Towards "dynamic domains": totally continuous cocomplete $\mathcal{Q}$-categories

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Abstract

It is common practice in both theoretical computer science and theoretical physics to describe the (static) logic of a system by means of a complete lattice. When formalizing the dynamics of such a system, the updates of that system organize themselves quite naturally in a quantale, or more generally, a quantaloid. In fact, we are lead to consider cocomplete quantaloid-enriched categories as fundamental mathematical structure for a dynamic logic common to both computer science and physics. Here we explain the theory of totally continuous cocomplete categories as generalization of the well-known theory of totally continuous suplattices. That is to say, we undertake some first steps towards a theory of "dynamic domains".

Key words: Quantaloid-enriched category, quantaloid-module, projectivity, small-projectivity, complete distributivity, total continuity, total algebraicity, dynamic domain, dynamic logic.

1 Introduction

Towards "dynamic domains".

It is common practice in both theoretical computer science and theoretical physics to describe the 'properties' of a 'system' by means of a complete lattice $\mathcal{L}$; this lattice is then thought of as the logic of the system. For example, the lattice of closed subspaces of a Hilbert space is the logic of properties of a quantum system; and, in computer science, a domain is the logics of observables of a computational system.

More recently, also another ordered structure has been recognized to play an important rôle in both physics and computer science: when formalizing

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the dynamics of a physical or computational system, it turns out that the 'updates' of a system – think of them as programs for a computational system, and property transitions for a physical system – organize themselves quite naturally in a quantale $\mathcal{Q}$ [Abramsky and Vickers, 1993; Coecke and Stubbe, 1999].

Having a complete lattice $\mathcal{L}$ of properties of a system and a quantale $\mathcal{Q}$ of updates, we give an operational meaning to each $f \in \mathcal{Q}$ by the so-called Principle of Causal Duality (explained in detail in [Stubbe, 2002] but going back to [Floyd, 1967; Hoare, 1969] for computational systems and [Coecke, Moore and Stubbe, 2001] for physical systems): we want every $f \in \mathcal{Q}$ to determine an adjoint pair of order-preserving morphisms $f^* \dashv f_*\colon \mathcal{L} \leftrightarrows \mathcal{L}$. So the left adjoint assigns to a given input $a \in \mathcal{L}$ its strongest consequence $f^*(a) \in \mathcal{L}$ under the action of f ('strongest postcondition'), and the right adjoint assigns to a given output $b \in \mathcal{L}$ the weakest cause $f_*(b) \in \mathcal{L}$ under the action of f ('weakest precondition'). Moreover we ask that $(g \circ f)^* = g^* \circ f^*$, $1^* = 1_{\mathcal{L}}$ and $(\bigvee_i f_i)^* = \bigvee_i f_i^*$ for every $f, g, (f_i)_i \in \mathcal{Q}$ (and $1 \in \mathcal{Q}$ is the unit for the monoid structure of $\mathcal{Q}$).

In fact, a complete lattice $\mathcal{L}$ and a quantale $\mathcal{Q}$ linked by the Principle of Causal Duality, tangle up in one simple mathematical structure: a cocomplete $\mathcal{Q}$-enriched category. Indeed, putting $\mathbb{A}_0 = \mathcal{L}$ as set of objects, the mapping

$$\mathbb{A}(-,-)\colon \mathbb{A}_0 \times \mathbb{A}_0 \longrightarrow \mathcal{Q}\colon (a,b) \mapsto \bigvee\{f \in \mathcal{Q} \mid f^*(a) \leq b\}$$

endowes $\mathbb{A}_0 = \mathcal{L}$ with a "$\mathcal{Q}$-valued implication" [Lawvere, 1973]: for $a, b \in \mathbb{A}_0 = \mathcal{L}$, the element $\mathbb{A}(a,b) \in \mathcal{Q}$ is the weakest (i.e. least deterministic) update that, for input a, guarantees output b. This in fact turns $\mathbb{A}$ into a $\mathcal{Q}$-enriched category. This $\mathcal{Q}$-category is tensored and cotensored due to the Principle of Causal Duality; and the underlying order of this $\mathcal{Q}$-category $\mathbb{A}$ being a suplattice, namely $\mathcal{L}$, implies together with the tensors and cotensors that $\mathbb{A}$ is cocomplete.

So, conclusively, we are lead to consider cocomplete $\mathcal{Q}$-categories as crucial mathematical structure in a *dynamic logic* as common mathematical foundation for dynamic phenomena in both computer science and physics. We will allow $\mathcal{Q}$ to be a quantaloid rather than a quantale, for this extra generality (allowing a 'typed dynamics') doesn't really complicate matters—even though one has to bring in some adjustments to pass from enrichement in a monoidal category (i.e. bicategory with one object) to enrichment in a bicategory (with possibly many objects). For the basic theory of $\mathcal{Q}$-enriched categorical structures, see [Stubbe 2004, 2005a, 2005b]; we keep all the notations introduced there. Those works contain the more "historical" references on the theory of quantaloid-enriched categories.

Our notation for the 2-category of $\mathcal{Q}$-categories and functors is $\mathsf{Cat}(\mathcal{Q})$; and further on $\mathsf{Cocont}(\mathcal{Q})$ denotes the 2-category of cocomplete $\mathcal{Q}$-categories and cocontinuous functors.

Modules or cocomplete categories?

There is an alternative and probably better known way of coupling a complete lattice $\mathcal{L}$ (static properties of some system) with a quantale $\mathcal{Q}$ (dynamics of that system): namely, by means of an action of the latter on the former. Such is a morphism $\alpha: \mathcal{L} \otimes \mathcal{Q} \longrightarrow \mathcal{L}$ in Sup, the category of suplattices and supmorphisms (i.e. complete lattices and mappings that preserve arbitrary suprema), satisfying axioms on the compatibility with the monoid structure of $\mathcal{Q}$. Then $\mathcal{L}$ is said to be a (right) $\mathcal{Q}$-module, and with the obvious notion of homomorphism between such modules over a fixed $\mathcal{Q}$, one obtains a (2-)category of $\mathcal{Q}$-modules.

Abramsky and Vickers [1993] (but see also [Resende, 2000] for a survey) apply the theory of $\mathcal{Q}$-modules to process semantics: taking into account that an informatic system may be affected by the way in which it is observed, they argue that the observable properties of an informatic system form a quantale (or even a quantaloid), and a module is then viewed as a generalization of a labelled transition system. Also in [Baltag *et al.*, 2004], modules on a quantale are used to cope with dynamic phenoma in computer science, in particular, to provide an algebraic semantics for epistemic actions and updates.

