Physics and Categories:

The Geneva School approach to the axiomatic foundations of physics

Isar Stubbe*

June 1999†

"En physique, on décrit un système par ses propriétés" (Piron, 1990).

The study of what is now commonly referred to as 'quantum structures' was initiated by the paper of G. Birkhoff and J. von Neumann (1936) in which they argue that "based on admittedly heuristic arguments, one can reasonably expect to find a calculus of propositions [in physical theories which, like quantum mechanics, do not conform classical logic] which is formally indistinguishable from the calculus of linear subspaces". Let me briefly recall their arguments.

Standardly, one represents the "states" of a quantum mechanical system as wave functions, elements of a Hilbert space $\mathcal{H}$. It is an essential ingredient of quantum theory that even a complete mathematical description of a quantum system does not in general enable one to predict with certainty the result of an experiment on that physical system and most pairs of observations cannot be made simultaneously (cf. Heisenberg's uncertainty principle). A finite number of compatible measurements $\alpha_1, ..., \alpha_n$ corresponds to self-adjoint operators on the Hilbert space $\mathcal{H}$ that determine mutually orthogonal closed linear subspaces $\mathcal{H}_i$, these in fact being the family of proper functions of the operator α_i, such that every wave function $f \in \mathcal{H}$ can be written uniquely as $f = c_1 \cdot f_1 + ... + c_n \cdot f_n$, where $f_i \in \mathcal{H}_i$ and c_i are scalars. An according "observation space" is then the $(x_1, ..., x_n)$-space of numerical outputs of the measurements $\alpha_1, ..., \alpha_n$, and the subsets of this observation space

*Postdoctoral Fellow of the Research Foundation Flanders (FWO), Department of Mathematics, University of Antwerp, Belgium. E-mail: isar.stubbe@ua.ac.be

†This text is the first chapter of the dissertation that I presented to obtain a "Diplôme d'Etudes Approfondies" in mathematics from the Université Catholique de Louvain under supervision of F. Borceux. I have made some minor adaptations to the form of the original text, however without changing its content (except for correcting some obvious mistakes), in January 2007. For more details on the lattice theory and the projective geometry involved here, see "Propositional systems, Hilbert lattices and generalized Hilbert spaces", by I. Stubbe and B. Van Steirteghem, to appear in 2007 as a chapter in the Handbook Quantum Logic (ed. D. Gabbay, D. Lehmann and K. Engesser), Elsevier.

are the “experimental propositions” concerning the physical system. Therefore, the following definition seems *reasonable*:

> The “mathematical representative” of a subset S of the observation space associated to the compatible observations $\alpha_1, ..., \alpha_n$ is: $span\{f \in \mathcal{H} \mid \exists\ (x_1, ..., x_n) \in S : \alpha_i f = x_i f\}$.

It follows immediately that such a “mathematical representative” is a closed linear subspace of the Hilbert space $\mathcal{H}$; the self-adjointness of the operators implies that “negation” of an experimental property corresponds to orthogonal complement of subspace; and implication of experimental propositions corresponds to inclusion of subspaces. Postulating that the intersection of two subspaces representing two experimental propositions is itself the representative of an experimental proposition it follows that also the closed linear sum of any two subspaces representing two experimental propositions represents an experimental proposition. Therefore, the closed linear subspaces of the Hilbert space $\mathcal{H}$ represent mathematically the propositional calculus for the physical system. Now defining a “physical quality” of a system as the equivalence class of all experimental propositions with the same mathematical representative, it is true that the propositional calculus of properties of a quantum mechanical system forms a complete orthomodular lattice. In other words — and modulo some technical “irreducibility arguments” that basically say something about a minimum dimension for $\mathcal{H}$ — the propositional calculus of quantum mechanics has the same structure as an abstract projective geometry.

However *reasonable* this exposition may be, it does not *prove* that the propositional calculus of a quantum theory is an abstract projective geometry; it only shows that in the particular Hilbert space formalism the closed linear subspaces could be understood as decoding “physical qualities” of the system that the Hilbert space describes. It was the objective of a group of researchers, referred to as the Geneva School, to provide for an axiomatic theory for the description of a (general) physical system, and to recover one way or another the standard Hilbert space description for quantum systems.

The central thesis of the Geneva School approach to the foundations of physics, as developed in (Jauch and Piron, 1969; Piron, 1976, 1977, 1990; Aerts, 1982) is two-folded. To begin with, it is their point of view that the mathematical structure appropriate to a physical theory should be motivated as much as possible by reflection on the physical meaning of the primitive notions of the theory; this *pragmatism* leads them to consider the notion of ‘property’, cf. definition 1.2, as fundamental. Secondly, much effort is spent in trying to encode subjective semantic intuitions into syntactic axioms, which are then nothing more than physically reasonable postulates; physical systems that do not meet the axioms are simply considered as not subjected to the theory, thus this is definitely an instance of *axiomatism.* Only recently it was pointed out how the use of categories for the formalization and abstraction of the theory serves this theory’s methodological power (Moore, 1995, 1999); we could speak of *category-ism.*

Of course, there is considerable interplay between these three aspects. In this chapter I have brought together various elements of the aforementioned publications to obtain a reasonably detailed introduction to the formalism of the Geneva School. In particular I have tried to stress the use of categories for stating and understanding the axioms of the Geneva formalism, and thus the theory and philosophy behind this formalism.

1. Primitive notions.

As in all branches of physics, we will assume the concept of physical system: it is some part of the ostensively external phenomenal world, supposed separated from its surroundings in the sense that its interaction with the environment can either be ignored or modeled in a simple way. Thus, in ascribing a characteristic to a physical system in a way yet to be made precise, we presuppose as given the phenomenon to which it pertains[1]. At each instance of time, a physical system has certain characteristics that make it unique among all its possible realizations. We will be speaking of a 'particular realization of the physical system', or particular physical system for short. For instance, 'my car' is a physical system, and at each moment it has a certain position and a certain momentum which — if we decide to work with classical mechanics to describe 'my car' — determine completely all of its other physical characteristics. Thus 'my car at rest in my garage' is a different realization than 'my car passing at 100 km/h a certain point on the highway'.

The characteristics of a physical system are called its 'properties', such a property is an inherent quality of this system, it is (part of) its nature. The notion of property coincides with the notion of 'element of reality' in the sense of (Einstein, Podolsky and Rosen, 1935), and the notion of 'physical quality' in the sense of (Birkhoff and von Neumann, 1936). A system has its properties independently of the fact that we do or do not know them (which means that we implicitly take the so called *realistic point of view*); should we however know all of its properties, then we know everything there is to know about this system. Therefore we will now look for a way to give a description of a system by means of its properties — we will in fact look for a mathematical representative for this very deep and difficult notion of 'property' in a way that is in accordance with "the scientific method".

A 'definite experimental project' relative to a physical system must be understood as an experimental procedure where we have defined in advance what would be the positive response should we perform this procedure; these conditions define the response 'yes'. If these conditions are not satisfied we assign the response 'no'. Often, definite experimental projects are referred to as 'measurements' or 'questions'. The outcome of such an experiment depends of course on the particularity of the physical system. A given definite experimental project is said to be 'certain' for a particular system if it is sure that the positive response would obtain should we

[1] This implies of course a degree of idealization that is almost always unreachable in practice... which shows again that only idealists would ever think of building a theory to help them understand reality!

perform the experiment on that particular physical system, it is ‘impossible’ if it is sure that the negative response would obtain. It is crucial that if a definite experimental project is certain (impossible) for a particular system, then it is so before we perform the experiment, and even if we decide not to perform it. Thus, certainty (impossibility) of a given definite experimental project is an objective feature of the physical system. An example: a piece of chalk is ‘breakable’ when, *if* we were to hit it with a hammer, it *would* break. Hence if it is ‘breakable’, then it is so even before we tried to break it, but the ‘breakability’ is nevertheless at all times operationally verifiable.

