Reculer pour mieux sauter: une analyse logico-catégorique du théorème du point fixe

Isar Stubbe

Colloquium EMA-ADA du 15 décembre 2022, Calais

From: **Bourel Christophe** christophe.bourel@univ-littoral.fr
Subject: [personnels-lmpa] Séminaire commun ADA-EMA le 15/12
Date: 5 December 2022 at 21:00
To: personnels-lmpa@liste.univ-littoral.fr, seminaires-lmpa@liste.univ-littoral.fr

CB

Bonjour à tous,

Nous avons le plaisir de vous annoncer la mise en place d'une nouvelle série d'exposés qui seront communs aux deux équipes du LMPA. Ces séminaires seront de type colloquium et seront donnés uniquement par des collègues du Laboratoire. L'objectif est que ces collègues puissent expliquer les idées et enjeux généraux de leur domaine de recherche à tous les membres du LMPA.

Dans ce contexte, nous avons le plaisir de vous annoncer la tenue du premier séminaire ADA-EMA la semaine prochaine le **jeudi 15/12** en salle **B014** à **14h**.

Il sera donné par **Isar Stubbe** et aura pour titre :

Reculer pour mieux sauter: une analyse logico-catégorique du théorème du point fixe.

Nous profiterons de ce séminaire pour organiser un "goûter de Noël" en salle B014 à partir de 15h30.

Bonne journée à tous,
Christophe, Lucile, Nicolas et Pierre-Louis

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goûter de Noël !!


POP
HIC
Franquin

From: **Carole Rosier** Carole.Rosier@univ-littoral.fr
Subject: Fwd: [personnels-lmpa] publication et diffusion de la "synthèse nationale et de prospective sur les mathématiques"
Date: 13 November 2022 at 18:43
To: personnels-lmpa personnels-lmpa@liste.univ-littoral.fr

CR

Bonsoir à toutes et à tous,

A la veille des assises des mathématiques qui auront lieu du 14 au 16 novembre à la maison de l'Unesco à Paris, je vous encourage à télécharger le document "Synthèse nationale et de prospective sur les mathématiques" (lien ci-dessous) qui donne en 3 volumes un panorama plus qu'intéressant de la recherche en mathématiques française.

Bien à vous,
Carole

VOLUME 1

RAPPORT PRINCIPAL

SYNTHÈSE NATIONALE ET DE PROSPECTIVE SUR LES MATHÉMATIQUES

TABLE DES MATIÈRES

1.2.4 L'UNITÉ DES MATHÉMATIQUES, À LA SOURCE DE LEUR UNIVERSALITÉ

Une des grandes idées mathématiques du XX^e siècle a été celle de structure, très liée à l'idée d'axiomatique : une structure[28] est définie par le système d'axiomes correspondant. On fait en général naître le programme structuraliste avec l'école d'algèbre allemande des années 1920 pour se développer ensuite, en particulier en France, avec la rédaction des « Éléments de mathématique » du groupe Bourbaki.

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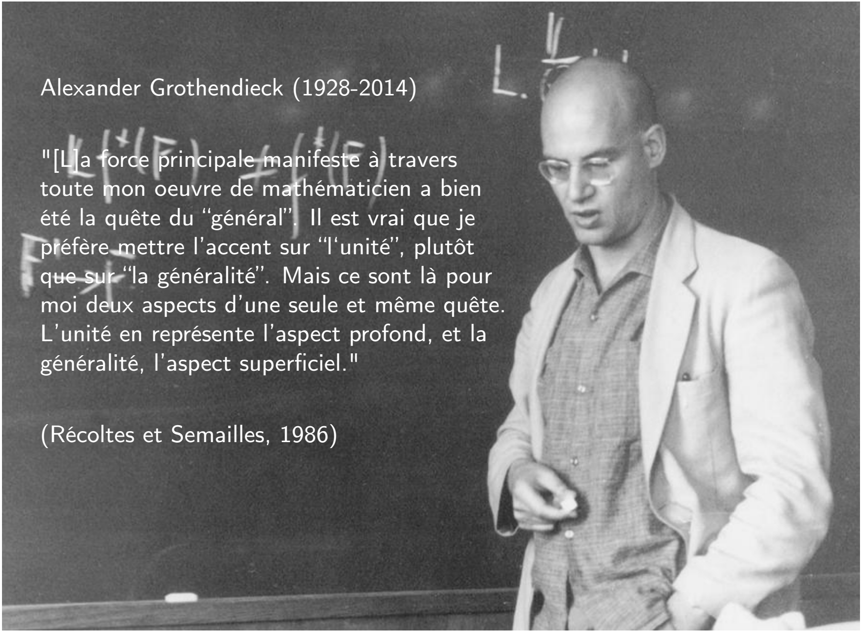
28. Comme celle de groupe, d'espace vectoriel, d'espace topologique, etc.

1.2.4 L'UNITÉ DES MATHÉMATIQUES, À LA SOURCE DE LEUR UNIVERSALITÉ

Partons d'une figure emblématique du programme général d'unification des mathématiques, Alexandre Grothendieck. Si les notions formalisées par Grothendieck sont au cœur d'une partie importante des mathématiques contemporaines, sa pensée, ses intuitions inabouties et ses programmes de recherche alimentent aujourd'hui encore de nombreux travaux, en France et ailleurs. Une partie de ses idées apparaît dans son « Esquisse d'un programme » en 1984, liant la topologie et la théorie des catégories.

1.2.4 L'UNITÉ DES MATHÉMATIQUES, À LA SOURCE DE LEUR UNIVERSALITÉ

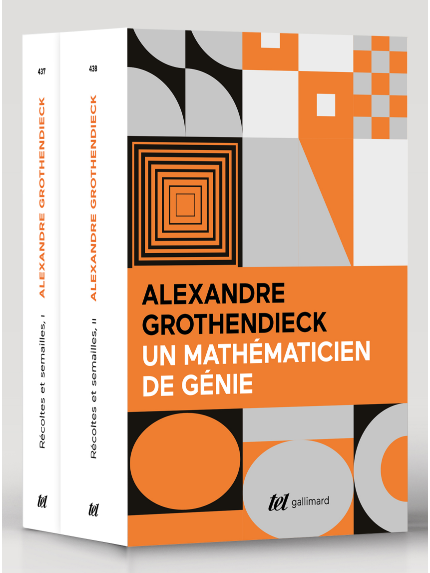
Ce type d'aventure mathématique, intellectuelle et scientifique, est à l'image de la richesse et de l'imprévu des mathématiques contemporaines. Ces développements, tant conceptuels que pragmatiques, n'auraient certainement pas été imaginés quelques décennies plus tôt et démontrent que les mathématiques ne peuvent être réduites à quelques idées applicatives ou quelques calculs explicites à un temps t, au gré de modes fluctuantes. C'est dans la rencontre entre la diversité de leur spectre et de leur unité profonde, qu'elles trouvent leur pleine mesure.



