A Topos Approach to the Formulation of Physical Theories

Category Theory 2008
Calais
26. June 2008

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"A theory is something nobody believes, except the person who made it. An experiment is something everybody believes, except the person who made it."

(Unknown)

Motivation

The motivation for this work comes from fundamental physics. We have very good physical theories of the world of atoms and smaller down to a scale of roughly 10^{-18} m (standard model, QT) and of gravity (general relativity, GR).

What is lacking is a unification or reconciliation of QT and GR in a theory of **quantum gravity** (QG) and **quantum cosmology** (QC).

Today, we have several approaches (string theory, loop quantum gravity, ...), but no predictive, experimentally testable theory.

Apart from technical questions, there are a number of deep conceptual problems. Two of them are:

- The mathematical formalism of quantum theory is usually interpreted in an **instrumentalist** manner.
- All physical structures used are based on the idea of a **continuum**. Their mathematical description uses the real numbers in a fundamental way.

The problem with instrumentalism

Instrumentalism means that the interpretation of the mathematical formalism of quantum theory depends on measurements, observers etc.

This is a well-known problem of quantum theory itself, but it becomes more severe in a future theory of QC or QG:

- If we treat space and time as quantum objects (whatever this will mean in detail), what could a measurement *of* space or *of* time mean? 'Where' and 'when' does such a measurement take place?
- In QC at least, we will have to treat the whole universe as a quantum system. Clearly, there is no observer external to the universe who could perform measurements.
- We need to overcome or circumvent the usual instrumentalism of quantum theory.
- A more **realist** formulation of QT is needed.

The problem with the continuum

It is commonly expected from extrapolations of existing physical theories that at very small scales (10^{-35} m) and very high energies (10^{19} GeV), where QG is important, the continuum picture of space-time will break down.

- This means that in QG, space-time will presumably not be described by a smooth manifold. Related to that, physical quantities need not necessarily have real numbers as values.
- In QG/QC, the continuum in the form of the real numbers and all structures build upon them (manifolds, Hilbert spaces, operators, path integrals, strings, loops...) will potentially play a much less prominent rôle than in QT and GR.
- More down to earth: due to the non-commutativity of physical quantities like position and momentum, the concept of a state space of a quantum system becomes problematic.
- Ideally, we would like a framework for the formulation of physical theories that does not fundamentally depend on the real numbers.

Topos theory as a new mathematical framework

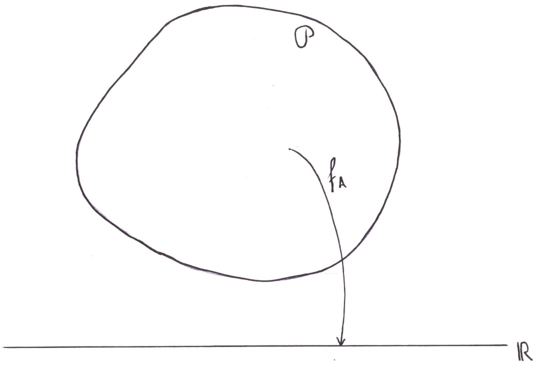
In this talk, I want to show how **topos theory** allows to formulate physical theories in a way that

- is ‘neo-realist’ in the sense that there is an analogue of a state space, and propositions about the values of physical quantities have truth-values, independent of measurements, observers etc., and
- the framework does not (fundamentally) depend on the real numbers.

Of course, a theory of quantum gravity is still a long way off. I will sketch some ingredients of the general framework and then show how ordinary (algebraic) quantum theory can be reformulated such that it fits into this scheme.

State spaces and Boolean logic

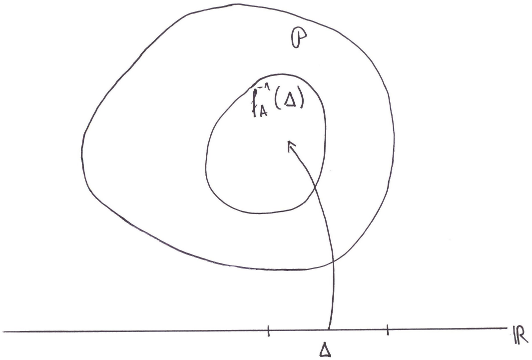
In classical physics, physical quantities/observables A are described by real-valued functions f_A on the state space $\mathcal{P}$, that is, mappings



Points of $\mathcal{P}$ are states. In a given state, all physical quantities have values.