To any given physical system we can now associate the collection Q of all definite experimental projects which could eventually be performed on it. This collection is preordered with bounds:

- for $\alpha, \beta \in Q$ we write $\alpha \preceq \beta$, and say that “α is stronger than β”, whenever β is certain in each case that α is certain;
- a maximal element $I \in Q$ is provided by the “trivial” definite experimental project that says: do whatever you wish with the system and assign the response ‘yes’;
- a minimal element $O \in Q$ is provided by the “absurd” definite experimental project that says: do whatever you wish with the system and assign the response ‘no’.

Now we can recognize two natural, everyday operations on Q. First of all, for a family A of definite experimental projects, define ΠA to be the definite experimental project that says “choose an $\alpha \in A$ and effectuate it”. Then ΠA is certain for a particular physical system if and only if each $\alpha \in A$ is certain. Secondly, for a given definite experimental project $\alpha \in Q$, set $\alpha^\sim$ to be the definite experimental project with the same experimental procedure as α but with ‘yes’ and ‘no’ interchanged. Then $\alpha^\sim$ is certain if and only if α is impossible, and *vice versa*. Note that $(\alpha^\sim)^\sim = \alpha$, and that the interaction between these two natural operations is precisely that $(\Pi_i \alpha_i)^\sim = \Pi_i \alpha_i^\sim$. Of course we can, and will, suppose that Q is closed under these operations Π and $^\sim$.

Caution: the certainty of α does not imply anything about the certainty of $\alpha^\sim$! Consider for instance the definite experimental projects O and $O\Pi I$, either of which is never certain, however $O = I^\sim$ and I is always certain, and $O\Pi I = (I\Pi O)^\sim$ and $I\Pi O$ is never certain. In particular, $\alpha^\sim$ is emphatically not the logical negation of α.

As with any preorder, it makes sense to define an equivalence relation $\approx$ on the collection Q: put $\alpha \approx \beta$ if $\alpha \preceq \beta$ and $\beta \preceq \alpha$. In words: Two experiments are said to be equivalent when either one of them is certain if and only if the other one is certain as well. With the aid of this equivalence relation on the collection of tests we can now give a mathematical representative for the notion of ‘property’ of a physical system.

Proposition 1.1 *The property of the physical system associated to the definite experimental project $\alpha \in Q$ is represented by the equivalence class $[\alpha] = \{\beta \in Q \mid \beta \approx \alpha\}$.*

With abuse of language we will often say that $[\alpha]$ "is" a property; it is of course understood that $[\alpha]$ is merely an equivalence class of tests whereas the corresponding property is the element of reality that is tested by any $\beta \in [\alpha]$.

Definition 1.2 *If α is certain for a particular physical system then the property $[\alpha]$ is called 'actual' for that particular system, otherwise it is called 'potential'.*

To come back to an example of the opening paragraph: Consider again as physical system 'my car'. When measuring the momentum of 'my car', I should be very precise in describing the definite experimental project. I should, for instance, prescribe the frame of reference that I take for my measurement. When I perform a measurement of the momentum with another frame of reference, the numerical outcome will in general be different. Does this mean that the physical quantity 'momentum' depends on the frame of reference that I choose? Not at all, the 'momentum' of 'my car' is an element of reality that exists even without it being measured, and the according actual property should therefore not be confused with the numerical output of an experiment verifying this property! But it is true that, for this example, all measurements of the 'momentum' — in whatever frame of reference — are equivalent!

A working hypothesis of the theory is the following.

Working hypothesis. *The quotient $Q/\approx \; = \; \{[\alpha] \mid \alpha \in Q\}$ is a set.*

All the foregoing allows us now to state the following proposition, giving the crucial mathematical structure of the theory.

Proposition 1.3 *Denoting $\mathcal{L} = Q/\approx$ we obtain that $\mathcal{L}$ is a complete lattice with as order relation $[\alpha] \leq [\beta] \Leftrightarrow \alpha \preceq \beta$ and as meet $\wedge_{\alpha \in A}[\alpha] = [\Pi A]$.*

As notations for top and bottom element, we will use $1 = [I]$ and $0 = [O]$. It is obvious that 1 is the property that is always actual for the system, and 0 is the "absurd" property, the property that is never actual. The order relation is the semantic implication of properties, and the meet expresses semantic conjunction (Piron, 1977). That is, to say that a property $a \in \mathcal{L}$ is less than a property $b \in \mathcal{L}$ is to say that whenever property a is actual for a particular physical system, then also property b is actual for that particular system. And to say that a property $a \wedge b$, for $a, b \in \mathcal{L}$, is actual for a certain particular physical system, is to say that both properties *a and b* are actual for this system. The join of properties is of course the meet of the set of all upper bounds (this set is non-empty because it contains 1), but admits of no direct physical meaning. In particular, it is not true that when a property $a \vee b$ is actual for a particular physical system, then either *a or b* is actual! This is already a big difference between a 'classical logic' and a 'quantum logic': in the latter it is very well possible that a superposition of propositions is true, without any of the superimposed propositions is. (A good comparison is the following: In geometry, two lines that intersect in one point determine a plane, but it is of course not true that any point of the plane is a point of one of these lines! Hence, in the lattice of subspaces of a vectorspace, the join is not a union — it is not an 'or' in the classical sense.)

Next we turn to the notion of 'state' of a physical system. Intuitively the state of a physical system is an abstract name for a particular realization of this system. Therefore the following definition.

Definition 1.4 *The 'state' of a particular physical system is the set ε of all the properties that are actual for this particular realization.*

Hence knowing the state of a system we know everything that can be obtained from it with certainty, thus we indeed know all the characteristics of that particular realization. The set of possible states of a system — note that by virtue of the working hypothesis the collection of states is indeed a set — is denoted by Σ. The fact that any possible state ε of a physical system is completely determined by its meet $p_\varepsilon = \wedge\varepsilon$ — simply because $\varepsilon = \{a \in \mathcal{L} \mid p_\varepsilon \leq a\}$ — allows for the identification of $p_\varepsilon = \wedge\varepsilon$ with ε. Therefore we can (and often will) speak of the state set $\Sigma = \{p_\varepsilon \mid \varepsilon \text{ is a state}\}$, which is now a subset of the lattice of properties[2]: $\Sigma \subseteq \mathcal{L}$. As such, Σ is order-generating for $\mathcal{L}$ in the sense of the following proposition.

Proposition 1.5 *$\forall a \in \mathcal{L}$: $a = \vee\{p \in \Sigma \mid p \leq a\}$.*

Proof : $a \geq \vee\{p \in \Sigma \mid p \leq a\}$ is obvious. On the other hand we have that, if the physical system is in a state ε such that a is actual then $p_\varepsilon \in \{p \in \Sigma \mid p \leq a\}$, hence $p_\varepsilon \leq \vee\{p \in \Sigma \mid p \leq a\}$, which means that $\vee\{p \in \Sigma \mid p \leq a\}$ is actual, so we can conclude that $a \leq \vee\{p \in \Sigma \mid p \leq a\}$. □

We already used the preorder on Q and the product operator Π to define and structure the collection of 'properties', now we will use the remaining operator $^\sim$ to give the set Σ an appropriate structure. We say that two states $p, q \in \Sigma$ are 'orthogonal', written $p \perp q$, if there exists an experiment $\alpha \in Q$ for which the positive outcome is certain if the physical system under consideration is in state p and the negative outcome is certain if the system is in state q; that is, α is certain for p and impossible for q. Taking as point of view that $\Sigma \subseteq \mathcal{L}$ this means exactly that there exists an $\alpha \in Q$ such that $p \leq [\alpha]$ and $q \leq [\alpha^\sim]$. We can record the following observation.