However, the (2-)category of modules on a quantaloid $\mathcal{Q}$ is (bi)equivalent to the (2-)category $\mathsf{Cocont}(\mathcal{Q})$ of cocomplete $\mathcal{Q}$-categories (see [Stubbe, 2004] for details)! Our explicit choice to work with cocomplete $\mathcal{Q}$-enriched categories rather than $\mathcal{Q}$-modules, even though they are mathematically equivalent structures, reflects a simple yet powerful idea: we explicitly put ourselves in the context of a *logic with truth values in* $\mathcal{Q}$ within which we develop our mathematics. The claim in this paper is then that, even in this universe of discourse gouverned by such a “dynamic logic”, it is possible to develop (a strong variant of) domain theory. And it is precisely because we have chosen to work with cocomplete $\mathcal{Q}$-categories instead of $\mathcal{Q}$-modules, that our presentation is so naturally a generalization of the (“classical”) results. (In section 8 we shall discuss the meaning of our results for module theory though.)

Totally continuous suplattices.

Suplattices are of course examples of cocomplete quantaloid-enriched categories: consider the two-element Boolean algebra $\mathbf{2}$ as a one-object quantaloid, then Sup is (biequivalent to) $\mathsf{Cocont}(\mathbf{2})$. That is to say, suplattices are dynamic logics... with a trivial dynamics! Given the importance of totally continuous suplattices in computer science (as a particular kind of domain), it is natural to ask in how far the “classical” theory of totally continuous suplattices generalizes to $\mathsf{Cocont}(\mathcal{Q})$. This presentation is all about giving an answer to that question. So let us first quickly recall the basics of the theory of totally continuous suplattices.

On any suplattice L one may define the so-called “way-below” relation: say that a is way-below b, and write $a \ll b$, when for every *directed* downset $D \subseteq L$, $b \leq \bigvee D$ implies $a \in D$. A suplattice is said to be continuous when every element is the supremum of all elements way-below it. The theory of continuous suplattices has connections with topology and analysis (as the adjective "continuous" would suggest), and applications in computer science (since they are examples of "domains"). The classical reference is [Gierz *et al.*, 1980].

As a (stronger) variant of the above, one may also define the "totally-below" relation on a suplattice L: say that a is totally-below b, and write $a \lll b$, when for *any* downset $D \subseteq L$, $b \leq \bigvee D$ implies $a \in D$. Of course L is now said to be totally continuous when every element is the supremum of all elements totally-below it; in this case L is also continuous. Our main reference on this subject is [Rosebrugh and Wood, 1994]. Let us recall some of the features of these structures.

(a) A suplattice L is totally continuous if and only if any supmorphism $f\colon L \longrightarrow M$ factors through any surjective supmorphism $g\colon K \twoheadrightarrow M$. This gives the totally continuous suplattices a universal status within the quantaloid Sup: they are precisely its projective objects.

(b) Totally continuous suplattices are precisely those suplattices for which the map sending a downset to its supremum has a left adjoint: the left adjoint to $\bigvee\colon \mathsf{Dwn}(L) \longrightarrow L\colon D \mapsto \bigvee D$ is namely the map $a \mapsto \{x \in L \mid x \lll a\}$. In other words, the supremum-map is required to preserve all infima; and so such a suplattice is also said to be completely distributive[2].

(c) The totally-below relation on a totally continuous suplattice is idempotent. Conversely, given a set equipped with an idempotent binary relation $(X, \prec)$, the subsets $S \subseteq X$ such that $x \in S$ if and only if there exists a $y \in S$ such that $x \prec y$, form a totally continuous suplattice. This correspondence underlies the 2-equivalence of the split-idempotent completion of Rel (whose objects are thus idempotent relations) and the full subcategory of Sup determined by the totally continuous suplattices.

(d) Given any ordered set $(X, \leq)$, the construction in (c) implies that $\mathsf{Dwn}(X)$ is a totally continuous suplattice. But it distinguishes itself in that every element of $\mathsf{Dwn}(X)$ is the supremum of "totally compact elements", i.e. elements that are totally below themselves. Such a suplattice is said to be totally algebraic; and in fact all totally algebraic suplattices are of the form $\mathsf{Dwn}(X)$ for some ordered set $(X, \leq)$. This correspondence underlies the 2-equivalence of the split-monad completion of Rel (whose objects are thus orders) and the full subcategory of Sup determined by the totally algebraic suplattices.

[2] [Rosebrugh and Wood, 1994] study precisely this notion under the name of *constructive* complete distributivity for suplattices in a topos $\mathcal{E}$. [Fawcett and Wood, 1990] prove that, when working with suplattices in Set (and thus disposing of the axiom of choice), this constructive complete distributivity coincides with complete distributivity in the usual sense of the word. See also [Wood, 2004].

Totally continuous cocomplete $\mathcal{Q}$-categories.

In how far does the "classical" theory of totally continuous suplattices generalize to $\mathsf{Cocont}(\mathcal{Q})$, the category of cocomplete $\mathcal{Q}$-enriched categories? The following answer is a combination of 3.1, 4.4, 5.1 and 6.4 below.

Theorem 1.1 *For a cocomplete $\mathcal{Q}$-category $\mathbb{A}$, the following are equivalent:*

(i) *$\mathbb{A}$ is projective in $\mathsf{Cocont}(\mathcal{Q})$,*

(ii) *$\mathbb{A}$ is completely distributive,*

(iii) *$\mathbb{A}$ is totally continuous,*

(iv) *$\mathbb{A} \simeq \mathcal{R}\mathbb{B}$ for some regular $\mathcal{Q}$-semicategory $\mathbb{B}$.*

And, as particular case of the above, the following are equivalent:

(i) *$\mathbb{A}$ is totally algebraic,*

(ii) *$\mathbb{A} \simeq \mathcal{P}\mathbb{C}$ for some $\mathcal{Q}$-category $\mathbb{C}$.*

Therefore, denoting $\mathsf{Cocont}_{\mathsf{tc}}(\mathcal{Q})$, respectively $\mathsf{Cocont}_{\mathsf{ta}}(\mathcal{Q})$, for the full sub-2-category of $\mathsf{Cocont}(\mathcal{Q})$ determined by its totally continuous objects, respectively totally algebraic objects, the following diagram, in which the horizontal equalities are biequivalences (corestrictions of the local equivalences encountered in (2) and (3) further on), and the vertical arrows are full 2-inclusions, commutes:

$$\begin{array}{ccc}
\mathsf{RSDist}(\mathcal{Q}) & = & \mathsf{Cocont}_{\mathsf{tc}}(\mathcal{Q}) \\
\uparrow & & \uparrow \\
\mathsf{Dist}(\mathcal{Q}) & = & \mathsf{Cocont}_{\mathsf{ta}}(\mathcal{Q})
\end{array}$$

That is to say, the crucial aspects of the theory of totally continuous suplattices recalled above all generalize neatly to cocomplete $\mathcal{Q}$-categories: it is possible to make sense of such notions as 'projectivity', 'complete distributivity', 'total continuity' and 'total algebraicity' in the context of cocomplete $\mathcal{Q}$-categories.