State spaces and Boolean logic

One can also consider inverse images of (Borel) subsets $\Delta \subseteq \mathbb{R}$:



Such a subset of the state space $\mathcal{P}$ corresponds to a **proposition** "$A \in \Delta$", that is, "the physical quantity A has a value lying in the set Δ".

State spaces, Boolean logic and realism

Each point of the state space $\mathcal{P}$ either lies in $f_A^{-1}(\Delta)$ or not, i.e., in the state represented by the point the corresponding proposition is either true or false.

The (Borel) subsets of state space form a **Boolean algebra**.

All this implies that classical physics is a **realist** theory. In a given state $s \in \mathcal{P}$, all physical quantities have values, and all propositions have truth-values. Logical formulas involving propositions can be manipulated according to the rules of a deductive system. These are the rules of classical, Boolean logic, which is closely tied to the use of sets:

- Stone '36: Every Boolean algebra is isomorphic to the Boolean algebra of clopen subsets of a suitable space.

Classical physics, by its very form as a theory based upon state spaces, which are *sets*, has a Boolean logical structure.

Algebraic quantum theory

Algebraic quantum theory describes a quantum physical system by

- a non-abelian **von Neumann algebra** $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$, where the self-adjoint operators $\widehat{A} : \mathcal{H} \to \mathcal{H}$ in $\mathcal{N}$ represent physical quantities,
- **states** on this algebra, i.e. positive linear functionals $\rho : \mathcal{N} \to \mathbb{C}$ of norm 1, and
- **propositions** of the form "$A \in \Delta$" are represented by projection operators $\hat{E}[A \in \Delta]$ in $\mathcal{N}$.

A particular kind of states are vector states $w_\psi(\_) = \langle \psi, \_\psi \rangle$, where $\psi \in \mathcal{H}$ is a unit vector. A vector state is called an **eigenstate of** $\widehat{A}$ if $\widehat{A}(\psi) = a\psi$.

Algebraic quantum theory

In any given state ρ, only a few physical quantities A have definite values, namely those for which ρ is an eigenstate of $\widehat{A}$.

Every proposition "$A \in \Delta$" has a probability of being true in the state ρ, given by

$$P(\text{“}A \in \Delta\text{”}, \rho) = \rho(\hat{E}[A \in \Delta]) \in [0, 1].$$

In general, a physical quantity A aquires a definite value only upon a measurement of A, so a proposition "$A \in \Delta$" aquires a truth-value 'true' or 'false' only upon measurement.

A measurement (of A) brings about a discontinuous change of ρ (into an eigenstate of $\widehat{A}$). The same initial state ρ can give rise to different final states.

The Kochen-Specker theorem

Problem: Is there a realist formulation of quantum theory similar to classical physics?

More concretely, is there a 'state space' for a quantum system such that physical quantities are real-valued functions on this space? We require that the self-adjoint operators in the von Neumann algebra $\mathcal{N}$ of physical quantities correspond to functions on the (hypothetical) state space.

Kochen, Specker 1967: If the von Neumann algebra $\mathcal{N}$ of physical quantities of a quantum system consists of all bounded operators on Hilbert space, $\mathcal{N} = \mathcal{B}(\mathcal{H})$, where $\dim \mathcal{H} \geq 3$, then there exists no state space model of QT (under very natural conditions). It is impossible to assign real values to all physical quantities at once.

AD 2005: This also holds for all von Neumann algebras $\mathcal{N}$ without summands of type I_1 and I_2, i.e., for all quantum systems with symmetries and/or superselection rules.

The Kochen-Specker theorem

The KS theorem is equivalent to the fact that in quantum theory we cannot consistently assign ‘true’ or ‘false’ to all propositions at once (or 1 resp. 0 to the projections corresponding to the propositions).