Alexander Grothendieck (1928-2014)

"[L]a force principale manifeste à travers toute mon oeuvre de mathématicien a bien été la quête du “général”. Il est vrai que je préfère mettre l'accent sur “l'unité”, plutôt que sur “la généralité”. Mais ce sont là pour moi deux aspects d'une seule et même quête. L'unité en représente l'aspect profond, et la généralité, l'aspect superficiel."

(Récoltes et Semailles, 1986)



Une idée de cadeau de fin d'année...

(Collection Tel, Gallimard; date de parution: 13-01-2022)

Bref—généraliser pour unifier—c'est

reculer pour mieux sauter.

Une analyse logico-catégorique

du théorème du point fixe

Fréchet: Metric spaces (1906)

Maurice Fréchet
(1878 – 1973)

SUR QUELQUES POINTS DU CALCUL FONCTIONNEL;

Par M. **Maurice Fréchet** (Paris) *).

*) Thèse présentée à la Faculté des Sciences de Paris pour obtenir le grade de Docteur ès Sciences.

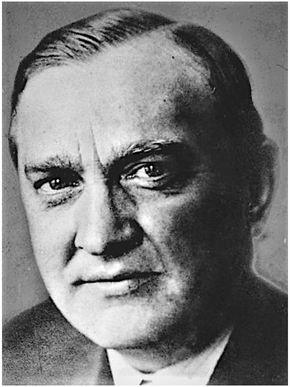
Adunanza del 22 aprile 1906.

49. *Introduction de l'écart.* — Lorsque nous appliquerons les résultats généraux de la Première Partie à des exemples concrets, nous reconnaitrons d'abord que, dans chaque cas, on peut faire correspondre à tout couple d'éléments A, B un nombre $(A, B) \geq 0$, que nous appellerons *l'écart des deux éléments* et qui jouit des deux propriétés suivantes: *a*) L'écart (A, B) n'est nul que si A et B sont identiques. *b*) Si A, B, C, sont trois éléments quelconques, on a toujours $(A, B) \leq (A, C) + (C, B)$.

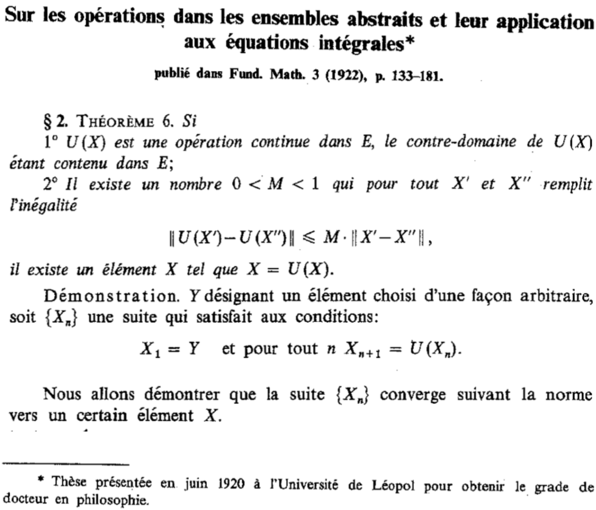
[...]

discerner si deux d'entre eux sont ou non identiques et tels, de plus, qu'à deux quelconques d'entre eux A, B, on puisse faire correspondre un nombre $(A, B) = (B, A) \geq 0$

Banach: Fixpoint theorem (1922)



Stefan Banach
(1892-1945)



Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*

publié dans Fund. Math. 3 (1922), p. 133–181.

§ 2. Théorème 6. *Si*

1° *$U(X)$ est une opération continue dans E, le contre-domaine de $U(X)$ étant contenu dans E;*

2° *Il existe un nombre $0 < M < 1$ qui pour tout X' et X'' remplit l'inégalité*

$$\|U(X') - U(X'')\| \leqslant M \cdot \|X' - X''\|,$$

il existe un élément X tel que $X = U(X)$.

Démonstration. Y désignant un élément choisi d'une façon arbitraire, soit $\{X_n\}$ une suite qui satisfait aux conditions:

$$X_1 = Y \quad \text{et pour tout } n \ X_{n+1} = U(X_n).$$

Nous allons démontrer que la suite $\{X_n\}$ converge suivant la norme vers un certain élément X.

* Thèse présentée en juin 1920 à l'Université de Léopol pour obtenir le grade de docteur en philosophie.

Lawvere: Metric spaces are categories (1973)

Bill Lawvere
(1937-...)

METRIC SPACES, GENERALIZED LOGIC,
AND CLOSED CATEGORIES

(Conferenza tenuta il 30 marzo 1973) *

By taking account of a certain natural generalization of category theory within itself, namely the consideration of strong categories whose hom-functors take their values in a given « closed category » $\mathcal{V}$ (not necessarily in the category $\mathcal{S}$ of abstract sets), we will show below that it is possible to regard a metric space as a (strong) category and that moreover by specializing the constructions and theorems of general category theory we can deduce a large part of *general* metric space theory.

Alors, le théorème du point fixe, est-il un
cas particulier d'un thèorème général
en théorie des catégories?!

Banach: Fixpoint theorem (modern version)

Let (X, d) be a complete metric space.

Let $f : X \to X$ be a contraction: $d(fx, fy) \leq k \cdot d(x, y)$ for some $0 < k < 1$.