In the context of theoretical computer science, [Abramsky and Jung, 1994] argue that a mathematical structure deserves to be called a "domain" when it is an algebraic structure that unites aspects of convergence and of approximation. A totally continuous cocomplete $\mathcal{Q}$-category does exactly that: it is cocomplete ("every presheaf converges") and is equipped with a well-behaved totally-below relation ("approximations from below"). The above results may then be "translated" into the domain theoretic lingo. For example, in section 5 domain theorists will recognize the construction of bases: 5.1 could be read as saying that " a cocomplete $\mathbb{A}$ is a domain if and only if it has a basis $\mathbb{B}$". So this work really has the flavour of "quantaloid-enriched domain theory"—or "dynamic domains".

Related work and future projects.

Clearly, totally continuous cocomplete $\mathcal{Q}$-categories are very strong structures; in particular can one argue that, having abandonned the notion of "directedness", their usefulness in computation is rather limited. So it is definitely an interesting project to investigate how a notion of "directedness" can be brought back in again. Certainly, other categorical generalizations of domain theory, in particular [Adámek and Rosický, 1994; Adámek, 1997], may be very inspiring; our difficulty here, however, is that we need to generalize a notion such as "directed (or filtered) colimit" to the case of categories enriched in a quantaloid. (But it seems that Gordon and Power [1997] and also Kelly and Schmitt [2005] have ideas on that subject that will get us on track.) By the way, remark that – precisely because we have chosen to work with the formalism of cocomplete $\mathcal{Q}$-categories rather than $\mathcal{Q}$-modules – we have a lot of ideas and techniques from (enriched) category theory that we can try to adapt to the situation at hand!

Another closely related, but at the same time very different work, is that of Wagner [1997]. Indeed, he unifies notions of "liminf convergence" in orders and metric spaces – and thus gives one setting for treating recursive domain equations by a generalized inverse limit theorem à la Scott – by means of categories enriched in a quantale. However, this base quantale is supposed to be *commutative* and its *top element* is supposed to be the *unit* for its multiplication. These very strong assumptions, especially the commutativity, are precisely what we want to avoid in our work: for we believe that it is an essential feature of a "dynamic logic" that its truth values (the possible updates of a system that constitute its dynamics) do not commute!

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2 Projective cocomplete $\mathcal{Q}$-categories

The forgetful 2-functor $\mathcal{U}\colon \mathsf{Cocont}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ admits a left 2-adjoint: the free cocompletion of a $\mathcal{Q}$-category $\mathbb{A}$ is the presheaf category $\mathcal{P}\mathbb{A}$. By a *free object* in $\mathsf{Cocont}(\mathcal{Q})$ we will mean a free object relative to the forgetful functor $\mathcal{U}$, i.e. an object equivalent to the presheaf category $\mathcal{P}\mathbb{A}$ on some $\mathcal{Q}$-category $\mathbb{A}$.

In fact, the free 2-functor $\mathcal{P}\colon \mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cocont}(\mathcal{Q})$ is the composition of two 2-functors. First every functor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ induces a left adjoint distributor (the "graph" of F),

(1) $$\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Dist}(\mathcal{Q})\colon \Big(F\colon\mathbb{A}\longrightarrow\mathbb{B}\Big)\mapsto\Big(\mathbb{B}(-,F-)\colon\mathbb{A}\longrightarrow\mathbb{B}\Big).$$

Then every distributor determines a cocontinuous functor between presheaf categories,

(2) $$\mathsf{Dist}(\mathcal{Q})\longrightarrow\mathsf{Cocont}(\mathcal{Q})\colon \Big(\Phi\colon\mathbb{A}\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{B}\Big)\mapsto\Big(\Phi\otimes-\colon\mathcal{P}\mathbb{A}\longrightarrow\mathcal{P}\mathbb{B}\Big).$$

The latter is locally an equivalence (actually, locally an isomorphism since $\mathsf{Dist}(\mathcal{Q})$ is a quantaloid and each $\mathcal{P}\mathbb{B}$ is skeletal). There are more details in [Stubbe, 2005a, **3.7, 6.12**].

The adjunction $\mathcal{P}\dashv\mathcal{U}$ works as follows: a functor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ from any $\mathcal{Q}$-category into a cocomplete $\mathcal{Q}$-category determines a cocontinuous functor $\langle F,Y_{\mathbb{A}}\rangle\colon\mathcal{P}\mathbb{A}\longrightarrow\mathbb{B}$ by (pointwise) left Kan extension of F along the Yoneda embedding for $\mathbb{A}$; and a cocontinuous functor $G\colon\mathcal{P}\mathbb{A}\longrightarrow\mathbb{B}$ into a cocomplete $\mathcal{Q}$-category determines a functor $G\circ Y_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathbb{B}$ by composition with the Yoneda embedding. In other words, for an $\mathbb{A}\in\mathsf{Cat}(\mathcal{Q})$, the Yoneda embedding $Y_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathcal{P}\mathbb{A}$ gives the unit of the adjunction; and for some $\mathbb{B}\in\mathsf{Cocont}(\mathcal{Q})$, the left Kan extension $\langle 1_{\mathbb{B}},Y_{\mathbb{B}}\rangle\colon\mathcal{P}\mathbb{B}\longrightarrow\mathbb{B}$ gives the counit. The latter sends a presheaf $\phi\in\mathcal{P}\mathbb{B}$ to the colimit $\mathrm{colim}(\phi,1_{\mathbb{B}})$, and will be denoted from now on as $\mathsf{sup}_{\mathbb{B}}\colon\mathcal{P}\mathbb{B}\longrightarrow\mathbb{B}$ (for "supremum" of course). Actually, $\mathsf{sup}_{\mathbb{B}}$ is left adjoint to $Y_{\mathbb{B}}$ in $\mathsf{Cat}(\mathcal{Q})$; since the latter is fully faithful, the former is surjective. We refer to [Stubbe, 2005a, sections 5 and 6] for details.

A *projective object* $\mathbb{A}$ in $\mathsf{Cocont}(\mathcal{Q})$ is one such that any arrow $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ factors (up to local isomorphism) through any surjection [3] $G\colon\mathbb{C}\twoheadrightarrow\mathbb{B}$. This definition is classical for ordinary categories [4], and it will come as no surprise that one can prove that (i) the retract of a projective object in $\mathsf{Cocont}(\mathcal{Q})$ is again projective, and (ii) free objects in $\mathsf{Cocont}(\mathcal{Q})$ are projective. It follows that $\mathsf{Cocont}(\mathcal{Q})$ *has enough projectives*, i.e. that every object in $\mathsf{Cocont}(\mathcal{Q})$ is the quotient of a projective object: there is always the surjection $\mathsf{sup}_{\mathbb{A}}\colon\mathcal{P}\mathbb{A}\twoheadrightarrow\mathbb{A}$.

