Conway and Iteration Semirings

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Outline

- Some categories with fixed points
- Conway and iteration theories
- Matrix theories and star semirings
- A Kleene type theorem
- Some characterizations of $\mathbb{N}^{rat}\langle\langle A^*\rangle\rangle$ and $\mathbb{N}_\infty{}^{rat}\langle\langle A^*\rangle\rangle$
- Open problems

Some categories with fixed point operators

Suppose $\mathcal{C}$ is a category with coproducts. A parameterized fixed point operation is a function

$$\mathcal{C}(X, X+Y) \xrightarrow{\dagger} \mathcal{C}(X, Y)$$

such that

$$f^\dagger \;=\; X \xrightarrow{f} X+Y \xrightarrow{\langle f^\dagger, \mathbf{1}_Y \rangle} Y.$$

In product form: $\mathcal{C}(X \times Y, X) \xrightarrow{\dagger} \mathcal{C}(Y, X)$ such that

$$f^\dagger \;=\; Y \xrightarrow{\langle f^\dagger, \mathbf{1}_Y \rangle} X \times Y \xrightarrow{f} X$$

Pfn($\mathcal{A}$)

- $\mathcal{A}$ is a collection of sets closed under finite coproducts $+$
- $Pfn(\mathcal{A})$ is category with objects $X \in \mathcal{A}$
- an arrow $f : X \longrightarrow Y$ is partial function
- if $f : X \longrightarrow X + Y$, $f^\dagger : X \longrightarrow Y$ is "do f while value is in X".

$$f^\dagger = X \xrightarrow{f} X + Y \xrightarrow{\langle f^\dagger,\ \mathbf{1}_Y \rangle} Y.$$

$F_\omega(\mathcal{A})$

- $\mathcal{A}$ is a collection of ω-complete posets closed under finite products $\times$

- $F_\omega(\mathcal{A})$ is category with objects $X \in \mathcal{A}$

- an arrow $f : X \longrightarrow Y$ is continuous function

- if $f : X \times Y \longrightarrow X$, $f^\dagger : Y \longrightarrow X$ is least x such that $f(x, y) = x$.

$$\begin{aligned} f^\dagger &= Y \xrightarrow{\langle f^\dagger, \mathbf{1}_Y \rangle} X \times Y \xrightarrow{f} X \\ f^\dagger(y) &= f(f^\dagger(y), y). \end{aligned}$$

$Fn(\mathcal{A})$

- $\mathcal{A}$ a collection of ω-complete categories closed under finite products

- objects in $Fn(\mathcal{A})$ are categories in $\mathcal{A}$

- an arrow in $Fn(\mathcal{A})$ is ω-continuous functor $f : A \longrightarrow B$

- if $f : A \times B \longrightarrow A$, for each $b \in B$,

$$\begin{aligned} f_b : A &\longrightarrow A \text{ is} \\ f_b &= f(-, b) \end{aligned}$$

- $f^\dagger : B \longrightarrow A$ on b is initial f_b-algebra

$$f_b(f^\dagger(b)) \longrightarrow f^\dagger(b).$$

Trees $\Sigma_\perp TR$

- Σ a ranked set ; $\perp \in \Sigma_0$.
- a morphism $1 \longrightarrow p$ is node labeled Σ-tree with internal nodes with n-successors labeled by letter in Σ_n; leaves labeled either by letter in Σ_0 or "variable" $x_1, \ldots, x_p$
- a morphism $n \longrightarrow p$ is n-tuple of trees $1 \longrightarrow p$.
- composition: tree substition
- if $f : 1 \longrightarrow 1 + p$, $f^\dagger : 1 \longrightarrow p$ is unique tree such that

$$f^\dagger = 1 \xrightarrow{f} 1 + p \xrightarrow{\langle f^\dagger, \mathbf{1}_p \rangle} p.$$

Matrices

The subsets of words on an alphabet X form a semiring 2^{X^*} where $A + B = A \cup B$ and $A \cdot B = \{uv : u \in A,\ v \in B\}$. Matrices over this semiring form a category $Mat(2^{X^*})$:

