Cocomplete $\mathcal{Q}$-categories are precisely the injectives wrt. fully faithful functors

Isar Stubbe

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This note records a variation on a remark made by Dirk Hofmann at an informal meeting with Maria Manuel Clementino and myself, in Coimbra's early spring of 2006. Dirk Hofmann argued that[1], in the context of [Clementino *et al.*, 2004], "a $(T,\mathcal{V})$-algebra is injective with respect to embeddings in $\mathsf{Alg}(T,\mathcal{V})$ if and only if it is complete". By definition, an *embedding* $f\colon\mathcal{A}\longrightarrow\mathcal{B}$ of $(T,\mathcal{V})$-algebras is a homomorphism such that the unit of the induced adjoint pair of bimodules is an equality. And $\mathcal{A}$ is a *complete* $(T,\mathcal{V})$-algebra if, for every $(T,\mathcal{V})$-bimodule $\phi\colon\mathcal{A}\mathrel{-\!\!\circ\!\!\rightarrow}\mathcal{B}$, there exists a $(T,\mathcal{V})$-homomorphism $f\colon\mathcal{B}\longrightarrow\mathcal{A}$ whose induced left adjoint bimodule equals the extension[2] $\{\phi,\mathsf{id}_{\mathcal{A}}\}$. Dirk Hofmann's argument applies to the case $T=\mathsf{id}$, i.e. to $\mathcal{V}$-categories; and the account below is then the obvious generalization to $\mathcal{Q}$-categories.

Although the statement below and my proofs for them are different from the ones that Dirk Hofmann gave, I do not claim any originality. Moreover I should mention that Mathieu Dupont helped writing up a part of the proof. Actually, I suppose that all this is very well known; but I couldn't immediately find a reference.

So from now on we're only interested in $\mathcal{Q}$-categories; all needed notions concerning these can be found in [Stubbe, 2005]. Moreover we say that a $\mathcal{Q}$-category $\mathbb{C}$ is *injective* if for every fully faithful functor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ and any other functor $G\colon\mathbb{A}\longrightarrow\mathbb{C}$, there exists a (not necessarily unique) $H\colon\mathbb{B}\longrightarrow\mathbb{C}$ such that $H\circ F\cong G$. Here is the theorem[3] that we shall prove.

Theorem 0.1 *For a $\mathcal{Q}$-category $\mathbb{C}$, the following are equivalent:*

1. *$\mathbb{C}$ is cocomplete,*

[1] Under certain conditions on both T and $\mathcal{V}$...

[2] ... namely, conditions that guarantee the existence of a locally ordered 2-category $\mathsf{Bim}(T,\mathcal{V})$ of bimodules between $(T,\mathcal{V})$-algebras, and extensions therein.

[3] It is "abstract nonsense" that any retract of an injective is injective; so we could have added a line to this theorem saying that "$\mathbb{C}$ is a retract of an injective". Further it is a fact that a $\mathcal{Q}$-category is cocomplete if and only if it is complete [Stubbe, 2005, 5.10], and so we could have added some "dual statements" to this theorem.

2. $\mathbb{C}$ *is injective,*

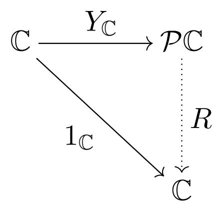
3. $Y_{\mathbb{C}}\colon \mathbb{C} \longrightarrow \mathcal{P}\mathbb{C}$ *has a left inverse,*

4. $Y_{\mathbb{C}}\colon \mathbb{C} \longrightarrow \mathcal{P}\mathbb{C}$ *has a left adjoint.*

Proof : The equivalence $(1 \Leftrightarrow 4)$ is in [Stubbe, 2005, 6.10], so it suffices to prove that $(1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4)$:

Any cocomplete $\mathcal{Q}$-category is injective: Consider functors between $\mathcal{Q}$-categories $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ and $G\colon \mathbb{A} \longrightarrow \mathbb{C}$, with F fully faithful and $\mathbb{C}$ cocomplete. The $\mathbb{B}(F-,-)$-weighted colimit of G provides a functor $H\colon \mathbb{B} \longrightarrow \mathbb{C}$, and from the general rules for computing a weighted colimit [Stubbe, 2005, 5.2] we can compute – with the aid of F's fully faithfulness in the second equality – that, for any $a \in \mathbb{A}$, $H(Fa) = \mathrm{colim}(\mathbb{B}(F-,Fa),G) = \mathrm{colim}(\mathbb{A}(-,a),G) \cong Ga$.

Injectivity implies that Yoneda has a left inverse: Suppose that $\mathbb{C}$ is injective; since the Yoneda embedding is fully faithful [Stubbe, 2005, 6.3], applying the injectivity of $\mathbb{C}$ in the diagram



gives a left inverse.

A left inverse for Yoneda is automatically its left adjoint: Suppose that

$$\mathbb{C} \underset{R}{\overset{Y_{\mathbb{C}}}{\rightleftarrows}} \mathbb{D}\ ,\ \ R \circ Y_{\mathbb{C}} \cong 1_{\mathbb{C}}$$

in $\mathsf{Cat}(\mathcal{Q})$. By the Yoneda lemma [Stubbe, 2005, 6.3], functoriality of R and $R \circ Y_{\mathbb{C}} \cong 1_{\mathbb{C}}$, we have for any $\phi \in \mathcal{P}\mathbb{C}$ and $c \in \mathbb{C}$ that

$$\phi(c) = \mathcal{P}\mathbb{C}(Y_{\mathbb{C}}c, \phi) \leq \mathbb{C}(RY_{\mathbb{C}}c, R(\phi)) = \mathcal{P}\mathbb{C}(Y_{\mathbb{C}}c, Y_{\mathbb{C}}R\phi) = Y_{\mathbb{C}}R\phi(c).$$

Thus $1_{\mathcal{P}\mathbb{C}} \leq Y_{\mathbb{C}} \circ R$, as wanted. □

References

[1] [Maria Manuel Clementino, Dirk Hofmann and Walter Tholen, 2004] One setting for all: metric, topology, uniformity, approach structure, *Appl. Categ. Struct.* **12**, pp. 127–154.

[2] [Isar Stubbe, 2005] Categorical structures enriched in a quantaloid: categories, distributors and functors, *Theory Appl. Categ.* **14**, pp. 1–45.