SHORT INTRODUCTION TO ENRICHED CATEGORIES

FRANCIS BORCEUX and ISAR STUBBE
Département de Mathématique,
Université Catholique de Louvain,
2 Ch. du Cyclotron, B-1348 Louvain-la-Neuve, Belgium.
e-mail: borceux@agel.ucl.ac.be — i.stubbe@agel.ucl.ac.be

This text aims to be a short introduction to some of the basic notions in ordinary and enriched category theory. With reasonable detail but always in a compact fashion, we have brought together in the first part of this paper the definitions and basic properties of such notions as limit and colimit constructions in a category, adjoint functors between categories, equivalences and monads. In the second part we pass on to enriched category theory: it is explained how one can "replace" the category of sets and mappings, which plays a crucial role in ordinary category theory, by a more general symmetric monoidal closed category, and how most results of ordinary category theory can be translated to this more general setting. For a lack of space we had to omit detailed proofs, but instead we have included lots of examples which we hope will be helpful. In any case, the interested reader will find his way to the references, given at the end of the paper.

1. Ordinary categories

When working with vector spaces over a field K, one proves such theorems as: for all vector spaces there exists a base; every vector space V is canonically included in its bidual V^{**}; every linear map between finite dimensional based vector spaces can be represented as a matrix; and so on. But where do the universal quantifiers take their value? What precisely does "canonical" mean? How can we formally "compare" vector spaces with matrices? What is so special about vector spaces that they can be based?

An answer to these questions, and many more, can be formulated in a very precise way using the language of category theory. All vector spaces and all linear maps form a "category" Vect_K, and the construction of the bidual of a vector space proves to be a "functor" $(-)^{**}\colon \mathsf{Vect}_K \to \mathsf{Vect}_K$. The inclusions $\sigma_V\colon V \hookrightarrow V^{**}$ being canonical means that they constitute a "natural transformation" $\sigma\colon 1_{\mathsf{Vect}_K} \Rightarrow (-)^{**}$, 1_{Vect_K} being the identity functor on Vect_K. The fact that computations with linear maps between finite dimensional vector spaces can be done "via matrices" translates categorically as an "equivalence" between FVect_K, the category of finite dimensional vector spaces, and $\mathsf{Matr}(K)$, the category of matrices. On the other hand, the fact that every vector space can be based translates categorically as an equivalence between Vect_K and the category of "free algebras" for the corresponding monad $T\colon \mathsf{Set} \to \mathsf{Set}$ on the category of sets and mappings.

We better begin, in section 1.1, by giving the correct definitions of the basic notions of category theory. Further on, in section 1.2, we discuss the universal constructions in a category. Passing to the notions of "adjoints" and "equivalences" in section 1.3, we conclude in section 1.4 with a brief introduction to the theory of "monads".

1.1. CATEGORIES, FUNCTORS AND NATURAL TRANSFORMATIONS

1.1.1 Definition *A category $\mathcal{C}$ consists of:*

- *a class $\mathcal{C}_0$ of "objects";*
- *for any $A, B \in \mathcal{C}_0$ a set $\mathcal{C}(A,B)$ of "morphisms from A to B";*
- *for any $A, B, C \in \mathcal{C}_0$ a "composition law" which is a mapping of sets $c_{A,B,C}\colon \mathcal{C}(A,B) \times \mathcal{C}(B,C) \to \mathcal{C}(A,C)\colon (f,g) \mapsto c_{A,B,C}(f,g) = g \circ f$;*
- *for any object $A \in \mathcal{C}_0$ an "identity morphism" $1_A \in \mathcal{C}(A,A)$;*

subject to the following axioms:

- *associativity for composition: for any $f \in \mathcal{C}(A,B)$, $g \in \mathcal{C}(B,C)$ and $h \in \mathcal{C}(C,D)$, $h \circ (g \circ f) = (h \circ g) \circ f$;*
- *identity for composition: for any $f \in \mathcal{C}(A,B)$, $f \circ 1_A = f = 1_B \circ f$.*

Given two categories $\mathcal{A}$ and $\mathcal{B}$, a functor $F\colon \mathcal{A} \to \mathcal{B}$ consists of:

- *a mapping $\mathcal{A}_0 \to \mathcal{B}_0\colon A \mapsto FA$;*
- *for any $A, B \in \mathcal{A}_0$ a mapping $\mathcal{A}(A,B) \to \mathcal{B}(FA,FB)\colon f \mapsto Ff$;*

subject to the following axioms:

- *preservation of composition: for any $f \in \mathcal{A}(A,B)$ and $g \in \mathcal{A}(B,C)$, $Fg \circ Ff = F(g \circ f)$;*
- *preservation of identities: for any $A \in \mathcal{A}_0$, $1_{FA} = F(1_A)$.*

For two functors $F, G\colon \mathcal{A} \to \mathcal{B}$, a natural transformation $\alpha\colon F \Rightarrow G\colon \mathcal{A} \to \mathcal{B}$ is a class of $\mathcal{B}$-morphisms $(\alpha_A\colon FA \to GA)_{A\in\mathcal{A}}$ such that for any $f \in \mathcal{A}(A,B)$, $\alpha_B \circ Ff = Gf \circ \alpha_A$.

For a morphism $f \in \mathcal{A}(A,B)$ the notation $f\colon A \to B$ is common. It is worth remarking that 1_A is the only morphism in $\mathcal{A}(A,A)$ that plays the role of an identity for the composition law. Given two functors $F, G\colon \mathcal{A} \to \mathcal{B}$, the "naturality" of a class $(\alpha_A\colon FA \to GA)_{A\in\mathcal{A}}$ can be expressed by the the commutativity of the following diagram, for any $f \in \mathcal{A}(A,B)$:

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\ GA & \xrightarrow[Gf]{} & GB \end{array}$$

Some obvious examples of categories include the following: Set is the category of sets and mappings, Rel is the category of sets and relations, Gr (Ab) is the category of groups (abelian groups) and group homomorphisms, Vect_K is the category of vector spaces over a field K and linear transformations, Sup is the category of complete lattices and sup-preserving maps. A monoid with unit $(M, \cdot, 1)$ is a category $\mathcal{M}$ with one object, say $\star$, such that $\mathcal{M}(\star, \star) = M$ in which, of course, the composition law is the multiplication of the monoid and the identity on $\star$ is the unit for that multiplication. Hence also any group can be viewed as a category with one object. A poset $(P, \leq)$ is a category $\mathcal{P}$ whose objects are the elements of P, and for which $\mathcal{P}(a, b)$ is a singleton if $a \leq b$ and is empty otherwise. More generally, a category of which each set of morphisms is either a singleton or empty, is called "thin", and besides posets the examples include also preordered sets, even preordered classes. A category $\mathcal{C}$ is "small" when its objects constitute a set. With the obvious definition for composition of functors $F \colon \mathcal{A} \to \mathcal{B}$ and $G \colon \mathcal{B} \to \mathcal{C}$, and the obvious definition for identity functor $1_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}$, it is now easily verified that there is a category Cat of all small categories and functors between them. It is crucial to consider only small categories, for otherwise $\mathsf{Cat}(\mathcal{A}, \mathcal{B})$ would not necessarily constitute a set.

Given a category $\mathcal{A}$, one can always consider a "dual" or "opposite" category $\mathcal{A}^{\mathsf{op}}$, by reversing the direction of the arrows. More specifically, $\mathcal{A}_0^{\mathsf{op}} = \mathcal{A}_0$ but $\mathcal{A}^{\mathsf{op}}(A, B) = \mathcal{A}(B, A)$ and accordingly $(g \circ f)^{\mathsf{op}} = f \circ g$. A "contravariant" functor $F \colon \mathcal{A} \to \mathcal{B}$ is by definition a (genuine) functor $F \colon \mathcal{A}^{\mathsf{op}} \to \mathcal{B}$. By way of contrast, a (genuine) functor is said to be "covariant". We specify contra- or covariance of a given functor only when confusion could arise, cf. 1.1.4. It is clear that if a statement expresses the existence of some objects or morphisms, or the equality of some composites of morphisms in any category, then the "dual statement", obtained by reversing arrows and suitably adjusting the composition, is also valid in any category. This is sometimes referred to as "the duality principle".

An obvious functor from Gr to Set is the one that maps any group onto the underlying set, and any group homomorphism onto the underlying map. Such type of functor is quite rightly called "forgetful". Another typical example would be the functor $\mathsf{Vect}_K \to \mathsf{Set}$ that "forgets" all about the linear algebra in Vect_K. Of a different nature is the "inclusion" of Ab in Gr, or likewise the "inclusion" of Set in Rel, both functorial. Note that a functor between two posets viewed as small thin categories corresponds precisely to an isotone mapping between those posets. Yet another example: Given a set A, there is a functor $A \times - \colon \mathsf{Set} \to \mathsf{Set}$ that maps any set X onto $A \times X$, the cartesian product of A and X, and any mapping $f \colon X \to Y$ onto the mapping $1_A \times f \colon A \times X \to A \times Y \colon (a, x) \mapsto (a, f(x))$.

Given two categories $\mathcal{A}$ and $\mathcal{B}$ and an object $B \in \mathcal{B}_0$ we will always write $\Delta_B \colon \mathcal{A} \to \mathcal{B}$ for the functor that maps any object to B and any morphism to 1_B. For any category $\mathcal{A}$ and any object A of $\mathcal{A}$, we can define a functor $\mathcal{A}(A, -) \colon \mathcal{A} \to \mathsf{Set}$ by putting $\mathcal{A}(A, -)(X) = \mathcal{A}(A, X)$ for an object X and $\mathcal{A}(A, -)(x) = x \circ -$ for a morphism $x \in \mathcal{A}(X, Y)$ — the latter prescription defines then indeed a mapping from $\mathcal{A}(A, X)$ to $\mathcal{A}(A, Y)$. This functor is said to be "represented by A". Along the same lines, one defines a contravariant representable functor $\mathcal{A}(-, A) \colon \mathcal{A} \to \mathsf{Set}$.

When considering two isotone mappings $f, g\colon P_1 \to P_2$ between two posets from the categorical point of view, thus as functors $F, G\colon \mathcal{P}_1 \to \mathcal{P}_2$ between thin small categories, then there exists a natural transformation $\alpha\colon F \Rightarrow G$ iff $f \leq g$ for the pointwise order; indeed, the condition of naturality of the class of morphisms $(\alpha_C\colon FC \to GC)_{C\in\mathcal{P}_2}$ is empty since $\mathcal{P}_2$ is thin, and the existence of such a class coincides with pointwise order.

Given a morphism $f\colon A \to B$ in a category $\mathcal{B}$, we can consider the two constant functors $\Delta_A, \Delta_B\colon \mathcal{A} \rightrightarrows \mathcal{B}$ on some category $\mathcal{A}$, and define the "constant natural transfo" $\Delta_f\colon \Delta_A \Rightarrow \Delta_B$ by putting every one of its components to be $f\colon A \to B$. Indeed, since the image of any $\mathcal{A}$–morphism by Δ_A (Δ_B) is the identity 1_A (1_B), the naturality condition of $\Delta_f\colon \Delta_A \Rightarrow \Delta_B$ reads $1_B \circ f = f \circ 1_A$, which is trivial.

On the other hand, for a morphism $f\colon A \to B$ in the category $\mathcal{A}$, we denote by $\mathcal{A}(f,-)\colon \mathcal{A}(B,-) \Rightarrow \mathcal{A}(A,-)$ the natural transformation between the functors represented by B and A, of which the component at an object $C \in \mathcal{A}_0$ is defined as $\mathcal{A}(f,-)_C = - \circ f\colon \mathcal{A}(B,C) \to \mathcal{A}(A,C)$. Naturality is in fact induced by the associativity of the composition in the category $\mathcal{A}$: Let $x\colon X \to Y$ be a morphism in $\mathcal{A}$, then commutativity of

$$\begin{array}{ccc} \mathcal{A}(B,X) & \xrightarrow{x\circ -} & \mathcal{A}(B,Y) \\ {\scriptstyle -\circ f}\downarrow & & \downarrow{\scriptstyle -\circ f} \\ \mathcal{A}(A,X) & \xrightarrow{x\circ -} & \mathcal{A}(A,Y) \end{array}$$

means that for every $g\colon B \to X$ we have $(x \circ g) \circ f = x \circ (g \circ f)$. Dually one defines $\mathcal{A}(-,f)\colon \mathcal{A}(-,A) \Rightarrow \mathcal{A}(-,B)$. In analogy to the term "representable functor" we could speak here of "representable natural transformation".