- an arrow $f : n \longrightarrow p$ is $n \times p$ matrix
- $f \cdot g$ matrix product
- if $f = [a\ b] : n \longrightarrow n + p$, $f^\dagger = [a^* b]$, where

$$a^* \ = \ \mathbf{1}_n + a + a^2 + \ldots .$$

Theories with fixed points

Each of the above examples are categories T with

- finite coproducts or products
- a parameterized fixed point operation
- In Pfn, Mat, F_ω, $f^\dagger$ is a least fixed point
- In $\Sigma_\perp TR$, $f^\dagger$ is a unique fixed point
- In Fn, $f^\dagger$ is an initial fixed point

Main Results

- (Bloom, Ésik) A large number of the computationally interesting structures have one of two equational theories IT, IT_0.

- $IT := Id(F_\omega(\mathcal{A})) = Id(Fn(\mathcal{A})) = Id(\Sigma_\perp TR)$

  IT is the iteration theory identities.

- $IT_0 = IT + \{x = y : x, y : 1 \to 0\} = Id(Pfn(\mathcal{A})) = Id(Mat(2^{X^*}))$,

- (Simpson, Plotkin) Every consistent iteration theory is either IT or IT_0.

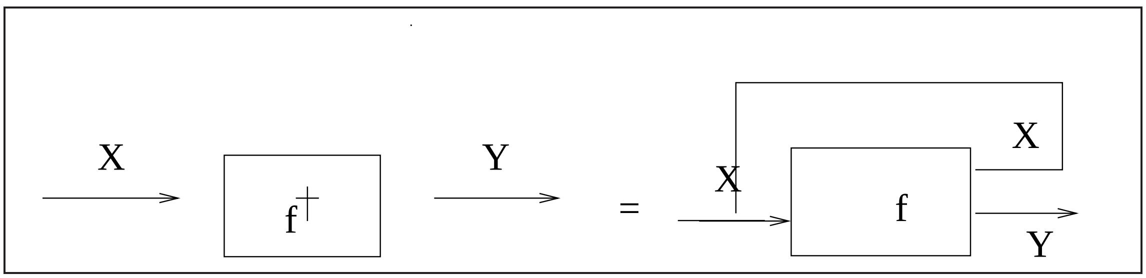
Axiomatization

Axioms for iteration theory identities fall into two groups.

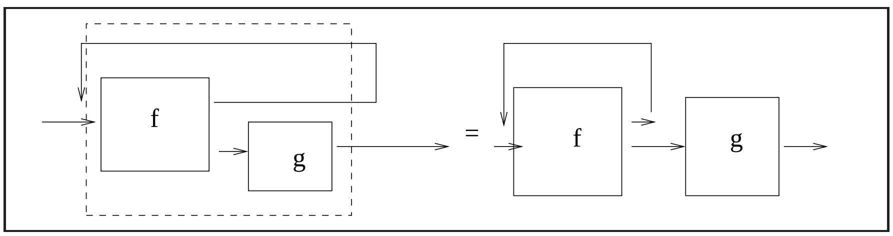
- The “Conway axioms”: fixed point, parameter, simple composition and double-dagger

- The group axioms.

Representing $f^\dagger$



Parameter identity



$$(f \cdot (\mathbf{1}_X \oplus g))^\dagger \;=\; f^\dagger \cdot g$$

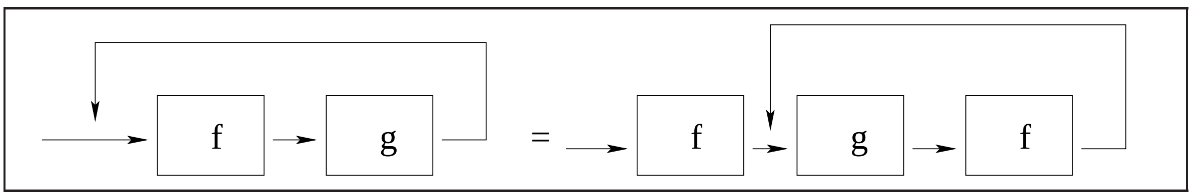
$$f : X \longrightarrow X + Y,\; g : Y \longrightarrow Z.$$