Proposition 2.1 *For a cocomplete $\mathcal{Q}$-category $\mathbb{A}$, the following are equivalent:*

(i) *$\mathbb{A}$ is a projective object in $\mathsf{Cocont}(\mathcal{Q})$,*

(ii) *$\mathsf{sup}_{\mathbb{A}}\colon\mathcal{P}\mathbb{A}\twoheadrightarrow\mathbb{A}$ has a section in $\mathsf{Cocont}(\mathcal{Q})$,*

(iii) *$\mathbb{A}$ is a retract of $\mathcal{P}\mathbb{A}$ in $\mathsf{Cocont}(\mathcal{Q})$,*

(iv) *$\mathbb{A}$ is a retract of a free object in $\mathsf{Cocont}(\mathcal{Q})$.*

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[3] All epimorphisms in $\mathsf{Cocont}(\mathcal{Q})$ are regular, and turn out to be precisely those functors which are *essentially surjective on objects*; therefore we speak of *surjections* when we mean epimorphisms in $\mathsf{Cocont}(\mathcal{Q})$.

[4] Usually one defines "projectivity" with respect to a preferred class of epimorphisms, giving rise to "regular projectivity", "strong projectivity", and whatnot. But every epimorphism in $\mathsf{Cocont}(\mathcal{Q})$ is regular, so we speak of "projectivity" *tout court*. See also section 8.

Proof : If $\mathbb{A}$ is a projective object in $\mathsf{Cocont}(\mathcal{Q})$, then there must be a factorization of $1_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathbb{A}$ through the surjection $\mathsf{sup}_{\mathbb{A}}\colon\mathcal{P}\mathbb{A}\twoheadrightarrow\mathbb{A}$. This proves that $\mathbb{A}$ is a retract of the free object $\mathcal{P}\mathbb{A}$. The remainder of the proof follows from the observations (i) and (ii) above. $\square$

3 Completely distributive cocomplete $\mathcal{Q}$-categories

A *(constructively*[5] *) completely distributive* cocomplete $\mathcal{Q}$-category $\mathbb{A}$ is one for which the left adjoint to the Yoneda embedding, $\mathsf{sup}_{\mathbb{A}}\colon\mathcal{P}\mathbb{A}\twoheadrightarrow\mathbb{A}$, has a further left adjoint. The terminology is classical for $\mathcal{Q}=\mathbf{2}$, i.e. for suplattices [Rosebrugh and Wood, 1994].

Proposition 3.1 *For a cocomplete $\mathcal{Q}$-category $\mathbb{A}$, the following are equivalent:*

(i) *$\mathbb{A}$ is completely distributive,*

(ii) *$\mathbb{A}$ is a projective object in $\mathsf{Cocont}(\mathcal{Q})$.*

Proof : Suppose that $L\dashv\mathsf{sup}_{\mathbb{A}}$ in $\mathsf{Cat}(\mathcal{Q})$. Then L is cocontinuous (because it is a left adjoint) and fully faithful (because $\mathsf{sup}_{\mathbb{A}}$ is surjective), so $\mathsf{sup}_{\mathbb{A}}\circ L\cong 1_{\mathbb{A}}$. That is to say, L is a section to $\mathsf{sup}_{\mathbb{A}}$ in $\mathsf{Cocont}(\mathcal{Q})$. Conversely, if $S\colon\mathbb{A}\longrightarrow\mathcal{P}\mathbb{A}$ is a cocontinuous section to $\mathsf{sup}_{\mathbb{A}}\colon\mathcal{P}\mathbb{A}\twoheadrightarrow\mathbb{A}$, then $\mathsf{sup}_{\mathbb{A}}\circ S\cong 1_{\mathbb{A}}$ implies $S\leq Y_{\mathbb{A}}$ (because $\mathsf{sup}_{\mathbb{A}}\dashv Y_{\mathbb{A}}$), and hence, for any $\phi\in\mathcal{P}\mathbb{A}$,

$$S\circ\mathsf{sup}_{\mathbb{A}}(\phi)\cong\mathrm{colim}(\phi,S)\leq\mathrm{colim}(\phi,Y_{\mathbb{A}})\cong\phi$$

(because S is cocontinuous). So $S\circ\mathsf{sup}_{\mathbb{A}}\leq 1_{\mathcal{P}\mathbb{A}}$, which proves it to be left adjoint to $\mathsf{sup}_{\mathbb{A}}$. $\square$

The above says that, for a cocomplete $\mathcal{Q}$-category $\mathbb{A}$, a cocontinuous section to $\mathsf{sup}_{\mathbb{A}}\colon\mathcal{P}\mathbb{A}\twoheadrightarrow\mathbb{A}$ is the same thing as a left adjoint. But there may be several non-cocontinuous sections for $\mathsf{sup}_{\mathbb{A}}$, e.g. the Yoneda embedding!

4 Totally continuous cocomplete $\mathcal{Q}$-categories

Given a completely distributive cocomplete $\mathcal{Q}$-category $\mathbb{A}$, the left adjoint to the surjection $\mathsf{sup}_{\mathbb{A}}\colon\mathcal{P}\mathbb{A}\twoheadrightarrow\mathbb{A}$ is a functor, say $T_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathcal{P}\mathbb{A}$, satisfying

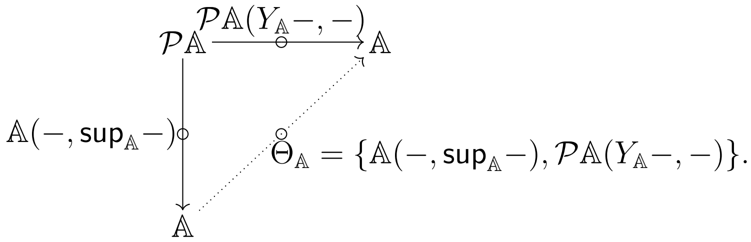
$$\mathcal{P}\mathbb{A}(T_{\mathbb{A}}-,-)=\mathbb{A}(-,\mathsf{sup}_{\mathbb{A}}-).$$

By the universal property of the presheaf category $\mathcal{P}\mathbb{A}$, this functor – like any functor from $\mathbb{A}$ to $\mathcal{P}\mathbb{A}$, for that matter – determines, and is determined by, a distributor $\Theta_{\mathbb{A}}\colon\mathbb{A}\,\mathrel{\circ\!\!\!\longrightarrow}\,\mathbb{A}$ through the formula $T_{\mathbb{A}}(a)(a')=\Theta_{\mathbb{A}}(a',a)$ [Stubbe, 2005a, **6.1**]. The elements of this distributor can be written as

