An introduction to quantaloid-enriched categories

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Abstract. This survey paper, specifically targeted at a readership of fuzzy logicians and fuzzy set theorists, aims to provide a gentle introduction to the basic notions of quantaloid-enriched category theory. We discuss at length the definitions of quantaloid, quantaloid-enriched category, distributor and functor, always giving several examples that – hopefully – appeal to the intended readership. To indicate the strength of this general theory, we explain in considerable detail how (co)limits are dealt with, and particularly how the Yoneda embedding of a quantaloid-enriched category in its free (co)completion comes to be. Our insistence on quantaloid-enrichment (rather than quantale-enrichment) is duly explained by examples requiring a notion of "partial elements" (sheaves, partial metric spaces). A final section glosses over some further topics, providing ample references for the interested reader.
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1. Historical background

Categories, functors and natural transformations were first defined by S. Eilenberg and S. MacLane [1945] as "a technical background for the intuitive notion of naturality", providing "opportunities for the comparison of constructions [...] in different branches of mathematics". In that paper they develop the basic notions of what we now call *category theory*, including e.g. limits and colimits, and give examples in homological algebra and algebraic topology. About a decade later, A. Grothendieck published his *Tôhoku paper* [1957] on homological algebra (but paving the way for algebraic geometry too). Particularly his definition of *Abelian category* shows how a category is not merely a convenient tool to speak about a collection of mathematical structures, but is in fact a versatile mathematical structure in itself. That is to say, we explicitly have here *categories as structures*, as opposed to *categories of structures*.

Some years later, J. Bénabou [1963] made an abstraction of the notion of tensor product, defining *catégories avec muliplication* (monoidal categories); Eilenberg and M. Kelly [1966] rather formalised the "internal hom" of a category, speaking of *closed categories*. Both Bénabou [1963] and Eilenberg and Kelly [1966] showed how such a monoidal/closed category $\mathcal{V}$ can serve as the base for $\mathcal{V}$*-enriched categories*. But it was B. Lawvere's [1973] paper, with its deep insights

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(enriched presheaves, Cauchy completion) and convincing examples (posets, metric spaces), that made $\mathcal{V}$-enriched categories part of the working mathematician's toolbox. Kelly's book [1982] became the standard reference on the subject.

Also in 1973, D. Higgs showed how the topos $\mathsf{Sh}(L)$ of sheaves on a locale L can be described equivalently as a category of "L-valued sets", i.e. sets equipped with an equality relation taking truth values in L, thus exhibiting its multi-valued intuitionistic logic. However, such an L-valued set is easily seen *not* to be an L-enriched category—indeed, a slightly more general notion is due. In fact, back in 1967, Bénabou had already defined *bicategories* and recognised that every such bicategory $\mathcal{W}$ can serve equally well as base for $\mathcal{W}$-enriched categories (which he called *polyads in* $\mathcal{W}$, for their kinship with *monads*). B. Walters [1981] then showed how any locale L gives rise to a bicategory $\mathcal{R}(L)$ in such a way that sheaves on L can be described as (particular) $\mathcal{R}(L)$-enriched categories; and in [1982], he generalised his argument to sheaves on a site. This encouraged R. Street [1981, 1983a, 1983b] to further develop the theory and applications of $\mathcal{W}$-enriched categories, often also with coauthors [Betti *et al.*, 1983; Carboni *et al.*, 1994].

This survey paper, written for a readership of fuzzy logicians and fuzzy set theorists, will be concerned with a particular instance of categories enriched in a bicategory, namely, where the base bicategory is a so-called *quantaloid* $\mathcal{Q}$: essentially, the local structure in the bicategory is posetal. This 'simplification' of the general theory still includes many important examples, such as ordered sets, (partial) metric spaces, sheaves, multi-valued logic and fuzzy sets; but it luckily does away with many a cumbersome "compatibility issue" so typical of bicategorical computations. A *quantale* is a one-object quantaloid (in other words, it is a monoidal closed poset), so the theory of quantaloid-enriched categories comprises the 'even simpler' theory of quantale-enriched categories—and indeed, the latter is already sufficiently general to include interesting examples such as ordered sets and metric spaces. However, we do insist on the use of quantaloid-enrichment: for one thing, quantaloids arise naturally as universal constructions, even when starting from a quantale; and for another, it is precisely with quantaloid-enrichment that we can elegantly express the notion of "partially/locally defined elements" (as well illustrated by the formulation of localic sheaves and of partial metric spaces as enriched categories).

Our modest ambition is only to explain and illustrate the basic definitions and a few emblematic results in quantaloid-enriched category theory, and we claim little or no originality for the mathematics contained in this paper. All of the concepts and results being by now quite standard notions in (enriched) category theory, we have not systematically traced their historical origins (apart from this short introduction). We do hope that the interested reader will find his or her way to the substantial literature on the subject.

2. Quantaloid-enriched categories

An ordered set $(X, \leq)$ can be thought of as a set X together with a binary predicate

$$X(-,-)\colon X \times X \longrightarrow \{0,1\}\colon (y,x) \mapsto X(y,x) := \begin{cases} 1 \text{ if } y \leq x \\ 0 \text{ otherwise} \end{cases}$$

satisfying, for all $x, y, z \in X$,

$$X(z,y) \wedge X(y,x) \leq X(z,x) \text{ and } 1 \leq X(x,x)$$

for transitivity and reflexivity[1]. The latter two conditions are equations in the Boolean algebra $\mathbf{2} = \{0, 1\}$ which only make use of its order structure, its intersection and its top element. Thus we can repeat this predicative definition of ordered set over any set of "truth values" $T = (T, \leq, \circ, 1)$ which is ordered and comes with a multiplication and neutral element. For convenience we first make some extra assumptions on the set of "truth values".

Definition 2.1 *A* **quantale**[2] $Q = (Q, \bigvee, \cdot, 1)$ *is a monoid* $(Q, \cdot, 1)$ *combined with a sup-lattice*[3] $(Q, \bigvee)$ *in such a way that, for all* $f, g, (f_i)_i, (g_j)_j \in Q$,

$$g \cdot (\bigvee_i f_i) = \bigvee_i (g \cdot f_i) \text{ and } (\bigvee_j g_j) \cdot f = \bigvee_j (g_j \cdot f).$$

A **homomorphism** $h: Q \longrightarrow Q'$ *between quantales is a monoid homomorphism which preserves suprema.*

Now we can formalise the notion of "a set equipped with a transitive and reflexive binary predicate taking values in a quantale":

Definition 2.2 *A* **category** $\mathbb{C}$ **enriched in a quantale** Q *(or* Q**-enriched category** $\mathbb{C}$*, or simply* Q**-category** $\mathbb{C}$*) is determined by*

`(obj)` *a set* $\mathbb{C}_0$ *of 'objects',*

`(hom)` *a 'hom function'* $\mathbb{C}(-,-): \mathbb{C}_0 \times \mathbb{C}_0 \longrightarrow Q: (y, x) \mapsto \mathbb{C}(y, x)$,

satisfying, for all $x, y, z \in \mathbb{C}_0$,

`(trans)` $\mathbb{C}(z, y) \cdot \mathbb{C}(y, x) \leq \mathbb{C}(z, x)$,

`(refl)` $1 \leq \mathbb{C}(x, x)$.

Of course, taking the 2-element Boolean algebra $\mathbf{2}$ as base quantale, the notion of $\mathbf{2}$-category[4] is precisely that of an ordered set. Here is another, more surprising example.

Example 2.3 (Metric spaces) The interval $[0, \infty]$ of positive real numbers with infinity added, underlies the quantale $([0, \infty], \bigwedge, +, 0)$. (So, crucially, we take the *opposite* of the natural partial order on $[0, \infty]$.) Strictly following the general definition given above, but adapting the terminology to common use, an $[0, \infty]$-category $\mathbb{C}$ consists of

`(obj)` a set $\mathbb{C}_0$ of 'points',

[1] What we call an 'order' is often called a 'preorder'. For a transitive, reflexive and anti-symmetric relation we shall use the term 'partial order' or 'anti-symmetric order'.

[2] C. Mulvey [1986] coined the term *quantale* as contraction of "quantum locale"; the study of locales as monoidal sup-lattices was initiated by [Joyal and Tierney, 1984].

[3] A sup-lattice $L = (L, \bigvee)$ is a partial order $(L, \leq)$ in which every $S \subseteq L$ has a supremum $\bigvee S$. This notion is obviously equivalent to that of a complete lattice, but with a bias toward the supremum as primitive ingredient. To wit, a sup-morphism between two sup-lattices is a map that preserves all suprema (and therefore also the order) but not necessarily the infima.

[4] This notion is not to be mistaken with that of a *2-dimensional category*, also called 2-category; the difference in the prefix' typeface – boldface $\mathbf{2}$ vs. normal 2 – is crucial.

(hom) a ‘metric’ $\mathbb{C}(-,-)\colon \mathbb{C}_0 \times \mathbb{C}_0 \longrightarrow [0,\infty]\colon (y,x) \mapsto \mathbb{C}(y,x)$,

satisfying

(trans) the ‘triangle inequality’ $\mathbb{C}(z,y) + \mathbb{C}(y,x) \geq \mathbb{C}(z,x)$,

(refl) the ‘point inequality’ $0 \geq \mathbb{C}(x,x)$.

This mathematical structure is a **generalised metric space** [Lawvere, 1973]. To define an (ordinary) metric space, a few requirements must be added: all distances must be *finite*, and the metric must be *symmetric* ($\mathbb{C}(y,x) = \mathbb{C}(x,y)$) and *separating* ($\mathbb{C}(x,y) \leq 0 \geq \mathbb{C}(y,x) \iff x = y$).

To wet the appetite of the intended readership of this article, we hasten to add another example (but see also Examples 2.14 and 3.6 further on).

Example 2.4 (t-norms) A **triangular norm**, or **t-norm** for short [Hájek, 1998], is a binary operator on the unit interval $[0,1]$, say

$$[0,1] \times [0,1] \longrightarrow [0,1]\colon (x,y) \mapsto x * y,$$

which is monotone in each variable (for the natural order on $[0,1]$) and such that $x * y = y * x$, $x * (y * z) = (x * y) * z$ and $x * 1 = x$. Such a t-norm is **left-continuous** (as a function in two variables) if and only if each $x * -\colon [0,1] \longrightarrow [0,1]$ is left-continuous, if and only if each $x * -\colon [0,1] \longrightarrow [0,1]$ admits a right adjoint (often called the *residuum* of the t-norm). Of course, by symmetry of the t-norm and completeness of $([0,1], \bigvee)$, this means precisely that both $x * -$ and $- * y$ distribute over arbitrary suprema in $[0,1]$. In other words, to give a left-continuous t-norm is *exactly* to specify the multiplication of a quantale structure on the sup-lattice $([0,1], \bigvee)$ which is *commutative and integral* (the latter meaning that the unit for the monoid is the top of the sup-lattice; this has also been called *strict two-sidedness* in the literature.)

For **continuous**[5] t-norms, a complete classification is known [Faucett, 1955; Mostert and Shields, 1957, Section 5, Theorem B], as follows. First one observes that the following formulas define continuous t-norms:

- the *product t-norm*: $x *_p y = xy$,
- the *Łukasiewicz t-norm*: $x *_l y = \max\{x + y - 1, 0\}$,
- the *minimum t-norm*: $x *_m y = \min\{x, y\}$.

Now, given any continuous t-norm $(x,y) \mapsto x * y$ on $[0,1]$, the set $E = \{x \in [0,1] \mid x * x = x\}$ of idempotents is a closed subset of $[0,1]$, so that its complement is a union of countably many non-overlapping open intervals $]a_i, b_i[$. Each closed interval $[a_i, b_i] \subseteq [0,1]$, endowed with the restriction of the given continuous t-norm on $[0,1]$, is isomorphic to either the Łukasiewicz t-norm or the product t-norm; and whenever $x, y \in [0,1]$ are not both contained in a single $]a_i, b_i[$, then $x * y = \min\{x, y\}$.

[5] This is indeed strictly stronger than mere left-continuity. For example, setting $x * y$ to $\min(x,y)$ if $x + y > 1$ and to 0 otherwise, defines a left-continuous but not right-continuous t-norm.

Left-continuous and continuous t-norms are important structures in fuzzy logic: taking the real unit interval as a model of possibly vague statements, conjunction is interpreted by a t-norm; the classic references include [Höhle, 1995; Hájek, 1998; Klement *et al.*, 2000; Gottwald, 2001]. A classification of merely left-continuous t-norms is not known (but attracts a great deal of attention in the fuzzy logic community)

Of course, Lawvere's quantale $([0,\infty], \bigwedge, +, 0)$ of positive real numbers is isomorphic to the product t-norm $([0,1], \bigvee, *_p, 1)$: the function $[0,\infty] \longrightarrow [0,1]\colon x \mapsto e^{-x}$ provides a bijective homomorphism. Categories enriched in $[0,1]$ with the product t-norm have been called *proximity spaces* (for e^{-x} expresses the *proximity* of two points at *distance* x). Categories enriched in $[0,1]$ endowed with any left-continuous t-norm, are sometimes called *fuzzy preorders* (relative to that t-norm); we leave it to the interested reader to spell out the details.

There are far too many examples to mention here; but we shall spell out two more classes of (non-commutative, non-integral) quantales that may appeal to the intuition of the logically inclined.