Another formulation of the parameter identity

$$\begin{array}{ccc}
\mathcal{C}(X, X+Y) & \xrightarrow{\dagger} & \mathcal{C}(X, Y) \\
\Big\downarrow {\scriptstyle \mathcal{C}(\mathbf{1}_X, \mathbf{1}_X \oplus g)} & & \Big\downarrow {\scriptstyle \mathcal{C}(\mathbf{1}_X, g)} \\
\mathcal{C}(X, X+Z) & \xrightarrow[\dagger]{} & \mathcal{C}(X, Z)
\end{array}$$

commutes, for any $g : Y \longrightarrow Z$.

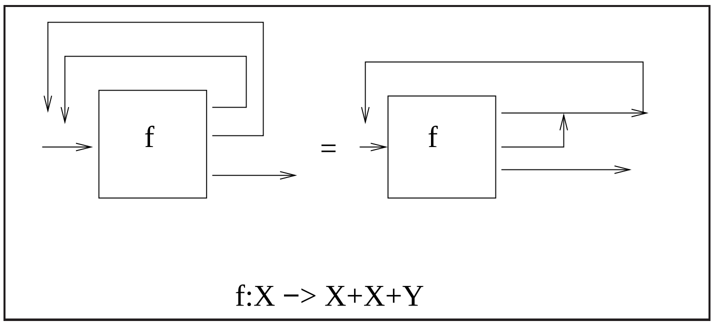
Simple Composition identity



$$(f \cdot g)^{\dagger} \;=\; f \cdot (g \cdot f)^{\dagger}$$

$$f : X \longrightarrow Y,\; g : Y \longrightarrow X.$$

Double dagger Identity



$$f^{\dagger\dagger} = (f \cdot (\langle \mathbf{1}_X, \ \mathbf{1}_X \rangle \oplus \mathbf{1}_Y))^{\dagger}$$

Traced monoidal categories

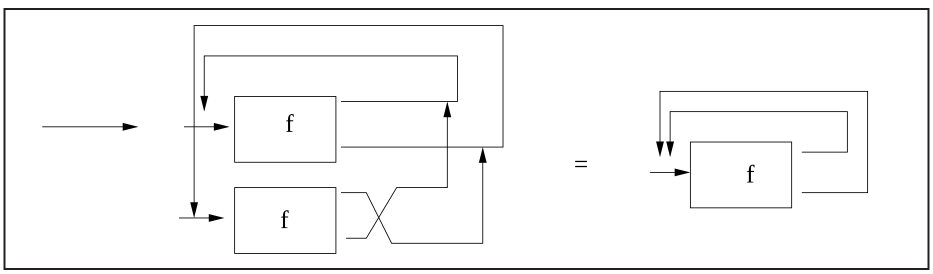
Conway theories are essentially the same as traced monoidal categories of Joyal, Street and Verity.

(Feedback replaces dagger.)

Iteration Theories

An iteration theory is a Conway theory satisfying all group identities.

For the two element group G_2:



$$1_2 \cdot \langle f \cdot \rho_1,\ f \cdot \rho_2 \rangle^{\dagger} \ = \ f^{\dagger\dagger}.$$

The G_2-identity

Star semirings

A semiring, or rig, S consists of

- a commutative monoid $(S, +, 0)$, and
- a monoid $(S, \cdot, 1)$, such that
- multiplication distributes over addition

$$\begin{aligned} x(y+z) &= xy + xz \\ (y+z)x &= yx + zx \\ 0 \cdot x &= x \cdot 0 \;=\; 0. \end{aligned}$$

- A star semiring is semiring with $^* : S \to S$.

Conway and iteration matrix theories

$Mat(S)$, the theory of matrices over a semiring S, is a Conway theory iff S is a star semiring satisfying

- the *sum star identity*:

$$(x+y)^* \;=\; (x^*y)^*x^*,$$

- the *product star identity*:

$$(xy)^* \;=\; 1 + x(yx)^*y.$$

Special cases: *zero and fixed point identity*:

$$\begin{aligned} 0^* &= 1 \\ x^* &= 1 + xx^* = 1 + x^*x. \end{aligned}$$

Definition. A **Conway semiring** is star semiring satisfying the sum and product star identities.