[5] We will not insist on the adjective "constructive" as do [Rosebrugh and Wood, 1994], because we think that, in the context of $\mathcal{Q}$-categories, no confusion will arise.

$$\begin{aligned}\Theta_{\mathbb{A}}(a',a) &= \mathcal{P}\mathbb{A}(Y_{\mathbb{A}}a', T_{\mathbb{A}}a)\\ &= \mathcal{P}\mathbb{A}(Y_{\mathbb{A}}a', -) \otimes \mathcal{P}\mathbb{A}(-, T_{\mathbb{A}}a)\\ &= \{\mathbb{A}(T_{\mathbb{A}}a, -), \mathcal{P}\mathbb{A}(Y_{\mathbb{A}}a', -)\}\\ &= \{\mathbb{A}(a, \mathsf{sup}_{\mathbb{A}}-), \mathcal{P}\mathbb{A}(Y_{\mathbb{A}}a', -)\}.\end{aligned}$$

That is to say, for a completely distributive cocomplete $\mathcal{Q}$-category $\mathbb{A}$ the distributor $\Theta_{\mathbb{A}}$ is the right extension of $\mathbb{A}(-, \mathsf{sup}_{\mathbb{A}}-)$ through $\mathcal{P}\mathbb{A}(Y_{\mathbb{A}}-, -)$ in $\mathsf{Dist}(\mathcal{Q})$:



But this right extension makes sense for *any* cocomplete $\mathcal{Q}$-category $\mathbb{A}$, so – whether $\mathbb{A}$ is completely distributive or not – we can *define* the distributor $\Theta_{\mathbb{A}}\colon \mathbb{A} \,{-\!\!\circ\!\!\!\rightarrow}\, \mathbb{A}$ to be this right extension, and denote $T_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$ for the functor corresponding with $\Theta_{\mathbb{A}}$ under the universal property of $\mathcal{P}\mathbb{A}$. In analogy with the case $\mathcal{Q} = \mathbf{2}$, we call the distributor $\Theta_{\mathbb{A}}\colon \mathbb{A} \,{-\!\!\circ\!\!\!\rightarrow}\, \mathbb{A}$ the *totally-below relation* on the cocomplete $\mathcal{Q}$-category $\mathbb{A}$; and the functor $T_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$ sends an object $a \in \mathbb{A}$ to the "presheaf of objects totally-below a". The calculation rules for weighted colimits [Stubbe, 2005a, **5.2**] make the following trivial.

Lemma 4.1 *For a cocomplete $\mathcal{Q}$-category $\mathbb{A}$, the following are equivalent:*

(i) *for every $a \in \mathbb{A}$, $\mathsf{sup}_{\mathbb{A}}(T_{\mathbb{A}}a) \cong a$,*

(ii) $\mathsf{sup}_{\mathbb{A}} \circ T_{\mathbb{A}} \cong 1_{\mathbb{A}}$,

(iii) $\mathrm{colim}(\Theta_{\mathbb{A}}, 1_{\mathbb{A}}) \cong 1_{\mathbb{A}}$.

A cocomplete $\mathcal{Q}$-category $\mathbb{A}$ is said to be *totally continuous* when it satisfies the equivalent conditions above; that is to say, "every object in $\mathbb{A}$ is the supremum of the objects totally-below it". We will see in 4.4 that "totally continuous" is synonymous with "completely distributive". But first we record two easy but helpful lemmas, the first of which literally is the "classical" definition of 'totally-below' (when we put $\mathcal{Q} = \mathbf{2}$)!

Lemma 4.2 *For a cocomplete $\mathcal{Q}$-category $\mathbb{A}$, the elements of the totally-below relation $\Theta_{\mathbb{A}}\colon \mathbb{A} \,{-\!\!\circ\!\!\!\rightarrow}\, \mathbb{A}$ are, for $a, a' \in \mathbb{A}$,*

$$\Theta_{\mathbb{A}}(a',a) = \bigwedge_{\phi \in \mathcal{P}\mathbb{A}} \{\mathbb{A}(a, \mathsf{sup}_{\mathbb{A}}\phi), \phi(a')\}.$$

Lemma 4.3 *For a cocomplete $\mathcal{Q}$-category $\mathbb{A}$ we have that the totally-below relation $\Theta_{\mathbb{A}}\colon \mathbb{A} \,{-\!\!\circ\!\!\!\rightarrow}\, \mathbb{A}$ satisfies $\Theta_{\mathbb{A}} \leq \mathbb{A}$ and $\Theta_{\mathbb{A}} \otimes \Theta_{\mathbb{A}} \leq \Theta_{\mathbb{A}}$.*

Proposition 4.4 *For a cocomplete $\mathcal{Q}$-category $\mathbb{A}$, the following are equivalent:*

(i) *$\mathbb{A}$ is completely distributive,*

(ii) *$\mathbb{A}$ is totally continuous.*

In this case, $T_{\mathbb{A}}$ is the left adjoint to $\mathsf{sup}_{\mathbb{A}}$ (and therefore also its cocontinuous section).

Proof : By 4.3 the functor $T_{\mathbb{A}}: \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$ satisfies $T_{\mathbb{A}} \circ \mathsf{sup}_{\mathbb{A}} \leq Y_{\mathbb{A}} \circ \mathsf{sup}_{\mathbb{A}} \leq 1_{\mathcal{P}\mathbb{A}}$ (whether $\mathbb{A}$ is completely distributive or not). So the second statement implies that $T_{\mathbb{A}} \dashv \mathsf{sup}_{\mathbb{A}}$, that is, $\mathbb{A}$ is completely distributive. Conversely, if $\mathbb{A}$ is completely distributive then, as argued in the beginning of this section, $T_{\mathbb{A}} \dashv \mathsf{sup}_{\mathbb{A}}$, so – by surjectivity of $\mathsf{sup}_{\mathbb{A}}$ – $\mathsf{sup}_{\mathbb{A}} \circ T_{\mathbb{A}} \cong 1_{\mathbb{A}}$. □

The totally-below relation on a (totally continuous) cocomplete $\mathcal{Q}$-category is an important tool. Its single most important property is the following.