Example 2.5 (Automata) If $(M, \cdot, 1)$ is any monoid, then the sup-lattice of subsets $(\mathcal{P}(M), \bigcup)$ has a natural quantale structure: for $A, B \subseteq M$ we define $A \cdot B := \{a \cdot b \mid a \in A, b \in B\}$; and the unit for this multiplication is $\{1\}$. This is, in fact, the construction of the *free quantale on a monoid*: it is the object-part of a left adjoint to the forgetful functor from the category of quantales to the category of monoids. Categories enriched in a free quantale relate to automata theory and process semantics [Betti, 1980; Abramsky and Vickers, 1993; Rosenthal, 1995]: the objects of a $\mathcal{P}(M)$-enriched category $\mathbb{C}$ are the *states* of an automaton, and the elements of M are the automaton's *labels* or *processes*. To have an $f \in \mathbb{C}(y, x)$ is then read as "having a process f to produce y from x"; often this is denoted as $f\colon x \rightsquigarrow y$.

Example 2.6 (Sup-endomorphisms) Let $(S, \bigvee)$ be any sup-lattice. The set $Q(S)$ of sup-endomorphisms on S, with pointwise suprema, composition as multiplication, and the identity on S as unit, forms a quantale[6]. (In particular, if X is any set and $(\mathcal{P}(X), \bigcup)$ is the sup-lattice of its subsets, then $Q(\mathcal{P}(X))$ is the quantale whose elements can be identified with binary relations on X.) The lattice S is then the object-set of a $Q(S)$-enriched category $\mathbb{S}$, when defining

$$\mathbb{S}(y, x) = \bigvee \{f \in Q(S) \mid f(y) \leq x\}.$$

This hom function is a canonical *quantale-valued implication* on S: it refines the order in the sense that $y \leq x$ if and only if $1_S \leq \mathbb{S}(y, x)$; and it satisfies a modus ponens in the sense that $\mathbb{S}(y, x)(y) \leq x$. (In Subsection 4.2 we discuss a further abstraction of this construction; see also Subsection 4.4 for a variation on this theme.)

However, for some naturally occuring situations the definition of quantale-enriched category is somewhat too narrow.

[6]By the way, $Q(S)$ is *not* the free quantale on the sup-lattice $(S, \bigvee)$. The latter, however, also exists; writing it as $\mathcal{F}(S)$, its underlying lattice is the direct sum of n-fold iterated tensor products of S: $\mathcal{F}(S) = \oplus_{n \in \mathbb{N}} S^{\otimes n}$. This definition of a free monoid by a "formal geometric series" is a standard procedure in the theory of symmetric monoidal closed categories [MacLane, 1998]; here it is applied to the category of sup-lattices and sup-morphisms.

Example 2.7 (The issue of partial elements) Consider, for the sake of argument, the collection

$$\mathsf{Part}(X,P)=\{f\colon S\longrightarrow P\mid S\subseteq X\}$$

of partially defined functions from a set X to an ordered set $P=(P,\leq)$. One way to order $\mathsf{Part}(X,P)$ is by putting $f\leq g$ to mean that $\mathsf{dom}(f)\subseteq\mathsf{dom}(g)$ and $f(x)\leq g(x)$ for all $x\in\mathsf{dom}(f)$; but clearly this does not always provide the best possible information about a pair of functions. Another idea is therefore to consider the predicate

$$[\cdot\leq\cdot]\colon\mathsf{Part}(X,P)\times\mathsf{Part}(X,P)\longrightarrow\mathcal{P}(X)\colon(f,g)\mapsto\{x\in\mathsf{dom}(f)\cap\mathsf{dom}(g)\mid f(x)\leq g(x)\}$$

assigning to a pair of partially defined functions (f,g) the largest part of X on which the pointwise order relation "$f\leq g$" holds. For indeed, the set $\mathcal{P}(X)$ of subsets of X underlies the quantale $(\mathcal{P}(X),\bigcup,\cap,X)$, and the above predicate does look like a 'hom-function' on the set $\mathsf{Part}(X,P)$ of 'objects'... but even though the transitivity axiom does hold, the reflexivity axiom need not hold in general: $X\subseteq[f\leq f]$ if and only if $f\in\mathsf{Part}(X,P)$ is totally defined!

So we see that the notion of Q-category $\mathbb{C}$, and particularly the required reflexivity of $\mathbb{C}$, excludes examples with "partially defined elements"—which are typical in sheaf theory and can therefore not be neglected.

There are now (at least) two possible modifications of the definition of Q-category. One way is simply to drop the reflexivity axiom in the definition of Q-category: one then speaks of a **Q-enriched semicategory** $\mathbb{C}$. But practice has shown that the resulting structure is too weak, and so the reflexivity axiom is rather replaced by the **regularity** axiom [Moens *et al.*, 2002; Garraway, 2005; Stubbe, 2005b]

$$\bigvee_{y\in\mathbb{C}_0}\mathbb{C}(z,y)\cdot\mathbb{C}(y,x)=\mathbb{C}(z,x).$$

In fact, for many important results this structure is still too weak, and the following stronger **total-regularity** axiom [Stubbe, 2005c] makes up for that:

$$\mathbb{C}(y,x)\cdot\mathbb{C}(x,x)=\mathbb{C}(y,x)=\mathbb{C}(y,y)\cdot\mathbb{C}(y,x).$$

It is immediate from the definitions that every Q-category is a totally regular Q-semicategory, which is necessarily a regular Q-semicategory, which of course is a Q-semicategory; and each of these inclusions is strict. Actually, the partially defined functions provide an example of a totally regular $\mathcal{P}(X)$-semicategory which is not a $\mathcal{P}(X)$-category.

Another way to overcome the issue of partial elements in a Q-category, while keeping the definition of enriched category, is by generalising the base of the enrichment: indeed, by considering categories enriched in a quantaloid instead of a quantale[7].

[7] Although the two approaches are related [Stubbe, 2005c]: the quantaloid of totally regular Q-semicategories and "regular distributors" is equivalent to the quantaloid of $\mathcal{R}(Q)$-categories and distributors (see Definition 3.1), where $\mathcal{R}(Q)$ is the split-idempotent completion of Q (as in Example 2.13). However, quantaloids arise naturally from quantales by universal constructions – as illustrated in several examples further on – and so it definitely pays off to pass to this level of generality.

Definition 2.8 *A* **quantaloid**[8] $\mathcal{Q}$ *is a category*[9] *for which each set* $\mathcal{Q}(X,Y)$ *of arrows between any two objects* X *and* Y *in* $\mathcal{Q}$ *is a sup-lattice, in such a way that, for all* $f, (f_i)_i \in \mathcal{Q}(X,Y)$ *and all* $g, (g_j)_j \in \mathcal{Q}(Y,Z)$,

$$g \circ (\bigvee_i f_i) = \bigvee_i (g \circ f_i) \text{ and } (\bigvee_j g_j) \circ f = \bigvee_j (g_j \circ f).$$

A **homomorphism** $H\colon \mathcal{Q} \longrightarrow \mathcal{Q}'$ *of quantaloids is a functor which preserves suprema of arrows.*

The intuition behind this definition should be clear: monoids stand to categories as quantales stand to quantaloids. Or, in other words, quantales are exactly quantaloids with a single object (which is usually not made explicit). In fact, complete lattices and sup-morphisms form a symmetric monoidal closed category Sup; quantales are precisely monoids in Sup, and quantaloids are precisely categories enriched in Sup. This is a very important technical point, but as this survey paper is meant to be a gentle self-contained introduction to quantaloid-enriched categories, we shall not be too concerned with this meta-level. We refer to [Kelly, 1982] for much, much more on categories enriched in a symmetric monoidal closed category.

The single most important property of a quantaloid $\mathcal{Q}$ is the following.

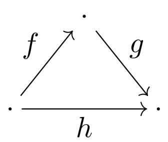
Proposition 2.9 *For any* $f\colon X \longrightarrow Y$ *and* Z *in a quantaloid* $\mathcal{Q}$, *both order-preserving maps*

$$- \circ f\colon \mathcal{Q}(Y,Z) \longrightarrow \mathcal{Q}(X,Z)\colon g \mapsto g \circ f \text{ and } f \circ -\colon \mathcal{Q}(Z,X) \longrightarrow \mathcal{Q}(Z,Y)\colon g \mapsto f \circ g$$

have right adjoints (because they are supremum-preserving functions between complete lattices); they are called **extension** *and* **lifting** *(along* f*) and written as*

$$\{f,-\}\colon \mathcal{Q}(X,Z) \longrightarrow \mathcal{Q}(Y,Z)\colon h \mapsto \{f,h\} \text{ and } [f,-]\colon \mathcal{Q}(Z,Y) \longrightarrow \mathcal{Q}(Z,X)\colon h \mapsto [f,h].$$

Schematically we may present these adjunctions as follows: for any triangle in $\mathcal{Q}$ like



we have that

$$g \circ f \le h \iff g \le \{f,h\} \iff f \le [g,h].$$

Other notations and terminology for these adjoints are used in the literature. For instance, $\{f,h\}$ is also written as $h \swarrow f$ and read as "h is implied by f", whereas $[g,h]$ would then be written as $g \searrow h$ and read as "g implies h". This makes all the more sense when one reads $g \circ f$ as a (non-commutative!) conjunction of g and f; the universal property of these "implications" then guarantees the "modus ponens" for both: $(h \swarrow f) \circ f \le h$ and $g \circ (g \searrow h) \le h$.

[8] This term was proposed by K. Rosenthal [1991, 1996], even though the structure itself had been studied explicitly before [Pitts, 1988].

[9] Most often we shall write the objects of $\mathcal{Q}$ as X, Y, Z, ... and its arrows as $f\colon X \longrightarrow Y$, $g\colon Y \longrightarrow Z$, ... Composition is written "from right to left", e.g. $g \circ f\colon X \longrightarrow Z$, and the compositional identities are written as $1_X\colon X \longrightarrow X$. This is of course a matter of convention, but especially the choice for the composition in $\mathcal{Q}$ is not unimportant for the further choice of notation in the definition of a $\mathcal{Q}$-enriched category.

The above proposition says in particular that a quantaloid is precisely a *locally posetal, locally complete and cocomplete, closed bicategory*. Importantly, *we can therefore use all bicategorical notions in any quantaloid*; for instance, an adjoint pair of arrows in $\mathcal{Q}$ consists of an $f\colon A \longrightarrow B$ and a $g\colon B \longrightarrow A$ such that $g \circ f \geq 1_A$ and $f \circ g \leq 1_B$.

The definition of a quantaloid-enriched category is now – almost – a copy of the previous definition of quantale-enriched category.

Definition 2.10 *A* **category** $\mathbb{C}$ **enriched in a quantaloid** $\mathcal{Q}$ *(or* $\mathcal{Q}$**-enriched category** $\mathbb{C}$*, or simply* $\mathcal{Q}$**-category** $\mathbb{C}$*) consists of*

`(obj)` *a set* $\mathbb{C}_0$ *of 'objects',*

`(type)` *a 'type function'* $t\colon \mathbb{C}_0 \longrightarrow \{\textit{objects of } \mathcal{Q}\}\colon x \mapsto tx$*,*

`(hom)` *a 'hom function'* $\mathbb{C}(-,-)\colon \mathbb{C}_0 \times \mathbb{C}_0 \longrightarrow \{\textit{arrows of } \mathcal{Q}\}\colon (y,x) \mapsto \Big(\mathbb{C}(y,x)\colon tx \longrightarrow ty\Big)$*,*

satisfying, for all $x, y, z \in \mathbb{C}_0$*,*

`(trans)` $\mathbb{C}(z,y) \circ \mathbb{C}(y,x) \leq \mathbb{C}(z,x)$*,*

`(refl)` $1 \leq \mathbb{C}(x,x)$*.*

Clearly, regarding a quantale as a one-object quantaloid, the type function in this definition becomes obsolete, and Definition 2.10 completely agrees with the earlier quantale-specific Definition 2.2. So ordered sets (**2**-categories), metric spaces ($[0,\infty]$-categories), t-norms (commutative and integral quantale structures on $[0,1]$) are included in quantaloid-enriched category theory. But how can we now solve the problem of "partial elements"?

Example 2.11 (Partial elements again) For X a set, let $\mathcal{Q}(X)$ be the quantaloid whose objects are the subsets of X, and in which the arrows from $S \subseteq X$ to $T \subseteq X$ are precisely all $U \subseteq S \cap T$. The composition law in $\mathcal{Q}(X)$ is given by intersection, and the identity on the object $S \subseteq X$ is $S\colon S \longrightarrow S$ itself. Note that $\{\text{objects of } \mathcal{Q}(X)\} = \mathcal{P}(X) = \{\text{arrows of } \mathcal{Q}(X)\}$. Now consider again the set $\mathsf{Part}(X,P)$ of partially defined functions from X to a partially ordered set P. The type function

$$\mathsf{dom}\colon \mathsf{Part}(X,P) \longrightarrow \mathcal{P}(X)\colon f \mapsto \mathsf{dom}(f)$$

and the hom function

$$[\cdot \leq \cdot]\colon \mathsf{Part}(X,P) \times \mathsf{Part}(X,P) \longrightarrow \mathcal{P}(X)\colon (f,g) \mapsto \{x \in \mathsf{dom}(f) \cap \mathsf{dom}(g) \mid f(x) \leq g(x)\}$$

define a genuine $\mathcal{Q}(X)$-category: the reflexivity axiom is the simple tautology $\mathsf{dom}(f) \subseteq [f \leq f]$.