Examples

- **Language semirings**:

  $(2^{X^*}, +, \cdot, 0, 1, ^*)$, where X is an alphabet, and for $A, B \subseteq X^*$,

$$\begin{aligned} A + B &= A \cup B;\ 0 = \emptyset;\ 1 = \{\epsilon\} \\ A \cdot B &= \{uv : u \in A,\ v \in B\} \\ A^* &= \bigcup_{k=0}^{\infty} A^k. \end{aligned}$$

More Examples

- $\mathbb{N}_\infty = \{0, 1, \ldots,\} \cup \{\infty\}$, with $0^* = 1$, $x^* = \infty$, otherwise.

- The boolean semiring $\mathbb{B} = \{0, 1\}$, with $x^* = 1 = 1 + x$.

Which Rings

are reducts of Conway semirings?

- The star fixed point identity:

$$x^* \;=\; 1 + x \cdot x^*$$

  implies

$$x^* \cdot (1 - x) \;=\; 1.$$

- Letting $x = 1$:

$$0 \;=\; 1.$$

Inductive definition of M^*

If M is $n \times n$, $n > 1$, write

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with a, d square. The Conway identities imply that M^* is

$$M^* = \begin{pmatrix} (a + bd^*c)^* & (a + bd^*c)^*bd^* \\ (d + ca^*b)^*ca^* & (d + ca^*b)^* \end{pmatrix}$$

Group identities for matrix theories

Suppose G is a group with underlying set

$\{1, 2, \ldots, n\}$.

The G identity is

$$(x_1 + \ldots + x_n)^* \;=\; e_1 M_G^* u_n$$

M_G is $n \times n$ matrix with entries in $\{x_1, \ldots, x_n\}$

$$M_G[i, j] \;=\; x_{i^{-1} \cdot j}$$

$i^{-1} \cdot j$ is computed in G.

$$e_1 \;=\; [1\ 0 \ldots\ 0].$$

u_n is $n \times 1$ matrix of 1's.

Example

When $G = G_2$ is the 2-element group,

$$M_G = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}$$

The G_2-identity is

$$(x_1 + x_2)^* = (x_1 + x_2 x_1^* x_2)^* (1 + x_1^* x_2).$$

Special case: $x_1 = 0$ and $x_2 = 1$,

$$1^* = 1^*(1 + 1) = 1^* + 1^*.$$

Iteration semirings

- Suppose S is a Conway semiring. $Mat(S)$ is an iteration theory iff S satisfies all group identities.

- Definition. An iteration semiring is a Conway semiring satisfying all group identities.

Closure Properties

If S is Conway or iteration semiring, so is

$$S^{n \times n}$$

for $n \geq 0$, the semiring of $n \times n$ matrices over S, with matrix product as product, pointwise sum, and star computed using the inductive formula.

Power series semirings

If S is Conway or iteration semiring, so is

$$S\langle\langle A^* \rangle\rangle$$

the semiring of formal power series. Elements are functions $A^* \longrightarrow S$.

- Notational convention:

  $$f = \sum_{u \in A^*} (f, u)u,$$

  where (f, u), the coefficient of u, is $f(u)$.

- Examples. For $a \in A, s \in S$:

  $$\begin{aligned} \tau_a(u) &= 1 \text{ if } u = a, \text{ 0 otherwise.} \\ \sigma_s(u) &= s \text{ if } u = \epsilon, \text{ 0 otherwise.} \end{aligned}$$

- The notational convention implies

  $$\begin{aligned} a &= \tau_a \\ s &= \sigma_s. \end{aligned}$$

Semiring operations in $S\langle\langle A^*\rangle\rangle$

- If

$$f = \sum_{u \in A^*} (f,u)u$$
$$g = \sum_{u \in A^*} (g,u)u$$

then

$$f + g := \sum_{u \in A^*} ((f,u) + (g,u))u$$
$$f \cdot g := \sum_{u \in A^*} \Big(\sum_{xy=u} (f,x) \cdot (g,y) \Big) u.$$

- Example If $f = 2 + 3a$, $g = 2b + ab$,

$$fg = 4b + \underline{8ab} + 3a^2b.$$

$ab = \epsilon \cdot (ab)$ or $a \cdot b$.

f^* in $S\langle\langle A^* \rangle\rangle$?