Proposition 4.5 *For a totally continuous cocomplete $\mathcal{Q}$-category $\mathbb{A}$, the totally-below relation $\Theta_{\mathbb{A}}: \mathbb{A} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{A}$ is a comonad in $\mathsf{Dist}(\mathcal{Q})$.*

Proof: For $a \in \mathbb{A}$, consider the presheaf $\Theta_{\mathbb{A}} \otimes \Theta_{\mathbb{A}}(-,a)$ on $\mathbb{A}$; by the calculation rules for weighted colimits [Stubbe, 2005a, **5.2**] and the result in 4.1,

$$\begin{aligned} \mathsf{sup}_{\mathbb{A}}\Big(\Theta_{\mathbb{A}} \otimes \Theta_{\mathbb{A}}(-,a)\Big) &\cong \mathrm{colim}\Big(\Theta_{\mathbb{A}}(-,a), \mathrm{colim}(\Theta_{\mathbb{A}}, 1_{\mathbb{A}})\Big) \\ &\cong \mathsf{sup}_{\mathbb{A}}\Big(\Theta_{\mathbb{A}}(-,a)\Big) \\ &\cong a. \end{aligned}$$

Putting $\phi = \Theta_{\mathbb{A}} \otimes \Theta_{\mathbb{A}}(-,a)$ in 4.2 gives

$$\Theta_{\mathbb{A}}(a',a) \leq \Big\{\mathbb{A}(a,a), \Theta_{\mathbb{A}}(a',-) \otimes \Theta_{\mathbb{A}}(-,a)\Big\}$$

which – since $1_{ta} \leq \mathbb{A}(a,a)$ – implies that $\Theta_{\mathbb{A}}(a',a) \leq \Theta_{\mathbb{A}}(a',-) \otimes \Theta_{\mathbb{A}}(-,a)$. This proves that $\Theta_{\mathbb{A}} \leq \Theta_{\mathbb{A}} \otimes \Theta_{\mathbb{A}}$, which together with 4.3 gives the result. □

The comultiplication of $\Theta_{\mathbb{A}}$ is often called its *interpolation property*. The result implies in particular that the totally-below relation on a totally continuous cocomplete $\mathcal{Q}$-category is idempotent.

5 Splitting the totally-below relation

Recall from [Stubbe, 2005b, **4.5**] that, considering regular $\mathcal{Q}$-semicategories and regular semidistributors,

(3) $$\mathsf{RSDist}(\mathcal{Q}) \longrightarrow \mathsf{Cocont}(\mathcal{Q}): \Big(\Phi: \mathbb{A} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{B}\Big) \mapsto \Big(\Phi \otimes -: \mathcal{R}\mathbb{A} \longrightarrow \mathcal{R}\mathbb{B}\Big)$$

is locally an equivalence. In particular, a cocontinuous functor $F: \mathcal{R}\mathbb{A} \longrightarrow \mathcal{R}\mathbb{B}$ determines (and is determined by) the regular semidistributor $\Phi: \mathbb{A} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{B}$ with elements $\Phi(b,a) = F(Y_{\mathbb{A}}(a))(b)$. Note that $\mathsf{Dist}(\mathcal{Q})$ is a full subquantaloid of $\mathsf{RSDist}(\mathcal{Q})$, and that the domain restriction of (3) to $\mathsf{Dist}(\mathcal{Q})$ is the local equivalence in (2): for a $\mathcal{Q}$-category $\mathbb{A}$, $\mathcal{R}\mathbb{A} = \mathcal{P}\mathbb{A}$.

Furthermore, [Stubbe, 2005b, **3.12**] says that, for each regular $\mathcal{Q}$-semicategory $\mathbb{B}$, the $\mathcal{Q}$-category $\mathcal{R}\mathbb{B}$ of regular presheaves on $\mathbb{B}$ is an essential (co)localization of a certain presheaf category. So certainly is $\mathcal{R}\mathbb{B}$ a projective object in $\mathsf{Cocont}(\mathcal{Q})$, i.e. a totally continuous cocomplete $\mathcal{Q}$-category (see 3.1 and 4.4). In fact, all totally continuous cocomplete $\mathcal{Q}$-categories are of the form $\mathcal{R}\mathbb{B}$, for some regular $\mathcal{Q}$-semicategory $\mathbb{B}$, as we show next.

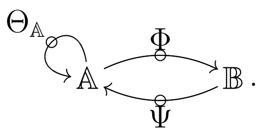
Proposition 5.1 *For a cocomplete $\mathcal{Q}$-category $\mathbb{A}$, the following are equivalent:*

(i) *$\mathbb{A}$ is totally continuous,*

(ii) *$\mathbb{A} \simeq \mathcal{R}\mathbb{B}$ in $\mathsf{Cocont}(\mathcal{Q})$ for some regular $\mathcal{Q}$-semicategory $\mathbb{B}$.*

In this case, the "$\mathbb{B}$" in the second statement is the regular $\mathcal{Q}$-semicategory, unique up to Morita equivalence[6], over which the totally-below relation on $\mathbb{A}$, $\Theta_{\mathbb{A}}\colon \mathbb{A} \multimap \mathbb{A}$, splits in $\mathsf{RSDist}(\mathcal{Q})$.

Sketch of proof: Suppose that $\mathbb{A}$ is a totally continuous cocomplete $\mathcal{Q}$-category. The totally-below relation $\Theta_{\mathbb{A}}\colon \mathbb{A} \multimap \mathbb{A}$ is an idempotent in $\mathsf{Dist}(\mathcal{Q})$ (see 4.5), hence an idempotent in $\mathsf{RSDist}(\mathcal{Q})$. But in the latter quantaloid idempotents split [Stubbe, 2005b, Appendix] so there must exist a regular $\mathcal{Q}$-semicategory, unique up to Morita equivalence, over which $\Theta_{\mathbb{A}}$ splits; let us denote such a splitting as



Applying (3) it can be calculated that $\mathbb{A}$ and $\mathcal{R}\mathbb{B}$ are equivalent categories. If now $\mathbb{A} \simeq \mathcal{R}\mathbb{B}'$ for some other regular $\mathcal{Q}$-semicategory $\mathbb{B}'$, then $\mathbb{B}$ and $\mathbb{B}'$ are Morita-equivalent, i.e. isomorphic in $\mathsf{RSDist}(\mathcal{Q})$, so $\Theta_{\mathbb{A}}$ also splits over $\mathbb{B}'$.

For the converse implication, we've argued above that $\mathcal{R}\mathbb{B}$ is totally continuous. And it follows from the first part of the proof that $\Theta_{\mathcal{R}\mathbb{B}}$ splits over $\mathbb{B}$. $\square$

It is an immediate consequence of this important proposition that, for a totally continuous cocomplete $\mathcal{Q}$-category $\mathbb{A}$, if $\Theta_{\mathbb{A}}\colon \mathbb{A} \multimap \mathbb{A}$ splits over some regular $\mathcal{Q}$-semicategory $\mathbb{B}$, then $\mathbb{A} \simeq \mathcal{R}\mathbb{B}$.

6 Totally algebraic cocomplete $\mathcal{Q}$-categories

[6] See [Stubbe, 2005b, section 4] for a discussion of "Morita equivalence" for regular $\mathcal{Q}$-semicategories.

