to hold if we require minimal number of cliques.

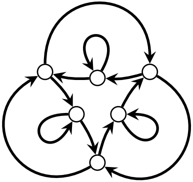
Surjectivity: realizable as subgraph

Proposition

Let H be a graph that is unrealizable. Then there exist at least one realizable graph H' such that H is an induced subgraph of H'.

$H =$ 

$H' =$ 



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Where:

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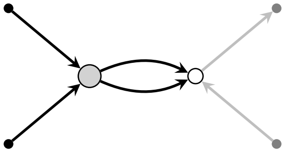
What are all digraph transformations that conserve hike graphs ?

Injectivity: Jumping



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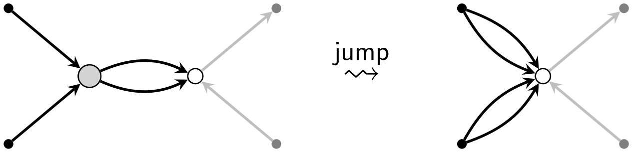

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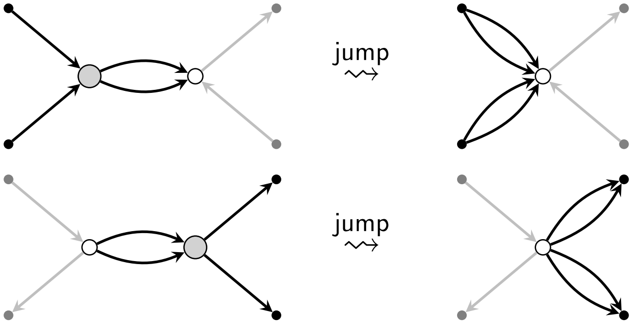

Proposition

Jumping conserves hike graphs.

Injectivity: Jumping

Corollary

- *Every realizable graph is realizable by a cubic digraph.*
- *Every realizable graph is realizable by a digraph where each vertex has at least 2 in-going vertices and 2 out-going vertices.*

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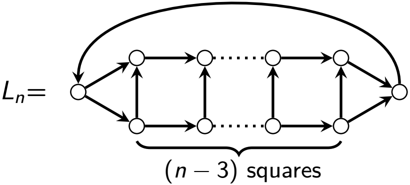
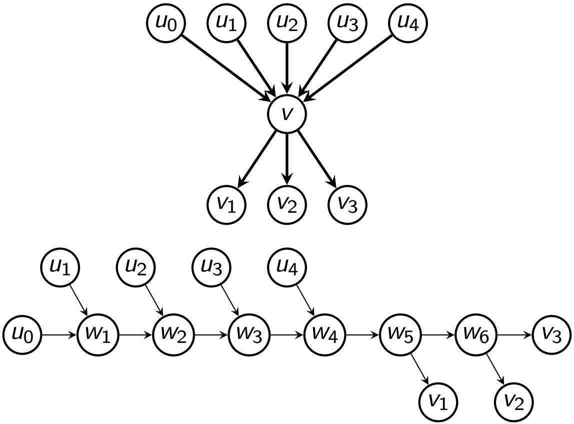
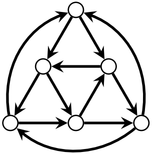


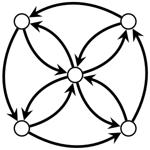
$\phi(L_n) = \phi(B_n) = K_n$

proof:



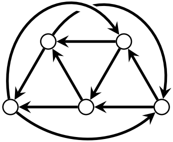
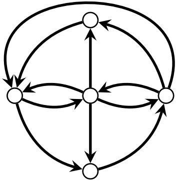
Jumping is not enough:



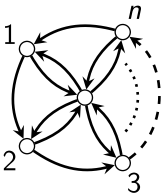


Injectivity: Different realizations of cliques

Counter examples for most graph properties.

$G_1=$  $G_2=$ 

$\phi(G_1) = \phi(G_2) = K_{12}$ and G_1 vertex-transitive and G_2 planar.

$F_n=$ 

$\phi(F_n) = K_{n^2+1} = \phi(L_{n^2+1})$.

Injectivity: restriction to bidirected graphs

G is bidirected if $v_1 \to v_2 \in G \iff v_2 \to v_1 \in G$.

Theorem (Giscard and Rochet, 2016)

The map ϕ restricted to bidirected graphs is injective with the exception of two bidirected graphs sharing the same image.

Injectivity: invariants of hike graphs

Adjacency matrix A of digraph G: $A_{i,j}$ = number of edges from i to j.

Proposition

Let G and G' be two ϕ-equivalent digraphs and let A and A′ be their adjacency matrices, respectively. Then:

- $\det(I - A) = \det(I' - A')$,
- $\text{perm}(I + A) = \text{perm}(I' + A')$.

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Conclusion

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$$T_n = \{a, b, c, d, x_1, \dots, x_n \mid ab = ba,\ bd = db\}\,.$$

T_0, T_1, T_3, T_5, T_9, T_{17} and T_{29} are not hike monoids

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$\Rightarrow$ Need 'theory of walks' independent from graphs.