Proposition 1.6 *$\perp$ is a symmetric, antireflective relation on Σ.*

The duality between the notions of state and property, as built-in in definition 1.4, is captured in the following map, called "cartan map" (after E. Cartan, who was

[2]As a deep consequence of this, a state p_ε of a system is an element of reality, whereas in principle ε isn't.

the first to introduce (in 1920) an *espace des états* in classical mechanics):

$$\mu : \mathcal{L} \to \mathcal{P}(\Sigma) : a \mapsto \{\varepsilon \in \Sigma \mid a \in \varepsilon\}. \tag{1}$$

It can be read off that a property $a \in \mathcal{L}$ is actual if the state of the system is contained in the set μa, so this map does indeed represent every property in $\mathcal{L}$ as a subset of the set of possible states Σ. But remark that a converse is not necessarily true: not every subset of Σ represents a property of the system! It is easily verified, for $a, b \in \mathcal{L}$, that $\mu a = \mu b$ exactly in the case where a is actual whenever b is, thus in the case that $a = b$. Further, a meet of properties $\bigwedge_{i \in I} a_i$ is actual whenever every a_i is so, which means that $\mu(\bigwedge_{i \in I} a_i) = \bigcap_{i \in I} \mu a_i$. This proves at once the following.

Proposition 1.7 *$\mu : \mathcal{L} \to \mathcal{P}(\Sigma)$ is a meet-preserving injection.*

Remark that *a fortiori* the cartan map is order preserving. But even more, for any $a, b \in \mathcal{L}$ it is true that $a \leq b \Leftrightarrow \mu a \subseteq \mu b$, by the very definition of the order relation of $\mathcal{L}$.

2. Axioms for general physical systems.

To summarize what has been recognized in the previous section: the properties pertaining to a physical system form a complete lattice, and the set of possible states of that system is equipped with a symmetric, antireflective orthogonality relation; these two descriptions of that physical system are dual, their duality being expressed by the cartan map. Now the axioms of the Geneva School will assure that the state space description of a physical system "contains the same information" as the property lattice description.

Axiom 2.1 *The states $p \in \Sigma \subseteq \mathcal{L}$ are 'atoms' of $\mathcal{L}$, that is: $0 < x \leq p \Rightarrow x = p$.*

It follows immediately from proposition 1.5 that $\mathcal{L}$ is a complete atomistic lattice. Physically, this means that if the considered system undergoes a change of state, some of its initially actual properties become potential, and some of its initially potential properties become actual.

Axiom 2.2 *For each possible state $p \in \Sigma$ of the system, there exists a definite experimental project $\alpha \in Q$ that is certain if and only if the system is in a state orthogonal to p.*

All definite experimental projects α that meet the requirements of axiom 2 for a fixed p, are obviously equivalent. We can thus define a corresponding property $p^{\#}$, and furthermore a map

$$' : \mathcal{L} \longrightarrow \mathcal{L} : a \mapsto a' = \bigwedge\{p^{\#} \mid p \leq a\}. \tag{2}$$

Remark that $p' = p^{\#}$.

Axiom 2.3 *The map* $' : \mathcal{L} \longrightarrow \mathcal{L} : a \mapsto a' = \bigwedge\{p^\# \mid p \leq a\}$ *is surjective.*

In words, the combination of axioms 2.2 and 2.3 means that any property $a \in \mathcal{L}$ is the "opposite" of some other property.

This map $' : \mathcal{L} \longrightarrow \mathcal{L}$ "interacts", due to proposition 1.7, with the cartan map, equation 1, as follows:

$$\begin{aligned} \mu(a') &= \mu(\textstyle\bigwedge\{p^\# \mid p \leq a\}) \\ &= \textstyle\bigcap\{\mu(p^\#) \mid p \in \mu a\} \\ &= \text{states orthogonal to every state in } \mu a \\ &=: (\mu a)^\perp. \end{aligned}$$

Surjectivity of $' : \mathcal{L} \longrightarrow \mathcal{L}$ then means precisely that for every $a \in \mathcal{L}$ there exists a $A \subseteq \Sigma$ such that $\mu a = A^\perp$. It is just a trivial observation that for any $A, B \subseteq \Sigma$ we have (i) $A \subseteq A^{\perp\perp}$ and (ii) $A \subseteq B \Rightarrow B^\perp \subseteq A^\perp$; but it implies that $A^{\perp\perp\perp} = A^\perp$, and further that the cartan map $\mu : \mathcal{L} \to \mathcal{P}(\Sigma)$ is surjective onto the biorthogonally closed subsets of Σ. With proposition 1.7 this means that $\mathcal{L}$ is even isomorphic to the lattice of biorthogonally closed subsets of Σ, i.e., those $A \subseteq \Sigma$ for which $A^{\perp\perp} = A$. Using this isomorphism one can show that regarding the map $' : \mathcal{L} \longrightarrow \mathcal{L}$ one has that $a \leq b \Rightarrow b' \leq a'$, also $a \wedge a' = 0$ and finally $a = a''$ — which means precisely that it is an orthocomplementation on $\mathcal{L}$. Finally remark how $\perp$ becomes a symmetric, antireflective relation on Σ that 'separates' the states in the sense that for all $p, q \in \Sigma$ such that $p \neq q$ there exists a $r \in \Sigma$ such that $p \perp r$ and $q \not\perp r$. Indeed, suppose that $\forall r \in \Sigma$, $r \perp p$ implies $r \perp q$, then in fact $p' \leq q'$. But then $q = q'' \leq p'' = p$, so $q = p$ since p is an atom and $q \neq 0$.

All this goes to prove the following.

Theorem 2.4 *For a physical system satisfying the axioms 2.1, 2.2 and 2.3, the cartan map determines an isomorphism between the lattice of properties* $\mathcal{L}$ *and the lattice of biorthogonally closed subsets of* $(\Sigma, \perp)$*, the latter being the set of possible states that comes equipped with a symmetric, antireflective orthogonality relation that separates its elements. In particular, these lattices are complete, atomistic and orthocomplemented.*

3. Categorical equivalence of State and Prop.

I now come to the abstraction, and foremost the categorization, of the foregoing. The whole idea is of course to extend the object-correspondence of theorem 2.4 to a categorical equivalence. This is a question of "finding the good morphisms". I will not dwell too much on the reason for this choice of morphisms, I will impose the definitions and take as justification that with these morphisms one can indeed build the required categorical equivalence. My point of view in this section is "purely mathematical" in the sense that I will state the relevant definitions and give the precise categorical equivalence without their physical connotations. In the next subsection I will then again work with the physical interpretation of this piece of mathematics.

The results presented in this section, in particular theorem 3.9, were first published in (Moore, 1995), but proven in a different manner. Whereas D.J. Moore chose to prove the fully faithfulness and essentially surjectivity of the functor $\mathcal{A}$, presented in lemma 3.8, I give explicitly the natural isomorphisms that constitute the equivalence, cf. lemma 3.5 and 3.6.

The first definition is almost "quoted" from theorem 2.4.

Definition 3.1 *A 'state space' is a couple* $(\Sigma, \perp)$ *where* Σ *is a set, the elements of which are called 'states', and* $\perp$ *is a binary relation, called 'orthogonality', that satisfies for* $s_i, s_j, s_k \in \Sigma$*:*

$$\begin{aligned} s_i \perp s_j &\Rightarrow s_j \perp s_i; \\ s_i \perp s_j &\Rightarrow s_i \neq s_j; \\ s_i \neq s_j &\Rightarrow \exists s_k : s_i \perp s_k, s_j \not\perp s_k. \end{aligned}$$

As examples we can record the following. Every ordinary set is a state space, when equipped with the trivial orthogonality relation (every two distinct elements are orthogonal). Considering a Hilbert space H, its rays form a state space with the usual orthogonality ($[\psi_1] \perp [\psi_2] \Leftrightarrow \langle \psi_1 \mid \psi_2 \rangle = 0$). These are in fact the paradigm examples, in a way that (hopefully) will be clear at the end of section 5: the former is typically the state space of a "classical physical system", the latter that of a "quantum physical system".