Example 2.12 (Underlying order, isomorphic objects) Each quantaloid-enriched category determines a family of (ordinary) ordered sets. Indeed, let $\mathbb{C}$ be a $\mathcal{Q}$-category, then for each object X in $\mathcal{Q}$, the set $\mathbb{C}_X = \{x \in \mathbb{C}_0 \mid tx = X\}$ is ordered by the relation

$$x \leq x' \stackrel{\text{def}}{\iff} 1_X \leq \mathbb{C}(x,x').$$

Taking the colimit in the category of ordered sets of all these $(\mathbb{C}_X, \leq)$, we obtain the **underlying order** $(\mathbb{C}_0, \leq)$ of the $\mathcal{Q}$-category $\mathbb{C}$—in which thus $x \leq x'$ precisely when $tx = tx'$ and $1_{tx} \leq \mathbb{C}(x, x')$. (Of course, if $P = (P, \leq)$ is an ordered set, and $\mathbb{P}$ is that same ordered set but now regarded as a **2**-category, then the underlying order of $\mathbb{P}$ is again $(P, \leq)$.) Whenever both $x \leq x'$ and $x' \leq x$, we say that x and x' are **isomorphic** and write this as $x \cong x'$. If, in $\mathbb{C}$, $x \cong x'$ implies $x = x'$, then we say that $\mathbb{C}$ is **skeletal** (even though **separated** would be a good term too, cf. the example of metric spaces).

Example 2.13 (Splitting idempotents, localic presheaves) An endomorphism $e: A \longrightarrow A$ in a quantaloid $\mathcal{Q}$ is said to be **idempotent** when $e \circ e = e$. Now consider the quantaloid $\mathcal{R}(\mathcal{Q})$ whose objects are the idempotents in $\mathcal{Q}$ and the sup-lattices of arrows are

$$\mathcal{R}(\mathcal{Q})(e, e') = \{f: \mathsf{dom}(e) \longrightarrow \mathsf{dom}(e') \text{ in } \mathcal{Q} \mid f \circ e = f = e' \circ f\},$$

with suprema as in $\mathcal{Q}$. Composition in $\mathcal{R}(\mathcal{Q})$ is computed as in $\mathcal{Q}$, but the identity arrow in $\mathcal{R}(\mathcal{Q})$ on an idempotent e is $e: e \longrightarrow e$ itself. Of course every identity arrow in $\mathcal{Q}$ is idempotent, and it is easy to see that $\mathcal{Q}(X, Y) = \mathcal{R}(\mathcal{Q})(1_X, 1_Y)$. This provides us with a full embedding of quantaloids

$$\mathcal{Q} \longrightarrow \mathcal{R}(\mathcal{Q}): \Big(f: X \longrightarrow Y\Big) \mapsto \Big(f: 1_X \longrightarrow 1_Y\Big)$$

exhibiting the quantaloid $\mathcal{R}(\mathcal{Q})$ as the **split-idempotent completion** of $\mathcal{Q}$ (in the sense of an appropriate universal property in the category of quantaloids and homomorphisms).

Suppose now given a locale $L = (L, \bigvee, \wedge, \top)$, that is, a quantale in which the multiplication is the binary infimum (and the unit is necessarily the top). Any presheaf $F: L^{\mathsf{op}} \longrightarrow \mathsf{Ord}$, taking values in the category of ordered sets and order-preserving functions, determines an $\mathcal{R}(L)$-category $\mathbb{C}_F$, with

(obj) $(\mathbb{C}_F)_0 = \uplus_{u \in L} F(u)$,

(type) $tx = u \iff x \in F(u)$,

(hom) $\mathbb{C}_F(y, x) = \bigvee\{w \leq tx \wedge ty \mid x_{|w} \leq_w y_{|w}\}$.

As usual, the notation $x_{|w}$ stands for the restriction of $x \in F(tx)$ to $F(w)$, and similarly we wrote $\leq_w$ for the order relation in $F(w)$. The hom of the enriched category $\mathbb{C}_F$ thus assigns to a pair (x, y) the "largest element of L on which x is smaller than y", or the "extent to which x is smaller than y". Indeed, this generalises the example of the partially defined functions, taking $L = \mathcal{P}(X)$ and $F(S) = \{\text{functions from } S \subseteq X \text{ to } P\}$.

There is an important variation on the above construction: if $F: L^{\mathsf{op}} \longrightarrow \mathsf{Set}$ is a presheaf taking values in the category of sets and functions, then the $\mathcal{R}(L)$-category $\mathbb{C}_F$ is *symmetric* $(\mathbb{C}_F(x, y) = \mathbb{C}_F(y, x))$ and *skeletal* $(\mathbb{C}_F(x, y) \geq 1_X \leq \mathbb{C}_F(y, x) \iff x = y)$.

Example 2.14 (Diagonals, divisibility, and partial metrics) A partial metric space, as defined by [Matthews, 1994], is a metric space in which the distance from a point to itself need not be zero. This means that such a partial metric space (X, d) is *not* a $[0, \infty]$-category, for the reflexivity axiom does not hold. But this problem can be fixed, again by considering an appropriate quantaloid constructed from the quantale $[0,\infty]$, as first indicated by U. Höhle and T. Kubiak [2011], but which we shall now put in a more general perspective.

First, if $\mathcal{Q}$ is any quantaloid, then we write $\mathsf{Sq}(\mathcal{Q})$ ("squares in $\mathcal{Q}$") for the new quantaloid whose objects are the arrows in $\mathcal{Q}$, and in which an arrow from $e\colon A_1 \longrightarrow A_2$ to $f\colon B_1 \longrightarrow B_2$ is a pair $(x_1\colon A_1 \longrightarrow B_1, x_2\colon A_2 \longrightarrow B_2)$ such that the square

$$\begin{array}{ccc} A_1 & \stackrel{x_1}{\dashrightarrow} & B_1 \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle f} \\ A_2 & \underset{x_2}{\dashrightarrow} & B_2 \end{array}$$

in $\mathcal{Q}$ commutes. Local suprema, composition and identities in $\mathsf{Sq}(\mathcal{Q})$ are all obvious. Clearly, $\mathcal{Q}$ is fully embedded in $\mathsf{Sq}(\mathcal{Q})$ by

$$\mathcal{Q} \longrightarrow \mathsf{Sq}(\mathcal{Q})\colon \Big(f\colon X \longrightarrow Y\Big) \mapsto \Big((f,f)\colon 1_X \longrightarrow 1_Y\Big).$$

Next, if two commutative squares

$$\begin{array}{ccc} A_1 & \stackrel{x_1}{\dashrightarrow} & B_1 \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle f} \\ A_2 & \underset{x_2}{\dashrightarrow} & B_2 \end{array} \quad \text{and} \quad \begin{array}{ccc} A_1 & \stackrel{y_1}{\dashrightarrow} & B_1 \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle f} \\ A_2 & \underset{y_2}{\dashrightarrow} & B_2 \end{array}$$

have the same diagonal, we shall write this as $(x_1,x_2) \sim (y_1,y_2)$. This is easily seen to be a *congruence* on $\mathsf{Sq}(\mathcal{Q})$, i.e. an equivalence relation which is compatible with composition, identities and suprema in $\mathsf{Sq}(\mathcal{Q})$. Therefore it follows that the *quotient* $\mathsf{Sq}(\mathcal{Q})/\!\sim$ is also a quantaloid; and $\mathcal{Q}$ still embeds fully in $\mathsf{Sq}(\mathcal{Q})/\!\sim$ because the equivalence relation on $\mathsf{Sq}(\mathcal{Q})(1_X,1_Y)$ is trivial.

For categories instead of quantaloids, the above constructions are well known, and their importance is related to the existence of *(proper) factorisation systems* [Grandis, 2002]. Specifically for a quantaloid $\mathcal{Q}$ however, the quotient $\mathsf{Sq}(\mathcal{Q})/\!\sim$ admits an appealing alternative description. Given arrows

$$\begin{array}{ccc} A_1 & & B_1 \\ {\scriptstyle e}\downarrow & \overset{d}{\searrow} & \downarrow{\scriptstyle f} \\ A_2 & & B_2 \end{array}$$

in $\mathcal{Q}$, the following two conditions are easily seen to be equivalent[10]:

(i) there exist $d_1\colon A_1 \longrightarrow B_1$ and $d_2\colon A_2 \longrightarrow B_2$ such that $f \circ d_1 = d = d_2 \circ e$,

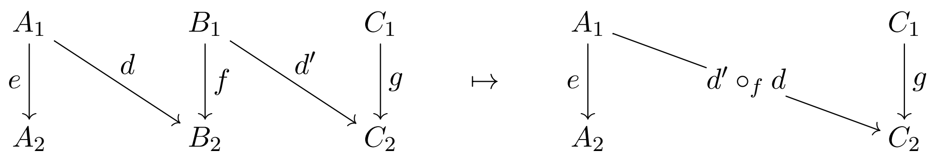
(ii) $f \circ (f \searrow d) = d = (d \swarrow e) \circ e$.

In this situation, we shall say that d is a **diagonal** from e to f. Writing $\mathcal{D}(\mathcal{Q})(e,f)$ for the set of diagonals from e to f, which inherits suprema from $\mathcal{Q}$, there is an isomorphism of sup-lattices

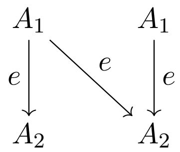
$$F_{e,f}\colon \mathsf{Sq}(\mathcal{Q})(e,f)/\!\sim \longrightarrow \mathcal{D}(\mathcal{Q})(e,f)\colon [(x_1,x_2)] \mapsto fx_1$$

[10]For the nontrivial implication: $d = f \circ x \leq f \circ (f \searrow d) \leq d$ holds because $f \circ x = d$ implies $x \leq f \searrow d$; and similarly $d = (d \swarrow e) \circ e$ follows from $y \circ e = d$.

whose inverse maps a diagonal $d\colon e \longrightarrow f$ to the equivalence class $[(f \searrow d, d \swarrow e)]$. This action on arrows can be made functorial: defining the composite of two diagonals to be



where $d' \circ_f d := (d' \swarrow f) \circ f \circ (f \searrow d)$ (which is further equal to $d' \circ (f \searrow d) = (d' \swarrow f) \circ d$ too) and the unit diagonal on an object $e\colon A_1 \longrightarrow A_2$ to be



we get the **quantaloid $\mathcal{D}(\mathcal{Q})$ of diagonals in** $\mathcal{Q}$ so that precisely

$$F\colon \mathsf{Sq}(\mathcal{Q})/\!\sim \longrightarrow \mathcal{D}(\mathcal{Q})\colon \Big([(x_1, x_2)]\colon e \longrightarrow f\Big) \mapsto \Big(fx_1\colon e \longrightarrow f\Big)$$

is an isomorphism of quantaloids. Let us underline that we thus have a full embedding

$$\mathcal{Q} \longrightarrow \mathcal{D}(\mathcal{Q})\colon \Big(f\colon X \longrightarrow Y\Big) \mapsto \Big(f\colon 1_X \longrightarrow 1_Y\Big)$$

(which has a universal property wrt. quantaloids equipped with a proper factorisation system).

If the given quantaloid $\mathcal{Q}$ enjoys certain "extra" properties, then this may have pleasant consequences for $\mathcal{D}(\mathcal{Q})$. For example, if $\mathcal{Q}$ is **integral**, in the sense that each identity arrow in $\mathcal{Q}$ is the biggest endomorphism on its (co)domain, then $\mathcal{D}(\mathcal{Q})$ is integral too[11]; and because $\mathcal{Q}$ fully embeds in $\mathcal{D}(\mathcal{Q})$, the converse holds as well. Similarly, the following conditions are equivalent[12]:

(i) $\mathcal{Q}$ is integral and for all $d, e\colon A \longrightarrow B$ in $\mathcal{Q}$, $e \circ (e \searrow d) = d \wedge e = (d \swarrow e) \circ e$.

(ii) $\mathcal{Q}$ is integral and for all $d \leq e\colon A \longrightarrow B$ in $\mathcal{Q}$, $e \circ (e \searrow d) = d = (d \swarrow e) \circ e$,

(iii) for all $d, e\colon A \longrightarrow B$ in $\mathcal{Q}$, $e \circ (e \searrow d) = d = (d \swarrow e) \circ e \iff d \leq e$,

(iv) for all $e\colon A \longrightarrow B$ in $\mathcal{Q}$, $\mathcal{D}(\mathcal{Q})(e, e) = \downarrow e$ (as sublattices of $\mathcal{Q}(A, B)$).

Extending (and slightly abusing) terminology of [Hájek, 1998], we shall call such a quantaloid $\mathcal{Q}$ **divisible**. For a *quantale* $Q = (Q, \bigvee, \cdot, 1)$ (viewed as a one-object quantaloid) these four conditions are furthermore equivalent to[13]:

[11] If $d, e\colon A \longrightarrow B$ in an integral $\mathcal{Q}$ are such that $d \in \mathcal{D}(\mathcal{Q})(e, e)$, then $d = (d \swarrow e)e \leq 1_B e = e$.