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For any $x \in X$,

- infer from contractivity that $x, fx, f^2x, \ldots$ is a Cauchy sequence:

$$\lim d(f^n x, f^m x) = 0$$

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$$0 = d(x^*, x^*) = \lim d(x^*, f^n x) \geq \lim d(fx^*, f^{n+1}x) = d(fx^*, x^*)$$

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Infer from contractivity that the fixpoint is unique:

$$fx^* = x^*, fy^* = y^* \Longrightarrow d(x^*, y^*) = d(fx^*, fy^*) \leq k \cdot d(x^*, y^*) \Longrightarrow d(x^*, y^*) = 0.$$

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Enriched categories (1)

A partially ordered set $(X, \leq)$ is

a binary relation "$\leq$" on X

such that, for all $x, y, z \in X$,

if $x \leq y$ and $y \leq z$ then $x \leq z$,

$x \leq x$,

if $x \leq y$ and $y \leq x$ then $x = y$.

A metric space (X, d) is

a function $d \colon X \times X \to \mathbb{R}$

such that, for all $x, y, z \in X$,

$d(x, y) \geq 0$,

$d(x, y) + d(y, z) \geq d(x, z)$,

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An ordered set $(X, \leq)$ is

a function $\chi_{\leq}\colon X \times X \to \{0, 1\}$

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in the ordered monoid $(\{0,1\}, \leq, \wedge, 1)$.

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A generalized metric space (X, d) is

a function $d \colon X \times X \to [0, \infty]$

such that, for all $x, y, z \in X$,

~~$d(x, y) \geq 0$~~,

$d(x, y) + d(y, z) \geq d(x, z)$,

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There are many more examples—notably in sheaf theory, non-commutative topology, monoidal topology, domain theory, quantum computing, automata theory...

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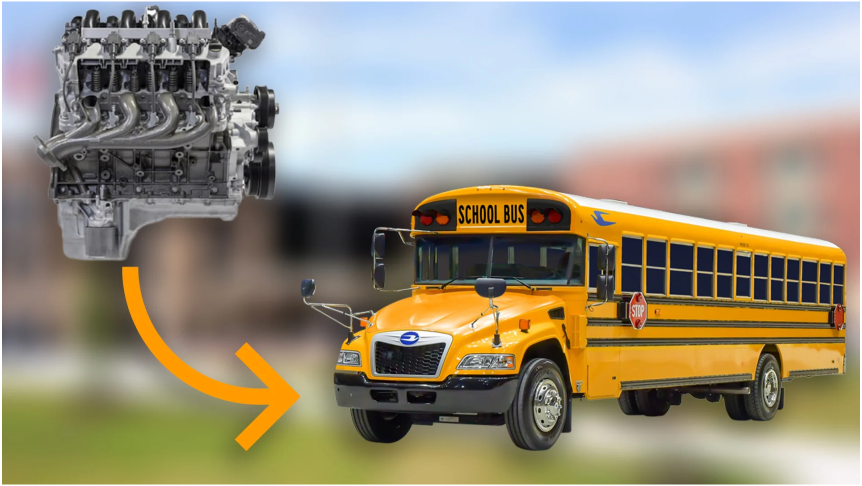

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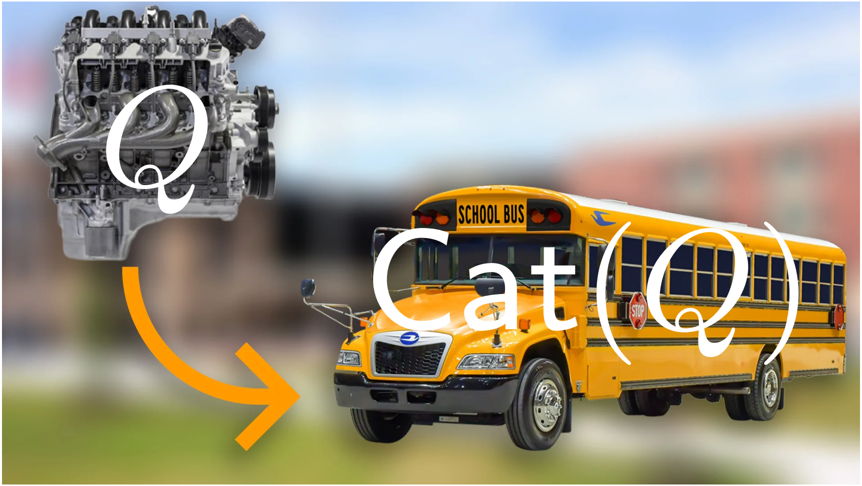

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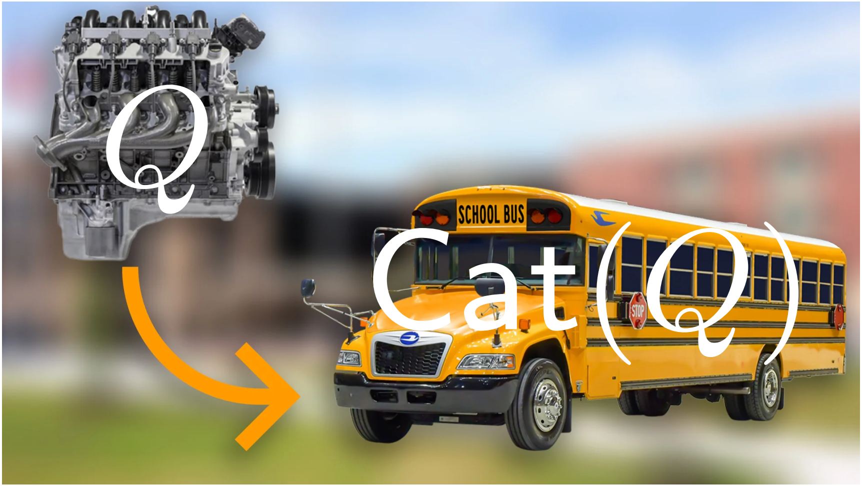