When F, G, H are functors from $\mathcal{A}$ to $\mathcal{B}$, and $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$ are natural transformations, then the formula $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$ defines a class of $\mathcal{B}$-morphisms $((\beta\circ\alpha)_A\colon FA \to HA)_{A\in\mathcal{A}}$ that constitutes a natural transformation $\beta\circ\alpha\colon F \to H$. Defining furthermore for any functor $F\colon \mathcal{A} \to \mathcal{B}$ an "identity natural transformation" $1_F\colon F \Rightarrow F$ of which all components are identity arrows, that is, $(1_F)_A = 1_{FA}$, it is now routine to check that, for any small category $\mathcal{A}$ and any category $\mathcal{B}$, there is a category $\mathsf{Fun}(\mathcal{A}, \mathcal{B})$ with as objects all functors from $\mathcal{A}$ to $\mathcal{B}$ and as morphisms all natural transfos between these functors. Again, smallness of $\mathcal{A}$ is necessary to make sure that the morphism-sets in this category are indeed sets!

The following proposition involves some calculations with representable functors. It establishes the crucial lemma for 1.1.4, which in turn is an omnipresent result in category theory.

1.1.2 Proposition (Yoneda lemma) *Consider a small category $\mathcal{A}$, a functor $F\colon \mathcal{A} \to \mathsf{Set}$, and an object $A \in \mathcal{A}$. Denote by $\mathsf{Nat}(\mathcal{A}(A,-), F)$ the set of natural transformations $\alpha\colon \mathcal{A}(A,-) \Rightarrow F\colon \mathcal{A} \to \mathsf{Set}$. There exists a bijection*

$$\theta_{F,A}\colon \mathsf{Nat}(\mathcal{A}(A,-), F) \to FA.$$

Further, we can define:

- *a functor* $N: \mathcal{A} \to \mathsf{Set}$ *by* $N(A) = \mathsf{Nat}(\mathcal{A}(A,-),F)$ *for an object* A *of* $\mathcal{A}$ *and* $N(f): \mathsf{Nat}(\mathcal{A}(A,-),F) \to \mathsf{Nat}(\mathcal{A}(B,-),F): \alpha \mapsto \alpha \circ \mathcal{A}(f,-)$ *for a morphism* $f: A \to B$*;*
- *a functor* $M: \mathsf{Fun}(\mathcal{A},\mathsf{Set}) \to \mathsf{Set}$ *by* $M(F) = \mathsf{Nat}(\mathcal{A}(A,-),F)$ *for an object* F *of* $\mathsf{Fun}(\mathcal{A},\mathsf{Set})$ *and* $M(\gamma): \mathsf{Nat}(\mathcal{A}(A,-),F) \to \mathsf{Nat}(\mathcal{A}(A,-),G): \alpha \mapsto \gamma \circ \alpha$ *for a morphism* $\gamma: F \to G$*;*
- *a functor* $\mathsf{ev}_A: \mathsf{Fun}(\mathcal{A},\mathsf{Set}) \to \mathsf{Set}$ *by* $\mathsf{ev}_A(F) = FA$ *for an object* F *of* $\mathsf{Fun}(\mathcal{A},\mathsf{Set})$ *and* $\mathsf{ev}_A(\gamma) = \gamma_A$ *for a morphism* $\gamma: F \Rightarrow G$ *(this functor is called "evaluation in* A*" for the obvious reason).*

The bijections $\theta_{F,A}: \mathsf{Nat}(\mathcal{A}(A,-),F) \to FA$ *constitute a natural transformation* $\theta_F: N \Rightarrow F$ *with components* $(\theta_F)_A = \theta_{F,A}$ *and a natural transformation* $\theta_A: M \Rightarrow \mathsf{ev}_A$ *with components* $(\theta_A)_F = \theta_{F,A}$*.*

Proof: For the definition of the bijection $\theta_{F,A}$: Given $\alpha: \mathcal{A}(A,-) \Rightarrow F$, define $\theta_{F,A}(\alpha) = \alpha_A(1_A) \in FA$; on the other hand, given $a \in FA$, define $\tau(a): \mathcal{A}(A,-) \Rightarrow F$ by its components $\tau(a)_B: \mathcal{A}(A,B) \to FB: f \mapsto Ff(a)$. Naturality of $\tau(a)$ is easily verified: For every morphism $g \in \mathcal{A}(B,C)$, the naturality condition expressed in the following diagram:

$$\begin{array}{ccc} \mathcal{A}(A,B) & \xrightarrow{\tau(a)_B} & FB \\ {\scriptstyle \mathcal{A}(A,g)}\downarrow & & \downarrow{\scriptstyle Fg} \\ \mathcal{A}(A,C) & \xrightarrow{\tau(a)_C} & FC \end{array}$$

reduces to: for all $f \in \mathcal{A}(A,B)$, $F(g \circ f)(a) = Fg(Ff(a))$, which is true by functoriality of $F:\mathcal{A} \to \mathsf{Set}$. So indeed $\tau(a) \in \mathsf{Nat}(\mathcal{A}(A,-),F)$. These assignments are inverse to each other: Starting from $a \in FA$ we have

$$\theta_{F,A}(\tau(a)) = \tau(a)_A(1_A) = F(1_A)(a) = 1_{FA}(a) = a;$$

starting from $\alpha: \mathcal{A}(A,-) \Rightarrow F$ we have for any $f \in \mathcal{A}(A,B)$

$$\begin{aligned} \tau(\theta_{F,A}(\alpha))_B(f) &= \tau(\alpha_A(1_A))_B(f) \\ &= Ff(\alpha_A(1_A)) \\ &= \alpha_B(\mathcal{A}(A,f)(1_A)) \qquad \text{(by naturality of } \alpha) \\ &= \alpha_B(f \circ 1_A) \\ &= \alpha_B(f). \end{aligned}$$

We leave to the reader the verification of the functoriality of $N: \mathcal{A} \to \mathsf{Set}$ and $M, \mathsf{ev}_A: \mathsf{Fun}(\mathcal{A},\mathsf{Set}) \to \mathsf{Set}$, as well as the naturality of $\theta_{F,A}$ as indicated in the proposition. □

Actually, even when $\mathcal{A}$ is a "large" category the bijections of 1.1.2 exist (hence in particular $\mathsf{Nat}(\mathcal{A}(A,-),F)$ is a set) and constitute a natural transformation $\theta_F: N \Rightarrow F$. But when $\mathcal{A}$ is not small, it makes no sense to define a "category" $\mathsf{Fun}(\mathcal{A},\mathsf{Set})$, let alone a natural transfo $\theta_A: M \Rightarrow \mathsf{ev}_A: \mathsf{Fun}(\mathcal{A},\mathsf{Set}) \to \mathsf{Set}$. Note that, for a small category $\mathcal{A}$, the set $\mathsf{Nat}(\mathcal{A}(A,-),F)$ is just a notation for

$\mathsf{Fun}(\mathcal{A},\mathsf{Set})(\mathcal{A}(A,-),F)$ since the latter is by definition the set of morphisms from the object $\mathcal{A}(A,-)$ to the object F in the category $\mathsf{Fun}(\mathcal{A},\mathsf{Set})$, hence precisely the natural transformations $\mathcal{A}(A,-) \Rightarrow F$.

Let us now fix some standard terminology.

1.1.3 Definition *A morphism $f: A \to B$ in a category $\mathcal{C}$ is a*

- *monomorphism if for every object C and every pair $g, h: C \to A$ of morphisms in $\mathcal{C}$, $f \circ g = f \circ h$ implies $g = h$;*
- *epimorphism if for every object D and every pair $s, t: C \to D$ of morphisms in $\mathcal{C}$, $s \circ f = t \circ f$ implies $s = t$;*
- *isomorphism if there exists a (necessarily unique) morphism $f^{-1}: B \to A$ such that $f \circ f^{-1} = 1_B$ and $f^{-1} \circ f = 1_A$.*

A functor $F: \mathcal{A} \to \mathcal{B}$

- *preserves monomorphisms (epimorphisms, isomorphisms) if the image Ff of any such morphism f is again such a morphism;*
- *reflects monomorphisms (epimorphisms, isomorphisms) if, when the image Ff is such a morphism, then f was such a morphism in the first place.*

Consider now a functor $F: \mathcal{A} \to \mathcal{B}$, and for each pair of objects $A, B \in \mathcal{A}_0$ the mapping $\mathcal{A}(A,B) \to \mathcal{B}(FA,FB): f \mapsto Ff$. The functor F is:

- *faithful when all these mappings are injective;*
- *full when all these mappings are surjective.*

Now consider the mapping $\mathcal{A}_0 \to \mathcal{B}_0$ associated to $F: \mathcal{A} \to \mathcal{B}$. F is:

- *injective on objects when this mapping is injective;*
- *surjective on objects when this mapping is surjective;*
- *essentially surjective on objects if every object in $\mathcal{B}$ is isomorphic to the image by F of an object in $\mathcal{A}$.*

Notice that the notions of monomorphism and epimorphism are "dual", in the sense that $f: A \to B$ is mono in $\mathcal{C}$ iff it is epi in $\mathcal{C}^{\mathsf{op}}$. An isomorphism is both mono and epi, but the converse does not hold: Consider the category with two objects, say A and B, in which there exists besides the identity morphisms exactly one other morphism $f: A \to B$. Then trivially f is both mono and epi, but never iso. (In fact, in every thin category every morphism is both mono and epi, but not necessarily iso.) The composition of two monomorphisms (epimorphisms, isomorphisms) is again such a map, and clearly identity morphisms are isomorphisms, hence also mono and epi. As notation, one often writes $f: A \rightarrowtail B$ for a monomorphism, $f: A \twoheadrightarrow B$ for an epimorphism, and $f: A \xrightarrow{\cong} B$ for an isomorphism.

A monomorphism (epi, iso) in Set, Gr, Sup is an injective (surjective, bijective) morphism. In Rng, the category of rings and ring homomorphisms, the inclusion of the integers in the rationals is epi, but clearly not surjective! It is also mono, so it once again shows that being mono and epi does not suffice to be iso. This can also be seen in the category Top of topological spaces and continuous maps, where the monomorphisms (epimorphisms) are exactly the continuous injections (surjections), but the homeomorphisms, which are the isomorphisms, are more than just continuous bijections.

The composition of two faithful (full, injective, surjective, essentially surjective) functors is again such a functor. A faithful functor reflects monomorphisms and epimorphisms. Obviously every functor preserves isomorphisms, and a full and faithful one also reflects iso's. A functor $F: \mathcal{A} \to \mathcal{B}$ is an isomorphism in Cat if it is full, faithful and injective and surjective on objects. If $\mathcal{A}$ is a small category, then an isomorphism in the category $\mathsf{Fun}(\mathcal{A}, \mathcal{B})$ is a natural transformation $\alpha: A \Rightarrow G$ of which every component $\alpha_A: FA \to GA$ is an isomorphism in $\mathcal{B}$; of course, $\alpha^{-1}: G \Rightarrow F$ is then the transformation with components $\alpha_A^{-1}: GA \to FA$. Even when $\mathcal{A}$ is large, we will say that a "natural isomorphism" $\alpha: G \overset{\sim}{\Rightarrow} F$ is a natural transformation all of whose components are isomorphisms (in the category $\mathcal{B}$), even though in this case there is no "category" $\mathsf{Fun}(\mathcal{A}, \mathcal{B})$ in which α can be an "isomorphism" in the sense of 1.1.3.

The forgetful functors $\mathsf{Gr} \to \mathsf{Set}$, $\mathsf{Ab} \to \mathsf{Gr}$ are both faithful. The latter is also full and injective on objects, the former obviously neither. Next consider the category FVect_K of finite dimensional vector spaces over a field K, and the category $\mathsf{Matr}(K)$ of which the objects are all natural numbers, and of which an arrow from n to m is a $m \times n$ matrix with elements from K. Then we can define a functor from $\mathsf{Matr}(K)$ to FVect_K that assigns to any natural number n the vectorspace K^n, and to any $m \times n$ matrix the linear application from $K^n \to K^m$ represented by this matrix. This functor is full, faithful and essentially surjective. (In 1.3.13 we will call such a functor an "equivalence".)

A category $\mathcal{A}$ is said to be a "subcategory" of a category $\mathcal{B}$ if $\mathcal{A}_0$ is a subclass of $\mathcal{B}_0$ and, for any $A, B \in \mathcal{A}_0$, $\mathcal{A}(A, B)$ is a subset of $\mathcal{B}(A, B)$, such that $\mathcal{A}$ is a category under the composition law and identities inherited from $\mathcal{B}$. Clearly, this situation gives rise to an injective and faithful inclusion functor $\mathcal{A} \hookrightarrow \mathcal{B}$. If this functor is also full, then $\mathcal{A}$ is said to be a "full subcategory" of $\mathcal{B}$. A full subcategory can thus be defined by specifying its class of objects. For instance, the category of all sets and all injections is a subcategory of Set, but is not a full subcategory. On the contrary, the category of all finite sets and all mappings is a full subcategory of Set. We have already noticed that Ab is a full subcategory of Gr. In the obvious way one can also consider the inclusion of Set in Rel, which is not full.