[12] Clearly (iii) and (iv) are equivalent; either of these conditions implying the integrality of $\mathcal{D}(\mathcal{Q})$, and therefore also of $\mathcal{Q}$, (iii) implies (ii). But (ii) implies (iv) too: integrality of $\mathcal{Q}$ implies integrality of $\mathcal{D}(\mathcal{Q})$, which means that $\mathcal{D}(\mathcal{Q})(e, e) \subseteq \downarrow e$ (as sublattices of $\mathcal{Q}(A, B)$); whereas the second part of (ii) says precisely that $\downarrow e \subseteq \mathcal{D}(\mathcal{Q})(e, e)$. Whereas (i) trivially implies (ii), the converse needs a bit of an argument. First observe that, for any $d, e\colon A \longrightarrow B$, we have $e(e \searrow d) \leq d \geq (d \swarrow e)e$ by the universal property of liftings/extensions, and $e(e \searrow d) \leq e1_A = e = 1_B e \geq (d \swarrow e)e$ if we use the integrality of $\mathcal{Q}$; so $e(e \searrow d) \leq d \wedge e \geq (d \swarrow e)e$ holds in any integral $\mathcal{Q}$. Under the hypothesis of (ii) we can compute for $d \wedge e \leq e$ that $e(e \searrow d) \geq e(e \searrow (d \wedge e)) = d \wedge e = ((d \wedge e) \swarrow e)e \leq (d \swarrow e)e$; thus (ii) implies (i). The converse implication is trivial.

[13] Obviously (v) implies (iv), and (i) implies (v). But let us stress that this fifth condition makes no sense for a general quantaloid $\mathcal{Q}$: "$e \wedge f$" is not defined whenever domain and codomain of e and f do not match!

(v) for all $e, f \in Q$, $\mathcal{D}(Q)(e,f) = \downarrow(e \wedge f)$ (as sublattices of Q).

So-called *BL-algebras* [Hájek, 1998] are examples of divisible quantales; *BL-chains* (linearly ordered BL-algebras) are precisely commutative, divisible, linearly ordered quantales. In fact, P. Hájek [1998] showed that a left-continuous t-norm $(x, y) \mapsto x * y$ (see Example 2.4) is continuous if and only if each $x * -\colon [0,1] \longrightarrow [0,1]$ is continuous, if and only if the corresponding commutative and integral quantale $([0,1], \bigvee, *, 1)$ is divisible. The results of W. Faucett's [1955] imply furthermore that any divisible quantale structure on $([0,1], \bigvee)$ is necessarily commutative. Thus, divisible quantale structures on the sup-lattice $([0,1], \bigvee)$ are precisely the same thing as continuous t-norms.

Let us now fix a quantale $Q = (Q, \bigvee, \cdot, 1)$: both the set of objects and the set of arrows of $\mathcal{D}(Q)$ are thus equal to Q. Strictly following the general definition, a $\mathcal{D}(Q)$-category $\mathbb{C}$ therefore consists of

`(obj)` a set $\mathbb{C}_0$,

`(type)` a function $t\colon \mathbb{C}_0 \longrightarrow Q\colon x \mapsto tx$,

`(hom)` a function $\mathbb{C}(-,-)\colon \mathbb{C}_0 \times \mathbb{C}_0 \longrightarrow Q\colon (y,x) \mapsto \mathbb{C}(y,x)$ such that $\mathbb{C}(y,x) \in \mathcal{D}(Q)(tx, ty)$,

satisfying

`(trans)` $\mathbb{C}(z,y) \circ_{ty} \mathbb{C}(y,x) \leq \mathbb{C}(z,x)$ (composition taken in $\mathcal{D}(Q)$),

`(refl)` $1_{tx} \leq \mathbb{C}(x,x)$ (identity taken in $\mathcal{D}(Q)$).

If Q is an *integral* quantale, then $tx = 1_{tx} = \mathbb{C}(x,x)$ in the integral quantaloid $\mathcal{D}(Q)$: this makes the type function implicit in the hom function, and the reflexivity axiom for $\mathbb{C}$ obsolete. And if, furthermore, Q is a *divisible* quantale, then $\mathbb{C}(y,x) \in \mathcal{D}(Q)(tx,ty)$ is equivalent to $\mathbb{C}(x,x) \leq \mathbb{C}(x,x) \wedge \mathbb{C}(y,y)$. Formulating the composition in $\mathcal{D}(Q)$ back into terms proper to Q, the above definition then boils down to the following: a $\mathcal{D}(Q)$-category $\mathbb{C}$ is a set $\mathbb{C}_0$ together with a function

$$\mathbb{C}(-,-)\colon \mathbb{C}_0 \times \mathbb{C}_0 \longrightarrow Q\colon (y,x) \mapsto \mathbb{C}(y,x)$$

satisfying

$$\mathbb{C}(x,y) \wedge \mathbb{C}(x,x) \leq \mathbb{C}(y,y) \quad \text{and} \quad (\mathbb{C}(z,y) \swarrow \mathbb{C}(y,y)) \cdot \mathbb{C}(y,x) \leq \mathbb{C}(z,x).$$

It is easy to check that $([0,\infty], \bigwedge, +, 0)$ is a divisible quantale. Adopting common terminology, a $\mathcal{D}([0,\infty])$-category $\mathbb{C}$ is precisely a set $\mathbb{C}_0$ of 'points' together with a 'metric'

$$\mathbb{C}(-,-)\colon \mathbb{C}_0 \times \mathbb{C}_0 \longrightarrow [0,\infty]$$

satisfying

$$\mathbb{C}(x,y) \geq \mathbb{C}(x,x) \vee \mathbb{C}(y,y) \quad \text{and} \quad \mathbb{C}(z,y) - \mathbb{C}(y,y) + \mathbb{C}(y,x) \geq \mathbb{C}(z,x).$$

In line with Example 2.3 we can call this structure a **generalised partial metric space**; imposing *finiteness*, *symmetry* and *separatedness*, we recover exactly S. Matthews' [1994] notion of partial metric space. This means that partial metrics are a part of general quantaloid-enriched category theory; so in particular all of that theory's completion theorems apply to partial metric spaces.

We leave it to the interested reader to experiment with categories enriched in $\mathcal{D}([0,1])$ when putting a (left-)continuous t-norm on the unit-interval.

Concerning the two previous constructions, $\mathcal{R}(\mathcal{Q})$ (splitting idempotents) and $\mathcal{D}(\mathcal{Q})$ (diagonals), on a given quantaloid $\mathcal{Q}$, it is worth underlining that $\mathcal{R}(\mathcal{Q})$ *is precisely the full subquantaloid of* $\mathcal{D}(\mathcal{Q})$ *whose objects are the idempotents*[14]. So there are full embeddings of quantaloids

$$\mathcal{Q} \longrightarrow \mathcal{R}(\mathcal{Q}) \longrightarrow \mathcal{D}(\mathcal{Q})$$

where the first is defined by $(f\colon A \longrightarrow B) \mapsto (f\colon 1_A \longrightarrow 1_B)$, and the second is simply an inclusion. The behaviour of these embeddings can be very different from one example to another. For instance, for a locale L (viewed as one-object quantaloid), $\mathcal{R}(L)$ is identical to $\mathcal{D}(L)$ (because all elements of L are idempotents), and we saw that this "enlargment" of L is exactly what is needed to formalise the partial elements of a (pre)sheaf on L as objects of a category enriched in $\mathcal{R}(L)$. On the other hand, splitting idempotents in the quantale $[0,\infty]$ creates a quantaloid $\mathcal{R}([0,\infty])$ which has only two objects (0 and ∞), thus only two additional types for the objects of an $\mathcal{R}([0,\infty])$-category: this construction is not adequate to formalise partial metric spaces, and the much larger quantaloid $\mathcal{D}([0,\infty])$ is needed.

3. Distributors and functors

In the previous section we defined categories enriched in a quantaloid $\mathcal{Q}$, the *objects* of our study. This section will be concerned with two kinds of *morphisms* between such enriched categories, namely distributors and functors.

From now on we fix a quantaloid $\mathcal{Q}$, which – to avoid set-theoretic problems – we assume to be small, that is, have a *set* of objects.

Definition 3.1 *A* $\mathcal{Q}$**-distributor**[15] $\Phi\colon \mathbb{C} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{D}$ *between two* $\mathcal{Q}$*-categories is*

(matr) *a 'matrix'* $\Phi\colon \mathbb{D}_0 \times \mathbb{C}_0 \longrightarrow \{\textit{arrows of } \mathcal{Q}\}\colon (y,x) \mapsto \Big(\Phi(y,x)\colon tx \longrightarrow ty\Big)$

satisfying, for all $x, x' \in \mathbb{C}_0$ *and* $y, y' \in \mathbb{D}_0$*,*

(act) *the 'action inequality'* $\mathbb{D}(y',y) \circ \Phi(y,x) \circ \mathbb{C}(x,x') \leq \Phi(y',x')$[16].

Two consecutive distributors $\Phi\colon \mathbb{C} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{D}$ *and* $\Psi\colon \mathbb{D} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{E}$ *compose by a 'matrix formula': the composite distributor is written as* $\Psi \otimes \Phi\colon \mathbb{C} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{E}$ *and its elements are, for* $x \in \mathbb{C}_0$ *and* $z \in \mathbb{E}_0$*,*

$$(\Psi \otimes \Phi)(z,x) = \bigvee_{y \in \mathbb{D}_0} \Psi(z,y) \circ \Phi(y,x).$$

[14] That is, for idempotents $e\colon A \longrightarrow A$ and $f\colon B \longrightarrow B$ in $\mathcal{Q}$, an arrow $d\colon A \longrightarrow B$ is a diagonal (i.e. there exist x, y such that $f \circ x = d = y \circ e$) if and only if $f \circ d = d = d \circ e$. For the not-immediately-trivial direction, remark that $fd = ffx = fx = d = ye = yee = de$.

[15] We use the terminology of [Bénabou, 1973]; the same concept has been named 'module' or 'bimodule' (particularly by the Australian category theorists) and 'profunctor' (particularly in the context of proarrow equipments [Wood, 1982]).

[16] The single inequality given here is equivalent to the combination of the two inequalities $\mathbb{D}(y',y) \circ \Phi(y,x) \leq \Phi(y',x)$ and $\Phi(y,x) \circ \mathbb{C}(x,x') \leq \Phi(y,x')$ (for 'left action' and 'right action').

The identity distributor $\mathsf{id}_{\mathbb{C}}\colon \mathbb{C} \xrightarrow{\circ} \mathbb{C}$ *has elements, for* $x, x' \in \mathbb{C}_0$*,*

$$\mathsf{id}_{\mathbb{C}}(x', x) = \mathbb{C}(x', x).$$

Two parallel distributors $\Phi, \Phi'\colon \mathbb{C} \xrightarrow{\circ} \mathbb{D}$ *are ordered 'elementwise':*

$$\Phi \le \Phi' \stackrel{\text{def}}{\iff} \Phi(y, x) \le \Phi'(y, x) \textit{ for all } (x, y) \in \mathbb{C}_0 \times \mathbb{D}_0.$$

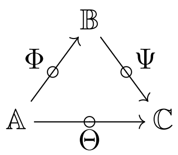
In this manner, distributors are the morphisms of a (large) quantaloid $\mathsf{Dist}(\mathcal{Q})$*.*

Example 3.2 (Universal properties) Each object X of the quantaloid $\mathcal{Q}$ determines a trivial one-object $\mathcal{Q}$-category $\mathbb{1}_X$ defined by $(\mathbb{1}_X)_0 = \{*\}$, $t* = X$ and $\mathbb{1}_X(*, *) = 1_X$. Each arrow $f\colon X \longrightarrow Y$ in $\mathcal{Q}$ determines a one-element distributor $(f)\colon \mathbb{1}_X \xrightarrow{\circ} \mathbb{1}_Y$ defined by $(f)(*, *) = f$. This produces a homomorphism of quantaloids

$$i\colon \mathcal{Q} \longrightarrow \mathsf{Dist}(\mathcal{Q})\colon \Big(f\colon X \longrightarrow Y\Big) \mapsto \Big((f)\colon \mathbb{1}_X \xrightarrow{\circ} \mathbb{1}_Y\Big)$$

which is injective on objects and on arrows; therefore we can (and often tacitly will) identify $\mathcal{Q}$ with its image in $\mathsf{Dist}(\mathcal{Q})$. Moreover, the above homomorphism has several universal properties: (i) it is the lax completion and the lax cocompletion of $\mathcal{Q}$ [Stubbe, 2005a, p. 42], (ii) it is the direct-sum-split-monad completion of $\mathcal{Q}$ [*ibid.*], (iii) it is the Sup-enriched cocompletion of $\mathcal{Q}$ for colimits weighted by principally generated modules [Heymans and Stubbe, 2009a, p. 52]. These universal properties are crucial in many problems, particularly those where "change of base" (see Subsection 4.1) is involved; for more details we refer to the references.