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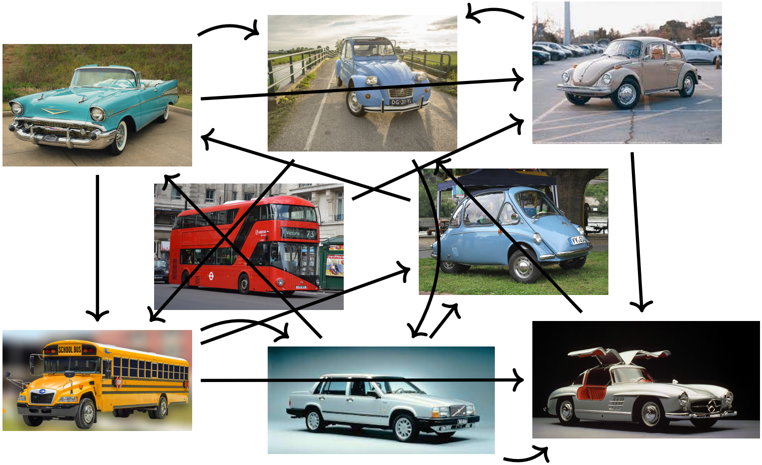

Make the "logic" Q part of the "algebra" Cat(Q).
Study common features of all Cat(Q).
Study the specifics of each Cat(Q).
Compare Cat(Q)'s, using general category theory (functors etc.).

Banach: Fixpoint theorem (modern version)

Let (X, d) be a complete metric space.

Let $f : X \to X$ be a contraction: $d(fx, fy) \leq k \cdot d(x, y)$ for some $0 < k < 1$.

(Note that f is a fortiori non-expansive.)

For any $x \in X$,

- infer from contractivity that $x, fx, f^2x, \ldots$ is a Cauchy sequence:

$$\lim d(f^n x, f^m x) = 0$$

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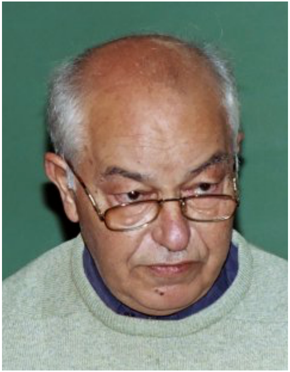
Bénabou: Distributors (= profunctors = modules) (1973)



Jean Bénabou
(1932-2022)

LES DISTRIBUTEURS
d'après le cours de "Questions spéciales de mathématique"

par
J. BENABOU *rédigé par Jean-Roger ROISIN*

Rapport n° 33, janvier 1973
Séminaires de Mathématique Pure

Nous supposerons maintenant que $\mathcal{V}$ est un cosmos c'est-à-dire une catégorie multiplicative symétrique fermée complète à gauche et à droite.

Une flèche de $\mathcal{A}$ vers $\mathcal{B}$, appelée un distributeur, est un $\mathcal{V}$-bifoncteur vers $\mathcal{V}$, contravariant en $\mathcal{B}$ et covariant en $\mathcal{A}$.

4.3. *Proposition.*

Dist($\mathcal{V}$) est une bicatégorie fermée.

Street: Absolute weights (1983)

Ross Street
(1945 - ...)

CAHIERS DE TOPOLOGIE
ET GÉOMÉTRIE DIFFÉRENTIELLE

Vol. XXIV-4 (1983)

ABSOLUTE COLIMITS IN ENRICHED CATEGORIES

by Ross STREET

What is it about an indexing type ϕ that ensures that every colimit indexed by ϕ is preserved by all functors? The present short note answers this question in the context of enriched categories. Appropriate references are listed at the end of the paper. The base for enrichment can be a bicategory W although the reader may take it to be a symmetric monoidal closed category should this be more commodious. We use the term *module* for what others have called *bimodule, profunctor, distributor.*

THEOREM. *Every colimit weighted by ϕ is absolute if and only if ϕ has a right adjoint in the bicategory of modules.*

The above gives another characterization of the Cauchy completion of an enriched category as consisting of the weightings (indexing types) for absolute colimits. Hence a *category is Cauchy complete if and only if it admits all absolute colimits.*

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A ideal relation $R\colon A \mathrel{-\!\!\ominus\!\!\rightarrow} B$ between ordered sets $(A, \chi_{\leq_A})$ and $(B, \chi_{\leq_B})$ is

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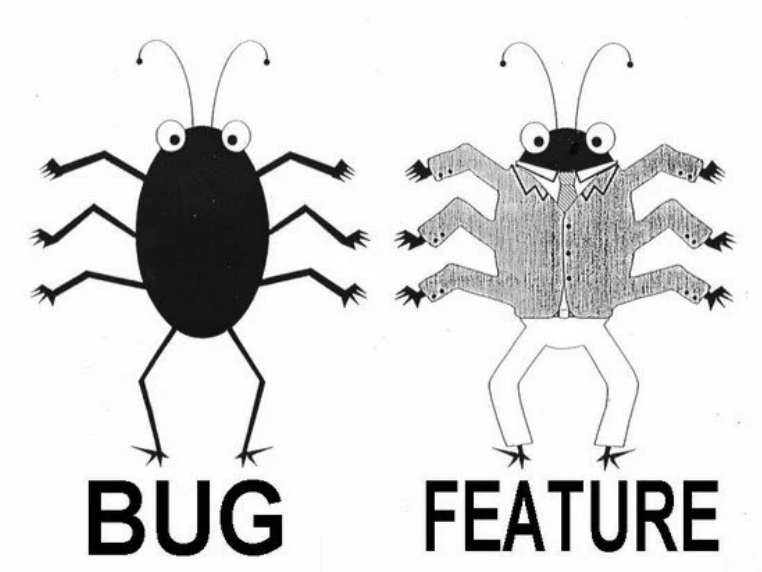
Conversely, not every left adjoint Q-distributor is the graph of a Q-functor!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A distributor $\Phi\colon \mathbb{A} \mathrel{\ooalign{$\longrightarrow$\cr\hidewidth$\circ$\hidewidth}} \mathbb{B}$ is **representable** if it is the graph of a functor $F\colon \mathbb{A} \to \mathbb{B}$:

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This is exactly the formula for Cauchy sequences in a metric space:

$$\lim d(x_n, x_m) \leq 0 \quad \text{in} \quad ([0, \infty], \bigwedge, +, 0).$$

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A category $\mathbb{C}$ is **Cauchy-complete** if any left adjoint distributor into $\mathbb{C}$ is representable.