As in section 4, we write $\Theta_{\mathbb{A}}\colon\mathbb{A}\mathrel{-\!\!\circ\!\!\!\rightarrow}\mathbb{A}$ for the totally-below relation on a given cocomplete $\mathcal{Q}$-category $\mathbb{A}$ (whether it is totally continuous or not), and the corresponding functor as $T_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathcal{P}\mathbb{A}$. An elementary calculation wil prove the following.

Lemma 6.1 *Let $\mathbb{A}$ be a cocomplete $\mathcal{Q}$-category. For an object $a\in\mathbb{A}$, the following are equivalent:*

(i) $1_{ta}\leq\Theta_{\mathbb{A}}(a,a)$,

(ii) *for all* $x\in\mathbb{A}$, $\mathbb{A}(x,a)\leq\Theta_{\mathbb{A}}(x,a)$,

(iii) *for all* $x\in\mathbb{A}$, $\mathbb{A}(a,x)\leq\Theta_{\mathbb{A}}(a,x)$,

(iv) $Y_{\mathbb{A}}(a)\leq T_{\mathbb{A}}(a)$.

In fact, the "$\leq$" may be replaced by "$=$" in all statements but the first.

An object $a\in\mathbb{A}$ of a cocomplete $\mathcal{Q}$-category satisfying the equivalent conditions in 6.1, is said to be *totally compact*. We will write $i\colon\mathbb{A}_{\mathsf{c}}\longrightarrow\mathbb{A}$ for the full subcategory of $\mathbb{A}$ determined by its totally compact objects; it is thus the so-called *inverter* of the 2-cell $T_{\mathbb{A}}\leq Y_{\mathbb{A}}\colon\mathbb{A}\rightrightarrows\mathcal{P}\mathbb{A}$ in $\mathsf{Cat}(\mathcal{Q})$, as we spell out next.

Proposition 6.2 *For any cocomplete $\mathcal{Q}$-category $\mathbb{A}$, the full embedding of the totally compact objects $i\colon\mathbb{A}_{\mathsf{c}}\longrightarrow\mathbb{A}$ satisfies $T_{\mathbb{A}}\circ i\cong Y_{\mathbb{A}}\circ i$, and any other functor $F\colon\mathbb{C}\longrightarrow\mathbb{A}$ such that $T_{\mathbb{A}}\circ F\cong Y_{\mathbb{A}}\circ F$, factors essentially uniquely through i. Moreover, if F is fully faithful, then so is its factorization through i.*

It follows straightforwardly that equivalent cocomplete $\mathcal{Q}$-categories, say $\mathbb{A}\simeq\mathbb{A}'$, have equivalent $\mathcal{Q}$-categories of totally compact objects, $\mathbb{A}_{\mathsf{c}}\simeq\mathbb{A}'_{\mathsf{c}}$.

For any cocomplete $\mathcal{Q}$-category $\mathbb{A}$, we can now *define* the distributor $\Sigma_{\mathbb{A}}\colon\mathbb{A}\mathrel{-\!\!\circ\!\!\!\rightarrow}\mathbb{A}$ to be precisely the comonad determined by the adjoint pair of distributors induced by the full embedding $i\colon\mathbb{A}_{\mathsf{c}}\longrightarrow\mathbb{A}$ of totally compact objects:

$$\Sigma_{\mathbb{A}}(a',a)=\mathbb{A}(a',i-)\otimes\mathbb{A}(i-,a).$$

Further we put $S_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathcal{P}\mathbb{A}$ to be the functor corresponding to $\Sigma_{\mathbb{A}}$ under the universal property of the presheaf category, i.e. $S_{\mathbb{A}}(a)=\Sigma_{\mathbb{A}}(-,a)$. A short calculation using 6.1 will show that, for a cocomplete $\mathcal{Q}$-category $\mathbb{A}$, $\Sigma_{\mathbb{A}}\leq\Theta_{\mathbb{A}}$. The following result, that for brevity's sake we state without proof, must be compared with 4.1.

Lemma 6.3 *For a cocomplete $\mathcal{Q}$-category $\mathbb{A}$, the following are equivalent:*

(i) $i\colon\mathbb{A}_{\mathsf{c}}\longrightarrow\mathbb{A}$ *satisfies* $\langle i,i\rangle\cong 1_{\mathbb{A}}$,

(ii) *for every* $a\in\mathbb{A}$, $\mathsf{sup}_{\mathbb{A}}(S_{\mathbb{A}}a)\cong a$,

(iii) $\mathsf{sup}_{\mathbb{A}}\circ S_{\mathbb{A}}\cong 1_{\mathbb{A}}$,

(iv) $\mathrm{colim}(\Sigma_{\mathbb{A}},1_{\mathbb{A}})\cong 1_{\mathbb{A}}$.

In this case, $\Sigma_{\mathbb{A}}=\Theta_{\mathbb{A}}$.

Mimicking the classical terminology of [Rosebrugh and Wood, 1994] once more, a cocomplete $\mathcal{Q}$-category is *totally algebraic* when it satisfies the equivalent conditions in 6.3; that is to say, "every object is the supremum of the (downclosure of the set of) totally compact objects below it".

It is immediate from 6.3 and 4.1 that "totally algebraic" implies "totally continuous", but the converse is not true. (For a counterexample, compare 5.1 and 6.4, with [Stubbe, 2005b, **4.7**].) Actually, a totally continuous cocomplete $\mathcal{Q}$-category $\mathbb{A}$ is totally algebraic if and only if it is totally continuous and $\Theta_{\mathbb{A}} = \Sigma_{\mathbb{A}}$.

The following should be compared with 5.1.

Proposition 6.4 *For a totally continuous cocomplete $\mathcal{Q}$-category $\mathbb{A}$, the following are equivalent:*

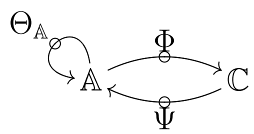
(i) *$\mathbb{A}$ is totally algebraic,*

(ii) *$\mathbb{A} \simeq \mathcal{P}\mathbb{A}_{\mathsf{c}}$,*

(iii) *$\mathbb{A} \simeq \mathcal{P}\mathbb{C}$ for some $\mathcal{Q}$-category $\mathbb{C}$.*

Sketch of proof : It follows directly from 6.3 that for a totally algebraic $\mathbb{A}$, $\Theta_{\mathbb{A}}(= \Sigma_{\mathbb{A}})$ splits over the $\mathcal{Q}$-category $\mathbb{A}_{\mathsf{c}}$; so 5.1 implies that $\mathbb{A} \simeq \mathcal{P}\mathbb{A}_{\mathsf{c}}$.