Because $\mathsf{Dist}(\mathcal{Q})$ is a quantaloid, it has **extensions and liftings**: for any triangle of $\mathcal{Q}$-distributors as in



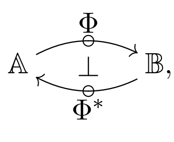
there exist unique distributors $\{\Phi, \Theta\}\colon \mathbb{B} \xrightarrow{\circ} \mathbb{C}$ and $[\Psi, \Theta]\colon \mathbb{A} \xrightarrow{\circ} \mathbb{B}$ for which

$$\Psi \otimes \Phi \le \Theta \iff \Psi \le \{\Phi, \Theta\} \iff \Phi \le [\Psi, \Theta].$$

It is not difficult to compute explicitly that

$$\{\Phi, \Theta\}(c, b) = \textstyle\bigwedge_{a \in \mathbb{A}_0} \{\Phi(b, a), \Theta(c, a)\} \text{ and } [\Psi, \Theta](b, a) = \textstyle\bigwedge_{c \in \mathbb{C}_0} [\Psi(c, b), \Theta(c, a)], \qquad (1)$$

but it is often much better practice to use the categorical meaning of these extensions and liftings (they are determined by adjoints!) than to make computations with these explicit formulas. Furthermore we can make use of all bicategorical notions in the quantaloid $\mathsf{Dist}(\mathcal{Q})$; for example, an adjunction of distributors between $\mathcal{Q}$-categories, say



means exactly that $\mathsf{id}_{\mathbb{A}} \le \Phi^* \otimes \Phi$ and $\Phi \otimes \Phi^* \le \mathsf{id}_{\mathbb{B}}$ in $\mathsf{Dist}(\mathcal{Q})$.

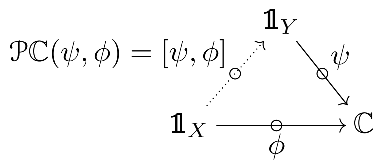
Example 3.3 (Presheaves, representables, sups and infs) For an object $X \in \mathcal{Q}$ and a $\mathcal{Q}$-category $\mathbb{C}$, a **contravariant $\mathcal{Q}$-presheaf of type X on** $\mathbb{C}$ is, by definition, a distributor $\phi\colon \mathbb{1}_X -\!\!\circ\!\!\!\rightarrow \mathbb{C}$. Reckoning that $\mathbb{1}_X$ has only one object and that its single hom-arrow is 1_X, such a contravariant presheaf is really

`(matr)` a function $\phi\colon \mathbb{C}_0 \longrightarrow \{\text{arrows of } \mathcal{Q} \text{ with domain } X\}\colon x \mapsto \Big(\phi(x)\colon X \longrightarrow tx\Big)$

`(act)` such that $\mathbb{C}(y,x) \circ \phi(x) \leq \phi(y)$, or equivalently $\mathbb{C}(y,x) \leq \{\phi(x), \phi(y)\}$, in $\mathcal{Q}$.

Note how, in the latter inequality, the x and y swap places: this accounts for the "contravariancy" of the presheaf ϕ.

Now fix a $\mathcal{Q}$-category $\mathbb{C}$, and consider the collection of all contravariant presheaves on $\mathbb{C}$. For any two such presheaves, say $\phi\colon \mathbb{1}_X -\!\!\circ\!\!\!\rightarrow \mathbb{C}$ and $\psi\colon \mathbb{1}_Y -\!\!\circ\!\!\!\rightarrow \mathbb{C}$, the lifting $[\psi, \phi]\colon \mathbb{1}_X -\!\!\circ\!\!\!\rightarrow \mathbb{1}_Y$ in the quantaloid $\mathsf{Dist}(\mathcal{Q})$ is a distributor with a single element, which can therefore be identified with an arrow from X to Y in $\mathcal{Q}$. This simple fact gives rise to the construction of the **category $\mathcal{P}\mathbb{C}$ of contravariant presheaves**: its objects are the contravariant presheaves on $\mathbb{C}$ (of all possible types); the type of a presheaf $\phi\colon \mathbb{1}_X -\!\!\circ\!\!\!\rightarrow \mathbb{C}$ is, obviously, X; and the hom $\mathcal{P}\mathbb{C}(\psi, \phi)$ is the (single element of the) lifting $[\psi, \phi]$:



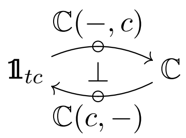
The underlying order (as in Example 2.12) of $\mathcal{P}\mathbb{C}$ is easily seen to coincide with the ordering of the presheaves as distributors (so for ϕ and ψ as in the diagram above, we have $\phi \leq \psi$ if and only if $X = Y$ and $\phi(c) \leq \psi(c)$ in $\mathcal{Q}(X, tc)$ for all $c \in \mathbb{C}$); in particular is $\mathcal{P}\mathbb{C}$ a skeletal $\mathcal{Q}$-category.

All the above can be repeated for **covariant $\mathcal{Q}$-presheaves on** $\mathbb{C}$, i.e. distributors of the form $\kappa\colon \mathbb{C} -\!\!\circ\!\!\!\rightarrow \mathbb{1}_X$ (for any X). The category of covariant presheaves on $\mathbb{C}$ is written as $\mathcal{P}^\dagger\mathbb{C}$, and the hom is given by $\mathcal{P}^\dagger\mathbb{C}(\lambda, \kappa) = \{\kappa, \lambda\}$. (Attention: the arguments κ and λ must be swapped, as a diagram of the situation will immediately make clear.)

For any $c \in \mathbb{C}_0$, it is a straightforward fact that

$$\mathbb{C}(-,c)\colon \mathbb{1}_{tc} -\!\!\circ\!\!\!\rightarrow \mathbb{C}\colon x \mapsto \mathbb{C}(x,c) \text{ and } \mathbb{C}(c,-)\colon \mathbb{C} -\!\!\circ\!\!\!\rightarrow \mathbb{1}_{tc}\colon x \mapsto \mathbb{C}(c,x)$$

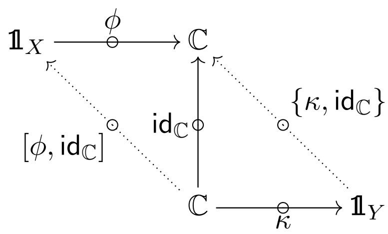
are a contravariant and a covariant presheaf of type tc on $\mathbb{C}$: these are the presheaves **represented by** $c \in \mathbb{C}$. Furthermore, they participate in an adjunction



in the quantaloid $\mathsf{Dist}(\mathcal{Q})$, that is to say, they satisfy the inequations

$$\mathsf{id}_{\mathbb{1}_{tc}} \leq \mathbb{C}(c,-) \otimes \mathbb{C}(-,c) \text{ and } \mathbb{C}(-,c) \otimes \mathbb{C}(c,-) \leq \mathsf{id}_{\mathbb{C}}.$$

Here is a typical illustration of the yoga that one can perform with these presheaves. Let $\phi\colon \mathbb{1}_X \,{\to}\!\!\!\!\!\!\circ\;\; \mathbb{C}$ be a contravariant presheaf, then the lifting of ϕ through $\mathsf{id}_\mathbb{C}\colon \mathbb{C} \,{\to}\!\!\!\!\!\!\circ\;\; \mathbb{C}$ produces a covariant presheaf $[\phi, \mathsf{id}_\mathbb{C}]\colon \mathbb{C} \,{\to}\!\!\!\!\!\!\circ\;\; \mathbb{1}_X$; and similarly, if $\kappa\colon \mathbb{C} \,{\to}\!\!\!\!\!\!\circ\;\; \mathbb{1}_X$ is a covariant presheaf, then the extension $\{\kappa, \mathsf{id}_\mathbb{C}\}\colon \mathbb{1}_Y \,{\to}\!\!\!\!\!\!\circ\;\; \mathbb{C}$ is a contravariant presheaf:



It can easily be checked that $[\mathbb{C}(-,c), \mathsf{id}_\mathbb{C}] = \mathbb{C}(c,-)$ and $\{\mathbb{C}(c,-), \mathsf{id}_\mathbb{C}\} = \mathbb{C}(-,c)$; that is to say, computing the lifting/extension of a representable presheaf on $\mathbb{C}$ along $\mathsf{id}_\mathbb{C}$ always produces a representable presheaf. *But it may happen that the lifting/extention of a not-necessarily representable presheaf still produces a representable.* If $\phi \in \mathcal{P}\mathbb{C}$ is such that $[\phi, \mathsf{id}_\mathbb{C}] = \mathbb{C}(c,-)$ in $\mathcal{P}^\dagger\mathbb{C}$, then we say that $c \in \mathbb{C}$ is the **supremum** of ϕ (and necessarily $c \in \mathbb{C}$ will be an essentially unique object with the same type as the presheaf ϕ); and similarly, if $\kappa \in \mathcal{P}^\dagger\mathbb{C}$ is such that $\{\kappa, \mathsf{id}_\mathbb{C}\} = \mathbb{C}(-,c)$ in $\mathcal{P}\mathbb{C}$, then $c \in \mathbb{C}$ is the **infimum** of ψ. Finally, $\mathbb{C}$ is said to be **cocomplete** if every $\phi \in \mathcal{P}\mathbb{C}$ has a supremum; and it is **complete** if every $\kappa \in \mathcal{P}^\dagger\mathbb{C}$ has an infimum.

Example 3.4 (Internal homs) For any object X of a quantaloid $\mathcal{Q}$, we can consider the $\mathcal{Q}$-categories $\mathcal{P}\mathbb{1}_X$ and $\mathcal{P}^\dagger\mathbb{1}_X$: a short computation will convince the reader that the former's objects are the $\mathcal{Q}$-arrows *into* X, whereas the latter's objects are the $\mathcal{Q}$-arrows *out of* X; and the homs are given by liftings and extensions in $\mathcal{Q}$ respectively. Applied to a base *quantale* Q, this gives the familiar fact that Q itself can be made into a Q-enriched category in two different ways: one way is to define $\mathbb{Q} := \mathcal{P}\mathbb{1}$, so that $\mathbb{Q}_0 = Q$ and $\mathbb{Q}(f,g) = [f,g] = f \searrow g$; another is to define $\mathbb{Q}' := \mathcal{P}^\dagger\mathbb{1}$, for which $\mathbb{Q}'_0 = Q$ and $\mathbb{Q}'(f,g) = \{g,f\} = f \swarrow g$. (But note that the underlying order of $\mathbb{Q}$ is exactly $(Q, \leq)$, whereas the underlying order of $\mathbb{Q}'$ is $(Q, \leq^{\mathsf{op}})$.)

For this reason, the lifting and extension in a base quantale or quantaloid are often referred to as its **internal hom functions**.

Example 3.5 (Ideal relations, up- and downclosed subsets) As indicated earlier, an ordered set $(C, \leq)$ defines, and is defined by, the **2**-category $\mathbb{C}$ with

`(obj)` $\mathbb{C}_0 = C$,

`(hom)` $\mathbb{C}(x,y) = 1$ iff $x \leq y$ (and 0 otherwise).

Because $\mathbf{2} = (\{0,1\}, \bigvee, \wedge, 1)$ is a quantale, there is no need to specify types; and because it is a commutative quantale, lifting and extension are identical, so we write both as $x \to y$.

Now, for ordered sets $(C, \leq)$ and $(D, \leq)$ viewed as **2**-categories $\mathbb{C}$ and $\mathbb{D}$, a distributor $\Phi\colon \mathbb{C} \,{\to}\!\!\!\!\!\!\circ\;\; \mathbb{D}$ is, according to the definition,

`(matr)` a matrix $\Phi\colon D \times C \longrightarrow \{0,1\}$

`(act)` such that $\mathbb{D}(y',y) \wedge \Phi(y,x) \wedge \mathbb{C}(x,x') \leq \Phi(y',x')$.

This says that Φ is exactly (the characteristic function of) an **ideal relation** between $(C,\leq)$ and $(D,\leq)$: a binary relation $\Phi\subseteq D\times C$ such that $y'\leq y\ \Phi\ x\leq x'$ implies $y'\Phi x'$.

In particular, letting $\mathbb{C}=\mathbb{1}$ be the trivial **2**-category (corresponding with the singleton ordered set), so considering in fact a contravariant presheaf on $\mathbb{D}$, the corresponding ideal relation reduces to (the characteristic function of) a downset of $(D,\leq)$. By the general theory, these downsets/presheaves are the objects of a **2**-category $\mathcal{P}\mathbb{D}$; explicitly, for $\phi\colon\mathbb{1}\looparrowright\mathbb{D}$ and $\psi\colon\mathbb{1}\looparrowright\mathbb{D}$, we can compute with the formulas in (1) that

$$\mathcal{P}\mathbb{D}(\psi,\phi)=\bigwedge_{d\in D}\psi(d)\to\phi(d).$$

This logical formula evaluates to the sentence: "for all $d\in D$, if $d\in\psi$ then $d\in\phi$". In other words, the **2**-category $\mathcal{P}\mathbb{D}$ is the set of downsets of $(D,\leq)$, ordered by *inclusion*. Similary, now letting $\mathbb{D}=\mathbb{1}$, the reader will easily find that covarant presheaves on $\mathbb{C}$ correspond with upsets of $(C,\leq)$; and $\mathcal{P}^{\dagger}\mathbb{C}$ is the set of upsets of $(C,\leq)$, ordered by *containment*.

Furthermore, a representable contravariant presheaf $\mathbb{D}(-,d)\colon\mathbb{1}\looparrowright\mathbb{D}$ is the characteristic function

$$D\longrightarrow\{0,1\}\colon y\mapsto\begin{cases}1 \text{ iff } y\leq d\\ 0 \text{ otherwise}\end{cases}$$

of the principal downset $\downarrow d$ of $(D,\leq)$. And, as can now be expected, a representable covariant presheaf is "the same thing" as an upset.