Any sequence $(x_n)_{n\in\mathbb{N}}$ in a Q-category $\mathbb{C}$ determines a pair of distributors

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These are adjoint if and only if

$$\bigvee_{N\in\mathbb{N}} \bigwedge_{m,n\geq N} \mathbb{C}(x_n, x_m) \geq 1 \quad \text{in} \quad (Q, \bigvee, \circ, 1).$$

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Cauchy-completeness is important in many areas, e.g. sheaf theory (gluing condition),
module theory (finitely generated projective modules, Morita equivalence),
set theory (axiom of choice), general categories (Karoubi envelope), and more.

Borceux: Cauchy completion (1986)

CAHIERS DE TOPOLOGIE
ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

Vol. XXVII-2 (1986)

Francis Borceux
(1948 - ...)

CAUCHY COMPLETION IN CATEGORY THEORY

by *Francis BORCEUX and Dominique DEJEAN*

This paper is to be considered as a survey article presenting an original and unified treatment of various results, scattered in the litterature. The reason for such a work is the grewing importance of everything concerned with the splitting of idempotents and the lack of a reference text on the subject. Most of the work devoted to Cauchy completion has been developped in the sophisticated context of bicategories : it's our decision to focuse on direct proofs in the context of classical category theory.

Theorem 2. *The following conditions are equivalent on a small category* $\mathbf{C}$.

(1) $\mathbf{C}$ *is Cauchy complete.*
(2) A distributor $1 \xrightarrow{\;\circ\;} \mathbf{C}$ *has a right adjoint iff it is a functor.*
(3) For every small category $\mathbf{A}$ *a distributor* $\mathbf{A} \xrightarrow{\;\circ\;} \mathbf{C}$ *has a right adjoint iff it is a functor.*

Example 3. *When* $\mathbf{V}$ *is the category* $\bar{\mathbb{R}}_+$ *defined by F.W. Lawvere (cf. [9]) the Cauchy completion of a metric space is its usual completion using Cauchy sequences.*

Banach: Fixpoint theorem (modern version)

Let (X, d) be a complete metric space.

Let $f : X \to X$ be a contraction: $d(fx, fy) \leq k \cdot d(x, y)$ for some $0 < k < 1$.

(Note that f is a fortiori non-expansive.)

For any $x \in X$,

- infer from contractivity that $x, fx, f^2x, \ldots$ is a Cauchy sequence:

$$\lim d(f^n x, f^m x) = 0$$

- infer from completeness that the sequence converges, say to x^*:

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- infer from non-expansiveness that $fx^* = x^*$:

$$0 = d(x^*, x^*) = \lim d(x^*, f^n x) \geq \lim d(fx^*, f^{n+1} x) = d(fx^*, x^*)$$

Infer from contractivity that the fixpoint is unique:

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A fixpoint theorem for Q-categories

(based on article with A. Benkhadra, to appear in the Cahiers)

Fixpoint theorem (1)

Proposition

Suppose that $F\colon \mathbb{C} \to \mathbb{C}$ *is a* Q*-functor on a Cauchy complete* $\mathbb{C}$.
If there is an $x \in \mathbb{C}_0$ *such that* $(F^n x)_{n\in\mathbb{N}}$ *is Cauchy, then* F *has a fixpoint.*

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Having both $1 \leq \mathbb{C}(x^*, Fx^*)$ and $1 \leq \mathbb{C}(Fx^*, x^*)$ means that $Fx^* \cong x^*$ in $\mathbb{C}$.

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Fixpoint theorem (2)

Definition

Say that $\varphi\colon Q \to Q$ is a **control function** and $F\colon \mathbb{C}_0 \to \mathbb{C}_0$ is a φ**-contraction**, if

- $\varphi(t) \geq t$ for all $t \in Q$,
- $\varphi(t) = t$ implies that either $t = 0$ or $1 \leq t$.
- $\mathbb{C}(Fx, Fy) \geq \varphi(\mathbb{C}(x, y))$ for any $x, y \in \mathbb{C}_0$.

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The "Banach case" for metric spaces: for

$$\varphi\colon [0, \infty] \to [0, \infty]\colon t \mapsto k \cdot t \quad \text{for some } 0 < k < 1$$

it is easily verified (recalling that $[0, \infty]$ comes with opposite order) that

$k \cdot t \leq t$,

$k \cdot t = t$ implies that either $t = \infty$ or $t = 0$,

so a function $f\colon X \to X$ on a (generalized) metric space (X, d) is a φ-contraction if

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$\mathbb{C}(Fx, Fy) \geq \varphi(\mathbb{C}(x, y))$ for any $x, y \in \mathbb{C}_0$.

The "Banach case" for metric spaces: for

$$\varphi\colon [0,\infty] \to [0,\infty]\colon t \mapsto k \cdot t \qquad \text{for some } 0 < k < 1$$

it is easily verified (recalling that $[0,\infty]$ comes with opposite order) that

$k \cdot t \leq t$,

$k \cdot t = t$ implies that either $t = \infty$ or $t = 0$,

so a function $f\colon X \to X$ on a (generalized) metric space (X, d) is a φ-contraction if

$d(fx, fy) \leq k \cdot d(x, y)$ for any $x, y \in X$.

There are other non-trivial examples, e.g. for probabilistic metric spaces:

$$\text{define } \varphi\colon \Delta \to \Delta \text{ by } \varphi(u)(t) = \begin{cases} \frac{1}{2}(u(t) + 1) & \text{if } 0 < t \leq \infty \\ 0 & \text{if } t = 0 \end{cases}$$

Fixpoint theorem (2)

Definition

Say that $\varphi\colon Q \to Q$ is a **control function** and $F\colon \mathbb{C}_0 \to \mathbb{C}_0$ is a φ**-contraction**, if

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Now suppose that $Fx^* \cong x^*$ and $Fy^* \cong y^*$, then

$$\mathbb{C}(x^*, y^*) = \mathbb{C}(Fx^*, Fy^*) \geq \varphi(\mathbb{C}(x^*, y^*)) \geq \mathbb{C}(x^*, y^*)$$

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If $\mathbb{C}$ is symmetric, then any two fixpoints of a φ-contraction are either isomorphic or in different summands of $\mathbb{C}$.

If $\mathbb{C}$ has no zero-homs, then any two fixpoints of a φ-contraction are always isomorphic.