For any $S \subseteq \Sigma$ defining as before $S^\perp = \{p \in \Sigma \mid \forall s \in S : p \perp s\}$, we obtain a T_1-closure by biorthocomplement on Σ, given explicitly by

$$\mathcal{C} : \mathcal{P}(\Sigma) \longrightarrow \mathcal{P}(\Sigma) : S \mapsto (S^\perp)^\perp = S^{\perp\perp}. \tag{3}$$

Thus, state spaces form a particular kind of closure spaces, and it is most natural to equip these with partially defined "continuous" morphisms.

Definition 3.2 *A morphism* $f : (\Sigma_1, \perp_1) \to (\Sigma_2, \perp_2)$ *from one state space to another is given by an underlying partially defined map* $f : \Sigma_1 \setminus K_1 \to \Sigma_2$ *such that* $f(\mathcal{C}_1(A) \setminus K_1) \subseteq \mathcal{C}_2(f(A \setminus K_1))$*.*

Of course, in this definition $\mathcal{C}_i$ refers to the closure associated to the state space $(\Sigma_i, \perp_i)$ as prescribed in equation 3. As is well-known, this definition is equivalent to asking that for any biorthogonally closed subset $F \subseteq \Sigma_2$ the set $K_1 \cup f^{-1}(F)$ is biorthogonally closed in Σ_1. The set K_1 on the complement of which a morphism is defined, is called 'kernel'. It is trivial that the kernel of a morphism is biorthogonally closed — simply put $F = \emptyset$.

The composition two such morphisms $f_1 : \Sigma_1 \setminus K_1 \to \Sigma_2$ and $f : \Sigma_2 \setminus K_2 \to \Sigma_3$ yields a morphism $f_2 \circ f_1 : \Sigma_1 \setminus K \to \Sigma_3$, its kernel is given by $K = K_1 \cup f_1^{-1}(K_2)$. Obviously identities are continuous, thus we can define a category **State** of state spaces and their morphisms. The epimorphisms of **State** are precisely the morphisms of which the underlying maps are surjections. The mono's have an injection with empty kernel as underlying map. To have an isomorphic morphism, the underlying map needs to be a bijection with empty kernel that maps a biorthogonally closed subset of its domain onto a biorthogonally closed subset of its codomain — explicitly, with notations as in the definition, $K_1 = \emptyset$, $f : \Sigma_1 \to \Sigma_2$ is a bijection and $f(\mathcal{C}_1(A)) = \mathcal{C}_2(f(A))$. So in this category "iso > mono + epi". The category **State** is of course concrete over **ParSet**, the category of sets and partially defined maps. The forgetful functor has a left adjoint, and on objects it accords to any set the state space with the trivial orthogonality relation. Thus free state spaces are precisely those sets in which any two elements are orthogonal.

Again with one eye on theorem 2.4 we define the following object.

Definition 3.3 *A 'property lattice' is a complete atomistic orthocomplemented lattice* $(\mathcal{L}, \leq, ')$.

Observe that quite trivially, the class of biorthogonally closed subsets $\mathcal{F}(\Sigma) = \{S \subseteq \Sigma \mid S^{\perp\perp} = S\}$ of a state space $(\Sigma, \perp)$ forms such a property lattice. Indeed, the order is given by $\subseteq$, the atoms are the singletons, and an orthocomplement is given by ${}^{\perp} : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma) : F \mapsto F^{\perp}$. Conversely it is true that, for any atomistic lattice $\mathcal{L}$, its set of atoms being denoted by $\mathcal{A}(\mathcal{L})$, the operator

$$\mathcal{C} : \mathcal{P}(\mathcal{A}(\mathcal{L})) \longrightarrow \mathcal{P}(\mathcal{A}(\mathcal{L})) : S \mapsto \{s \in \mathcal{A}(\mathcal{L}) \mid s \leq \bigvee S\} \tag{4}$$

is a T_1-closure; if moreover the lattice is orthocomplemented (it is thus a property lattice) then the closure of equation 4 coincides with the closure by biorthocomplement that one can define (according equation 3) on the following state space:

$$(\mathcal{A}(\mathcal{L}), \perp) \text{ in which } p \perp q \text{ iff } p \leq q'. \tag{5}$$

As we will soon see, and as could be expected from theorem 2.4, it is true that $\Sigma \cong \mathcal{A}(\mathcal{F}(\Sigma))$ and $\mathcal{F}(\mathcal{A}(\mathcal{L})) \cong \mathcal{L}$ (the former being an isomorphism in **State**, the latter an isomorphism of lattices in the usual sense, i.e., an order preserving bijection, which will prove to be an isomorphism in a category of property lattices yet to be defined).

This whole situation resembles very much the more familiar situation of projective geometries: CL.-A. Faure and A. Frölicher (1993) defined a category of 'projective geometries' (= matroids with a modular lattice of fixpoints) with collineations as morphisms, and proved the categorical equivalence with a category of so-called projective lattices, i.e., complete atomistic meet-continuous lattices. (Meet-continuity means that for any directed subset $D \subseteq \mathcal{L}$ — a subset such that for every $x_1, x_2 \in D$ there exists a $x_3 \in D$ such that both $x_1 \leq x_3$ and $x_2 \leq x_3$ — and any $x \in \mathcal{L}$ we have that $x \wedge (\vee D) = \vee(x \wedge D)$ where $x \wedge D = \{x \wedge d \mid d \in D\}$. There are several equivalent expressions.) For an overview, see (Stubbe, 1998). D.J. Moore (1995) "translated" this categorical equivalence to the case of state spaces and property lattices, inheriting the definitions for morphisms.

Definition 3.4 *A morphism* $f : (\mathcal{L}_1, \leq_1, {}'_1) \rightarrow (\mathcal{L}_2, \leq_2, {}'_2)$ *from one property lattice to another is given by an underlying join preserving map* $f : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ *that maps atoms of* $\mathcal{L}_1$ *to either atoms of* $\mathcal{L}_2$ *or its bottom element* 0_2*.*

Hence, in particular, $f(0_1) = 0_2$ by considering the join of the empty set. Of course, composites exist and identity maps meet the conditions of this definition, and we can define a category **Prop** of property lattices and their morphisms. Remark that it is true that bijective, order-preserving maps between property lattices determine the isomorphisms in this category.

And now we are ready to state the categorical equivalence that is the "categorization" of theorem 2.4, in the sense that the object correspondence is an abstraction of the cartan map between states and properties. We will do this with a series of lemmas.

Lemma 3.5 *For any state space* $(\Sigma, \perp)$ *there is an isomorphism*

$$\eta_\Sigma : \Sigma \rightarrow \mathcal{A}(\mathcal{F}(\Sigma)) : x \mapsto \{x\}. \tag{6}$$

Proof : The kernel of η_Σ is trivially empty and bijectiveness of η_Σ is due to the atomisticity of $\mathcal{F}(\Sigma)$. Let $A \subseteq \Sigma$, then:

$$\begin{aligned} \eta_\Sigma(\mathcal{C}(A)) &= \{\eta_\Sigma(a) \mid a \in \mathcal{C}(A)\} \\ &= \{\{a\} \mid a \in \mathcal{C}(A)\} \\ &\stackrel{*}{=} \{\{a\} \mid \{a\} \leq \vee(\eta_\Sigma(A))\} \\ &= \mathcal{C}'(\eta_\Sigma(A)), \end{aligned}$$

where $\mathcal{C}$ denotes the T_1-closure by biorthocomplement on Σ and $\mathcal{C}'$ denotes the T_1-closure on $\mathcal{A}(\mathcal{F}(\Sigma))$ cf. equation 4. In $*$ we used that $\vee(\eta_\Sigma(A)) = \vee\{\eta_\Sigma(a) \mid a \in A\} = \vee\{\{a\} \mid a \in A\} = \mathcal{C}(A)$. $\square$