- If $f(\epsilon) = 0$, there is unique f^* satisfying

$$f^* = 1 + f \cdot f^*$$
$$f^* = 1 + \sum_u (\sum_{xy=u} (f, x)(f^*, y))u.$$

- If $f = s + g$, $g(\epsilon) = 0$, the sum star identity implies

$$f^* = (s+g)^* = (s^*g)^*s^*.$$

Some iteration and Conway semirings

- Each of

$$\mathbb{B} = \{0, 1\},\ \mathbb{N}_\infty,\ 2^{X^*}$$

is an iteration semiring.

- The initial Conway semiring S_0 has elements

$$0, 1, 2, \ldots, k(1^*)^p,\ \text{and}\ 1^{**}.$$

- In any iteration semiring

$$1^* + 1^* \ = \ 1^*.$$

- The initial iteration semiring S_1 has elements

$$0, 1, 2, \ldots, (1^*)^p,\ 1^{**}.$$

(Countably) complete semirings

S is (countably) complete if for any (countable) set I, $\sum_{i \in I} s_i$ exists, and has the usual properties. We may *define*

$$s^* := 1 + s + s^2 + \ldots$$

and the resulting star semiring is an iteration semiring.

Example: $\mathbb{N}_\infty$ is countably complete. In $\mathbb{N}_\infty$,

$$x^* = \begin{cases} 1 & x = 0 \\ \infty & \text{otherwise.} \end{cases}$$

For each $n \geq 1$,

- $S_0^{n\times n}$ is a Conway, but not an iteration semiring, and

- For each alphabet A, $S_0\langle\langle A^*\rangle\rangle$ is a Conway, but not an iteration semiring.

Rational power series

For a Conway semiring S and alphabet A,

- $S^{rat}\langle\langle A^*\rangle\rangle$, the rational series, is the *least sub star semiring* of $S\langle\langle A^*\rangle\rangle$ containing each series

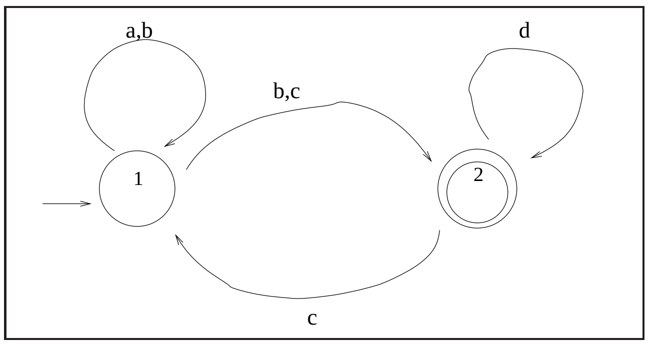
  $$\sigma_s, \text{ and } \tau_a,$$

  for $s \in S$, $a \in A$.

Example: regular sets on A

The series in $\mathbb{B}^{rat}\langle\langle A^*\rangle\rangle$ may be identified with least collection of subsets of A^* containing the singletons $\{a\}$, for $a \in A$, the empty set, the set $\{\epsilon\}$, closed under binary union, product and star, i.e., the regular sets.

Automata via matrices We may model finite automata as matrices:



$$\alpha = [1\ \ 0],\ \ M = \begin{pmatrix} a+b & b+c \\ c & d \end{pmatrix},\ \ \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

α, β are matrices over $\mathbb{B}$; M is matrix whose entries are linear combinations of letters.

The behavior of (α, M, β) is

$$\alpha \cdot M^* \cdot \beta$$

in $\mathbb{B}\langle\langle A^* \rangle\rangle$.

S-weighted Automata

For a Conway semiring S, and alphabet A, SA is the set of all finite series in $S\langle\langle A^*\rangle\rangle$ of the form

$$s_1a_1 + \ldots + s_na_n,$$

with $s_i \in S,\ a_i \in A$.