Banach: Fixpoint theorem (modern version)

Let (X, d) be a complete metric space.

Let $f : X \to X$ be a contraction: $d(fx, fy) \leq k \cdot d(x, y)$ for some $0 < k < 1$.

(Note that f is a fortiori non-expansive.)

For any $x \in X$,

- infer from contractivity that $x, fx, f^2x, \ldots$ is a Cauchy sequence:

$$\lim d(f^n x, f^m x) = 0$$

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$$0 = d(x^*, x^*) = \lim d(x^*, f^n x) \geq \lim d(fx^*, f^{n+1}x) = d(fx^*, x^*)$$

Infer from contractivity that the fixpoint is unique:

$$fx^* = x^*, fy^* = y^* \Longrightarrow d(x^*, y^*) = d(fx^*, fy^*) \leq k \cdot d(x^*, y^*) \Longrightarrow d(x^*, y^*) = 0$$

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Proposition

Suppose that $F\colon \mathbb{C} \to \mathbb{C}$ is a φ-contraction on a Q-category.
Suppose that Q is a ***continuous lattice*** *and $\varphi\colon Q \to Q$ is a* ***lower-semicontinuous function****.*
For any $x \in \mathbb{C}_0$ such that $\mathbb{C}(Fx, x) \neq 0 \neq \mathbb{C}(x, Fx)$, the sequence $(F^n x)_{n\in\mathbb{N}}$ is Cauchy.

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These conditions are met by the previously mentioned examples—in particular the "Banach" case.

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These conditions are met by the previously mentioned examples—in particular the "Banach" case.

(The theorem holds under weaker conditions, but it makes the statement more technically involved, so skipped here for convenience.)

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Suppose that $F\colon \mathbb{C} \to \mathbb{C}$ is a φ-contraction on a Q-category.
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For any $x \in \mathbb{C}_0$ such that $\mathbb{C}(Fx, x) \neq 0 \neq \mathbb{C}(x, Fx)$, the sequence $(F^n x)_{n\in\mathbb{N}}$ is Cauchy.

Proof. Putting $C_{x,f} := \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} \bigwedge_{m\geq N} \mathbb{C}(f^n x, f^m x) \in Q$, we recall from Subsection 1.2 that $\phi_{x,f} \dashv \psi_{x,f}$ if and only if $C_{x,f} \geq 1$. We shall show that $C_{x,f} \not\geq 1$ leads to a contradiction.

(i) Picking an $x \in \mathbb{C}_0$ such that $\mathbb{C}(x, fx) \neq 0 \neq \mathbb{C}(fx, x)$, we put $c_n := \mathbb{C}(f^n x, f^{n+1} x) \in Q$ for all $n \in \mathbb{N}$. By assumption, $0 < c_0 \leq 1$ and the conditions on φ imply that $c_0 \leq \varphi(c_0) \leq c_1$. Repeating the argument we find that $c_n \leq \varphi(c_n) \leq c_{n+1}$, so the sequence is increasing and strictly above 0. Therefore we can compute, using the conditions on φ, that:

$$\begin{aligned}
\bigvee_{N\in\mathbb{N}} c_N &= \bigvee_{N\in\mathbb{N}} c_{N+1} \\
&= \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} c_{n+1} \\
&\geq \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} \varphi(c_n) \\
&\geq \varphi(\bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} c_n) \\
&= \varphi(\bigvee_{N\in\mathbb{N}} c_N) \\
&\geq \bigvee_{N\in\mathbb{N}} c_N
\end{aligned}$$

We thus find a fixpoint of φ which is not 0, so it must satisfy $1 \leq \bigvee_{N\in\mathbb{N}} c_N$.

(ii) Similarly, the sequence $(a_n := \mathbb{C}(f^{n+1}x, f^n x))_{n\in\mathbb{N}}$ must also satisfy $1 \leq \bigvee_{\in\mathbb{N}} a_n$.

(iii) Next, suppose that $1 \not\leq C_{f,x}$; by continuity of the underlying complete lattice of Q, this means that there exists an $\epsilon \ll 1$ such that $\epsilon \not\leq C_{f,x}$ (and so in particular $\epsilon \neq 0$). Using the definition of $C_{f,x}$ as a sup-inf, we may infer:

$$\begin{aligned}
\epsilon \not\leq \bigvee_{k\in\mathbb{N}} \left(\bigwedge_{n\geq k} \bigwedge_{m\geq k} \mathbb{C}(f^n x, f^m x) \right) &\Longrightarrow \forall k \in \mathbb{N} : \epsilon \not\leq \bigwedge_{n\geq k} \bigwedge_{m\geq k} \mathbb{C}(f^n x, f^m x) \\
&\Longrightarrow \forall k \in \mathbb{N}, \exists n_k, m_k \geq k : \epsilon \not\leq \mathbb{C}(f^{n_k} x, f^{m_k} x)
\end{aligned}$$

In the last line above, it cannot be the case that $m_k = n_k$, because otherwise $\mathbb{C}(f^{n_k}x, f^{n_k}x) \geq 1$ (by the "identity" axiom for the Q-category $\mathbb{C}$), which would then also be above $\epsilon \ll 1$. So suppose that $n_k < m_k$, then we can replace m_k by

$$m'_k := \min\{m > n_k \mid \epsilon \not\leq \mathbb{C}(f^{n_k}x, f^m x)\}$$

and so we still have $\epsilon \not\leq \mathbb{C}(f^{n_k}x, f^{m'_k}x)$, but now we know also that $\epsilon \leq \mathbb{C}(f^{n_k}x, f^{m_k-1}x)$. Similarly, if $n_k > m_k$ then we may replace n_k by

$$n'_k := \min\{n > m_k \in \mathbb{N} \mid \epsilon \not\leq \mathbb{C}(f^n x, f^{m_k} x)\}$$

and we still have $\epsilon \not\leq \mathbb{C}(f^{n'_k}x, f^{m_k}x)$, but now we know also that $\epsilon \leq \mathbb{C}(f^{n'_k-1}x, f^{m_k}x)$. That is to say, we can always pick $n_k, m_k \geq k$ to ensure that

$$\epsilon \not\leq \mathbb{C}(f^{n_k}x, f^{m_k}x) \text{ and } \begin{cases} \text{either} & \mathbb{C}(f^{n_k}x, f^{m_k-1}x) \geq \epsilon \quad (A) \\ \text{or} & \mathbb{C}(f^{n_k-1}x, f^{m_k}x) \geq \epsilon \quad (B) \end{cases}$$