1.1.4 Proposition (Yoneda embedding) *For every small category $\mathcal{A}$, both*

- *the contravariant functor $Y^*: \mathcal{A} \to \mathsf{Fun}(\mathcal{A}, \mathsf{Set})$, defined for an object A and a morphism $f: A \to B$ in $\mathcal{A}$ as $Y^*(A) = \mathcal{A}(A, -)$, $Y^*(f) = \mathcal{A}(f, -)$;*
- *the covariant functor $Y_*: \mathcal{A} \to \mathsf{Fun}(\mathcal{A}, \mathsf{Set})$, defined for an object A and a morphism $f: A \to B$ in $\mathcal{A}$ as $Y_*(A) = \mathcal{A}(-, A)$, $Y_*(f) = \mathcal{A}(-, f)$*

are full and faithful.

Proof: To show that for any $A, B \in \mathcal{A}$ there is a bijective correspondence between $\mathcal{A}(A, B)$ and $\mathsf{Nat}(\mathcal{A}(B, -), \mathcal{A}(A, -))$, apply the Yoneda lemma for $\mathcal{A}(A, -)$ and B. A similar argument goes to prove the "covariant" case. □

1.2. UNIVERSAL CONSTRUCTIONS IN A CATEGORY

In a poset $(P, \leq)$, viewed as a thin category, the infimum $l = inf\{x_i \mid i \in I\}$ of a family is, by definition, an element $l \in P$ provided with morphisms $\lambda_i: l \to x_i$ (that is, $l \leq x_i$ for each $i \in I$) such that, if another element $k \in P$ is provided with morphisms $\kappa_i: k \to x_i$ (that is, $k \leq x_i$ for each $i \in I$) then there is a (necessarily unique) morphism $f: k \to l$ (that is, $k \leq l$). Transposing this idea to the more general context of a category $\mathcal{C}$, not necessarily thin, yields the notion of "limit in $\mathcal{C}$". Of course, in a thin category, all existing diagrams are commutative; in the general contex the expected commutativity conditions must be specified. This is done in particular by requesting the "naturality" of the families $(\lambda_i)_{i \in I}$, $(\kappa_i)_{i \in I}$ in the following definition.

1.2.5 Definition *Given a (covariant) functor $F: \mathcal{D} \to \mathcal{C}$, a cone on F is a natural transformation $\lambda: \Delta_L \Rightarrow F$ for some object $L \in \mathcal{C}$. A limit of F is a "universal" cone in the following sense: for any other cone $\kappa: \Delta_K \Rightarrow F$ there exists a unique constant natural transformation $\Delta_f: \Delta_K \Rightarrow \Delta_L$ such that $\lambda \circ \Delta_f = \kappa$. The category $\mathcal{C}$ is (finitely) complete when all (finite) small limits exist, that is, when for every (finite) small category $\mathcal{D}$ and every functor $F: \mathcal{D} \to \mathcal{C}$, the limit of F exists.*

The dual notions of cocone and colimit are obtained by reversing the directions of the natural transformations in the above; equivalently, a colimit of $F: \mathcal{D} \to \mathcal{C}$ is a limit of $F: \mathcal{D}^{\mathsf{op}} \to \mathcal{C}^{\mathsf{op}}$. Accordingly, one speaks of a (finitely) cocomplete category.

At first sight it may seem more natural to define a complete category as "a category which has all (even large) limits", for the notion of large limit makes sense. But in fact, a "complete category" in this sense is necesseraly thin. Therefore, the pertinent definition is indeed the one with a "smallness condition".

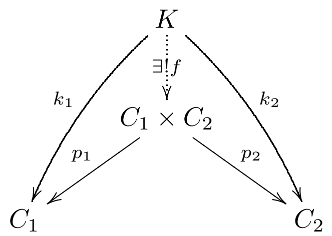
The following proposition explains at once the term "universal".

1.2.6 Proposition *If a functor has a (co)limit, then this (co)limit is unique "up to isomorphism".*

Proof: Suppose that, given $F: \mathcal{D} \to \mathcal{C}$, there are two limits, say $\lambda: \Delta_L \Rightarrow F$ and $\lambda': \Delta_{L'} \Rightarrow F$. Then the factorizations $\Delta_f: \Delta_{L'} \Rightarrow \Delta_L$ and $\Delta_{f'}: \Delta_L \Rightarrow \Delta_{L'}$ prove to be inverses to each other: Consider $\lambda: \Delta_L \Rightarrow F$ merely as a cone on F, then one factorization of $\lambda: \Delta_L \Rightarrow F$ through $\lambda: \Delta_L \Rightarrow F$ is $1_{\Delta_L}: \Delta_L \Rightarrow \Delta_L$, and another is $\Delta_f \circ \Delta_{f'}$, hence by unicity of the factorization already $\Delta_f \circ \Delta_{f'} = 1_{\Delta_L}$. Likewise one shows that $\Delta_{f'} \circ \Delta_f = 1_{\Delta_{L'}}$. The proof for colimits proceeds "dually". □

Let us make these abstract notions somewhat more practical. If we think of the category $\mathcal{D}$ as an abstract diagram of which the vertices are the objects and the edges are the morphisms of $\mathcal{D}$, with maybe some commutativity rules encoded by the composition in $\mathcal{D}$, then the functor $F: \mathcal{D} \to \mathcal{C}$ will produce in $\mathcal{C}$ a diagram of shape $\mathcal{D}$, in which the same commutativity rules apply. Therefore we often speak of "a limit of this or that diagram in $\mathcal{C}$". Here are some examples of particular interest.

Take for $\mathcal{D}$ a non-empty small "discrete category", that is, a set thought of as a category with no other morphisms that the identity morphisms. The image of a functor $F: \mathcal{D} \to \mathcal{C}$ is then just a family of objects of $\mathcal{C}$, indexed by the objects of $\mathcal{D}$, say $(C_D)_{D \in \mathcal{D}}$. A limit of a diagram of this kind, if it exists, is called a "product" and its colimit is a "coproduct" (or "sum"). The notation for the product is $\prod_D C_D$, or $C_1 \times C_2$ for binary products, and the $\mathcal{C}$–morphisms $(\Delta_{\Pi_D C_D}(D) \to FD)_{D \in \mathcal{D}}$ that constitute the natural transformation $\Delta_{\Pi_D C_D} \Rightarrow F$ are called "projections", written $(p_D: \prod_D C_D \to C_D)_{D \in \mathcal{D}}$. In principle these projections constitute a *natural* transformation, but the naturality condition is empty because there are no non–trivial arrows in the diagram $\mathcal{D}$. So far we have described a cone over the diagram. Its universality means that for any other such "object with projections", say $(K, k_D: K \to C_D)_{D \in \mathcal{D}}$, there exists a unique constant natural factorization $\Delta_f: \Delta_K \Rightarrow \Delta_{\Pi_D C_D}$, that is, there exists a unique $\mathcal{C}$–morphism $f: K \to \prod_D C_D$ such that both triangles in the following diagram commute (the diagram shows the case of the binary product, $\mathcal{D} = \{1, 2\}$):



Dually, the coproduct is denoted $\coprod_D C_D$, the according morphisms are "coprojections". The picture that goes with this is just the dual of the diagram above (for a binary coproduct).

In Set, the product of A and B is just their cartesian product $A \times B$ with the projections $p_A: A \times B \to A: (a, b) \mapsto a$ and $p_B: A \times B \to B: (a, b) \mapsto b$. The coproduct of two sets is their disjoint union, together with the obvious inclusions as coprojections. Also Cat has products and coproducts: $\mathcal{A} \times \mathcal{B}$ is the category with $(\mathcal{A} \times \mathcal{B})_0 = \mathcal{A}_0 \times \mathcal{B}_0$, and $(\mathcal{A} \times \mathcal{B})\big((A, B), (A', B')\big) = \mathcal{A}(A, A') \times \mathcal{B}(B, B')$; coproducts are disjoint unions. In Gr the coproduct of a family of groups is, what is called in group theory, their "free product"; in Ab, coproducts are simply "direct sums". In fact, both Gr and Ab have products, given by cartesian product with pointwise operations, but in Ab the product of a given finite family is isomorphic to the coproduct of that family. In that respect, Sup is much like Ab: all products "are" coproducts, they are given respectively by cartesian product with pointwise order operations and a "direct sum". A counterexample: whereas in Ban_∞, the category of real Banach spaces and bounded linear applications, all finite products exist, the product of an infinite family of objects does not exist in general (for instance, $\mathbf{R} \times \mathbf{R} \times ...$ does not exist). In a poset P, thought of as a thin small category $\mathcal{P}$, the product of a family of elements $(a_i)_{i \in I}$ is, when it exists, precisely the infimum of this family; likewise, if it exists, their coproduct is their supremum.

It can easily be verified that, given a family of objects $(C_i)_{i \in I}$ in a category $\mathcal{C}$ and a partition $(J_k)_{k \in K}$ of I, one has that $\prod_{i \in I} C_i \cong \prod_{k \in K} \prod_{j \in J_k} C_j$.

The analogue holds for coproducts, of course. It should be noted that in general "products do not distribute over coproducts": Consider four sets A_1, A_2, B_1, B_2, then $(A_1 \times A_2) \coprod (B_1 \times B_2)$ is different from $(A_1 \coprod B_1) \times (A_2 \coprod B_2)$ in Set, if only by a cardinality argument.

If $\mathcal{D}$ is the empty category, then the limit of this diagram in $\mathcal{C}$, if it exists, is called a "terminal object" of $\mathcal{C}$: it is just one object (or any object isomorphic to that object, cf. 1.2.6), usually denoted $\mathbf{1}$, such that for every other object C in $\mathcal{C}$ there exists exactly one morphism $C \to \mathbf{1}$ in $\mathcal{C}$. Dually, the colimit of the empty diagram is called an "initial object"; it is denoted $\mathbf{0}$. In some categories $\mathbf{1} = \mathbf{0}$, and then it is called a "zero object". For example, Set has as terminal object a (thus any) singleton $\{\star\}$ and as initial object the empty set. In Gr there is a zero object which is simply the group with one element.

A different type of (co)limit, is the "(co)equalizer" of two morphisms in a category $\mathcal{C}$. By definition, this is the (co)limit of the diagram that besides identities is $\bullet \rightrightarrows \bullet$. Thus, applying 1.2.5, the equalizer of two arrows $f, g: A \rightrightarrows B$ in a category $\mathcal{C}$ is an object K together with an arrow $k: K \to A$ such that $f \circ k = g \circ k$, and the pair (K, k) has the universal property. Dually, a coequalizer is a universal pair $(Q, q: B \to Q)$ of an object and a morphism in $\mathcal{C}$ for the property that $q \circ f = q \circ g$. The notations $\mathsf{Ker}(f, g)$ and $\mathsf{Coker}(f, g)$ are standard for respectively equalizer and coequalizer. If an equalizer (coequalizer) exists, then it is a monomorphisms (epimorphism). In every category, for every morphism $f: A \to B$, $\mathsf{Ker}(f, f)$ exists and is simply 1_A; likewise, $\mathsf{Coker}(f, f) = 1_B$. A morphism that is both an epimorphism and an equalizer is an isomorphism; dually, a morphism that is both a monomorphism and a coequalizer is an isomorphism as well.

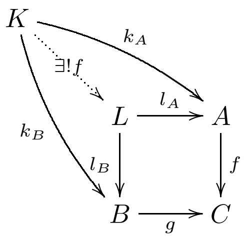
In Set, for two morphisms $f, g: A \rightrightarrows B$, the equalizer is given by the set $\{a \in A \mid f(a) = g(a)\}$ with the obvious inclusion mapping; the coequalizer is the quotient of B by the equivalence relation generated by $\{(f(a), g(a)) \mid a \in A\}$ with the quotient map. This construction for $\mathsf{Ker}(f, g)$ is also valid in "concrete categories" such as Gr, Top, Ban_∞, where the object part of $\mathsf{Ker}(f, g)$ then inherits its structure (as group, as topological space, as Banach space) from A. In the category Ab, the coequalizer of a group homomophism $f: A \to B$ and the zero homomorphism $0: A \to B$ is precisely the quotient of B by the subgroup $f(A)$; more generally there is the formula $\mathsf{Coker}(f, g) = \mathsf{Coker}(f - g, 0)$. An analogue holds in Vect_K. In Top the coequalizer of two arrows is constructed as in Set and provided with the quotient topology. In Gr and Rng one calculates $\mathsf{Coker}(f, g)$ as the quotient of B by the congruence relation (= smallest equivalence relation closed under the considered algebraic operations) generated by all pairs $\{(f(a), g(a)) \mid a \in A\}$. In any thin category there are of course no non-trivial parallel arrows; therefore all (co)equalizers exist and are identities.