An automaton is triple (α, M, β), where

- α is $1 \times n$ matrix over S
- M is $n \times n$ matrix over SA
- β is $n \times 1$ matrix over S.

The behavior of (α, M, β) is the series in $S\langle\langle A^*\rangle\rangle$:

$$|(\alpha, M, \beta)| \;=\; \alpha \cdot M^* \cdot \beta.$$

Recognizable series

- A series f is recognizable if

$$f \ = \ |(\alpha, M, \beta)|$$

  for some automaton (α, M, β).

- The collection of all recognizable series is denoted

$$S^{rec}\langle\langle A^* \rangle\rangle.$$

Examples

- $s = \sigma_s \in S$ is $|(s, 0, 1)| = s \cdot 0^* \cdot 1$

- τ_a is $|\alpha, M, \beta|$ when

$$\alpha = (1\ 0)\,,\ \ M = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix},\ \ \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} \alpha M^* \beta &= (1\ 0) \cdot \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= a \end{aligned}$$

A Kleene theorem for Conway semirings

Theorem(Bloom, Ésik, Kuich)

$$S^{rat}\langle\langle A^* \rangle\rangle = S^{rec}\langle\langle A^* \rangle\rangle.$$

Some free semirings

- $\mathbb{B}^{rat}\langle\langle A^*\rangle\rangle$ is freely generated by A in class of iteration semirings satisfying $1^* = 1$. [Krob]

- $\mathbb{B}^{rat}\langle\langle A^*\rangle\rangle$ is isomorphic to star semiring of regular subsets of A^*; thus the axioms of iteration semirings together with $1^* = 1$ give an equational axiomatization of the regular sets.

Partial Conway semirings

An ideal I in a semiring S is a subset I containing 0, satisfying

$$S \cdot I \ \cup \ I \cdot S \ \cup \ I + I \subseteq I$$

A partial Conway semiring is a semiring with a partial star operation defined on an ideal $I = I(S)$ in S such that the Conway identities hold when restricted to elements in I:

$$\begin{aligned} (x+y)^* &= (x^*y)^*x^* \\ (xy)^* &= 1 + x(yx)^*y, \end{aligned}$$

for $x, y \in I$.

The Kleene theorem also holds for partial Conway semirings.

Examples

- For any semiring S, $S\langle\langle A^* \rangle\rangle$ is a partial Conway semiring, with f^* defined on all series f such that $f(\epsilon) = 0$. In fact, for such series, f^* is the unique solution to

  $$\xi \;=\; 1 + f \cdot \xi.$$

- $\mathbb{N}$ is partial iteration semiring, with $I(N) = \{0\}$.

  (If $n > 0$, what is $n^* = 1 + n \cdot n^*$?)

A characterization of $\mathbb{N}^{rat}\langle\langle A^* \rangle\rangle$

Theorem.(Bloom, Ésik) $\mathbb{N}^{rat}\langle\langle A^* \rangle\rangle$ is freely generated by A in the class of all partial iteration semirings, i.e., for any partial iteration semiring S and any function $h : A \to I(S)$, there is a unique semiring morphism $h^{\#} : \mathbb{N}^{rat}\langle\langle A^* \rangle\rangle \to S$ such that if $(f, \epsilon) = 0$, then

$$h^{\#}(f) \in I(S) \text{ and } h^{\#}(f^*) = (h^{\#}(f))^*.$$

We will outline the proof, which uses the Kleene theorem: rational series are the behaviors of automata.

Corollary. If S is an iteration semiring, any function $h : A \to S$ extends uniquely to a star semiring morphism $\mathbb{N}^{rat}\langle\langle A^*\rangle\rangle \to S$.

The extension

- Given a partial iteration semiring S and a function $f : A \to I(S)$, we define the image of the automaton (α, M, β) in $\mathbb{N}\langle\langle A^* \rangle\rangle$ in the obvious way: e.g., if $M[i,j] = 3a + 4b$, then $fM[i,j] = 3f(a) + 4f(b)$.