A straightforward computation (again with the formulas in (1)) also shows that, for any presheaf/downset $\phi\in\mathcal{P}\mathbb{D}$,

$$[\phi,\mathsf{id}_{\mathbb{D}}](y)=\bigwedge_{d\in D}\phi(d)\to\mathsf{id}_{\mathbb{D}}(d,y),$$

which is to say that $y\in[\phi,\mathsf{id}_{\mathbb{D}}]$ if and only if "for all $d\in D$, if $d\in\phi$ then $d\leq y$". In plain English: the covariant presheaf $[\phi,\mathsf{id}_{\mathbb{D}}]\colon\mathbb{D}\looparrowright\mathbb{1}$ is the upset containing all upper bounds of the contravariant presheaf $\phi\colon\mathbb{1}\looparrowright\mathbb{D}$.

Combining the previous computations, we can thus conclude that $[\phi,\mathsf{id}_{\mathbb{D}}]$ is representable by $d\in\mathbb{D}$ if and only if d is the least upper bound of ϕ.

Similarly one checks that, for any upset $\kappa\in\mathcal{P}^{\dagger}\mathbb{C}$, the extension $\{\kappa,\mathsf{id}_{\mathbb{C}}\}$ is representable if and only if κ has a greatest lower bound. Therefore, cocomplete **2**-categories coincide with ordered sets in which all suprema exist; and complete **2**-categories are those ordered sets in which all infima exist. Further on, in Example 3.11, we shall see that – in general – a $\mathcal{Q}$-category is cocomplete if and only if it is complete.

Example 3.6 (Discrete categories, fuzzy sets) For any quantaloid $\mathcal{Q}$, a $\mathcal{Q}$-category $\mathbb{S}$ is said to be **discrete** if, for any $x,y\in\mathbb{S}_0$,

$$\mathbb{S}(x,y)=\begin{cases}1_X \text{ whenever } x=y \text{ and } tx=X,\\ 0_{X,Y} \text{ otherwise.}\end{cases}$$

(Here $0_{X,Y}$ denotes the bottom element of the hom-sup-lattice $\mathcal{Q}(X,Y)$.) That is to say, such a discrete $\mathcal{Q}$-category is precisely a set S, typed by a function $t\colon S\longrightarrow\{\text{objects of }\mathcal{Q}\}$, viewed in a trivial fashion as $\mathcal{Q}$-category[17]. But the enriched co- and contravariant presheaves on a discrete $\mathcal{Q}$-category may be highly non-trivial! In fact, it is our understanding that this is at the origin of fuzzy set theory.

To make our point, first consider a discrete **2**-category $\mathbb{S}$; it is completely determined by giving its set $S = \mathbb{S}_0$ of objects. A contravariant/covariant presheaf ϕ on $\mathbb{S}$ is, as computed in the previous Example 3.5, a downset/upset of S—but because $\mathbb{S}$ is discrete, neither down- nor upclosedness is meaningful, so ϕ ends up being simply (the characteristic function of) a subset of S. As a consequence, $\mathcal{P}\mathbb{S}$ "is" the set of subsets of $S = \mathbb{S}_0$, ordered by inclusion, and $\mathcal{P}^\dagger\mathbb{S}$ "is" the set of subsets, ordered by containment—and neither of these is discrete!

Similarly, fix a left-continuous t-norm on $[0, 1]$ (see the earlier Example 2.4), and consider a discrete $[0, 1]$-enriched category $\mathbb{S}$: it is simply a **crisp set** S (the type function is again obsolete, since we work over a base quantale). Because $\mathbb{S}$ is discrete, the notions of co- and contravariant presheaf on $\mathbb{S}$ reduce again to $[0, 1]$-valued functions on S: a presheaf on $\mathbb{S}$ is thus a **fuzzy set** (on S). However, by the general theory explained above, it follows that fuzzy sets (on S) are the objects of two $[0, 1]$-categories $\mathcal{P}\mathbb{S}$ and $\mathcal{P}^\dagger\mathbb{S}$—and non-discrete ones at that! For instance, for $\phi, \psi \in \mathcal{P}\mathbb{S}$ we can compute with the formula in (1) that

$$\mathcal{P}\mathbb{S}(\psi, \phi) = \bigwedge_{s\in S} \psi(s) \to_* \phi(s),$$

where $\to_*$ is now the residuum of the given left-continuous t-norm $*$ on $[0, 1]$. So we find here precisely (the truth degree of) the **graded inclusion of fuzzy sets** [Gottwald, 2001, Definition 18.2.3]. Moreover, the underlying ordered set of $\mathcal{P}\mathbb{S}$ produces exactly the binary ordering relation on fuzzy sets [Gottwald, 2001, Equation (18.6)]. (In the same way, the reader can check that $\mathcal{P}^\dagger\mathbb{S}$ is the collection of fuzzy sets on S, ordered by "graded containment".)

More generally, a distributor between discrete **2**-categories is precisely a relation, whereas a distributor between $[0, 1]$-categories (with a left-continuous t-norm on $[0, 1]$) is the same thing as a **fuzzy relation**; distributor-composition subsumes the composition of ordinary/fuzzy relations [Gottwald, 2001, Section 18.3].

The upshot of these observations is that the general results on quantaloid-enriched categories apply to the particular case of fuzzy sets; and conversely, that well worked out examples in fuzzy set theory may be a starting point for new developments in quantaloid-enriched category theory.

We now turn to the second type of morphism between $\mathcal{Q}$-categories.

Definition 3.7 *A* **$\mathcal{Q}$-functor** $F: \mathbb{C} \longrightarrow \mathbb{D}$ *between two $\mathcal{Q}$-categories is*

`(map)` *an 'object map'* $F: \mathbb{C}_0 \longrightarrow \mathbb{D}_0: x \mapsto Fx$

satisfying, for all $x, x' \in \mathbb{C}_0$,

`(type)` *the 'type equality'* $t(Fx) = tx$,

`(fun)` *the 'functor inequality'* $\mathbb{C}(x', x) \leq \mathbb{D}(Fx', Fx)$.

[17] Indeed, there is an appropriate "forgetful" functor from $\mathcal{Q}$-categories to $\mathcal{Q}$-typed sets, whose left adjoint sends a $\mathcal{Q}$-typed set to the "free" discrete $\mathcal{Q}$-category.

Functors $F: \mathbb{A} \longrightarrow \mathbb{B}$ and $G: \mathbb{B} \longrightarrow \mathbb{C}$ are composed in the obvious way to produce a new functor $G \circ F: \mathbb{A} \longrightarrow \mathbb{C}$, and the identity object map provides for the identity functor $1_{\mathbb{A}}: \mathbb{A} \longrightarrow \mathbb{A}$. Thus functors are the morphisms of a (large) category $\mathsf{Cat}(\mathcal{Q})$.

A priori there is no extra structure in $\mathsf{Cat}(\mathcal{Q})$, contrary to $\mathsf{Dist}(\mathcal{Q})$ which is a quantaloid and therefore comes with a lot of extra structure and properties. But we shall now see how $\mathsf{Cat}(\mathcal{Q})$ embeds in $\mathsf{Dist}(\mathcal{Q})$, and therefore inherits some of that structure.

Proposition 3.8 *Every functor $F: \mathbb{A} \longrightarrow \mathbb{B}$ determines an* **adjoint pair of distributors**

$$\mathbb{A} \underset{F^*}{\overset{F_*}{\underset{\perp}{\rightleftarrows}}} \mathbb{B}$$

defined by $F_*(b,a) = \mathbb{B}(b, Fa)$ and $F^*(a,b) = \mathbb{B}(Fa, b)$. That is to say, in the quantaloid $\mathsf{Dist}(\mathcal{Q})$ we have the inequalities

$$\mathsf{id}_{\mathbb{A}} \leq F^* \otimes F_* \text{ and } F_* \otimes F^* \leq \mathsf{id}_{\mathbb{B}}.$$

Loosely speaking, we say that F_* is the **graph** of the functor F, and F^* is its **cograph**. Taking graphs and cographs extends to a pair of functors, one covariant and the other contravariant:

$$\mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q}): \Big(F: \mathbb{A} \longrightarrow \mathbb{B}\Big) \mapsto \Big(F_*: \mathbb{A} {-\!\!\circ\!\!\rightarrow} \mathbb{B}\Big),$$

$$\mathsf{Cat}(\mathcal{Q})^{\mathsf{op}} \longrightarrow \mathsf{Dist}(\mathcal{Q}): \Big(F: \mathbb{A} \longrightarrow \mathbb{B}\Big) \mapsto \Big(F^*: \mathbb{B} {-\!\!\circ\!\!\rightarrow} \mathbb{A}\Big).$$

With this, we make $\mathsf{Cat}(\mathcal{Q})$ a *locally ordered category* by defining, for any parallel pair of functors $F, G: \mathbb{A} \longrightarrow \mathbb{B}$,

$$F \leq G \overset{\text{def}}{\iff} F_* \leq G_* \iff F^* \geq G^*.$$

Whenever $F \leq G$ and $G \leq F$, we write $F \cong G$ and say that these functors are isomophic.

Example 3.9 (Adjoints, equivalences) An **adjunction**

$$\mathbb{A} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathbb{B}$$

in $\mathsf{Cat}(\mathcal{Q})$ is a pair of functors satisfying $1_{\mathbb{A}} \leq G \circ F$ and $F \circ G \leq 1_{\mathbb{B}}$. This is equivalent to the requirement that $\mathbb{B}(Fx, y) = \mathbb{A}(x, Gy)$.

A functor $F: \mathbb{A} \longrightarrow \mathbb{B}$ is **fully faithful** when $\mathbb{A}(x', x) = \mathbb{B}(Fx', Fx)$, that is, when $F^* \otimes F_* = \mathsf{id}_{\mathbb{A}}$; and it is **essentially surjective** when for each $y \in \mathbb{B}_0$ there exists an $x \in \mathbb{A}_0$ such that $Fa \cong b$, or equivalently, when $F_* \otimes F^* = \mathsf{id}_{\mathbb{B}}$. Further, $F: \mathbb{A} \longrightarrow \mathbb{B}$ is an **equivalence** if there exists a $G: \mathbb{B} \longrightarrow \mathbb{A}$ such that $G \circ F \cong 1_{\mathbb{A}}$ and $F \circ G \cong 1_{\mathbb{B}}$; it is equivalent to say that F is fully faithful and essentially surjective.

If $F: \mathbb{A} \longrightarrow \mathbb{B}$ is a functor whose object map is an inclusion, then $\mathbb{A}$ is a **subcategory** of $\mathbb{B}$; and if F is moreover fully faithful, then $\mathbb{A}$ is a **full subcategory**. In other words, a full subcategory $\mathbb{A}$ of $\mathbb{B}$ is completely determined by a selection of objects $\mathbb{A}_0 \subseteq \mathbb{B}_0$, endowed with the same hom function as on $\mathbb{B}$.

In a similar fashion we can go on to define **monads** and **comonads** in on a $\mathcal{Q}$-category, **Kan extensions** of functors between $\mathcal{Q}$-categories, and so on; for more on this we refer to the literature.

Example 3.10 (Duality) If $\mathcal{Q}$ is a quantaloid, then so is its opposite $\mathcal{Q}^{\mathsf{op}}$, for it is obtained by formally reversing the direction of the arrows in $\mathcal{Q}$. It thus makes perfect sense to define $\mathcal{Q}^{\mathsf{op}}$-categories too. But whenever $\mathbb{C}$ is a $\mathcal{Q}$-category, we can consider its formal opposite too: define $\mathbb{C}^{\mathsf{op}}$ to have the same objects and types as $\mathbb{C}$, but put $\mathbb{C}^{\mathsf{op}}(y,x) := \mathbb{C}(x,y)$; this produces a $\mathcal{Q}^{\mathsf{op}}$-category $\mathbb{C}^{\mathsf{op}}$. The reader will have no difficulty in verifying that this extends to distributors and functors, to produce isomorphisms

$$\mathsf{Dist}(\mathcal{Q}) \cong \mathsf{Dist}(\mathcal{Q}^{\mathsf{op}})^{\mathsf{op}} \text{ and } \mathsf{Cat}(\mathcal{Q}) \cong \mathsf{Cat}(\mathcal{Q}^{\mathsf{op}})^{\mathsf{co}},$$

where the "co" means that we formally reverse the order between the arrows (but keep the direction of the arrows). This now gives us a duality principle, as follows. When a notion, say a *widget*, is defined for general quantaloid-enriched categories, it has an incarnation in $\mathcal{Q}$-categories and an incarnation in $\mathcal{Q}^{\mathsf{op}}$-categories. Translating the latter back in terms of $\mathcal{Q}$-categories via the above isomorphisms, produces the **dual notion** to the original widget, usually (but not always) called *cowidget*. For instance, the infimum-supremum definitions are dual to each other in this precise sense. In fact, the covariant-contravariant presheaf constructions are dual, in the sense that $\mathcal{P}^{\dagger}\mathbb{C}$ is identical to $(\mathcal{P}(\mathbb{C}^{\mathsf{op}}))^{\mathsf{op}}$. Finally, it is an exercise to check that a contravariant presheaf $\phi\colon \mathbb{1}_X \to \mathbb{C}$, i.e. an arrow in $\mathsf{Dist}(\mathcal{Q})$, is precisely the same thing as a functor $F\colon \mathbb{C}^{\mathsf{op}} \longrightarrow \mathcal{P}\mathbb{1}_X$ in $\mathsf{Cat}(\mathcal{Q}^{\mathsf{op}})$.