There are several other important particular examples of limits and colimits, but (co)products and (co)equalizers prove to be the "generic" examples, in the sense of the following proposition.

1.2.7 Proposition *A category is (co)complete iff each set-indexed family of objects has a (co)product and each pair of parallel morphisms has a (co)equalizer.*

Proving this proposition in the non-trivial direction for a general $\mathcal{C}$ would lead us too far. Instead, let us verify its validity on an example: Knowing now that products and equalizers exist in Set, let us consider a functor $F\colon \mathcal{D} \to \mathsf{Set}$, where $\mathcal{D}$ is, besides identity morphisms, given by $\bullet \longrightarrow \bullet \longleftarrow \bullet$, and let us construct its limit (which is called a "pullback").

The image of F will be $A \xrightarrow{f} C \xleftarrow{g} B$, a diagram of sets and mappings. In principle, according to 1.2.5, a cone on this diagram in Set consists of a set L and three mappings, say $l_A\colon L \to A$, $l_B\colon L \to B$ and $l_C\colon L \to C$, such that $f \circ l_A = l_B$ and $g \circ l_C = l_B$ — which is precisely the naturality of a natural transformation $\Delta_L \Rightarrow F$ with components (l_A, l_B, l_C). In other terms, such a cone consists of a set L and two morphisms l_A, l_B such that $f \circ l_A = g \circ l_C$. This cone is a limit if it has the universal property, which means that any other such cone $(K; k_A, k_B)$ must factor uniquely through L:



But in fact, this set L can easily be described directly as:

$$L = \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

Hence L is the subset of the product $A \times B$ whose elements equalize the arrows $f \circ p_A$ and $g \circ p_B$, where now the morphisms p_A and p_B are the respective projections from $A \times B$ onto A and B; that is, L is the equalizer

$$L \rightarrowtail A \times B \overset{f \circ p_A}{\underset{g \circ p_B}{\rightrightarrows}} C.$$

With 1.2.7 we can now conclude that Set is complete and cocomplete; the same goes for Ab, Rng, Sup, etc. A poset viewed as thin small category is complete iff it is a complete lattice iff it is cocomplete as category.

We conclude this paragraph with some observations on functors that commute with limits.

1.2.8 Definition *A functor $F\colon \mathcal{A} \to \mathcal{B}$*

- *preserves (co)limits when, for every small category $\mathcal{D}$ and every functor $G\colon \mathcal{D} \to \mathcal{A}$, the image by F of the (co)limit of G, if this exists, is the (co)limit of $F \circ G$;*
- *reflects (co)limits when, for every small category $\mathcal{D}$ and every functor $G\colon \mathcal{D} \to \mathcal{A}$, if the image of a (co)cone on G is a (co)limit of $F \circ G$, then this (co)cone is the (co)limit of G.*

By 1.2.7 it is clear that, for a (co)complete category $\mathcal{A}$, $F: \mathcal{A} \to \mathcal{B}$ preserves (co)limits iff it does so for (co)products and (co)equalizers. (In fact, the result can be improved: Only the existence of (co)products in $\mathcal{A}$ is required.) Since we have by a direct argument that the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$ preserves equalizers and products (their calculation is "the same" in both these categories), we have that this functor preserves all limits.

Further it is true that a limit preserving functor $F: \mathcal{A} \to \mathcal{B}$, defined on a complete category $\mathcal{A}$, that reflects isomorphisms, also reflects limits. Therefore, the forgetful $\mathsf{Ab} \to \mathsf{Set}$ is also a limit reflecting functor. By calculation one shows that any full and faithful functor reflects limits. As a consequence, the "forgetful" functor $\mathsf{Ab} \to \mathsf{Gr}$ reflects limits; it also preserves them.

Any representable functor $\mathcal{A}(A, -): \mathcal{A} \to \mathsf{Set}$ preserves limits (even "large" limits); it requires a mere calculation to verify this. Remark that, when $F: \mathcal{A} \to \mathcal{B}$ is a contravariant functor, thus corresponds with a covariant functor $\mathcal{A}^{\mathsf{op}} \to \mathcal{B}$, if the latter preserves limits then the former will transform colimits into limits; therefore any contravariant representable functor $\mathcal{A}(-, A): \mathcal{A} \to \mathsf{Set}$ transforms colimits into limits.

Next consider the functor category $\mathsf{Fun}(\mathcal{A}, \mathcal{B})$ for a small category $\mathcal{A}$ and a complete category $\mathcal{B}$, and any functor $F: \mathcal{D} \to \mathsf{Fun}(\mathcal{A}, \mathcal{B})$ on any small category $\mathcal{D}$. The limit of F exists, and can be computed "pointwise": The limit of F is a functor $L: \mathcal{A} \to \mathcal{B}$ the value of which at an object $A \in \mathcal{A}$ is the limit of the functor $F_A: \mathcal{D} \to \mathcal{B}$ determined by $F_A(D) = F(D)(A)$. (However it is possible that $\mathsf{Fun}(\mathcal{A}, \mathcal{B})$ is complete while $\mathcal{B}$ is not! More details can be found in any handbook.) As a consequence, for any small category $\mathcal{A}$, the category $\mathsf{Fun}(\mathcal{A}, \mathsf{Set})$ is complete, and the Yoneda embeddings $\mathcal{A} \to \mathsf{Fun}(\mathcal{A}, \mathsf{Set})$ of 1.1.4 reflect limits, because they are both full and faithful. It is also true that the covariant embedding $Y_*: \mathcal{A} \to \mathsf{Fun}(\mathcal{A}, \mathsf{Set})$, mapping A onto $\mathcal{A}(-, A)$, preserves limits: Consider a functor $F: \mathcal{D} \to \mathcal{A}$, with limit $(L, \lambda: \Delta_L \Rightarrow F)$, then we must show that $(\mathcal{A}(-, L), (\mathcal{A}(-, l_D))_{D \in \mathcal{D}})$, where the l_D are the components of the natural transformation $\lambda: \Delta_L \Rightarrow F$, is the limit of $Y_* \circ F$. But Set being complete, it is sufficient to know that for every $A \in \mathcal{A}$, $(\mathcal{A}(A, L), (\mathcal{A}(A, l_D))_{D \in \mathcal{D}})$ is the limit of the functor $\mathcal{A}(A, -) \circ F: \mathcal{C} \to \mathsf{Set}$, because then a pointwise computation will give us the limit of F. This last assertion is true because L is the limit of F and every representable $\mathcal{A}(A, -)$ preserves limits.

This explains at once why the Yoneda embeddings are so important in Category Theory: They allow the covariant embedding of a small category $\mathcal{A}$ in the complete category $\mathsf{Fun}(\mathcal{A}, \mathsf{Set})$ in which the computation of limits is "easy" because all calculations actually take place in Set; the embedding then allows to "reflect" the limit back to the category $\mathcal{A}$.

1.3. ADJUNCTIONS AND EQUIVALENCES

Once more we take the theory of sets with order as a motivating example. For an isotone mapping $f: (P, \leq) \to (Q, \leq)$, a reverse isotone map $g: (Q, \leq) \to (P, \leq)$ is commonly called the "Galois adjoint" of (or "Galois dual" to) f if, for all $x \in P$

and all $y \in Q$:
$$g(y) \leq x \Leftrightarrow y \leq f(x).$$
Of course, g needn't exist, but if it exists, the pair (g, f) is referred to as a "Galois pair". From the categorical perspective, when writing $\mathcal{P}$, resp. $\mathcal{Q}$, for the poset P, resp. Q, viewed as thin category, this condition says exactly that, for any object x in $\mathcal{P}$ and any object y in $\mathcal{Q}$, the morphism sets $\mathcal{P}(g(y), x)$ and $\mathcal{Q}(y, f(x))$ are either both empty or both a singleton; in any case,
$$\mathcal{P}(g(y), x) \cong \mathcal{Q}(y, f(x)) \text{ in } \mathsf{Set}.$$
It is well known that a Galois pair (f, g) is also caracterized by the following inequations, for all $x \in P$ and $y \in Q$:
$$y \leq f(g(y)) \text{ and } g(f(x)) \leq x.$$
Recalling that natural transformations between isotone mappings are the "categorical translation" of the pointwise order of these maps, we can restate this as:
$$\text{there exist natural transformations } \eta\colon 1_{\mathcal{Q}} \Rightarrow f \circ g \text{ and } \varepsilon\colon g \circ f \Rightarrow 1_{\mathcal{P}}.$$
Now what does this Galois dual, if it exists, mean? Simply put, the map $g\colon Q \to P$ selects, for every element $y \in Q$, the smallest element of P whose image through f is still bigger than (or equal to) y. Again using the idiom of categories, we can say that for every object $y \in \mathcal{Q}$ we want to select an object $g(y) \in \mathcal{P}$ together with a $\mathcal{P}$–morphism $\eta_y\colon y \to f(g(y))$ (that is, $y \leq f(g(y))$) such that for any other such pair of an object $x \in \mathcal{P}$ and a morphism $\xi\colon y \to f(x)$ (that is, for any other x such that $y \leq f(x)$) there exists a (necessarily unique) morphism $\varphi\colon g(y) \to x$ (that is, $g(y) \leq x$). Such a couple $(g(y), \eta_y)$ is what we will call in 1.3.11 a "reflection of y along f".

In principle, a dual definition is possible as well: Calling then g as above the "left Galois adjoint" of f, by reversing all "less or equal than" signs in the above one can speak of a "right Galois adjoint" of f, which is then, if it exists, an isotone map $\tilde{g}$ such that $(f, \tilde{g})$ is a Galois pair.

It is the aim of definition 1.3.10 and the propositions thereafter to develop a theory of "adjoint functors" between (not necessarily thin) categories. Of course, the situation for thin categories is particularly simple because any diagram in such a category commutes; it should come as no surprise that we have to ask for a "compatibility" of the natural transformations $\eta\colon 1_{\mathcal{Q}} \Rightarrow f \circ g$ and $\varepsilon\colon g \circ f \Rightarrow 1_{\mathcal{P}}$ if we replace $\mathcal{P}$ and $\mathcal{Q}$ by more general categories. To properly express these requirements, we need the following technicality.

1.3.9 Proposition *Consider three categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, two pairs of functors $F, G\colon \mathcal{A} \rightrightarrows \mathcal{B}$ and $R, S\colon \mathcal{B} \rightrightarrows \mathcal{C}$, and two natural transformations $\alpha\colon F \Rightarrow G$ and $\gamma\colon R \Rightarrow S$. We can define a natural transformation $\gamma * \alpha\colon R \circ F \Rightarrow S \circ G$, called the "Godement product" of α and γ, by putting its components to be*
$$(\gamma * \alpha)_A = \gamma_{GA} \circ R(\alpha_A) = S(\alpha_A) \circ \gamma_{FA}.$$

If moreover two functors $H: \mathcal{A} \to \mathcal{B}$ and $T: \mathcal{B} \to \mathcal{C}$ with natural transformations $\beta: G \Rightarrow H$ and $\delta: S \Rightarrow T$ are given, then the "interchange law" holds, that is,

$$(\beta \circ \alpha) * (\delta \circ \gamma) = (\delta * \beta) \circ (\gamma * \alpha).$$

The straightforward proof is left to the reader. Note that the definition of the Godement product uses the naturality of $\gamma: R \Rightarrow S: \mathcal{B} \to \mathcal{C}$ applied to the $\mathcal{B}$–morphism $\alpha_A : FA \to GA$.

1.3.10 Definition *A functor $F: \mathcal{A} \to \mathcal{B}$ is right adjoint to a functor $G: \mathcal{B} \to \mathcal{A}$ (and G is left adjoint to F), notation $G \dashv F: \mathcal{A} \to \mathcal{B}$, if there exist natural transformations $\eta: 1_{\mathcal{B}} \Rightarrow F \circ G$ and $\varepsilon: G \circ F \Rightarrow 1_{\mathcal{A}}$ such that the following diagrams, of which the vertices are functors and the edges are natural transformations, commute:*

$$\begin{array}{ccc} 1_{\mathcal{B}} \circ F & \xRightarrow{\eta * 1_F} & F \circ G \circ F \\ & \diagdown\!\!\diagdown & \Downarrow {\scriptstyle 1_F * \varepsilon} \\ & & F \circ 1_{\mathcal{A}} \end{array} \qquad \begin{array}{ccc} G \circ 1_{\mathcal{B}} & \xRightarrow{1_G * \eta} & G \circ F \circ G \\ & \diagdown\!\!\diagdown & \Downarrow {\scriptstyle \varepsilon * 1_G} \\ & & 1_{\mathcal{A}} \circ G \end{array}$$

A functor can have several adjoints, but if $G_1 \dashv F$ and $G_2 \dashv F$ then there exists a natural isomorphism $\alpha: G_1 \overset{\sim}{\Rightarrow} G_2$. (Recall that such a natural isomorphism α is a natural transformation of which every component α_A is an isomorphism.)