Suppose now that $\mathbb{A} \simeq \mathcal{P}\mathbb{C}$ for some $\mathcal{Q}$-category $\mathbb{C}$; by 5.1 we know that $\mathbb{A}$ is totally continuous and that there is a splitting



of the comonad $\Theta_{\mathbb{A}}$ in $\mathsf{Dist}(\mathcal{Q})$. Then in particular $\Psi \dashv \Phi$, and therefore – since any cocomplete $\mathcal{Q}$-category is Cauchy complete[7] – there exists a functor $F\colon \mathbb{C} \longrightarrow \mathbb{A}$ such that $\Psi = \mathbb{A}(-, F-)$ and $\Phi = \mathbb{A}(F-, -)$. Using this fact, a calculation will show that $\Theta_{\mathbb{A}} = \Sigma_{\mathbb{A}}$ and hence $\mathbb{A}$ is totally algebraic. $\square$

From this proof it follows that a cocomplete $\mathbb{A}$ is totally algebraic if and only if there exist a $\mathcal{Q}$-category $\mathbb{C}$ and a fully faithful functor $F\colon \mathbb{C} \longrightarrow \mathbb{A}$ such that $\Theta_{\mathbb{A}}$ is the comonad determined by the adjunction $\mathbb{A}(-, F-) \dashv \mathbb{A}(F-, -)$ in $\mathsf{Dist}(\mathcal{Q})$; and that in this case *every* splitting of $\Theta_{\mathbb{A}}$ in $\mathsf{Dist}(\mathcal{Q})$ is of this kind.

7 Cauchy completions revisited

Already in the proof of 6.4, the theory of Cauchy complete $\mathcal{Q}$-categories comes lurking around the corner. Without details or proof, we exhibit a more explicit link.

[7] See [Stubbe, 2005a, section 7] for a presentation of the theory of Cauchy complete $\mathcal{Q}$-categories.

Proposition 7.1 *For a $\mathcal{Q}$-category $\mathbb{C}$, the category $(\mathcal{P}\mathbb{C})_{\mathsf{c}}$ of totally compact objects in $\mathcal{P}\mathbb{C}$ is (equivalent to) the Cauchy completion $\mathbb{C}_{\mathsf{cc}}$ of $\mathbb{C}$.*

It follows now from 6.4 and 7.1 that for a totally algebraic cocomplete $\mathcal{Q}$-category $\mathbb{A}$, the full subcategory $\mathbb{A}_{\mathsf{c}}$ of totally compact objects is Cauchy complete: because $\mathbb{A} \simeq \mathcal{P}\mathbb{C}$ implies $\mathbb{A}_{\mathsf{c}} \simeq (\mathcal{P}\mathbb{C})_{\mathsf{c}} \simeq \mathbb{C}_{\mathsf{cc}}$, and a category which is equivalent to a Cauchy complete category is Cauchy complete itself.

8 In terms of modules

The locally ordered category $\mathsf{Cocont}(\mathcal{Q})$ is biequivalent to $\mathsf{QUANT}(\mathcal{Q}^{\mathsf{op}}, \mathsf{Sup})$, the quantaloid of (right) $\mathcal{Q}$-modules. This is really a part of the theory of tensored and cotensored $\mathcal{Q}$-categories; [Stubbe, 2004, section 4] contains the details. It is then a matter of fact that the projective objects in $\mathsf{Cocont}(\mathcal{Q})$ correspond to those in $\mathsf{QUANT}(\mathcal{Q}^{\mathsf{op}}, \mathsf{Sup})$ under this biequivalence.

Proposition 8.1 *Let $\mathbb{A}$ and $\mathcal{F}$ be a cocomplete $\mathcal{Q}$-category and a $\mathcal{Q}$-module that correspond to each other under the biequivalence*

$$\mathsf{Cocont}(\mathcal{Q}) \simeq \mathsf{QUANT}(\mathcal{Q}^{\mathsf{op}}, \mathsf{Sup}),$$

then the following are equivalent:

(i) *$\mathbb{A}$ is a projective object of $\mathsf{Cocont}(\mathcal{Q})$,*

(ii) *$\mathcal{F}$ is a projective object of $\mathsf{QUANT}(\mathcal{Q}^{\mathsf{op}}, \mathsf{Sup})$.*

Since $\mathsf{QUANT}(\mathcal{Q}^{\mathsf{op}}, \mathsf{Sup})$ is a (large) quantaloid (in particular – and in contast to $\mathsf{Cocont}(\mathcal{Q})$ – its local order is reflexive), an object $\mathcal{F}$ is projective if and only if the representable homomorphism

$$\mathsf{QUANT}(\mathcal{Q}^{\mathsf{op}}, \mathsf{Sup})(\mathcal{F}, -) \colon \mathsf{QUANT}(\mathcal{Q}^{\mathsf{op}}, \mathsf{Sup}) \longrightarrow \mathsf{Sup} \tag{4}$$

preserves epimorphisms. (This is really a straightforward reformulation of the definition of "projectivity" that was given in section 2.) A seemingly stronger notion is of much importance in the theory of (Sup-)enriched categories: after [Kelly, 1982], a *small-projective object* $\mathcal{F} \in \mathsf{QUANT}(\mathcal{Q}^{\mathsf{op}}, \mathsf{Sup})$ is one for which the representable homomorphism in (4) preserves all small weighted colimits. Clearly a small-projective object in $\mathsf{QUANT}(\mathcal{Q}^{\mathsf{op}}, \mathsf{Sup})$ is also projective—but also the converse holds! Without proofs we indicate the intermediate steps that are required to achieve this result.

First we need a handy description of the projective $\mathcal{Q}$-modules.

Lemma 8.2 *The projective objects of $\mathsf{QUANT}(\mathcal{Q}^{\mathsf{op}}, \mathsf{Sup})$ are precisely the retracts of direct sums of representable modules.*

Then we can make the link with small-projectives in $\mathsf{QUANT}(\mathcal{Q}^{\mathsf{op}}, \mathsf{Sup})$. It is proved in [Kelly, 1982, **5.26**] (in the more general context of $\mathcal{V}$-enriched categories) that representable $\mathcal{Q}$-modules are small-projective; and [Kelly, 1982,

5.25] shows that retracts of small-projective $\mathcal{Q}$-modules are small-projective themselves. In the specific case of Sup-enrichment, using that in any quantaloid sums and products coincide, we may also prove the following.

Lemma 8.3 *A direct sum of small-projective $\mathcal{Q}$-modules is again small-projective.*

Because a small-projective is always projective, 8.2, 8.3 and the theorems in [Kelly, 1982] recalled above, imply the following.

Proposition 8.4 *For a $\mathcal{Q}$-module $\mathcal{F}$, the following are equivalent:*

(i) *$\mathcal{F}$ is a projective object,*

(ii) *$\mathcal{F}$ is a retract of a direct sum of representable $\mathcal{Q}$-modules,*

(iii) *$\mathcal{F}$ is a small-projective object.*

Via 8.1 this says something about projective objects in $\mathsf{Cocont}(\mathcal{Q})$ too.

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