Lemma 3.6 *For any property lattice* $(\mathcal{L}, \leq, ')$ *there is an isomorphism*

$$\varepsilon_\mathcal{L} : \mathcal{F}(\mathcal{A}(\mathcal{L})) \rightarrow \mathcal{L} : F \mapsto \vee F. \tag{7}$$

Proof : It is obvious that $\varepsilon_\mathcal{L} : \mathcal{F}(\mathcal{A}(\mathcal{L}))$ is a morphism in **Prop**. We'll show that

$$\varepsilon_\mathcal{L}^{-1} : \mathcal{L} \rightarrow \mathcal{F}(\mathcal{A}(\mathcal{L})) : x \mapsto \{a \in \mathcal{A}(\mathcal{L}) \mid a \leq x\}$$

is its (two-sided) inverse. First of all, the map $\varepsilon_\mathcal{L}^{-1}$ is well-defined, because $\forall\ x \in \mathcal{L}$:

$$\begin{aligned} \mathcal{C}(\varepsilon_\mathcal{L}^{-1}(x)) &= \{a \in \mathcal{A}(\mathcal{L}) \mid a \leq \vee\varepsilon_\mathcal{L}^{-1}(x)\} \\ &= \{a \in \mathcal{A}(\mathcal{L}) \mid a \leq x\} \\ &= \varepsilon_\mathcal{L}^{-1}(x), \end{aligned}$$

hence the codomain is indeed $\mathcal{F}(\mathcal{A}(\mathcal{L}))$. Further, it preserves joins, because $\forall\, T \subseteq \mathcal{L}$:

$$\begin{array}{rcl} \varepsilon_{\mathcal{L}}^{-1}(\vee T) & = & \{a \in \mathcal{A}(\mathcal{L}) \mid a \leq \vee T\} \\ & = & \vee(\cup_{t\in T}\{a \in \mathcal{A}(\mathcal{L}) \mid a \leq t\}) \\ & = & \vee \varepsilon_{\mathcal{L}}^{-1}(T). \end{array}$$

And also the condition of "mapping atoms onto atoms or the bottom" is verified, because $\forall\, a \in \mathcal{A}(\mathcal{L})$:

$$\begin{array}{rcl} \mathcal{C}(\varepsilon_{\mathcal{L}}^{-1}(a)) & = & \mathcal{C}(a) \\ & = & \{a\} \\ & \in & \mathcal{A}(\mathcal{F}(\mathcal{A}(\mathcal{L}))). \end{array}$$

By atomisticity of the lattices involved and equation 4 is true that $\varepsilon_{\mathcal{L}}$ is indeed the inverse of $\varepsilon_{\mathcal{L}}^{-1}$. $\square$

Lemma 3.7 *The following actions defines a functor:*

$$\begin{array}{rcl} \mathcal{F} & : & \mathbf{State} \to \mathbf{Prop} \\ & : & \left\{ \begin{array}{lcl} (\Sigma, \perp) & \mapsto & (\mathcal{F}(\Sigma), \subseteq, {}^{\perp}) \\ f : \Sigma_1 \setminus K_1 \to \Sigma_2 & \mapsto & \mathcal{F}(f) : \mathcal{F}(\Sigma_1) \to \mathcal{F}(\Sigma_2) : F \mapsto \mathcal{C}_2(f(F \setminus K_1)) \end{array} \right. \end{array}$$

where $\mathcal{C}_2$ is the closure by biorthocomplement associated to $(\Sigma_2, \perp_2)$.

Proof : Let us first of all verify that the action on morphisms works. The image of a map is join-preserving: Let $\mathcal{T} \subseteq \mathcal{F}(\Sigma_1)$, then

$$\begin{array}{rcl} \mathcal{F}(f)(\vee \mathcal{T}) & = & \mathcal{C}_2(f(\vee \mathcal{T} \setminus E)) \\ & = & \mathcal{C}_2(f(\mathcal{C}_1(\cup \mathcal{T}) \setminus E)) \\ & \stackrel{*}{=} & \mathcal{C}_2(f(\cup \mathcal{T} \setminus E)) \\ & = & \mathcal{C}_2(\cup_{T\in\mathcal{T}} f(T \setminus E)) \\ & = & \mathcal{C}_2(\cup_{T\in\mathcal{T}} \mathcal{C}_2(f(T \setminus E))) \\ & = & \vee_{T\in\mathcal{T}} \mathcal{C}_2(f(T \setminus E)) \\ & = & \vee_{T\in\mathcal{T}} \mathcal{F}(f)(T) \end{array}$$

where $*$ uses for $\subseteq$ the "continuity" of f and for the converse $\supseteq$ that $\cup \mathcal{T} \setminus E \subseteq \mathcal{C}_1(\cup \mathcal{T}) \setminus E$. The image of a map f does indeed maps atoms onto atoms or onto the bottom element: Let $a \in \Sigma_1$, then

$$\begin{array}{rcl} \mathcal{F}(f)(\{a\}) & = & \mathcal{C}_2(f(\{a\} \setminus K_1)) \\ & = & \left\{ \begin{array}{l} \emptyset \text{ if } a \in K_1 \\ \mathcal{C}_2(\{f(a)\}) \stackrel{*}{=} \{f(a)\} \text{ otherwise} \end{array} \right. \end{array}$$

where $*$ uses that $a \in \Sigma_1 \setminus K_1$ implies that $f(a) \in \Sigma_2 \cong \mathcal{A}(\mathcal{F}(\Sigma_2))$.

Now for the functorality of $\mathcal{F}$. Given a composable pair of morphisms $f_1 : \Sigma_1 \setminus K_1 \to \Sigma_2$ and $f : \Sigma_2 \setminus K_2 \to \Sigma_3$ it follows straightforwardly, but with headachy notations, that:

$$\begin{aligned}
(\mathcal{F}(f_2) \circ \mathcal{F}(f_1))(F) &= \mathcal{F}(f_2)(\mathcal{C}_2(f_1(F \setminus K_1))) \\
&= \mathcal{C}_3(f_2(\mathcal{C}_2(f_1(F \setminus K_1)) \setminus K_2)) \\
&\stackrel{*}{=} \mathcal{C}_3(\mathcal{C}_3(f_2(f_1(F \setminus K_1) \setminus K_2))) \\
&= \mathcal{C}_3(f_2(f_1(F \setminus K_1) \setminus K_2)) \\
&\stackrel{**}{=} \mathcal{C}_3(f_2(f_1(F \setminus (K_1 \cup f_1^{-1}(K_2))))) \\
&= \mathcal{F}(f_2 \circ f_1)(F)
\end{aligned}$$

where $*$ uses for $\subseteq$ the continuity of f_2 and $\supseteq$ follows from $f_1(F \setminus K_1) \setminus K_2 \subseteq \mathcal{C}_2(f_1(F \setminus K_1)) \setminus K_2$. For $**$ we need the following reasoning:

$$\begin{aligned}
a \in f_1(F \setminus K_1) \setminus K_2 &\Leftrightarrow \exists b \in F \setminus K_1 : f_1(b) = a, f_1(b) \notin K_2 \\
&\Leftrightarrow \exists b \in F \setminus (K_1 \cup f_1^{-1}(K_2)) : f_1(b) = a \\
&\Leftrightarrow a \in f_1(F \setminus (K_1 \cup f_1^{-1}(K_2))).
\end{aligned}$$