Example 3.11 (Classification of distributors, Yoneda Lemma) Consider again the category $\mathcal{P}\mathbb{C}$ of contravariant presheaves on a $\mathcal{Q}$-category $\mathbb{C}$: it has an important **classifying property**. Indeed, a functor $F\colon \mathbb{A} \longrightarrow \mathcal{P}\mathbb{C}$ sends every object $a \in \mathbb{A}$ to a contravariant presheaf $F(a)\colon \mathbb{1}_{ta} \to \mathbb{C}$; it is easy to check that this can be reinterpreted as a distibutor $\Phi_F\colon \mathbb{A} \to \mathbb{C}$ whose elements are $\Phi_F(c,a) = F(a)(c)$. Conversely, if $\Phi\colon \mathbb{A} \to \mathbb{C}$ is any distributor, then fixing an $a \in \mathbb{A}$ produces a contravariant presheaf $\Phi(-,a)\colon \mathbb{1}_{ta} \to \mathbb{C}$; actually, this makes for a functor $F_\Phi\colon \mathbb{A} \longrightarrow \mathcal{P}\mathbb{C}\colon a \mapsto \Phi(-,a)$. These procedures are each others' inverse, and determine an isomorphism of ordered sets

$$\mathsf{Cat}(\mathcal{Q})(\mathbb{A}, \mathcal{P}\mathbb{C}) \cong \mathsf{Dist}(\mathcal{Q})(\mathbb{A}, \mathbb{C}),$$

exhibiting how $\mathcal{P}\mathbb{C}$ *classifies distributors with codomain* $\mathbb{C}$. The identity distributor $\mathsf{id}_\mathbb{C}\colon \mathbb{C} \to \mathbb{C}$ in particular corrpesponds with a functor, written $Y_\mathbb{C}\colon \mathbb{C} \longrightarrow \mathcal{P}\mathbb{C}$ and called the **(contravariant) Yoneda embedding**, which sends an object $c \in \mathbb{C}$ precisely to the representable presheaf $\mathbb{C}(-,c)\colon \mathbb{1}_{tc} \to \mathbb{C}$. Because $Y_\mathbb{C}(c) = \mathbb{C}(-,c)$ is left adjoint to $\mathbb{C}(c,-)$ in $\mathsf{Dist}(\mathcal{Q})$, it is easy to compute, for any $\phi \in \mathcal{P}\mathbb{C}$, that

$$\mathcal{P}\mathbb{C}(Y_\mathbb{C}(c), \phi) = [\mathbb{C}(-,c), \phi] = \mathbb{C}(c,-) \otimes \phi = \phi(c).$$

This result is known as the **Yoneda Lemma**; it implies that $Y_\mathbb{C}$ is a fully faithful functor (thus deserving the name "embedding").

The analogous classifying property of $\mathcal{P}^\dagger\mathbb{C}$ is expressed by the isomorphism

$$\mathsf{Cat}(\mathcal{Q})(\mathbb{A},\mathcal{P}^\dagger\mathbb{C}) \cong \mathsf{Dist}(\mathcal{Q})(\mathbb{C},\mathbb{A})^{\mathsf{op}},$$

where the "op" on the right hand side stands for the reversal of the order; the (covariant) Yoneda embedding is $Y^\dagger_\mathbb{C}\colon \mathbb{C}\longrightarrow\mathcal{P}^\dagger\mathbb{C}\colon c\mapsto\mathbb{C}(c,-)$; and the (covariant) Yoneda Lemma says that, for any $\kappa\in\mathcal{P}^\dagger\mathbb{C}$,

$$\mathcal{P}^\dagger\mathbb{C}(\kappa, Y^\dagger_\mathbb{C}(c)) = \kappa(c).$$

It is now a standard fact that $\mathbb{C}$ is cocomplete if and only if $Y_\mathbb{C}\colon\mathbb{C}\longrightarrow\mathcal{P}\mathbb{C}$ has a (necessarily surjective) left adjoint, which we then write as $\sup_\mathbb{C}\colon\mathcal{P}\mathbb{C}\longrightarrow\mathbb{C}$, for it maps a contravariant presheaf $\phi\in\mathcal{P}\mathbb{C}$ to its "supremum":

$$\mathcal{P}\mathbb{C} \underset{Y_\mathbb{C}}{\overset{\sup_\mathbb{C}}{\underset{\perp}{\rightleftarrows}}} \mathbb{C}$$

Similarly, $Y^\dagger_\mathbb{C}\colon\mathbb{C}\longrightarrow\mathcal{P}^\dagger\mathbb{C}\colon c\mapsto\mathbb{C}(c,-)$ has a right adjoint, written $\inf_\mathbb{C}\colon\mathcal{P}^\dagger\mathbb{C}\longrightarrow\mathbb{C}$ and read as "infimum", if and only if $\mathbb{C}$ is complete. Also the computation of "upper bounds" and "lower bounds" is functorial: precisely, there are adjoint functors

$$\mathcal{P}\mathbb{C} \underset{L}{\overset{U}{\underset{\perp}{\rightleftarrows}}} \mathcal{P}^\dagger\mathbb{C}$$

where $U(\phi)=[\phi,\mathsf{id}_\mathbb{C}]$ and $L(\kappa)=\{\kappa,\mathsf{id}_\mathbb{C}\}$. It follows that $\mathbb{C}$ is cocomplete if and only if it is complete: essentially because $\sup_\mathbb{C}(\phi)=\inf_\mathbb{C}(U(\phi))$ and $\inf_\mathbb{C}(\kappa)=\sup_\mathbb{C}(L(\kappa))$.

Of course not every distributor is a left adjoint distributor, and not every left adjoint distributor is the graph of a functor. This motivates the following terminology.

Definition 3.12 *If for a (necessarily left adjoint) distributor $\Phi\colon\mathbb{A}\mathrel{\circ\!\!\!\rightarrow}\mathbb{B}$ between $\mathcal{Q}$-categories there exists a (necessarily essentially unique) functor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ such that $\Phi=F_*$, then we say that Φ* **is represented by** *F, or simply that Φ* **is representable**. *(It is of course equivalent to require the right adjoint of Φ to be equal to F^*.)*

Studying the representability of distributors is at the heart of $\mathcal{Q}$-categorical algebra!

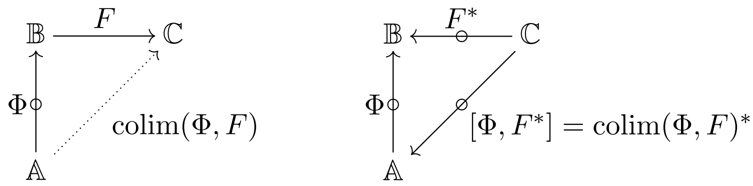
Example 3.13 (Weighted limits and colimits) For $c\in\mathbb{C}$ an object, of type $X\in\mathcal{Q}$ say, of a $\mathcal{Q}$-category $\mathbb{C}$, there is a functor "pointing at" c as follows:

$$\Delta c\colon \mathbb{1}_X\longrightarrow\mathbb{C}\colon *\mapsto c.$$

The adjoint graph and cograph of this functor are precisely the representable presheaves: $(\Delta c)_* = \mathbb{C}(-,c)$ and $(\Delta c)^*=\mathbb{C}(c,-)$. Therefore, $c\in\mathbb{C}$ is the supremum of some contravariant presheaf $\phi\in\mathcal{P}\mathbb{C}$ if and only if the lifting $[\phi,\mathsf{id}_\mathbb{C}]\in\mathcal{P}^\dagger\mathbb{C}$ is represented by Δc. Similarly, $c\in\mathbb{C}$ is the infimum of $\kappa\in\mathcal{P}^\dagger\mathbb{C}$ if and only if Δc represents $\{\kappa,\mathsf{id}_\mathbb{C}\}\in\mathcal{P}\mathbb{C}$.

This situation is suitably generalised as follows.

For a distributor $\Phi\colon \mathbb{A} \,\text{-}\!\!\circ\!\!\to \mathbb{B}$ and a functor $F\colon \mathbb{B} \longrightarrow \mathbb{C}$, all between $\mathcal{Q}$-categories, the **Φ-weighted colimit of** F is (whenever it exists) the functor $\mathrm{colim}(\Phi, F)\colon \mathbb{A} \longrightarrow \mathbb{C}$ that represents the lifting $[\Phi, F^*]$:



Thus, the supremum $\sup_{\mathbb{C}}(\phi)$ of $\phi \in \mathcal{P}\mathbb{C}$ is exactly the same thing as $\mathrm{colim}(\phi, 1_{\mathbb{C}})$. On the other hand, $\mathrm{colim}(\Phi, F)(a)$ is exactly the same thing as $\sup_{\mathbb{C}}(F_* \otimes \Phi(-, a))$. This makes clear that a $\mathcal{Q}$ category $\mathbb{C}$ is cocomplete (recall, we defined this to mean that $\mathbb{C}$ has all suprema) if and only if all weighted colimits in $\mathbb{C}$ exist.

To say, in the above situation, that a functor $G\colon \mathbb{C} \longrightarrow \mathbb{D}$ **preserves** the weighted colimit $\mathrm{colim}(\Phi, F)$, is to say that $G \circ \mathrm{colim}(\Phi, F) \cong \mathrm{colim}(\Phi, G \circ F)$. If G preserves all colimits that happen to exist in $\mathbb{C}$, then it is said to be **cocontinous**. It is a classical result that every left adjoint functor is cocontinuous; in general the converse need not hold, but it does precisely when $\mathbb{C}$ is cocomplete. As a consequence, the category $\mathsf{Cocont}(\mathcal{Q})$ of cocomplete $\mathcal{Q}$-categories and *cocontinuous functors*, is precisely the category of cocomplete $\mathcal{Q}$-categories and *left adjoint* functors.

The dual notion to weighted colimit is that of **weighted limit**, and a functor that preserves weighted limits, is said to be **continuous**; we leave the details to the reader. General weighted limits and colimits in a $\mathcal{Q}$-category $\mathbb{C}$ can be computed from very particular ones, namely so-called tensors and cotensors, conical limits and colimits, and order-suprema and infima. We refer to [Stubbe, 2006] for more details.

The application of all the above to **2**-categories, i.e. ordered sets, produces many familiar notions; we leave this to the reader. To end this section, we spell out an inspiring example from metric spaces, with a surprising connection to sheaf theory.

Example 3.14 (Cauchy completeness) Since not every left adjoint distributor need be representable, it makes sense to define a $\mathcal{Q}$-category $\mathbb{B}$ to be **Cauchy complete** whenever every left adjoint distributor into $\mathbb{B}$ is representable [Lawvere, 1973; Street, 1981]. That is to say, $\mathbb{B}$ is Cauchy complete if and only if, for every other $\mathcal{Q}$-category $\mathbb{A}$, there is an isomorphism of ordered sets

$$\mathsf{Cat}(\mathcal{Q})(\mathbb{A}, \mathbb{B}) \xrightarrow{\cong} \mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))(\mathbb{A}, \mathbb{B})\colon F \mapsto F_*,$$

where we write $\mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))$ for the subcategory of $\mathsf{Dist}(\mathcal{Q})$ with all the same objects but only the left adjoint arrows (also called "maps", hence the notation). Consequently, writing now $\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})$ for the full subcategory of $\mathsf{Cat}(\mathcal{Q})$ containing only the Cauchy complete $\mathcal{Q}$-categories but with all functors between them, and similarly $\mathsf{Dist}_{\mathsf{cc}}(\mathcal{Q})$, we have an equivalence

$$\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}) \simeq \mathsf{Map}(\mathsf{Dist}_{\mathsf{cc}}(\mathcal{Q})).$$

Better still, it can be shown that every $\mathcal{Q}$-category is isomorphic in $\mathsf{Dist}(\mathcal{Q})$ to a Cauchy complete one, making $\mathsf{Dist}_{\mathsf{cc}}(\mathcal{Q})$ equivalent to $\mathsf{Dist}(\mathcal{Q})$, so that actually

$$\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}) \simeq \mathsf{Map}(\mathsf{Dist}(\mathcal{Q})).$$

The origin of the terminology is that, for a metric space viewed as a $[0,\infty]$-category, the categorical notion of Cauchy completeness turns out to be equivalent to the usual metric notion [Lawvere, 1973]. For ordered sets viewed as **2**-categories, the reader will easily verify that *every* **2**-category is Cauchy complete—which might explain why a common generalisation of metric spaces and ordered sets had not been made explicit earlier.

There is however another important situation where Cauchy completeness is crucial. Indeed, in Example 2.13 we saw how a presheaf $F\colon L^{\mathsf{op}}\longrightarrow\mathsf{Set}$ on a locale L determines a symmetric, skeletal $\mathcal{R}(L)$-enriched category $\mathbb{C}_F$. It turns out that F is a *sheaf*[18] if and only if $\mathbb{C}_F$ is Cauchy complete [Walters, 1981]. That is to say, the topos $\mathsf{Sh}(L)$ of sheaves on a locale (with natural transformations as morphisms) is equivalent to the category $\mathsf{Cat}_{\mathsf{skel,sym,cc}}(\mathcal{R}(L))$ of skeletal, symmetric and Cauchy complete $\mathcal{R}(L)$-categories (and functors between them). This result can be extended to sheaves on an arbitrary site [Walters, 1982].