- If s is the behavior of (α, M, β) define $f^{\#}(s) = \alpha(fM)^*\beta$.

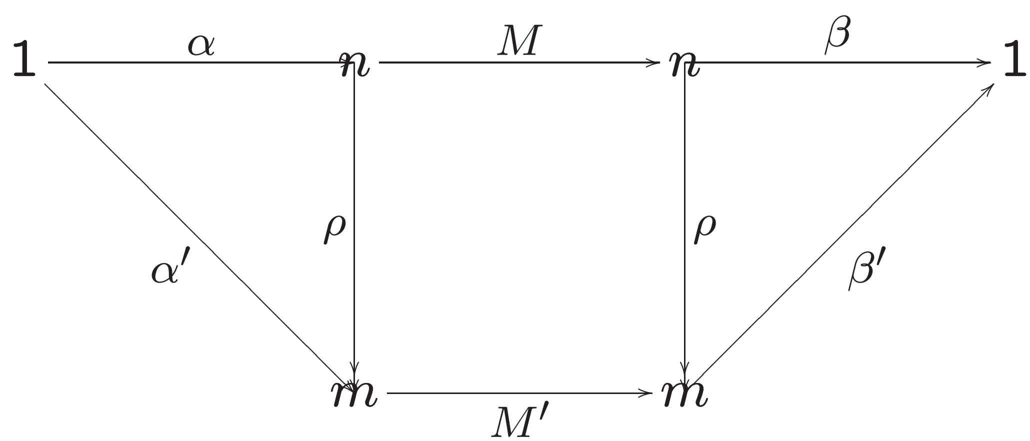
- Must show if two automata have the same behavior, their image is the same.

Simulations

In the classical case, $\mathbb{B}^{rat}\langle\langle A^*\rangle\rangle$, if two automata have the same behavior, their accessible and coaccessible parts have a common homomorphic image. A similar fact holds for $\mathbb{N}^{rat}\langle\langle A^*\rangle\rangle$.

Suppose $\mathfrak{A} = (\alpha, M, \beta)$ and $\mathfrak{B} = (\alpha', M', \beta')$

A simulation $\rho : \mathfrak{A} \to \mathfrak{B}$ is: $n \times m$ matrix over $\mathbb{N}$ such that



Let $\sim$ be the least equivalence on automata such that if $\rho : \mathfrak{A} \to \mathfrak{B}$ is a simulation, then $\mathfrak{A} \sim \mathfrak{B}$.

Theorem. (Béal, Lombardy, Sakarovitch). Two automata $\mathfrak{A}$, $\mathfrak{B}$ have the same behavior iff $\mathfrak{A} \sim \mathfrak{B}$.

Where are the group axioms?

They are used to prove that if

$$\begin{array}{ccc} n & \xrightarrow{M} & n \\ \rho \downarrow & & \downarrow \rho \\ m & \xrightarrow[M']{} & m \end{array}$$

commutes, then

$$\begin{array}{ccc} n & \xrightarrow{M^*} & n \\ \rho \downarrow & & \downarrow \rho \\ m & \xrightarrow[(M')^*]{} & m \end{array}$$

does also.

A characterization of ${\mathbb{N}_\infty}^{rat}\langle\langle A^* \rangle\rangle$

- If S is a countably complete star semiring, and $s^* = \sum_{i=0}^{\infty} s^i$, then S satisfies

$$\begin{aligned} 1^* \cdot 1^* &= 1^*; \\ x \cdot 1^* &= 1^* \cdot x; \\ 1^* \cdot (1^* \cdot x)^* &= 1^* \cdot x^*. \end{aligned}$$

- $\mathbb{N}_\infty$ is initial in $\mathcal{V}$.

- Theorem (Bloom, Ésik) ${\mathbb{N}_\infty}^{rat}\langle\langle A^* \rangle\rangle$ is freely generated by A in the variety $\mathcal{V}$ of all iteration semirings satisfying these three extra identities.

An A-term t

$$t = 0 \mid 1 \mid a \mid t + t \mid t \cdot t \mid t^*.$$

for $a \in A$.