Of course identities are mapped onto identities. □

Lemma 3.8 *The following action defines a functor:*

$$\begin{aligned}
\mathcal{A} &: \mathbf{Prop} \to \mathbf{State} \\
&: \begin{cases} (\mathcal{L}, \leq, ') & \mapsto (\mathcal{A}(\mathcal{L}), \perp) \\ f : \mathcal{L}_1 \to \mathcal{L}_2 & \mapsto \mathcal{A}(f) : \mathcal{A}(\mathcal{L}_1) \setminus K_1 \to \mathcal{A}(\mathcal{L}_2) : s \mapsto f(s) \end{cases}
\end{aligned}$$

where $K_1 = \{a \in \mathcal{A}(\mathcal{L}_1) \mid f(a) = 0\}$*.0*

Proof: Concerning the action on morphisms, let us verify the "continuity" of $\mathcal{A}(f) : \mathcal{A}(\mathcal{L}_1) \setminus K_1 \to \mathcal{A}(\mathcal{L}_2)$. Using equation 4 over and over again, let $A \subseteq \mathcal{A}(\mathcal{L}_1)$:

$$\begin{aligned}
f(\mathcal{C}_1(A) \setminus E) &= f(\{a \in \mathcal{A}(\mathcal{L}_1) \mid a \leq \vee A\} \setminus E) \\
&= \{f(a) \mid a \in \mathcal{A}(\mathcal{L}_1) \setminus E, a \leq \vee A\} \\
&\subseteq \{f(a) \mid a \in \mathcal{A}(\mathcal{L}_1), f(a) \neq 0, f(a) \leq f(\vee A)\} \\
&\subseteq \{f(a) \mid a \in \mathcal{A}(\mathcal{L}_1), f(a) \neq 0, f(a) \leq \vee f(A)\} \\
&\subseteq \{b \in \mathcal{A}(\mathcal{L}_2) \mid b \leq \vee(f(A \setminus E))\} \\
&= \mathcal{C}_2(f(A \setminus E)).
\end{aligned}$$

Functorality is obvious. Remark that for a composable pair of morphisms of complete atomistic lattices $f_1 : \mathcal{L}_1 \to \mathcal{L}_2$ and $f_2 : \mathcal{L}_2 \to \mathcal{L}_3$, then the kernel of $\mathcal{A}(f_2 \circ f_1) = \mathcal{A}(f_2) \circ \mathcal{A}(f_1)$ is given by $K_1 \cup (\mathcal{A}(f_1))^{-1}(K_2)$, where K_i is the kernel of $\mathcal{A}(f_i)$. Preservation of identities is even more obvious. □

And now finally, as a conclusion for this section...

Theorem 3.9 *The isomorphisms of lemmas 3.5 and 3.6 are the components of natural transformations* $1 \cong \mathcal{A} \circ \mathcal{F}$ *and* $\mathcal{F} \circ \mathcal{A} \cong 1$*, which goes to say that* **State** *is categorically equivalent to* **Prop** *by means of these functors* $\mathcal{F}$ *and* $\mathcal{A}$*.*

Proof : Let $f : \mathcal{L}_1 \to \mathcal{L}_2$ be a morphism of complete atomistic lattices, consider$F \in \mathcal{F}(\mathcal{A}(\mathcal{L}_1))$. Then $F = \{a \in \mathcal{A}(\mathcal{L}_1) \mid a \leq \vee F\}$. Observe that

$$\begin{aligned}
(\varepsilon_{\mathcal{L}_2} \circ \mathcal{F}(\mathcal{A}(f)))(F) &= \vee(\mathcal{F}(\mathcal{A}(f))(F)) \\
&= \vee\{a \in \mathcal{A}(\mathcal{L}_2) \mid a \leq \vee(\mathcal{A}(f)(F \setminus K_1))\} \\
&= \vee(\mathcal{A}(f)(F \setminus K_1)) \\
&= \vee\{f(a) \mid a \in F \setminus K_1\} \\
&= \vee\{f(a) \mid a \in F\} \\
&= \vee f(F) \\
&= \vee f(F) \\
&= f(\vee F) \\
&= (f \circ \varepsilon_{\mathcal{L}_1})(F),
\end{aligned}$$

where we have introduced $K_1 = \{a \in \mathcal{L}_1 \mid f(a) = 0\}$, proving exactly the naturality of the transformation $\varepsilon : \mathcal{F} \circ \mathcal{A} \Rightarrow 1$ with components $\varepsilon_{\mathcal{L}} : \mathcal{F}(\mathcal{A}(\mathcal{L})) \xrightarrow{\sim} \mathcal{L}$. Analogously one can verify the naturality of the transformation $\eta : 1 \Rightarrow \mathcal{A} \circ \mathcal{F}$ with components $\eta_X : X \xrightarrow{\sim} \mathcal{A}(\mathcal{F}(X))$. □

4. Two remarks.

The equivalence of **Prop** and **State** displays of course the equivalence of the description of a physical system by means of its properties and its description by means of its states. As the property lattice is constructed from the primitive notion of 'definite experimental project', via the categorical equivalence one could also explain the closure by biorthocomplement on the state space, cf. equation 3, from this primitive notion. This was done in (Aerts, 1994; Valckenborgh, 1997). For completeness' sake I will briefly recall the core of their argumentation. For an $\alpha \in Q$ one writes eig_α('yes') for the set of states $p \in \Sigma$ for which the outcome 'yes' is certain if the test α would be performed on the physical system in state p — read eig_α('yes') as: the set of 'eigenstates' corresponding to the value 'yes' for the experiment α . The notation eig_α('no') is then obvious. The "interaction" with the cartan map, equation 1.7, is that eig_α('yes') $= \mu([\alpha])$; with respect to the product operator Π on Q we can say that $eig_{\Pi A}$('yes') $= \cap_{\alpha \in a} eig_\alpha$('yes'); and for the operator $^\sim$ it is true that eig_α('yes') $= eig_{\alpha^\sim}$('no'). Two states $p, q \in \Sigma$ are then orthogonal if precisely $p \in eig_\alpha$('yes') and $q \in eig_\alpha$('no') for some $\alpha \in Q$. Now it can easily be verified that

$$\mathcal{C} : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma) : S \mapsto \cap\{E \mid E \text{ is an eigentate-set that contains } S\} \tag{8}$$

is a closure operator. The axioms 2.1, 2.2 and 2.3 are then equivalent in this "closure approach" to asking that this "eigenclosure" of equation 8 is T_1 and coincides with the closure by biorthocomplement, cf. equation 3.

Secondly remark that the proofs in the previous section never make explicit use of the fact that a property is orthocomplemented; they rely entirely on the fact that such a lattice is atomistic. Correspondingly, it is not as much that a state space is a set equipped with a particular orthogonality as well that this orthogonality determines a T_1-closure, that plays a crucial role. Indeed, the "same" lemmas 3.5 to 3.8 and the "same" theorem 3.9 prove that the category $\mathbf{T_1Space}$ of T_1-closure spaces and partially defined continuous maps is equivalent to **JCALat**, the category of complete atomistic lattices and join-preserving maps that send atoms onto atoms or the bottom; then **Prop** $\cong$ **State** is a "full subequivalence" of $\mathbf{T_1Space} \cong \mathbf{JCALat}$. This reveals that axiom 2.1 is of a different nature than (the combination of) axioms 2.2 and 2.3: Whereas the former is necessary to ensure that the state space description of a physical entity has exactly "the same content" as its property lattice description, the latter are more of a further specification of that content.

5. A universal construction, and more axioms.

So we have constructed a categorical setting for the foundations of physics. But we shouldn't forget where we came from, and what we're here for: we want to build a theory for the description of physical systems, in particular quantum physical systems. In this section we'll go back to the physical interpretation of the objects of the categories **State** and **Prop**, and we'll see how a universal construction, in itself a mathematical construction, can be understood in the physical context. Notably it is precisely this universal construction that paves the way for the statement and understanding of the axiom for 'quantum entities'.

First a definition, concerning property lattices.