The following proposition is then a characterization of adjointness of functors that generalizes what we already know for isotone maps between posets.

1.3.11 Proposition *The following are equivalent:*

1. *$G \dashv F: \mathcal{A} \to \mathcal{B}$;*

2. *there exist bijections $\theta_{A,B}: \mathcal{A}(GB, A) \to \mathcal{B}(B, FA)$, for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, which constitute a natural isomorphism between both expressions, seen as functors defined on $\mathcal{B}^{\mathsf{op}} \times \mathcal{A}$;*

3. *there exists a natural transformation $\eta: 1_{\mathcal{B}} \Rightarrow F \circ G$ such that for every $B \in \mathcal{B}$, (GB, η_B) is a "reflection" of B along F, that is, for every other pair (A, a) of an object $A \in \mathcal{A}$ and morphism $a: B \to FA$ in $\mathcal{B}$ there exists a unique $\mathcal{A}$–morphism $f: GB \to A$ such that $Ff \circ \eta_B = a$;*

4. *there exists a natural transformation $\varepsilon: G \circ F \Rightarrow 1_{\mathcal{A}}$ such that for every $A \in \mathcal{A}$, (FA, ε_A) is a "coreflection" of A along G, that is, the dual of the universal property in part 3 holds.*

Proof: (1 $\Rightarrow$ 2) Defining $\theta_{A,B}(a) = Fa \circ \eta_B$ for a morphism $a \in \mathcal{A}(GB, A)$, and conversely $\theta^{-1}(b) = \varepsilon_A \circ Gb$ for $b \in \mathcal{B}(A, FB)$, one uses the naturality of both η and ε to prove that these maps are indeed inverse to each other; the naturality of $\theta_{A,B}$ as indicated in the proposition is now easily verified. (2 $\Rightarrow$ 3) One proves that, given $B \in \mathcal{B}$, its reflection along F is given by $(GB, \theta_{GB,B}(1_{GB}))$, where in fact the $\mathcal{B}$–morphisms $\theta_{GB,B}(1_{GB}): B \to F(G(B))$ are components of the required natural transformation $1_{\mathcal{B}} \Rightarrow F \circ G$. ($3 \Rightarrow 1$) The component at an object $A \in \mathcal{A}$ of the required natural transformation $\varepsilon: G \circ F \Rightarrow 1_{\mathcal{A}}$ is obtained as follows: Since (FA, η_{FA}) is a reflection of FA along F, considering the pair $(A, 1_{FA})$, that satisfies the conditions as in (3), produces a unique morphism $\varepsilon_A : G(FA) \to A$ such that $F\varepsilon_A \circ \eta_B = 1_{FA}$ (which is already one of the "triangles" of definition 1.3.10). It remains to show that ε is indeed natural, and that the other "triangle" commutes as well, which is routine. (4) As for the equivalence with part 4, it follows by duality with part 3. □

Now for some examples. The forgetful functor $\mathsf{Ab} \to \mathsf{Set}$ has a left adjoint functor that associates to any set X just the coproduct in Ab of X copies of the abelian group of integers. Indeed, calling this group A_X, an explicit formula is:

$$A_X = \{(z_x)_{x \in X} \mid z_x \in \mathbf{Z}, \{x \mid z_x \neq 0\} \text{ is finite } \},$$

such that X is, as set, included in A_X simply by mapping an element x_0 to the sequence where z_{x_0} is 1 and the other are 0. Moreover, given an abelian group $(A, +)$ and a mapping $f: X \to A$, there is a unique factorization $g: A_X \to A$ given by $g((z_x)_{x \in X}) = \sum_{x \in X} z_x f(x)$. By 1.3.11 we have indeed described a left adjoint to the forgetful functor. Analogously, the forgetful $\mathsf{Rng} \to \mathsf{Set}$ has a left adjoint that maps a set X onto the ring of polynomials with integer coefficients, of which the variables are the elements of X. In general, a left adjoint $F: \mathsf{Set} \to \mathcal{A}$ to a "forgetful" functor $U: \mathcal{A} \to \mathsf{Set}$ is said to be a "free" functor, or the "free construction (of groups, rings, etc.)". Other such examples include: The free constructed group gives $F \dashv U: \mathsf{Gr} \to \mathsf{Set}$, the powerset construction gives $F \dashv U: \mathsf{Sup} \to \mathsf{Set}$, and so on.

The functor $- \times A: \mathsf{Set} \to \mathsf{Set}$, discussed earlier in this text, has a right adjoint, given by "exponentiation by A": For any two sets X, Y, there is an obvious bijective correspondence between maps $f: X \times A \to Y$ and maps $\tilde{f}: X \to Y^A$, that is, $\mathsf{Set}(X \times A, Y) \xrightarrow{\cong} \mathsf{Set}(X, Y^A)$, which by 1.3.11 means that $- \times A \dashv (-)^A$. Remark that $(-)^A = \mathsf{Set}(A, -)$. Along the same lines one can prove that, for a given small category $\mathcal{A}$, there is an adjunction $- \times \mathcal{A} \dashv \mathsf{Fun}(\mathcal{A}, -): \mathsf{Cat} \to \mathsf{Cat}$.

"Adjunctions can be composed", is the slogan by which the following is meant: Two adjunctions $G \dashv F: \mathcal{A} \to \mathcal{B}$, $S \dashv R: \mathcal{B} \to \mathcal{C}$, imply a third one, namely $(G \circ S) \dashv (R \circ F): \mathcal{A} \to \mathcal{C}$. This is evident by considering the canonical bijections $\mathcal{A}((G \circ S)(C), A) \xrightarrow{\cong} \mathcal{B}(SC, FA) \xrightarrow{\cong} \mathcal{C}(C, (R \circ F)(A))$, using 1.3.11. Denoting by Top the category of topological spaces and continuous maps, and by Comp the subcategory of compact Hausdorff spaces, the inclusion $\mathsf{Comp} \hookrightarrow \mathsf{Top}$ has a left adjoint, which is the Stone–Čech compactification. Also the forgetful $\mathsf{Top} \to \mathsf{Set}$ has a left adjoint, taking the discrete topology on a set (taking the chaotic topology on a set provides for a right adjoint), and therefore the forgetful $\mathsf{Comp} \to \mathsf{Set}$ has a left adjoint.

Functors with an adjoint have some "good" properties, for instance, a functor $F: \mathcal{A} \to \mathcal{B}$ with left adjoint preserves all limits which turn out to exist in $\mathcal{A}$; dually, a functor with right adjoint preserves all existing colimits. (For a proof, see any handbook on Category Theory.) Therefore the forgetful functors $\mathsf{Ab} \to \mathsf{Set}$,

$\mathsf{Gr} \to \mathsf{Set}$, $\mathsf{Rng} \to \mathsf{Set}$, $\mathsf{Sup} \to \mathsf{Set}$ and so on preserve limits, and their adjoints preserve colimits. And an isotone mapping with Galois adjoint preserves existing infima, its adjoint preserves suprema. But how do we know whether a functor has an adjoint?

Let us once more turn to isotone mappings between posets. If for such a mapping $f: (P, \leq) \to (Q, \leq)$, the poset P happens to be a complete lattice and f preserves infima, then the Galois adjoint exists; indeed, it is given by the easy formula:

$$g: Q \to P : y \mapsto inf\{x \in P \mid y \leq f(x)\}.$$

As a matter of fact, one considers all elements of P of which the image through f is bigger than an element $y \in Q$, and since this collection forms a subset of P, it makes sense to take its infimum; the infimum preserving map f will then take a value at this infimum that is bigger than y and by construction this infimum is the smallest such element. The following theorem generalizes this idea to the case of categories and functors.

1.3.12 Theorem (Adjoint functor theorem) *Let $\mathcal{A}$ be a complete category, $\mathcal{B}$ just any category, $F: \mathcal{A} \to \mathcal{B}$ a functor. The following are equivalent:*

1. *F has a left adjoint;*
2. *F preserves small limits and F satisfies the "solution set condition", that is, for any object B in $\mathcal{B}$ there exists a set $S_B \subseteq \mathcal{A}_0$ such that, for any $A \in \mathcal{A}_0$ and any morphism $b: B \to FA$ in $\mathcal{B}$ there exists an object $A' \in S_B$, a morphism $a: A' \to A$ in $\mathcal{A}$ and a morphism $b': B \to FA'$ in $\mathcal{B}$ such that $F(a) \circ b' = b$.*

A detailed proof of this theorem is beyond the scope of this introductory text; however it can be found in the references given at the end of the paper. Moreover, by duality one can rephrase the theorem so as to become a criterion for the existence of a right adjoint. Remark that, for ($1 \Rightarrow 2$), by 1.3.11, part 3, we can take for any $B \in \mathcal{B}$ the reflection along F as singleton "solution set". Of course, the solution set condition is trivial if $\mathcal{A}$ is a small category, as is the case for posets.

To conclude this paragraph, we present a definition upon which we already touched briefly when considering the full, faithful and essentially surjective functor from $\mathsf{Matr}(K)$ to FVect_K. These categories are clearly not isomorphic, but they are still "very much the same thing".

1.3.13 Definition *Two categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent if there exists a full, faithful and essentially surjective functor $F: \mathcal{A} \to \mathcal{B}$.*

1.3.14 Proposition *The following are equivalent:*

1. *$\mathcal{A}$ and $\mathcal{B}$ are equivalent;*
2. *there exist a functor $F: \mathcal{A} \to \mathcal{B}$ and a functor $G: \mathcal{B} \to \mathcal{A}$ and two (arbitrary) natural isomorphisms $1_{\mathcal{B}} \overset{\sim}{\Rightarrow} F \circ G$ and $G \circ F \overset{\sim}{\Rightarrow} 1_{\mathcal{A}}$;*

3. *there exists a functor $F: \mathcal{A} \to \mathcal{B}$ with left adjoint $G: \mathcal{B} \to \mathcal{A}$ such that the two canonical natural transformations $\eta: 1_{\mathcal{B}} \Rightarrow F \circ G$ and $\varepsilon: G \circ F \Rightarrow 1_{\mathcal{A}}$ of the adjunction are natural isomorphisms;*

4. *there exists a full and faithful functor $F: \mathcal{A} \to \mathcal{B}$ with a full and faithful right adjoint $G: \mathcal{B} \to \mathcal{A}$.*

For a lack of space, we omit the proof. Let us just indicate that part 2 of this proposition says that "equivalence" is indeed a weaker notion than "isomorphism" of categories: the latter would mean that there exist functors $F: \mathcal{A} \to \mathcal{B}$ and $G: \mathcal{B} \to \mathcal{A}$ such that $1_{\mathcal{B}} = F \circ G$ and $G \circ F = 1_{\mathcal{A}}$, in particular $G = F^{-1}$. But equivalence just means that $1_{\mathcal{B}} \cong F \circ G$ and $G \circ F \cong 1_{\mathcal{A}}$, that is, F has an inverse "up to a natural isomorphism". Part 3 of the proposition can be specified: if $\eta: 1_{\mathcal{B}} \overset{\simeq}{\Rightarrow} F \circ G$ and $\varepsilon: G \circ F \overset{\simeq}{\Rightarrow} 1_{\mathcal{A}}$, then there exists $\varepsilon': G \circ F \overset{\simeq}{\Rightarrow} 1_{\mathcal{A}}$ such that η, ε' satisfy the "triangular equalities" of definition 1.3.10; or dually, there exists $\eta': 1_{\mathcal{B}} \overset{\simeq}{\Rightarrow} F \circ G$ such that η', ε satisfy these equalities. Often, when working with an equivalence expressed by an adjoint pair of functors as in part 3 of the proposition, one speaks of an "adjoint equivalence".

Two equivalent categories share some categorical properties, such as for instance (co)completeness.