4. Some more $\mathcal{Q}$-categorical algebra

In this last section, we shall indicate some further topics in (our own work on) quantaloid-enriched category theory, without much details but with references, so that the interested fuzzy logician or fuzzy set theorist may perhaps apply some of these techniques to his or her own situation.

4.1. Change of base

A **lax morphism** $F\colon\mathcal{Q}\longrightarrow\mathcal{R}$ between two quantaloids is defined to send arrows $f\colon X\longrightarrow Y$ in $\mathcal{Q}$ to arrows $Ff\colon FX\longrightarrow FY$ in $\mathcal{R}$ in such a way that: $f\leq g$ implies $Ff\leq Fg$, $Fg\circ Ff\leq F(g\circ f)$ and $1_{FX}\leq F1_X$. If now $\mathbb{C}$ is a $\mathcal{Q}$-category, then it is straightforward to define an $\mathcal{R}$-category $F\mathbb{C}$ with the same object set as $\mathbb{C}$ but with homs given by $F\mathbb{C}(y,x)=F(\mathbb{C}(y,x))$. This construction extends to distributors and functors, producing a 2-functor $\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{R})$ and a lax morphism $\mathsf{Dist}(\mathcal{Q})\longrightarrow\mathsf{Dist}(\mathcal{R})$, both referred to as **change of base** functors.

If $H\colon\mathcal{Q}\longrightarrow\mathcal{R}$ is any quantaloid homomorphism, then for any $A,B\in\mathcal{Q}$ there is a sup-morphism $H_{A,B}\colon\mathcal{Q}(A,B)\longrightarrow\mathcal{R}(HA,HB)\colon f\mapsto Hf$, which necessarily has a right adjoint (in the category of ordered sets and order-preserving functions), say $H^*_{A,B}\colon\mathcal{R}(HA,HB)\longrightarrow\mathcal{Q}(A,B)$. Using these "local adjunctions", it is easily verified that $H^*_{B,C}(g)\circ H^*_{A,B}(f)\leq H^*_{A,C}(g\circ f)$ and $1_A\leq H^*_{A,A}(1_{HA})$. In other words, whenever H is *bijective on objects*, all these local right adjoints assemble to define a lax morphism $H^*\colon\mathcal{R}\longrightarrow\mathcal{Q}$. This is in particular the case for any homomorphism between quantales, so there are many examples of this situation.

Considering the quantale **2** as a one-object quantaloid, for any quantaloid $\mathcal{Q}$ we can define the lax morphism $\mathcal{Q}\longrightarrow\mathbf{2}$ which sends every arrow bigger than an identity in $\mathcal{Q}$ to the non-zero arrow in **2**, and all other arrows to zero. The change of base $\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathbf{2})$ then precisely sends a $\mathcal{Q}$-category to its underlying order (of Example 2.12).

As another example, recall that a probability measure μ on (the subsets of) a set S is a function $\mu\colon\mathcal{P}(S)\longrightarrow[0,1]$ such that $\mu(\emptyset)=0$, $\mu(S)=1$ and $\mu(\bigcup_i X_i)=\sum_i\mu(X_i)$ for any countable family of pairwise disjoint subsets of S. It is a standard exercise to show that μ is a

[18] That is to say, F satisfies the gluing condition: for every $u=\bigvee_i u_i$ in L and every $(x_i\in F(u_i))_i$ such that $x_{i|u_i\wedge u_j}=x_{j|u_i\wedge u_j}$ (for all i,j) there exists a unique $x\in F(u)$ such that $x_{|u_i}=x_i$ (for all i).

monotone function for which $\mu(X) + \mu(Y) - \mu(X \cap Y) \leq 1$ holds too: but this precisely says that $\mu\colon (\mathcal{P}(S), \bigcup, \cap, S) \longrightarrow ([0,1], \bigvee, *_l, 1)$ is a lax morphism from a complete boolean algebra to the unit interval with the Łukasiewicz t-norm (defined in Example 2.4).

Finally, remark that every homomorphism of quantaloids is trivially also a lax morphism. Given a quantaloid $\mathcal{Q}$, we saw in Examples 2.13 and 2.14 how $\mathcal{Q}$ embeds in $\mathcal{R}(\mathcal{Q})$ (the split-idempotent completion), which further embeds in $\mathcal{D}(\mathcal{Q})$ (the diagonal completion); these embeddings therefore determine change of base functors which straightforwardly embed $\mathsf{Cat}(\mathcal{Q})$ into $\mathsf{Cat}(\mathcal{R}(\mathcal{Q}))$ and further into $\mathsf{Cat}(\mathcal{D}(\mathcal{Q}))$. But – more surprisingly perhaps – there are also two lax morphisms from $\mathcal{D}(\mathcal{Q})$ to $\mathcal{Q}$, namely:

$$I\colon \mathcal{D}(\mathcal{Q}) \longrightarrow \mathcal{Q}\colon \Big(d\colon e \longrightarrow f\Big) \mapsto \Big(f \searrow d\colon A_1 \longrightarrow B_1\Big)$$

$$J\colon \mathcal{D}(\mathcal{Q}) \longrightarrow \mathcal{Q}\colon \Big(d\colon e \longrightarrow f\Big) \mapsto \Big(e \swarrow d\colon A_2 \longrightarrow B_2\Big)$$

Therefore, there are also two change of base functors which associate to any $\mathcal{D}(\mathcal{Q})$-category (which may have "partial/local objects") a $\mathcal{Q}$-category (which only have "total/global objects"). (And both I and J restrict to lax functors from $\mathcal{R}(\mathcal{Q})$ to $\mathcal{Q}$ as well, producing two more change of base functors from $\mathcal{R}(\mathcal{Q})$-categories to $\mathcal{Q}$-categories.)

Change of base functors can help to express useful information. For instance, it is a fact that the underlying order of a cocomplete $\mathcal{Q}$-category $\mathbb{C}$ (i.e. the image of $\mathbb{C}$ by the change of base induced by the lax morphism $\mathcal{Q} \longrightarrow \mathbf{2}$ discussed above) is a sup-lattice; but the converse need not hold. So it is interesting to study under which conditions $\mathbb{C}$ is cocomplete whenever its underlying order is known to be a sup-lattice. The answer is: $\mathbb{C}$ must be tensored and cotensored [Stubbe, 2006, Theorem 2.13]. Other examples, geared towards categories enriched in a divisible quantale, can be found in [Tao *et al.*, 2012]; and see [Hofmann and Reis, 2013] for an application in probabilistic metric spaces. For a (very) general theory of "change of base", see [Verity, 1992].

4.2. Modules

Let $Q = (Q, \bigvee, \cdot, 1)$ be a quantale. A **(right) Q-module** $M = (M, \bigvee, \otimes)$ is, by definition, a complete lattice $(M, \bigvee)$ together with a binary function (called a '(right) action' of Q on M)

$$M \times Q \longrightarrow M\colon (m, f) \mapsto m \otimes f$$

satisfying a number of conditions: it must be a sup-morphism in each variable, $(m \otimes f) \otimes g = m \otimes (f \cdot g)$ must hold for every $m \in M$ and $f, g \in Q$, and $(m \otimes 1) = m$ must hold for every $m \in M$. A **module homomorphism** $\alpha\colon M \longrightarrow N$ between two such modules is a sup-morphism which is equivariant for the actions; they compose and can be compared in the obvious manner, so that they are arrows of a (large) quantaloid $\mathsf{Mod}(Q)$.

Each such Q-module M determines a Q-category $\mathbb{M}$ as follows:

`(obj)` $\mathbb{M}_0 = M$,

`(hom)` $\mathbb{M}(n, m) = \bigvee\{f \in Q \mid n \otimes f \leq m\}$.

Moreover, the action of a module homomorphism $\alpha: M \longrightarrow N$ is indeed functorial between the Q-categories $\mathbb{M}$ and $\mathbb{N}$. The Q-categories and functors so obtained are *exactly* the cocomplete categories with the cocontinuous functors: that is to say, there is an equivalence

$$\mathsf{Mod}(Q) \simeq \mathsf{Cocont}(Q).$$

(In a similar fashion, left Q-modules correspond with complete Q-categories.)

All this generalises to '(right/left) modules on quantaloids', and the interplay between $\mathcal{Q}$-modules and $\mathcal{Q}$-categories can be refined [Stubbe, 2006]. This has proven fruitful, particularly in applications to domain theory [Stubbe, 2007] and to sheaf theory [Heymans and Stubbe, 2009a, 2009b] (and see also the references contained therein).

4.3. Cocompletion KZ-doctrines

If $\mathcal{W}$ be a class of distributors ("weights") between $\mathcal{Q}$-categories, then we say that a $\mathcal{Q}$-category $\mathbb{C}$ is **$\mathcal{W}$-cocomplete** if, for any $\Phi \in \mathcal{W}$, all Φ-weighted colimits exist in $\mathbb{C}$ [Albert and Kelly, 1988]. For $\mathcal{W} = \{\text{all distributors}\}$ we already saw that a $\mathcal{W}$-cocomplete $\mathbb{C}$ is simply said to be cocomplete, or "freely cocomplete". However, if $\mathcal{W} = \{\text{all left adjoint distributors}\}$ then $\mathbb{C}$ is $\mathcal{W}$-cocomplete if and only if it is Cauchy complete [Street, 1983b]. Thus, the more subtle notion of $\mathcal{W}$-cocompletion unifies Examples 3.13 and 3.14 of the previous section; and in fact there are many other interesting examples [Kelly and Schmitt, 2005; Hofmann and Waszkiewicz, 2011].

For a so-called saturated class $\mathcal{W}$ of weights (i.e. a class of weights satisfying a mild technical condition), it is possible to compute the **$\mathcal{W}$-cocompletion of** $\mathbb{C}$ – to be understood as the "smallest" $\mathcal{W}$-cocomplete $\mathcal{Q}$-category that contains $\mathbb{C}$ – in a fairly straightforward way: it is the full subcategory of $\mathcal{P}\mathbb{C}$ whose objects are those contravariant presheaves on $\mathbb{C}$ that lie in $\mathcal{W}$. Thus we see that the **free cocompletion** of a $\mathcal{Q}$-category $\mathbb{C}$ is exactly the presheaf category $\mathcal{P}\mathbb{C}$; and its **Cauchy completion** $\mathbb{C}_{\mathsf{cc}}$ is the full subcategory of $\mathcal{P}\mathbb{C}$ whose objects are those presheaves which have a left adjoint (also called Cauchy presheaves). Such a $\mathcal{W}$-cocompletion procedure always extends to a monad on $\mathsf{Cat}(\mathcal{Q})$ whose algebras are precisely the $\mathcal{W}$-cocomplete $\mathcal{Q}$-categories. All **cocompletion doctrines** that so arise can be classified [Stubbe, 2010].

4.4. Symmetry

An **involution** on a quantaloid $\mathcal{Q}$ is a homomorphism $\mathcal{Q}^{\mathsf{op}} \longrightarrow \mathcal{Q}: (f: X \longrightarrow Y) \mapsto (f^{\circ}: Y \longrightarrow X)$ such that $f^{\circ\circ} = f$. There are many examples of involutive quantaloids. Of course, every commutative quantale is trivially involutive. An important class of non-trivial examples is provided by so-called *quantaloids of closed cribles* [Walters, 1982; Heymans and Stubbe, 2012a], i.e. quantaloids naturally associated with Grothendieck topologies. Yet another class of examples can be constructed as follows: Suppose that S is a sup-lattice with a *duality*, i.e. a supremum-preserving map $d: S \longrightarrow S^{\mathsf{op}}$ such that $d(x) = d^*(x)$ and $d(d(x)) = x$ for all $x \in S$, where d^* is the right adjoint to d as order-preserving map. The quantale $Q(S)$ of Example 2.6 then has a natural involution [Mulvey and Pelletier, 1992]: for $f \in Q(S)$ put $f^{\circ} := d^{\mathsf{op}} \circ (f^*)^{\mathsf{op}} \circ d$ (where f is left adjoint to f^* as order-preserving map).

If $\mathcal{Q}$ is involutive, then a $\mathcal{Q}$-category $\mathbb{C}$ is **symmetric** if $\mathbb{C}(y, x) = \mathbb{C}(x, y)^{\circ}$. In this situation, every $\mathcal{Q}$-category $\mathbb{C}$ can be symmetrised: define $\mathbb{C}_s$ to have the same objects and types as $\mathbb{C}$, but with

$$\mathbb{C}_s(y,x) = \mathbb{C}(y,x) \wedge \mathbb{C}(x,y)^{\circ}.$$

This procedure is functorial: the object correspondence $\mathbb{C} \mapsto \mathbb{C}_s$ extends to a comonad on $\mathsf{Cat}(\mathcal{Q})$, whose coalgebras are precisely the symmetric $\mathcal{Q}$-categories.

If $T\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ is a cocompletion doctrine, then we say that $\mathcal{Q}$ is ***T*-bilateral** if there is a distributive law from the monad T over the symmetrisation comonad. Roughly speaking, this says that "the T-cocompletion of a symmetric $\mathcal{Q}$-category is again symmetric". Heymans and Stubbe [2011] characterised the **Cauchy-bilateral** quantaloids (so it is known for which $\mathcal{Q}$ the Cauchy completion of a symmetric $\mathcal{Q}$-category is always Cauchy complete), which is important for applications in sheaf theory [Betti and Walters, 1982; Heymans and Stubbe, 2012b], but a general characterisation of T-bilateral quantaloids is an open problem.

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