Definition 5.1 *A property $a \in \mathcal{L}$ is called 'classical' if for each atom p either $p \leq a$ or $p \leq a'$.*

In words, this means that for any particular realization of the physical system of which $\mathcal{L}$ is the propositional calculus either the property a is actual or its orthocomplement is actual. In any case, a classical property has a defined value in advance for every realization. Note that 0 and 1 are classical properties for any property lattice: $p \in \mathcal{L} \Rightarrow p \leq 1 = 0'$. The following is a standard result.

Proposition 5.2 *For a state space $(\Sigma, \perp)$ and the corresponding property lattice $(\mathcal{L}, \leq, ')$ the following conditions are equivalent:*

1. *each property $a \in \mathcal{L}$ is classical;*
2. *$\mathcal{L}$ is distributive, hence boolean;*
3. *any two distinct states in Σ are orthogonal.*

With this proposition, we can now identify the axiom for classical physical systems.

Axiom for classical systems. *For every* $a \in \mathcal{L}$*, there exists a definite experimental project* $\alpha \in a$ *such that for any state of the considered physical system either* α *is certain or* $\alpha^{\sim}$ *is certain. (Such a definite experimental project is said to be 'classical'.)*

In words, this axiom asserts that every measurement on the physical system has a certain outcome, which is precisely the case for classical mechanics. If $a \in \mathcal{L}$ is a property that contains a classical experimental project α then $[\alpha^{\sim}] = a'$, hence a is a classical property in the sense of definition 5.1. (But remark that even if a is a classical property, the existence of a classical experimental project $\alpha \in a$ is not guaranteed!) The axiom for classical systems implies axioms 2.1, 2.2 and 2.3, and by theorem 2.4 and proposition 5.2 assures that the property lattice is a boolean algebra, isomorphic to $\mathcal{P}(\Sigma)$. In particular, the propositional calculus of classical mechanics is a boolean algebra.

But this axiom is, as is well-known, not always valid. For instance, in a Stern-Gerlach experiment, if a silver atom enters the apparatus with a spin that is orthogonal to the direction that the apparatus measures, the probability that the silver atom comes out of the apparatus with spin *up* is equal to the probability that it comes out with spin *down*. Thus the definite experimental project that says "Perform the Stern-Gerlach experiment and assign the positive response if the atom comes out with spin *up* and otherwise assign the negative response" is not classical! However, the notion of classical property is still very useful for proving and understanding a universal decomposition for property lattices and state spaces (universal in the categorical sense).

The 'center' Z of a property lattice $\mathcal{L}$ is the set of its classical properties. A property lattice is called 'irreducible' if its only classical properties are its top and bottom element. The following lemma asserts that for any physical system there exists a "macroscopic" description that satisfies the axiom for classicallity.

Lemma 5.3 *Z is a distributive property sublattice of $\mathcal{L}$, with its own atoms that are all of the typical form $\alpha_p = \wedge\{a \in Z \mid p \leq a\}$, and the same top and bottom as $\mathcal{L}$. Further, for every atom α of Z the 'segment' $[0, \alpha] = \{a \in \mathcal{L} \mid a \leq \alpha\}$ is an irreducible property lattice for the induced order and relative orthocomplementation $a^* = a' \wedge \alpha$.*

Proof : It is straightforward that if $a \in \mathcal{L}$ is a classical property, then also a', and given a family $\{a_j \in \mathcal{L} \mid j \in J\}$ of classical properties, the meet $\bigwedge\{a_j \in \mathcal{L} \mid j \in J\}$ is also classical. Which already shows that Z is a complete lattice with orthocomplements. One now straightforwardly verifies that the elements of the form $\alpha_p = \wedge\{a \in Z \mid p \leq a\}$ are indeed exactly the atoms of Z, and for every classical property a one has that $a = \bigvee\{\alpha_p \mid \alpha_p \leq a\}$. Thus Z is atomistic, and by proposition 5.2 it is distributive. For the second part of the lemma, one must verify that for the segments, which are trivially complete atomistic lattices, the given formula $a^* = a' \wedge \alpha$ indeed defines an orthocomplementation. Once it is thus achieved that al these segments are property lattices, it is trivial that they are irreducible. □

The set of atoms of Z will be denoted by Ω — thus $\Omega = \mathcal{A}(Z)$. As $\Sigma = \mathcal{A}(\mathcal{L})$ is the state space associated to the property lattice $\mathcal{L}$, to the irreducible property lattice $[0, \alpha]$, for any $\alpha \in \Omega$, one can associate a state space $\Sigma_\alpha = \mathcal{A}([0, \alpha]) = \{p \in \Sigma \mid p \leq \alpha\}$. It is then at once clear that $\Sigma = \cup_{\alpha\in\Omega}\Sigma_\alpha$, this union in fact being disjoint. (From here on it will always be assumed that the index α ranges over Ω.) And by straightforward calculations one can show for $p, q \in \Sigma$ that $p \perp q$ if either $p \in \Sigma_{\alpha_1}$ and $q \in \Sigma_{\alpha_2}$ for $\alpha_1 \neq \alpha_2$ or $p \perp_\alpha q$ in case that $p, q \in \Sigma_\alpha$. In fact the following can be shown (Moore, 1995).

Proposition 5.4 *In the category* **State***,* $(\Sigma, \perp)$ *is the coproduct of the* $(\Sigma_\alpha, \perp_\alpha)$*.*

Proof : Using that A is biorthogonal in Σ if and only if each $A_\beta = A \cap \Sigma_\beta$ is biorthogonal in Σ_β, it is clear that the "inclusions"

$$s_\beta : \Sigma_\beta \hookrightarrow \Sigma : p \mapsto p \tag{9}$$

are "continuous" morphisms with empty kernel, thus constituting that Σ is a cocone of the Σ_β. For any other cocone diagram $s'_\beta : \Sigma_\beta \setminus K_\beta \to \Sigma'$ the map $f : \Sigma \setminus K \to \Sigma'$ with kernel $K = \cup_\beta K_\beta$ that maps $p \in (\Sigma_\beta \setminus K_\beta) \subseteq (\Sigma \setminus K)$ onto $s'_\beta(p)$ is the unique factorization of the s'_β over Σ in **State**. □

Through the categorical equivalence of **State** and **Prop**, see theorem 3.9, this yields that $\mathcal{L} \cong \amalg_\alpha[0, \alpha]$. In fact, given a family of property lattices $\mathcal{L}_i$, its coproduct $\amalg_i\mathcal{L}_i$ can be defined to be the cartesian product $\times_i\mathcal{L}_i$ with componentwise order, meet and join, and orthocomplement. The inclusion morphisms are then

$$s_j : \mathcal{L}_j \to \times_i\mathcal{L}_i : x \to (x_i) \text{ where } \begin{cases} x_j = x \\ x_{i\neq j} = 0 \end{cases} \tag{10}$$

and for any cocone $s'_j : \mathcal{L}_j \to \mathcal{L}'$ the unique factorization over $\times_i\mathcal{L}_i$ is $f : \times_i\mathcal{L}_i \to \mathcal{L}' : (x_i) \mapsto \bigvee\{s'_i(x_i)\}$. As can be verified, all the morphisms involved are indeed join-preserving and mapping atoms onto atoms or the bottom element. Applying this to the decomposition of a given property lattice into its irreducible components, we can explicitly write down that

$$f : \amalg_\alpha[0, \alpha] \xrightarrow{\sim} \mathcal{L} : (x_\alpha) \mapsto \bigvee_\alpha\{x_\alpha\} \tag{11}$$

is an isomorphism of lattices, with inverse

$$f^{-1} : \mathcal{L} \xrightarrow{\sim} \amalg_\alpha[0, \alpha] : x \mapsto (x_\alpha) \text{ where } x_\alpha = \bigvee\{p \in \Sigma_\alpha \mid p \leq x\}. \tag{12}$$

A short remark is in order here. As **Prop** is a subcategory of **JCLat** — the category of complete lattices and join-preserving maps — and the latter is a category with biproducts, the obvious candidate for a product in the former would be the coproduct. But in **JCLat** it so happens that the unique factorization of a cone over the (bi)product is a join-preserving map that needn't send atoms onto atoms or the bottom element, hence such a factorization is not an element of **Prop**. Notably, it is the fact that **Prop** is not a quantaloid (there is no join of morphisms) that causes this "shortcoming".