Now denote, for each such pick of $n_k, m_k \geq k \in \mathbb{N}$,

$$d_k := \mathbb{C}(f^{n_k}x, f^{m_k}x);$$

and let us insist that $\epsilon \not\leq d_k$ for all $k \in \mathbb{N}$. In case condition (A) holds for d_k, then in particular $m_k > n_k$ so $m_k \geq 1$, and we can use the "composition" axiom in $\mathbb{C}$ to get

$$\begin{aligned}
\epsilon \circ c_{m_k-1} &\leq \mathbb{C}(f^{n_k}x, f^{m_k-1}x) \circ \mathbb{C}(f^{m_k-1}x, f^{m_k}x) \\
&\leq \mathbb{C}(f^{n_k}x, f^{m_k}x) \\
&= d_k
\end{aligned}$$

In case condition (B) holds for d_k we can similarly prove that

$$a_{n_k-1} \circ \epsilon \leq d_k.$$

Hence, using in $(*)$ that a continuous lattice is always meet-continuous, and that both sequences

$$\left(\bigwedge\{d_k \mid k \geq N \text{ and } (A) \text{ holds}\}\right)_{N\in\mathbb{N}} \text{ and } \left(\bigwedge\{d_k \mid k \geq N \text{ and } (B) \text{ holds}\}\right)_{N\in\mathbb{N}}$$

are increasing, we may compute that

$$\begin{aligned}
\bigvee_{N\in\mathbb{N}} \bigwedge_{k\geq N} d_k &= \bigvee_{N\in\mathbb{N}_0} \bigwedge_{k\geq N} d_k \\
&= \bigvee_{N\in\mathbb{N}_0} \left(\bigwedge\{d_k \mid k \geq N \text{ and } (A) \text{ holds}\} \wedge \bigwedge\{d_k \mid k \geq N \text{ and } (B) \text{ holds}\}\right) \\
&\overset{(*)}{=} \left(\bigvee_{N\in\mathbb{N}_0} \bigwedge\{d_k \mid k \geq N \text{ and } (A) \text{ holds}\}\right) \\
&\qquad \wedge \left(\bigvee_{N\in\mathbb{N}_0} \bigwedge\{d_k \mid k \geq N \text{ and } (B) \text{ holds}\}\right) \\
&\geq \left(\bigvee_{N\in\mathbb{N}_0} \bigwedge\{\epsilon \circ c_{m_k-1} \mid k \geq N \text{ and } (A) \text{ holds}\}\right) \\
&\qquad \wedge \left(\bigvee_{N\in\mathbb{N}_0} \bigwedge\{a_{n_k-1} \circ \epsilon \mid k \geq N \text{ and } (B) \text{ holds}\}\right) \\
&\geq \left(\bigvee_{N\in\mathbb{N}} \bigwedge_{m\geq N} \epsilon \circ c_m\right) \wedge \left(\bigvee_{N\in\mathbb{N}} \bigwedge_{m\geq N} a_n \circ \epsilon\right) \\
&\geq \left(\epsilon \circ (\bigvee_{N\in\mathbb{N}} \bigwedge_{m\geq N} c_m)\right) \wedge \left((\bigvee_{N\in\mathbb{N}} \bigwedge_{m\geq N} a_n) \circ \epsilon\right) \\
&= \left(\epsilon \circ (\bigvee_{N\in\mathbb{N}} c_N)\right) \wedge \left((\bigvee_{N\in\mathbb{N}} a_N) \circ \epsilon\right) \\
&= (\epsilon \circ 1) \wedge (1 \circ \epsilon) \\
&= \epsilon
\end{aligned}$$

So, even though $\epsilon \not\leq d_k$ (for all $k \in \mathbb{N}$), we do have that $0 \neq \epsilon \leq \bigvee_{N\in\mathbb{N}} \bigwedge_{k\geq N} d_k$.

(iv) Using the "composition" axiom in $\mathbb{C}$, we have for every $k \geq N \in \mathbb{N}$ (recall that $n_k, m_k \geq k$ too) that

$$d_k \geq c_{n_k} \circ \mathbb{C}(f^{n_k+1}x, f^{m_k+1}x) \circ a_{m_k} \geq c_{n_k} \circ \varphi(d_k) \circ a_{m_k} \geq c_N \circ \varphi(d_k) \circ a_N$$

and so we may compute that

$$\begin{aligned}
\bigvee_{N\in\mathbb{N}} \bigwedge_{k\geq N} d_k &\geq \bigvee_{N\in\mathbb{N}} \bigwedge_{k\geq N} (c_N \circ \varphi(d_k) \circ a_N) \\
&\geq \bigvee_{N\in\mathbb{N}} \left(c_N \circ (\bigwedge_{k\geq N} \varphi(d_k)) \circ a_N\right) \\
&\overset{(*)}{=} \left(\bigvee_{N\in\mathbb{N}} c_N\right) \circ \left(\bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} \varphi(d_k)\right) \circ \left(\bigvee_{N\in\mathbb{N}} a_N\right) \\
&= 1 \circ \left(\bigvee_{N\in\mathbb{N}} \bigwedge_{k\geq N} \varphi(d_k)\right) \circ 1 \\
&\geq \varphi(\bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} d_k) \\
&\geq \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} d_k
\end{aligned}$$

where in $(*)$ we used once more the argument involving increasing sequences (explained in a previous footnote), but now for three sequences instead of two. This means that $\bigvee_{N\in\mathbb{N}} \bigwedge_{k\geq N} d_k$ is a fixpoint of φ which – as we showed earlier – is not 0, so we must have $1 \leq \bigvee_{N\in\mathbb{N}} \bigwedge_{k\geq N} d_k$.