1.3.15 Proposition *If $\mathcal{A}$ is a (finitely) (co)complete category, then so is any category equivalent to $\mathcal{A}$.*

Proof: Let $F: \mathcal{A} \to \mathcal{B}$ be an equivalence, with $G: \mathcal{B} \to \mathcal{A}$ left adjoint to F as in 1.3.14. For any (finite) small diagram $D: \mathcal{D} \to \mathcal{B}$ we know by (finitary) completeness of $\mathcal{A}$ that $G \circ D: \mathcal{D} \to \mathcal{A}$ has a limit, which is preserved by F since F has a left adjoint. But that limit of $F \circ G \circ D$ is isomorphic to the limit of D just because $F \circ G$ is isomorphic to $1_{\mathcal{B}}$. (Dually for cocompleteness.) □

1.4. MONADS

A monoid $(M, \cdot, 1)$ is, of course, a set M equipped with an associative binary operation $(x, y) \mapsto x \cdot y$ that admits a unit 1. Therefore, in a monoid "one can compute the product of all finite sequences", the empty product being just the unit 1. More precisely: Write $T(M)$ for the set of finite sequences of elements of M, and $\varepsilon_M: M \to T(M)$, resp. $\mu_M: T(T(M)) \to T(M)$, for the obvious inclusion of M in $T(M)$, resp. the concatenation of a finite sequence of finite sequences of elements of M to a finite sequence of elements of M. Then, saying what the product of a finite sequence is, is giving a map $m: T(M) \to M$, that is suitably compatible with ε_M and μ_M, in the sense of the definition below.

1.4.16 Definition *A monad on a category $\mathcal{C}$ is a triple (T, ε, μ) where $T: \mathcal{C} \to \mathcal{C}$ is a functor, and $\varepsilon: id_{\mathcal{C}} \Rightarrow T$ and $\mu: T \circ T \Rightarrow T$ are natural transformations such that the following diagrams commute:*

$$
\begin{array}{ccccc}
T & \overset{\varepsilon * 1_T}{\Longrightarrow} & T \circ T & \overset{1_T * \varepsilon}{\Longleftarrow} & T \\
& \diagdown\!\!\diagdown & \Downarrow \mu & /\!\!/ & \\
& & T & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
T \circ T \circ T & \overset{\mu * 1_T}{\Longrightarrow} & T \circ T \\
{\scriptstyle 1_T * \mu}\Downarrow & & \Downarrow{\scriptstyle \mu} \\
T \circ T & \underset{\mu}{\Longrightarrow} & T
\end{array}
$$

An algebra on this monad is a pair (C, c) of an object C in $\mathcal{C}$ and a morphism $c: T(C) \to C$ of $\mathcal{C}$ such that the following diagrams commute:

$$
\begin{array}{ccc}
C & \xrightarrow{\varepsilon_C} & T(C) \\
& \diagdown\!\!\diagdown & \downarrow{\scriptstyle c} \\
& & C
\end{array}
\qquad\qquad
\begin{array}{ccc}
T(T(C)) & \xrightarrow{\mu_C} & T(C) \\
{\scriptstyle Tc}\downarrow & & \downarrow{\scriptstyle c} \\
T(C) & \xrightarrow[c]{} & C
\end{array}
$$

A morphism $f: (C, c) \to (D, d)$ of algebras is a morphism $f: C \to D$ of $\mathcal{C}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
T(C) & \xrightarrow{Tf} & T(C) \\
{\scriptstyle c}\downarrow & & \downarrow{\scriptstyle d} \\
C & \xrightarrow{f} & D
\end{array}
$$

Back to the example of monoids. Let $T: \mathsf{Set} \to \mathsf{Set}$ be the endofunctor that associates to any set M the set of finite sequences of M–elements, and has the obvious action on morphisms; let the maps ε_M and μ_M be as described before. Then indeed these maps constitute natural transformations, and indeed the first pair of diagrams commute. An algebra on this monad is now exactly a set equipped with a "multiplication" $m: T(M) \to M$, that is compatible with ε_M, and that is associative with unit, by the second pair of diagrams. By the last diagram in the definition above, a morphism of monoids is a map $f: M \to N$ in Set that "respects the multiplication". With a slogan one might thus say that "this monad recognizes all things monoid in Set".

Other examples include: For any set X, denote by $T(X)$ the set of formal finite linear combinations of elements of X with coefficients in a field K (modulo the usual equivalence relation); with the obvious action on morphisms one obtains a functor $T: \mathsf{Set} \to \mathsf{Set}$, that is a monad with the obvious definitions for the maps ε_X and μ_X to constitute the required natural transformations. An algebra $(X, x: T(X) \to X)$ on this monad is then a vector space over the field K, the map x assigning consistently to each formal linear combination a particular "outcome" in X. A morphism of algebras corresponds in this particular example with a linear application. Or consider the powerset functor on Set, mapping a set onto its powerset and with the obvious action on morphisms; then $\varepsilon_X: X \to 2^X: x \mapsto \{x\}$ and $\mu_X: 2^{2^X} \to 2^X: \mathcal{T} \mapsto \cup\mathcal{T}$, and a set X is an algebra iff it is a complete sup–lattice, a morphism of algebras being a supremum preserving map. And a monad on a poset, viewed as thin category, is precisely a closure operator, i.e., an idempotent isotone mapping, the algebras now being the "closed" elements, i.e., the fixpoints.

1.4.17 Proposition *For a monad (T, ε, μ) on a category $\mathcal{C}$, the algebras and their morphisms constitute a category, written $\mathcal{C}^T$, called the "Eilenberg-Moore category". The forgetful functor $U: \mathcal{C}^T \to \mathcal{C}$ is faithful, reflects isomorphisms and has a left adjoint $F: \mathcal{C} \to \mathcal{C}^T$, such that $U \circ F = T$.*

Proof: The first claim is obvious. Also faithfulness of U is obvious. Straightforward calculations show that, if $f: (C, c) \to (D, d)$ is such that f is an isomorphism in $\mathcal{C}$, then its inverse f^{-1} in $\mathcal{C}$ is a morphism of algebras $f^{-1}: (D, d) \to (C, c)$ that is inverse to f in $\mathcal{C}^T$. To prove that U has a left adjoint, one proves that the $\mathcal{C}^T$-object $(T(C), \mu_C)$ – recall that $\mu_C: (T \circ T)(C) \to T(C)$ – together with the morphism $\varepsilon_C: C \to T(C)$ constitutes the reflection of C along U, this for every object C in $\mathcal{C}$. By construction we now have that $U \circ F = T$. □

For the vector space monad $T: \mathsf{Set} \to \mathsf{Set}$, the algebras form a category equivalent to Vect_K; the adjoint to the forgetful $\mathsf{Vect}_K \to \mathsf{Set}$ is precisely the "free construction" of a vector space. The Eilenberg-Moore category associated to the powerset functor on Set is equivalent to Sup. One says that Vect_K and Sup are "monadic over Set". For a poset viewed as thin category, the category of algebras is precisely the subposet of closed elements; the adjoint of the forgetful (the inclusion of the closed elements in the poset) is the closure itself.

With 1.3.11 we can now write down explicitly that $F: \mathcal{C} \to \mathcal{C}^T$ maps an object C and a morphism $f: C \to D$ of $\mathcal{C}$ respectively to $(T(C), \mu_C)$ and $T(f): (T(C), \mu_c) \to (T(D), \mu_D)$.

1.4.18 Definition *With notations as in 1.4.16 and 1.4.17, we say that an algebra is free when it is isomorphic to one of the form $F(C) = (T(C), \mu_C)$.*

1.4.19 Proposition *The full subcategory of $\mathcal{C}^T$ generated by the free algebras, denoted $\mathcal{F}_T$, is equivalent to the following category, denoted $\mathcal{C}_T$ and called the "Kleisli category": objects are those of $\mathcal{C}$, a morphism $f: C \to D$ in $\mathcal{C}_T$ is a morphism $f: C \to T(D)$ in $\mathcal{C}$, composition of two such $\mathcal{C}_T$–morphisms $f: A \to B$, $g: B \to C$ is given by $A \xrightarrow{f} T(B) \xrightarrow{Tg} T(T(C)) \xrightarrow{\mu_C} T(C)$ and the identity on an object C of $\mathcal{C}_T$ is $\varepsilon_C: C \to T(C)$ in $\mathcal{C}$.*

Proof: It is easily checked that $\mathcal{C}_T$ is indeed a category. Further one defines a functor $\mathcal{C}_T \to \mathcal{F}_T$ by the following assignments: an object C of $\mathcal{C}_T$ is mapped onto $(T(C), \mu_C)$ and a morphism $f: C \to D$ in $\mathcal{C}_T$ (therefore $f: C \to T(D)$ in $\mathcal{C}$) is mapped onto $\mu_D \circ T(f)$. By choice of $\mathcal{F}_T$ this functor is essentially surjective on objects. Calculations show that this functor is also fully faithful, thus an equivalence. □

When considering once more the vector space monad $T: \mathsf{Set} \to \mathsf{Set}$, we have that Set^T is equivalent to Set_T because every vector space V is free, it being isomorphic to $T(B)$ for B a base of V.

1.4.20 Corollary *Denoting by $V: \mathcal{C}_T \to \mathcal{C}$ the composite of the equivalence $\mathcal{C}_T \to \mathcal{F}_T$ with the inclusion $\mathcal{F}_T \hookrightarrow \mathcal{C}^T$ and the forgetful $\mathcal{C}^T \to \mathcal{C}$, we obtain a faithful,* *isomorphism-reflecting functor which has a left adjoint $G: \mathcal{C} \to \mathcal{C}_T$ that is the identity on objects and maps a morphism $f: C \to D$ to $\varepsilon_D \circ f: C \to D$.*

Given a monad (T, ε, μ) on a category $\mathcal{C}$ we have produced two adjunctions:

- the "Eilenberg-Moore adjunction" $F \dashv U: \mathcal{C}^T \to \mathcal{C}$;
- the "Kleisli adjunction" $G \dashv V: \mathcal{C}_T \to \mathcal{C}$.

Also a converse is true, namely, every adjoint pair produces a monad.

1.4.21 Proposition *Let $L \dashv R: \mathcal{X} \to \mathcal{C}$ constitute an adjoint pair, let the unit and counit of this adjunction be $\alpha: id_{\mathcal{C}} \Rightarrow R \circ L$ and $\beta: L \circ R \Rightarrow id_{\mathcal{X}}$. Putting $T = R \circ L$, $\varepsilon = \alpha$ and $\mu = id_R * \beta * id_L$ defines a monad (T, ε, μ) on $\mathcal{C}$. Moreover, there exists a full functor $J: \mathcal{X} \to \mathcal{C}^T$ and a fully faithful functor $K: \mathcal{C}_T \to \mathcal{X}$ such that $U \circ J \cong R$, $R \circ K \cong V$ and $J \circ K$ is isomorphic to the canonical inclusion $\mathcal{C}_T \hookrightarrow \mathcal{C}^T$. (Notations as in 1.4.17, 1.4.19 and 1.4.20).*

Proof: By calculation one verifies the axioms for a monad, cf. 1.4.16. The functor $J: \mathcal{X} \to \mathcal{C}^T$ is defined to map an object X of $\mathcal{X}$ onto $(R(X), R(\beta_X))$ and a morphism $x: X \to Y$ of $\mathcal{X}$ to $R(x): R(X) \to R(Y)$. On the other hand, the functor $K: \mathcal{C}_T \to \mathcal{X}$ is defined to map an object C of $\mathcal{C}_T$ onto $K(C) = L(C)$ and a morphism $f: C \to D$ of $\mathcal{C}_T$ onto $K(f) = \beta_{L(D)} \circ L(f): L(C) \to L(D)$. One verifies that this defines indeed functors with the indicated domain and codomain. Further, the isomorphisms referred to in the proposition are true by construction. Since $J \circ K: \mathcal{C}_T \to \mathcal{C}^T$ is isomorphic to the canonical inclusion (that is the composite of the equivalence $\mathcal{C}_T \to \mathcal{F}_T$ and the obvious inclusion $\mathcal{F}_T \hookrightarrow \mathcal{C}^T$, cf. 1.4.19), and this inclusion is full and faithful, it follows that J is full and K is faithful. Any morphism $h \in \mathcal{X}(L(C), L(D))$ corresponds by the adjunction $L \dashv R$ to a morphism $f \in \mathcal{C}(C, RL(D))$ (cf. 1.3.11), that is a morphism $f: D \to D$ in $\mathcal{C}_T$ such that $K(f) = h$. This means that K is full. □

The reader can easily apply this proposition to all the examples of adjoint functors listed in subsection 1.2.

So every monad produces adjoint pairs, and every adjoint pair produces a monad. The next proposition, that can be proved by calculation, shows how the ends meet.