The atoms of Z are called "superselection rules" for historical reasons, and whenever $p, q \in \Sigma$ are such that $p \in \Sigma_{\alpha_1}$ and $q \in \Sigma_{\alpha_2}$ for $\alpha_1 \neq \alpha_2$, these states are said "to be separated by a superselection rule". The physical interpretation is the following: if an experimentator only bothers for the classical experiments that are relevant for the physical system that he studies, he will of course obtain a classical description of that system, which is then precisely Z. This will describe the "macroscopical" appearance of that physical system. But in doing so, atoms p, q of $\mathcal{L}$ that determine the same atom of Z, $\alpha_p = \alpha_q$, are identified: the experimenter cannot distinguish all "microscopical" realizations of the physical system. This point of view reveals that in any case a classical description of a physical system is possible, but may nevertheless be only a "first order approximation" of the real situation. In other words, every physical system has classical degrees of freedom and quantal degrees of freedom.

Now we are ready to give the axiom that we need on top of axioms 2.1, 2.2 and 2.3 for the identification of quantum systems, more precisely the Hilbert space quantum mechanics. With the benefit of hindsight — see property 5.5 — we state the following.

Axiom for quantum entities. *The property lattice* $(\mathcal{L}, \leq, \,')$ *satisfies:*

1. *weak modularity: if* $a, b \in \mathcal{L}, a \leq b$ *then* $a \vee (b \wedge a') = b$*;*
2. *covering law: if* $a \in \mathcal{L},\ p \in \mathcal{A}(\mathcal{L})$ *and* $a \wedge p = 0$ *then* $a \leq x \leq (a \vee p) \Rightarrow x = a$ *or* $x = a \vee p$*.*

Is the above axiom verified by every physical system in quantum physics? The answer is no: the so-called "composite of two separated quantum systems" can never satisfy either the weak modularity nor the covering law unless at least one of the composites is in fact classical, as proven by D. Aerts (1982). The reason for this is in fact that the axiom for quantum systems imposes that it is impossible to have states that are separated by a superselection rule and are not orthogonal, which is exactly what happens in the case of the separated systems. More explicitly, let $(\Sigma_1, \perp_1)$ and $(\Sigma_2, \perp_2)$ be state spaces, associated respectively to property lattices $\mathcal{L}_1$ and $\mathcal{L}_2$, then define the "Aerts-productspace" $(\Sigma, \perp)$ by setting Σ as cartesian product of the Σ_i and $(s_1, s_2) \perp (t_1, t_2)$ if $s_i \perp_i t_i$ for at least one $i \in \{1, 2\}$. The property lattice $\mathcal{L}$ associated to $(\Sigma, \perp)$ then verifies the quantum axiom above if at most one $\mathcal{L}_i$ does. Aerts showed that this situation occurs exactly if the property lattice $\mathcal{L}$ describes the compound physical system formed by the two subsystems that have as description respectively $\mathcal{L}_1$ and $\mathcal{L}_2$, in the case that these components are "separated". This means, in three words, that there are experimental projects on the compound system that consist in performing simultaneously an experimental project α_1 on the first component and an experimental project α_2 on the second, with the extra feature that the outcome of α_1 (resp. α_2) is by no means affected by the simultaneous performance of α_2 (resp. α_1). Nevertheless, the axiom is true for all known 'quantum entities', i.e., quantum systems on which the experimental projects act *as a whole*. These are also known under the name of 'indivisible' quantum systems, 'simple systems', or 'particles'.

By an 'orthomodular lattice' is meant a lattice that is orthocomplemented and weakly modular. A proof of the following can be found in (Piron, 1976).

Proposition 5.5 *The irreducible components of any complete orthomodular atomistic lattice satisfying the covering law are themselves complete orthomodular atomistic lattices that satisfy the covering law. Any irreducible such lattice of rank at least 4 is canonically isomorphic to the lattice of all closed subspaces of a "generalized Hilbert space".*

A generalized Hilbert space $(V, F, {}^*, \langle\ |\ \rangle)$ consists, by definition, of the following data: a vectorspace $(V, +)$ over a field $(F, +, \cdot)$, an involutive anti-automorphism ${}^* : F \to F$ – which means that $(\alpha + \beta)^* = \alpha^* + \beta^*$, $(\alpha \cdot \beta)^* = \beta^* \cdot \alpha^*$ and $(\alpha^*)^* = \alpha$ – and a definite Hermitian form $\langle\ |\ \rangle : V \times V \to F$ – which means that $\langle f + \alpha \cdot g \mid h\rangle = \langle f \mid h\rangle + \alpha \cdot \langle g \mid h\rangle$, $\langle f \mid g\rangle^* = \langle g \mid f\rangle$ and $\langle f \mid f\rangle = 0 \Rightarrow f = 0$; and one asks that the following "orthomodularity axiom" holds: for $S \subseteq V$, if $S^{\perp\perp} = S$ then $S + S^{\perp} = V$ (where, of course, $S^{\perp} = \{f \in V \mid \forall g \in S : \langle f, g\rangle = 0\}$).

Just to give an idea of the construction which accomplishes the proof of theorem 5.5: For an irreducible complete orthomodular atomistic lattice $\mathcal{L}$ satisfying the covering law, the atoms $\Sigma = \mathcal{A}(\mathcal{L})$ define an irreducible projective geometry, in which a point is an element of Σ (a state of the quantum system) and a line defined by two different points p, q is the subset of states contained in the join $p \vee q$. It was proven by C. Piron (1976) that for an $\mathcal{L}$ of this kind, of rank at least 4, for the corresponding irreducible projective geometry $\mathcal{A}(\mathcal{L})$ there exists a field F with involution * and there exists a vector space V over this field with a Hermitian form $\langle\ |\ \rangle$ such that $\mathcal{L}$ is isomorphic to the closed subspaces of V — it is the orthocomplementation of $\mathcal{L}$ that is carried over to the vector space realization of the projective geometry to define the definite Hermitian form. (Piron's theorem generalizes the "same" result that was proven by Birkhoff and von Neumann (1936) for the case where $\mathcal{L}$ is of *finite* rank.)

In fact, as is proven in (Amemiya and Araki, 1967), if the field is the real, complex of quaternionic field equipped with the respective standard involution, then the generalized Hilbert space can only be a "genuine" Hilbert space — this justifies the terminology. It can be shown (Solèr, 1995) that this is the case if the vector space (V, F) contains an orthonormal sequence $(f_i)_{i\in\mathbb{N}}$ — orthonormal with respect to the Hermitian form, of course.

And now we come to the highlight of this chapter. Putting all pieces of the puzzle together, we can summarize our *démarches* in the following celebrated representation theorem due to C. Piron.

Theorem 5.6 *For any – sufficiently large – propositional system satisfying axioms 2.1, 2.2 and 2.3 and the axiom for quantum systems, the state space can be canonically realized as a family of generalized Hilbert spaces indexed by the set of superselection rules.*

This does not only motivate the habitual use of Hilbert spaces to describe simple systems in quantum mechanics, it also affirms the connection between quantum mechanics and projective geometry as put forward in (Birkhoff and von Neumann, 1936) — cf. the opening paragraph of this chapter. Moreover, the axioms 2.1, 2.2 and 2.3 plus the axiom for quantum systems have been shown to be consistent in the sense that a model has been constructed in which the definite experimental projects, that are after all the beginning of the whole story, are represented by certain operators on a Hilbert space (Cattaneo and Nisticó, 1991).

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