(v) Since $\epsilon \ll 1 \leq \bigvee_{N\in\mathbb{N}} \bigwedge_{k\geq N} d_k$, and the latter supremum is directed, there must exist an $N_0 \in \mathbb{N}$ such that $\epsilon \leq \bigwedge_{k\geq N_0} d_k$. Yet, we established earlier that $\epsilon \not\leq d_k$ for all $k \in \mathbb{N}$. This is the announced contradiction. □

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Let (X, d) be a complete metric space.

Let $f : X \to X$ be a contraction: $d(fx, fy) \leq k \cdot d(x, y)$ for some $0 < k < 1$.

(Note that f is a fortiori non-expansive.)

For any $x \in X$,

- infer from contractivity that $x, fx, f^2x, \ldots$ is a Cauchy sequence:

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Fixpoint theorem (4)

Theorem

Suppose that $F\colon \mathbb{C} \to \mathbb{C}$ is a φ-contraction on a Cauchy complete Q-category. Suppose that Q is a continuous lattice and $\varphi\colon Q \to Q$ is a lower-semicontinuous function.

If there exists an $x \in \mathbb{C}_0$ such that $\mathbb{C}(Fx, x) \neq 0 \neq \mathbb{C}(x, Fx)$, then F has a fixpoint.

If $\mathbb{C}$ is symmetric, then any two fixpoints of F are either isomorphic or in different summands of $\mathbb{C}$; if $\mathbb{C}$ has no zero-homs, then any two fixpoints of F are isomorphic.

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Examples:

$Q = (\{0,1\}, \bigvee, \wedge, 1)$: the theorem trivializes for ordered sets.

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- $Q = ([0,\infty], \bigwedge, +, 0)$: a generalized Banach fixpoint theorem for generalized metric spaces, allowing for non-linear contractions (see also (Boyd and Wong, 1969)).

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- $Q = ([0,1], \bigvee, *, 1)$: a new fixpoint theorem for fuzzy orders, to be compared with e.g. (Coppola et al., 2008).

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Suppose that Q *is a continuous lattice and* $\varphi\colon Q \to Q$ *is a lower-semicontinuous function.*

If there exists an $x \in \mathbb{C}_0$ *such that* $\mathbb{C}(Fx, x) \neq 0 \neq \mathbb{C}(x, Fx)$*, then* F *has a fixpoint.*

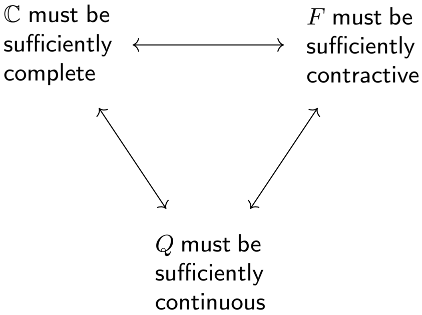
If $\mathbb{C}$ *is symmetric, then any two fixpoints of* F *are either isomorphic or in different summands of* $\mathbb{C}$*; if* $\mathbb{C}$ *has no zero-homs, then any two fixpoints of* F *are isomorphic.*

Examples:

- $Q = (\{0, 1\}, \bigvee, \wedge, 1)$: the theorem trivializes for ordered sets.
- $Q = ([0, \infty], \bigwedge, +, 0)$: a generalized Banach fixpoint theorem for generalized metric spaces, allowing for non-linear contractions (see also (Boyd and Wong, 1969)).
- $Q = ([0, 1], \bigvee, *, 1)$: a new fixpoint theorem for fuzzy orders, to be compared with e.g. (Coppola et al., 2008).
- $Q = (\Delta, \bigvee, *, e)$: a new fixpoint theorem for probabilistic metric spaces, encompassing certain known results (Hadžić and Pap, 2001).

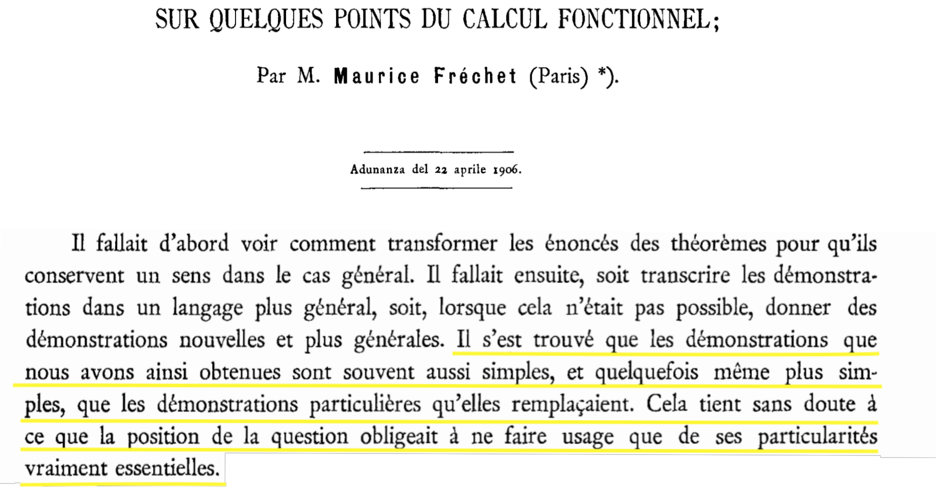
Take-away message: an equilibrum of three

To formulate a fixpoint theorem for a φ-contraction $F\colon \mathbb{C} \to \mathbb{C}$ on a Q-category,



Our theorem captures known examples and produces new results. Yet, the literature abounds with fixpoint theorems. Further study is necessary!

Closing remark from Fréchet (1906):



SUR QUELQUES POINTS DU CALCUL FONCTIONNEL;

Par M. **Maurice Fréchet** (Paris) *).

Adunanza del 22 aprile 1906.

Il fallait d'abord voir comment transformer les énoncés des théorèmes pour qu'ils conservent un sens dans le cas général. Il fallait ensuite, soit transcrire les démonstrations dans un langage plus général, soit, lorsque cela n'était pas possible, donner des démonstrations nouvelles et plus générales. Il s'est trouvé que les démonstrations que nous avons ainsi obtenues sont souvent aussi simples, et quelquefois même plus simples, que les démonstrations particulières qu'elles remplaçaient. Cela tient sans doute à ce que la position de la question obligeait à ne faire usage que de ses particularités vraiment essentielles.

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Bonne fin d'année 2022, et bon début d'année 2023 !