1.4.22 Proposition *For a monad (T, ε, μ) on a category $\mathcal{C}$, both the Eilenberg-Moore adjunction and the Kleisli adjunction reproduce the monad (T, ε, μ) via the construction of 1.4.21.*

There exist several theorems that say when a given functor $R: \mathcal{X} \to \mathcal{C}$ is "monadic", that is, when there exists a monad (T, ε, μ) on $\mathcal{C}$ such that $\mathcal{X}$ is equivalent to $\mathcal{C}^T$ via the construction of 1.4.21. However important these theorems are, they are beyond the scope of this introductory text. Other important topics related to monads that we can not explain here include: calculation of limits and colimits in categories of algebras, the "adjoint lifting theorem", descent theory, and many more. The reader can find out more about all this in the references given at the end of this paper.

2. Enriched category theory

Close inspection of definition 1.1.1 shows that a category $\mathcal{C}$ is a collection of objects $A, B, C, \ldots$ such that with any two objects A and B is associated an object $\mathcal{C}(A,B)$ of Set, with any three objects A, B and C is associated a composition morphism of Set, $c_{A,B,C}\colon \mathcal{C}(A,B) \times \mathcal{C}(B,C) \to \mathcal{C}(A,C)$, and with any object A is associated a morphism of Set, $u_A\colon \{\star\} \to \mathcal{C}(A,A)$, which selects that morphism of $\mathcal{C}(A,A)$ that will play the role of identity on A, of course such that some adequate axioms hold. In the same idiom, a functor $F\colon \mathcal{A} \to \mathcal{B}$ maps an object $A \in \mathcal{A}$ on an object $F(A) \in \mathcal{B}$ and has an action on sets of morphisms given by morphisms in Set, $F_{A,A'}\colon \mathcal{A}(A,A') \to \mathcal{B}\big(F(A), F(A')\big)$. Finally a natural transformation $\alpha\colon F \Rightarrow G$ between functors $F, G\colon \mathcal{A} \rightrightarrows \mathcal{B}$ can be viewed as a collection of morphisms $\alpha_A\colon \{\star\} \to \mathcal{B}\big(F(A), G(A)\big)$ such that a "naturality condition" holds. It is now our aim to replace in the above the category Set by a more abstract category $\mathcal{V}$ that mimics just enough properties of Set so as to do "category theory". It turns out that already with a symmetric monoidal category $\mathcal{V}$, introduced in section 2.1, one can develop "$\mathcal{V}$-enriched category theory" as in section 2.2. However, to build a theory that is in many ways parallel to ordinary category theory, one needs closedness of $\mathcal{V}$, as discussed in the concluding section 2.3.

2.1. SYMMETRIC MONOIDAL CATEGORIES

In view of the previous introduction, one would be tempted to replace Set by a category $\mathcal{V}$ with products; this is certainly an interesting possible generalization, which provides many interesting examples. But if we choose for $\mathcal{V}$ the category of real vector spaces, which has products, a $\mathcal{V}$-enriched category $\mathcal{C}$ has now vector spaces $\mathcal{C}(A,B)$, $\mathcal{C}(B,C)$ of morphisms and certainly we do not want the composition $\mathcal{C}(A,B) \times \mathcal{C}(B,C) \to \mathcal{C}(A,C)$ to be linear, that is, to satisfy

$$\Big(\sum_{i=1}^{n} r_i g_i\Big) \circ \Big(\sum_{i=1}^{n} s_i f_i\Big) = \sum_{i=1}^{n} r_i s_i (g_i \circ f_i).$$

We want instead the more standard relation

$$\Big(\sum_{i=1}^{n} r_i g_i\Big) \circ \Big(\sum_{j=1}^{m} s_j f_j\Big) = \sum_{i,j} r_i s_j (g_i \circ f_j),$$

that is, the bilinearity of the composition. But this bilinearity reduces to the linearity of the corresponding morphism $\mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \to \mathcal{C}(A,C)$.

2.1.23 Definition *A symmetric monoidal category $\mathcal{V}$ is a category $\mathcal{V}$ provided with a bifunctor $-\otimes -\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ which is associative, symmetric and admits a unit I. More precisely, there are natural isomorphisms $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$, $A \otimes B \cong B \otimes A$, $A \otimes I \cong A$ which satisfy coherence axioms, equivalent to the* *fact that every diagram constructed from these isomorphisms and identities, is necessarily commutative.*

A minimal list of coherence axioms can be found in any reference on the subject. It is of course a deep theorem to prove that a finite list of coherence axioms implies the commutativity of all possible diagrams mentioned in the previous definition (see [3] for a proof).

Among the examples, we have certainly all categories admitting binary products and a terminal object 1, which is the unit of the product. Among these we have in particular all the preordered sets admitting binary infima and a top element. Another generic class of examples is obtained by taking the actual tensor product as monoidal structure: abelian groups, modules over a commutative ring, graded (or differential) modules over a graded (or differential) commutative ring, Banach spaces with their projective tensor product, locally convex spaces, and so on. The category Sup of complete lattices and join preserving maps is yet another example; the tensor product of X and Y in Sup is obtained by taking the set of formal expressions $\bigvee_{i\in I} x_i \otimes y_i$, with I a set, $x_i \in X$ and $y_i \in Y$, and performing the quotient by the congruence generated by $x \otimes (\bigvee_{i\in I} y_i) \cong \bigvee_{i\in I}(x \otimes y_i)$ and $(\bigvee_{i\in I} x_i) \otimes y \cong \bigvee_{i\in I}(x_i \otimes y)$.

To conclude this section, consider the "forgetful functor" $\mathcal{V}(I,-)\colon \mathcal{V} \to \mathsf{Set}$. In the case $\mathcal{V} = \mathsf{Set}$, $I = \{\star\}$ and $\mathcal{V}(I,-)$ is isomorphic to the identity. In the case of abelian groups, with the tensor product as monoidal structure, one has $I = \mathbf{Z}$, the group of integers. For a group A, the morphisms $\mathbf{Z} \to A$ are determined by their value on 1, thus are in bijection with the elements of A; therefore $\mathcal{V}(I,-)$ is isomorphic to the ordinary forgetful functor $\mathsf{Ab} \to \mathsf{Set}$. An analogous conclusion holds for modules over a commutative ring R, replacing now $\mathbf{Z}$ by R.

Now we are ready to introduce category theory enriched in such a symmetric monoidal category $\mathcal{V}$.

2.2. ENRICHED CATEGORIES

2.2.24 Definition *Let $\mathcal{V}$ be a symmetric monoidal category. A $\mathcal{V}$–category consists in*

- *a class $\mathcal{C}_0$ of objects;*
- *for all objects $A, B \in \mathcal{C}_0$, an object $\mathcal{C}(A,B) \in \mathcal{V}$ called the "object of morphisms from A to B";*
- *for all objects $A, B, C \in \mathcal{C}_0$, a "composition" morphism in $\mathcal{V}$;* $c_{A,B,C}\colon \mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \to \mathcal{C}(A,C)$;
- *for every object $A \in \mathcal{C}$, a $\mathcal{V}$–morphism "identity on A" $u_A\colon I \to \mathcal{C}(A,A)$.*

Those data must satisfy diagrammatically the axioms expressing the associativity of the composition and the unit property; namely, the commutativity of

$$
\begin{array}{ccccc}
\big(\mathcal{C}(A,B)\otimes\mathcal{C}(B,C)\big)\otimes\mathcal{C}(C,D) & & \xrightarrow{\quad\cong\quad} & & \mathcal{C}(A,B)\otimes\big(\mathcal{C}(B,C)\otimes\mathcal{C}(C,D)\big) \\
\downarrow{\scriptstyle c_{A,B,C}\otimes 1} & & & & \downarrow{\scriptstyle 1\otimes c_{B,C,D}} \\
\mathcal{C}(A,C)\otimes\mathcal{C}(C,D) & \xrightarrow[c_{A,C,D}]{} & \mathcal{C}(A,D) & \xleftarrow[c_{A,B,D}]{} & \mathcal{C}(A,B)\otimes\mathcal{C}(B,D)
\end{array}
$$

where the isomorphism is the associativity one, and that of

$$
\begin{array}{ccccc}
I\otimes\mathcal{C}(A,B) & \xleftarrow{\cong} & \mathcal{C}(A,B) & \xrightarrow{\cong} & \mathcal{C}(A,B)\otimes I \\
\downarrow{\scriptstyle u_A\otimes 1} & & \| & & \downarrow{\scriptstyle 1\otimes u_B} \\
\mathcal{C}(A,A)\otimes\mathcal{C}(A,B) & \xrightarrow[c_{A,A,B}]{} & \mathcal{C}(A,B) & \xleftarrow[c_{A,B,B}]{} & \mathcal{C}(A,B)\otimes\mathcal{C}(B,B)
\end{array}
$$

Among the examples of $\mathcal{V}$-categories, we obtain in fact all the ordinary categories as Set-enriched categories. Categories like modules on a ring or vector spaces over a field are enriched in Ab, the category of abelian groups, with the tensor product as monoidal structure. The category of Hilbert spaces is enriched in that of Banach spaces, with the projective tensor product as monoidal structure. A category enriched in Sup is called a "quantaloid". Now consider the positive reals $[0,\infty[$ as a category, where a single arrow exists from r to s when $r \geq s$. The addition of reals provides a symmetric monoidal structure on that category, with 0 as unit. Every metric space (X,d) can be viewed as a category $\mathcal{X}$ enriched in the monoidal category of positive reals: just choose the elements of X as objects of $\mathcal{X}$ and define $\mathcal{X}(x,y) = d(x,y)$, the distance between those two points. The axioms for a $\mathcal{V}$-category reduce to $d(x,y)+d(y,z) \geq d(x,z)$ and $d(x,x) = 0$.

Given a $\mathcal{V}$-category $\mathcal{C}$, it is routine to verify the existence of a dual $\mathcal{V}$-category $\mathcal{C}^{\mathsf{op}}$, with the same objects and "reversed objects of morphisms", $\mathcal{C}^{\mathsf{op}}(A,B) = \mathcal{C}(B,A)$. In the same spirit, it is easy to check that given two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, we get a new category $\mathcal{A}\otimes\mathcal{B}$ with objects the pairs (A,B) of an object $A\in\mathcal{A}$ and an object $B\in\mathcal{B}$, and $(\mathcal{A}\otimes\mathcal{B})\big((A,B),(A',B')\big) = \mathcal{A}(A,A')\otimes\mathcal{B}(B,B')$. Writing down the straightforward details will in particular emphasize the necessity of the symmetry axiom in the definition of a monoidal category.

2.2.25 Definition *Let $\mathcal{V}$ be a symmetric monoidal category and $\mathcal{A}$, $\mathcal{B}$ two $\mathcal{V}$-categories. A $\mathcal{V}$-functor $F\colon\mathcal{A}\to\mathcal{B}$ consists in giving*

- *for each object $A\in\mathcal{A}_0$, an object $F(A)\in\mathcal{B}_0$;*
- *for all objects $A, A'\in\mathcal{A}_0$, a $\mathcal{V}$–morphism $F_{A,A'}\colon\mathcal{A}(A,A')\to\mathcal{B}\big(F(A),F(A')\big)$.*

Those data are required to satisfy diagrammatically the axioms of functoriality, namely the commutativity of

$$
\begin{array}{ccc}
\mathcal{A}(A,A')\otimes\mathcal{A}(A',A'') & \xrightarrow{F_{A,A'}\otimes F_{A',A''}} & \mathcal{B}\big(F(A),F(A')\big)\otimes\mathcal{B}\big(F(A'),F(A'')\big) \\
\downarrow{\scriptstyle c_{A,A',A''}} & & \downarrow{\scriptstyle c_{F(A),F(A'),F(A'')}} \\
\mathcal{C}(A,A'') & \xrightarrow[F_{A,A''}]{} & \mathcal{B}\big(F(A),F(A'')\big)
\end{array}
$$

and the equality $F_{A,A}\circ u_A = u_{F(A)}$.

2.2.26 Definition *Let $\mathcal{V}$ be a symmetric monoidal category, $\mathcal{A}$, $\mathcal{B}$ two $\mathcal{V}$-categories and $F, G\colon \mathcal{A} \rightrightarrows \mathcal{B}$ two $\mathcal{V}$-functors. A $\mathcal{V}$-natural transformation $\alpha\colon F \Rightarrow G$ consists in giving, for each object $A \in \mathcal{A}$, a morphism $\alpha_A\colon I \to \mathcal{B}\big(F(A), G(A)\big)$. Those data are required to satisfy diagrammatically the axiom of naturality, that the reader will easily write down.*

We leave also to the reader the routine definitions of composites of $\mathcal{V}$-functors or $\mathcal{V}$-natural transformations, and the description of the identity functors and identity natural transformations.

