CW-complexes in the category of exterior spaces

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    2. The homotopy constructions produce pro-spaces that many times can not be geometrically interpreted as regular spaces.

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- $\mathbb{R}_+ = [0, \infty)$.
- A proper map $\alpha : \mathbb{R}_+ \to X$ is called *base ray* in X. (We will also consider *base sequences*, i.e. proper maps $\mathbb{N} \to X$)

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An *exterior space* $(X, \varepsilon \subset \tau)$ consists of a topological space (X, τ) together with a nonempty family of open subsets ε (called *externology*) satisfying

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- Let X be a topological space. Then we can consider the *cocompact externology* $\varepsilon_{cc} = \{X - K : K \text{ is closed and compact}\}$. The corresponding exterior space is denoted by X_{cc}. This construction gives rise to a full embedding

$$(-)_{cc} : \mathbf{P} \hookrightarrow \mathbf{E}$$

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- We shall denote by $[X, Y]$, $[(X, \alpha), (Y, \beta)]^{\mathbb{R}_+}$, or $[(X, \alpha), (Y, \beta)]^{\mathbb{N}}$ the corresponding homotopy brackets.

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Definition (Exterior CW-complex)

A *relative exterior CW-complex* (X, A) is an exterior space X together with a filtration of exterior spaces $A = X^{-1} \subset X^0 \subset X^1 \subset \ldots \subset X_n \subset \ldots$, for which X is its colimit and, for each $n \geq 0$, X^n is obtained from X^{n-1} as the exterior pushout

$$\begin{array}{ccc} \sqcup_{\gamma\in\Gamma}\mathfrak{S}^{n-1}_\gamma & \xrightarrow{\{\varphi_\gamma\}_{\gamma\in\Gamma}} & X^{n-1} \\ \downarrow & & \downarrow \\ \sqcup_{\gamma\in\Gamma}\mathfrak{D}^{n}_\gamma & \xrightarrow{\{\psi_\gamma\}_{\gamma\in\Gamma}} & X^{n} \end{array}$$

via the attaching maps $\varphi_\gamma : \mathfrak{S}^{n-1} \to X^{n-1}$.

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- Given an exterior CW-complex X and a classical finite CW-complex K of dimension m, $X \bar{\times} K$ admits an exterior CW-structure.
- Let (X, A) be a locally finite, finite dimensional relative CW-complex in which, for any k, X has no k-cells or has infinite countable many k-cells. Then (X_{cc}, A_{cc}) is an exterior relative CW-complex.

Exterior homotopy groups

We consider $\mathbb{N}$-spheres and $\mathbb{N}$-discs as objects in $\mathbf{E}^{\mathbb{N}}$ via the base sequence

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Exterior homotopy groups

We consider $\mathbb{N}$-spheres and $\mathbb{N}$-discs as objects in $\mathbf{E}^{\mathbb{N}}$ via the base sequence

$$\eta : \mathbb{N} \to \mathfrak{S}^k \subset \mathfrak{D}^{k+1}, \quad \eta(n) = (n, *).$$

Remember: $\mathfrak{S}^k = \mathbb{N}\bar{\times}S^k$; $\quad \mathfrak{D}^k = \mathbb{N}\bar{\times}D^k$

Definition

The k-th Brown-Grossman exterior homotopy group of $(X, \alpha) \in \mathbf{E}^{\mathbb{R}_+}$, $k \geq 0$, is defined as $\pi_k^{\mathcal{B}}(X, \alpha) = [(\mathfrak{S}^k, \eta), (X, \alpha_{|\mathbb{N}})]^{\mathbb{N}}$.

Remarks:

- We can also consider exterior homotopy groups for $(X, \alpha) \in \mathbf{E}^{\mathbb{N}}$
- We can easily generalize to homotopy groups

$$\pi_k^{\mathcal{B}}(X, A, \alpha) = [(\mathfrak{D}^k, \mathfrak{S}^{k-1}, \eta), (X, A, \alpha)]^{\mathbb{N}}$$

- In the proper homotopy setting, i.e., whenever $(X, \alpha) \in \mathbf{P}^{\mathbb{R}_+}$, the exterior homotopy groups of (X_{cc}, α_{cc}) are the (global) Brown-Grossmann proper homotopy groups.

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- An exterior map $f : Y \to Z$ is an *exterior n-equivalence* or simply *e-n-equivalence* if for any exterior base sequence $\alpha : \mathbb{N} \to Y$

$$f_* : \pi_k^{\mathcal{B}}(Y, \alpha) \xrightarrow{\cong} \pi_k^{\mathcal{B}}(Z, f\alpha) \text{ is isomorphism for } 0 \leq k \leq n-1$$

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- We can establish: (X, A) *e-n-connected*; p-n-equivalences (p-n-connected)
- In the proper setting, a space $X \in \mathbf{P}_\infty$ is e-0-connected iff it is one-ended.

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Theorem (Exterior Whitehead)

Let $f : (X, \mathbb{R}_+) \to (Y, \mathbb{R}_+)$ *be an e-n-equivalence (*$n \leq \infty$*) between relative exterior CW-complexes of dimension at most n. Then,* f *is an exterior homotopy equivalence rel.* $\mathbb{R}_+$.

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Theorem (Exterior Cellular Approximation)

Given an exterior map $f : (X, A) \to (Y, B)$ between exterior relative CW-complexes, there exists and exterior and cellular map $g : (X, A) \to (Y, B)$ for which $g \simeq f$ rel. A.

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Given an e-0-connected space $(X, \alpha) \in \mathbf{E}^{\mathbb{R}_+}$ there exists a exterior ∞-equivalence

$$w : \widehat{X} \xrightarrow{\sim} X$$

in which $(\widehat{X}, \mathbb{R}_+)$ is a relative exterior CW-complex.

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Let $X = X_1 \cup X_2$, $A = X_1 \cap X_2$ in which (X_1, A) and (X_2, A) are exterior cofibred pairs; X_1, X_2 and A are e-0-connected; (X_1, A) is e-$(n-1)$-connected, and (X_2, A) is e-$(m-1)$-connected, $m, n \geq 1$. Then

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- Given a proper map $j : A \longrightarrow X$ between Hausdorff locally compact spaces, then j is a proper cofibration if and only if j_{cc} is an exterior cofibration.

Consequences in proper homotopy theory

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Theorem (Proper Whitehead)

If $f:(X,\mathbb{R}_+)\to(Y,\mathbb{R}_+)$ is a p-n-equivalence between locally finite CW-complexes with finite dimension less than n and for each $0\leq k\leq d$ either X (respec. Y) has no k-cells or X (respec. Y) has an infinite countable number of k-cells, then f is a proper homotopy equivalence rel. $\mathbb{R}_+$.

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Remark: There is no a proper version of a CW-approximation theorem. If $(X,\alpha) \in \mathbf{P}^{\mathbb{R}_+}$ (X one-ended) the construction of the relative exterior CW-complex $(\widehat{X},\mathbb{R}_+)$ gives a non cocompact exterior space.

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