The series denoted by t is $|t|$.

Outline of proof

In $\mathcal{V}$, each A-term is equivalent to one of form

$$t_c + t_I + 1^* s,$$

where

- $t_c \in \mathbb{N}$,
- t_I is "ideal term";

$$t_I = \ 0 \mid a \mid t_I + t_I \mid t_I \cdot t_I \mid t_I \cdot t_I^*$$

- s is any A-term
- $t_c \neq 0 \implies (|s|, \epsilon) = 0.$

Claim: $|s| = |t|$ iff $s = t$ holds in $\mathcal{V}$.

Fact. If $f = |t|$, for an *ideal term* t, then $(f, \epsilon) = 0$ and $(f, u) \in \mathbb{N}$, otherwise. Also, $f \in \mathbb{N}^{rat}\langle\langle A^* \rangle\rangle$.

First two steps

- If $|s_0 + s_I + 1^* s| = |t_0 + t_I + 1^* t|$, then

$$|s_0| = |t_0|,\ |s_I| = |t_I|,\ |1^* s| = |1^* t|.$$

- $|s_0| = |t_0|$ are in $\mathbb{N}$; $\mathbb{N}_\infty$ initial in $\mathcal{V}$ implies $s_0 = t_0$ in $\mathcal{V}$.

- $|s_I| = |t_I|$ belong to $\mathbb{N}^{rat}\langle\langle A^* \rangle\rangle$, which is free in all partial iteration semirings. Thus, $s_I = t_I$ in $\mathcal{V}$.

Last step

is to show that if $|1^*s| = |1^*t|$, then $1^*s = 1^*t$ in $\mathcal{V}$.

- Lemma. Let $\mathcal{W}$ be variety of iteration star semirings satisfying $1^* = 1$. An identity

  $$1^*s = 1^*t$$

  holds in $\mathcal{V}$ iff it holds in $\mathcal{W}$.

- $\mathbb{B}^{rat}\langle\langle A^*\rangle\rangle$ is freely generated by A in $\mathcal{W}$ [Krob].

- $|1^*s|$ is a rational series all of whose nonzero coefficients are 1^*. Thus $1^*s = 1^*t$ in $\mathbb{B}^{rat}\langle\langle A^*\rangle\rangle$, and hence in $\mathcal{W}$ by Krob. Thus, by the Lemma, $1^*s = 1^*t$ in $\mathcal{V}$.

Inductive semirings

An ordered semiring is semiring equipped with a partial order $\leq$ preserved by $+, \cdot$.

A star semiring is inductive if

$$\begin{aligned} aa^* + 1 \quad &\leq \quad a^* \\ ax + b \leq x \quad &\Longrightarrow \quad a^*b \leq x \\ xa + b \leq x \quad &\Longrightarrow \quad ba^* \leq x. \end{aligned}$$

Fact:(Ésik, Kuich) Any inductive semiring is an iteration semiring satisfying $1^* = 1^{**}$.

Examples of inductive semirings

- $Mat(2^{X^*})$ where $A \leq B \iff A \subseteq B$

- $\mathbb{N}_\infty$ where $n < n + 1 < \infty$

- ${\mathbb{N}_\infty}^{rat}\langle\langle A^*\rangle\rangle$ is inductive semiring, when ordered by sum order:

$$f \leq g \iff g = f + h,$$

  for some $h \in {\mathbb{N}_\infty}^{rat}\langle\langle A^*\rangle\rangle$

Non-examples of inductive semrings

- S_0, the initial Conway semiring
- S_1, the initial iteration semiring
- $S_0\langle\langle A^*\rangle\rangle$, $S_1\langle\langle A^*\rangle\rangle$.

A second characterization

Theorem: (Bloom, Ésik) ${\mathbb{N}_\infty}^{rat}\langle\langle A^*\rangle\rangle$ is freely generated by A in class of inductive semirings satisfying $x \cdot 1^* \leq 1^* \cdot x$.

Two Open problems

- Is there an algorithm to decide, given star semiring terms r, r' whether $r = r'$ in all Conway semirings?

- What is a concrete description of the free Conway and iteration semirings?

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