In the example of metric spaces, observe that a $[0, \infty[$-functor $F\colon (X, d) \to (X', d')$ between metric spaces is a contraction $f\colon X \to X'$, that is, a mapping satisfying $d\big((f(x), f(x')\big) \geq d(x, x')$. Such a mapping is of course continuous.

It is immediate to observe that applying the forgetful functor $\mathcal{V}(I, -)$ of the previous section to $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations yields underlying Set-based, thus ordinary, categories, functors and natural transformations. We conclude with the definition of $\mathcal{V}$-adjoint functors.

2.2.27 Definition *Let $\mathcal{V}$ be a symmetric monoidal category and $\mathcal{A}$, $\mathcal{B}$ two $\mathcal{V}$-categories. Two $\mathcal{V}$-functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ are $\mathcal{V}$-adjoint when there exist $\mathcal{V}$-natural transformations $\eta\colon \mathsf{id}_{\mathcal{B}} \Rightarrow F \circ G$ and $\varepsilon\colon G \circ F \Rightarrow \mathsf{id}_{\mathcal{A}}$ which satisfy the same triangular identities as in definition 1.3.10.*

2.3. SYMMETRIC MONOIDAL CLOSED CATEGORIES

Ordinary category theory is Set-based category theory ... and Set is itself a category, thus a Set-based category. But a symmetric monoidal category $\mathcal{V}$ has a priori no reason to be itself a $\mathcal{V}$-category. For example, finite products induce a symmetric monoidal structure on the category of groups; but in the non-abelian case, there is no way of defining a relevant "group of morphisms" $\mathsf{Gr}(G, G')$ between two groups. A symmetric monoidal category $\mathcal{V}$ is closed when, "in a natural way", it is itself a $\mathcal{V}$-category.

2.3.28 Definition *A symmetric monoidal category $\mathcal{V}$ is closed when, for every object B of $\mathcal{V}$, the functor $- \otimes B\colon \mathcal{V} \to \mathcal{V}$ has a right adjoint, which we denote by $[B, -]$. One has thus natural bijections $\mathcal{V}\big(A \otimes B, C\big) \cong \mathcal{V}\big(A, [B, C]\big)$.*

Putting $A = [B, C]$ in the previous formula, the identity on $[B, C]$ induces by adjunction an "evaluation morphism" $\mathsf{ev}_B\colon [B, C] \otimes B \to C$.

2.3.29 Proposition *In a symmetric monoidal category $\mathcal{V}$, the following data provide $\mathcal{V}$ with the structure of a $\mathcal{V}$-category:*

- *the objects of $\mathcal{V}$;*
- *for all B, C in $\mathcal{V}$, the object $[B, C] \in \mathcal{V}$;*

- *for all A, B, C in $\mathcal{V}$, the composition $[A,B]\otimes[B,C]\to[A,C]$ which corresponds by adjunction and symmetry to the composite*

$$[A,B]\otimes A\otimes[B,C]\xrightarrow{ev_A\otimes 1}B\otimes[B,C]\xrightarrow{ev_B}C$$

- *for each A in $\mathcal{V}$, the unit $I\to[A,A]$ which corresponds by adjunction to the isomorphism $I\otimes A\cong A$.*

In particular, the previous proposition allows considering $\mathcal{V}$-functors $\mathcal{C}\to\mathcal{V}$, for every $\mathcal{V}$-category $\mathcal{C}$, and $\mathcal{V}$-natural transformations between them. Among these functors we have the representable ones, $\mathcal{C}(A,-)$, and by duality the $\mathcal{V}$-functors $\mathcal{C}(-,A)$.

In the category of sets, the classical formula $\mathsf{Set}\big(A\times B,C\big)\cong\mathsf{Set}\big(A,\mathsf{Set}(B,C)\big)$ indicates that the set of mappings from B to C exhibits the expected closed structure. A symmetric monoidal closed structure, in which the monoidal structure is the cartesian product, is called a "cartesian closed category". As pointed out earlier in this text, also Cat, the category of small categories and functors, is a cartesian closed category.

For every small category $\mathcal{C}$, the category $\mathcal{V}=[\mathcal{C},\mathsf{Set}]$ of Set-valued functors and natural transformations is cartesian closed. The product of two functors F and G is computed pointwise. Given another functor H and choosing for F the representable functor $F=\mathcal{C}(A,-)$, the expected cartesian closedness forces, by the Yoneda lemma $[G,H](A)=\mathsf{Nat}\big(\mathcal{C}(A,-),[G,H]\big)\cong\mathsf{Nat}\big(\mathcal{C}(A,-)\times G,H\big)$. Choosing this last formula as a definition of $[G,H](A)$ yields indeed the expected cartesian closed structure. By duality, an analogous result holds for the categories $[\mathcal{C}^{\mathsf{op}},\mathsf{Set}]$ of contravariant functors. The case of simplicial sets is a famous particular case, obtained by choosing $\mathcal{C}=\Delta$.

Next choose a topological space $(X,\mathcal{T})$ and view the poset $\mathcal{T}$ of open subsets as a small category, with inclusions of open subsets as morphisms. Binary products in the category $\mathcal{T}$ are just binary intersections and the terminal object is X. It is easily seen that the right adjoint $[V,-]$ to the functor $-\cap V$ is given by $[V,W]=\bigcap\{U\in T\mid U\cap V\subseteq W\}$. Thus the category $\mathcal{T}$ is cartesian closed.

If we choose for $\mathcal{V}$ the category of abelian groups, Ab, or the category of modules on a commutative ring R, Mod_R, the classical isomorphism $\mathcal{V}\big(A\otimes B,C\big)\cong\mathcal{V}\big(A,\mathcal{V}(B,C)\big)$ indicates that the category is symmetric monoidal closed, when $\mathcal{V}(B,C)$ is provided with the pointwise operations. An analogous observation holds for the category Sup of sup–lattices. Those who are familiar with graded or differential modules will immediately recognize that they provide symmetric monoidal closed structures as well. An analogous conclusion holds in the case of Banach spaces, with now $\mathcal{V}(B,C)$ the Banach space of linear bounded mappings. The category $[0,\infty[$ of section 2.2 is symmetric monoidal as well, with $[s,t]=\mathsf{max}\{t-s,0\}$.

When the base category $\mathcal{V}$ is symmetric monoidal closed, complete and cocomplete, all classical theorems of category theory have their enriched counterpart.

2.3.30 Proposition *Let $\mathcal{V}$ be a complete symmetric monoidal closed category and $\mathcal{A}$, $\mathcal{B}$ two $\mathcal{V}$-categories, with $\mathcal{A}$ small. In those conditions, the category of $\mathcal{V}$-functors from $\mathcal{A}$ to $\mathcal{B}$ and $\mathcal{V}$-natural transformations between them is itself provided with the structure of a $\mathcal{V}$-category.*

Given two $\mathcal{V}$-functors $F, G\colon \mathcal{A} \rightrightarrows \mathcal{B}$, the object $\mathsf{Nat}(F,G) \in \mathcal{V}$ of $\mathcal{V}$-natural transformations is defined as an equalizer

$$\mathsf{Nat}(F,G) \rightarrowtail \prod_{A \in \mathcal{A}} \mathcal{B}\big(F(A), G(A)\big) \rightrightarrows \prod_{A, A' \in \mathcal{A}} \Big[\mathcal{A}(A,A'), \mathcal{B}\big(F(A), G(A')\big)\Big]$$

where the parallel arrows mimic diagrammatically the two composites whose equality – forced by the equalizer – expresses the naturality. This allows at once a Yoneda lemma:

2.3.31 Proposition *Consider a complete symmetric monoidal closed category $\mathcal{V}$. Let $\mathcal{C}$ be a small $\mathcal{V}$-category and $F\colon \mathcal{C} \to \mathsf{Set}$ a $\mathcal{V}$-functor. The isomorphism $F(A) \cong \mathsf{Nat}\big(\mathcal{C}(A,-), F\big)$ holds in $\mathcal{V}$ for every object $A \in \mathcal{C}$.*

Next the case of adjoint functors:

2.3.32 Proposition *Let $\mathcal{V}$ be a symmetric monoidal closed category and $\mathcal{A}$, $\mathcal{B}$ two $\mathcal{V}$ categories. Two $\mathcal{V}$-functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ are $\mathcal{V}$-adjoint when there exists a $\mathcal{V}$-natural isomorphism $\mathcal{A}\big(G(B), A\big) \cong \mathcal{B}\big(B, F(A)\big)$, where both sides of the formula are viewed as $\mathcal{V}$-functors $\mathcal{B}^{\mathsf{op}} \otimes \mathcal{A} \to \mathcal{V}$.*

We conclude by introducing $\mathcal{V}$-completeness and $\mathcal{V}$-cocompleteness. We observe first that in the case of an ordinary cocomplete category $\mathcal{C}$, the isomorphisms

$$\mathcal{C}\Big(\coprod_{i \in I} A, B\Big) \cong \prod_{i \in I} \mathcal{C}(A,B) \cong \mathsf{Set}\big(I, \mathcal{C}(A,B)\big)$$

indicate that the functor from Set to $\mathcal{C}$ that maps an object I to $\coprod_{i \in I} A$ is left adjoint to the representable functor $\mathcal{C}(A,-)$. By duality, when $\mathcal{C}$ is complete, the functor from Set to $\mathcal{C}^{\mathsf{op}}$, taking I to $\prod_{i \in I} A$ is left adjoint to the representable functor $\mathcal{C}^{\mathsf{op}}(A,-) = \mathcal{C}(-,A)$. Thus the existence of left adjoints to the representable functors reduces to the existence of "copowers" or "powers" in $\mathcal{C}$ indexed by an object I of the base category Set. Therefore the following definition.

2.3.33 Definition *Let $\mathcal{V}$ be a complete and cocomplete symmetric monoidal closed category; let $\mathcal{A}$ be a $\mathcal{V}$–category.*

- *$\mathcal{A}$ is tensored when each $\mathcal{V}$-representable functor $\mathcal{A}(A,-)\colon \mathcal{A} \to \mathcal{V}$ has a left $\mathcal{V}$-adjoint;*
- *$\mathcal{A}$ is cotensored when each $\mathcal{V}$-representable functor $\mathcal{A}(-,A)\colon \mathcal{A}^{\mathsf{op}} \to \mathcal{V}$ has a left $\mathcal{V}$-adjoint.*

Given objects $V \in \mathcal{V}$ and $A \in \mathcal{C}$, we write $V \otimes A \in \mathcal{C}$ for their tensor and $\{V, A\} \in \mathcal{C}$ for their cotensor, when these exist. Those objects are thus characterized by the existence of $\mathcal{V}$-natural isomorphisms

$$\mathcal{C}\big(V \otimes A, B\big) \cong \big[V, \mathcal{C}(A, B)\big], \quad \mathcal{C}\big(B, \{V, A\}\big) \cong \big[V, \mathcal{C}(B, A)\big].$$

A $\mathcal{V}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves tensors or cotensors when $F(V \otimes A) \cong V \otimes F(A)$ or $F\big(\{V, A\}\big) \cong \big\{V, F(A)\big\}$. It is immediate to notice that $\mathcal{V}$ itself is tensored and cotensored, with $V \otimes W$ the usual tensor product and $\{V, W\} = [V, W]$. And by definition of a cotensor, each $\mathcal{V}$-representable functor $\mathcal{A}(B, -)\colon \mathcal{A} \to \mathcal{V}$ preserves cotensors, while the contravariant $\mathcal{V}$-representable functors $\mathcal{A}(-, B)$ transform tensors in cotensors.

2.3.34 Definition *Let $\mathcal{V}$ be a complete and cocomplete, symmetric monoidal closed category.*

- *A $\mathcal{V}$-category $\mathcal{C}$ is $\mathcal{V}$-complete when it admits cotensors and its underlying* Set*-category is complete in the usual sense.*
- *A $\mathcal{V}$-category $\mathcal{C}$ is $\mathcal{V}$-cocomplete when it admits tensors and its underlying* Set*-category is cocomplete in the usual sense.*

Of course, a $\mathcal{V}$-functor is said to preserve $\mathcal{V}$-limits when it preserves cotensors, and the underlying Set-functor preserves ordinary limits. And dually for $\mathcal{V}$-colimits. The previous notions allow natural generalizations of the main theorems of category theory, for example:

2.3.35 Theorem *Let $\mathcal{V}$ be a complete symmetric monoidal closed category. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{V}$-categories and $F\colon \mathcal{A} \to \mathcal{B}$ a $\mathcal{V}$-functor. If*

- *$\mathcal{A}$ is $\mathcal{V}$-complete;*
- *F preserves $\mathcal{V}$-limits;*
- *the underlying* Set*-functor satisfies the "solution set condition" (see 1.3.12);*

then F admits a left $\mathcal{V}$-adjoint.

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