This means that we cannot use Boolean logic to describe quantum systems in a realist manner.

Ordinary Quantum Logic

Birkhoff, von Neumann 1936: lattice $L(\mathcal{H})$ of closed subspaces of Hilbert space $\mathcal{H}$ describes the logic of quantum systems. (More generally, the lattice of projections in the von Neumann algebra $\mathcal{N}$ is considered.)

- At first sight, this is similar to a classical propositional calculus with the Hilbert space $\mathcal{H}$ taking the rôle of the quantum state space analogue.
- Severe interpretational problem: if $\dim \mathcal{H} > 1$, then $L(\mathcal{H})$ is **non-distributive**. Example: the "quantum breakfast"

$$E \wedge (B \vee S) \neq (E \wedge B) \vee (E \wedge S).$$

- There are many further developments in **quantum logic**, but these are somewhat detached from physics.
- In particular, a viable deductive system is lacking.

The central idea

Given a physical system S, look for suitable objects $\Sigma, \mathcal{R}$ in a category such that

- Σ, the **state object**, takes the rôle of state space, but it need not be a set,
- subobjects of Σ represent propositions "$A \in \Delta$",
- $\mathcal{R}$, the **quantity-value object**, is where physical quantities 'take their values'; this need not be the real numbers $\mathbb{R}$,
- physical quantities are represented by arrows $\breve{A} : \Sigma \to \mathcal{R}$.

We want the subobjects of Σ to have a logical structure, while allowing for something more general than Boolean logic. We also want a deductive system. This suggests the use of **topoi**.

For a classical description of a physical system S, the topos Sets of sets and functions is used. For quantum theory, a suitable, physically motivated topos will be used. In future theories of QG, other topoi will play a rôle.

Contexts or Weltanschauungen

Which topos to use for quantum theory?

- There is no model of QT in which all physical quantities have values at once.
- Not surprisingly, there is no problem for *abelian* algebras. The operators in an abelian C^*-algebra can be written as continuous functions (Gel'fand transforms) on the Gel'fand spectrum.
- Commutative subalgebras of $\mathcal{N}$ are called **contexts**. They are like 'classical snapshots' of the quantum system.
- Some kind of *contextual* model of QT is needed (but with good control over the relations between contexts).

The context category

All constructions we use work for an arbitrary von Neumann algebra $\mathcal{N}$.

We consider the category $\mathcal{V}(\mathcal{N})$ of non-trivial unital abelian von Neumann subalgebras of the algebra $\mathcal{N}$ of physical quantities. This is a partially ordered set under inclusion and is called the **context category**.

This category 'knows about' the relations between the abelian subalgebras of $\mathcal{N}$: if $V_1, V_2 \in \mathcal{V}(\mathcal{N})$ and $V := V_1 \cap V_2$, then there are arrows $i_{VV_1} : V \to V_1$ and $i_{VV_2} : V \to V_2$.

Going from an abelian algebra V to a smaller algebra $V' \subset V$ is a process of **coarse-graining**: V' contains less self-adjoint operators and less projections than V, so we can describe less physics in it.

The projection lattice $\mathcal{P}(\mathcal{N})$ of a von Neumann algebra $\mathcal{N}$ is complete. The projection lattice $\mathcal{P}(V)$ of an abelian von Neumann algebra V is a complete Boolean algebra.

Gel'fand spectrum and Gel'fand transformation

Let V be an abelian C^*-algebra. The **Gel'fand spectrum** $\underline{\Sigma}_V$ of V is the set of all algebra homomorphisms $V \to \mathbb{C}$, equipped with the weak* topology, which makes it a compact Hausdorff space. If V is a von Neumann algebra, then $\underline{\Sigma}_V$ is extremely disconnected.

The **Gel'fand transformation** is the mapping

$$\begin{aligned} V &\longrightarrow C(\underline{\Sigma}_V) \\ \widehat{A} &\longmapsto \overline{A}, \end{aligned}$$

where, for all $\lambda \in \underline{\Sigma}_V$, we have $\overline{A}(\lambda) := \lambda(\widehat{A})$. If $\widehat{A}$ is self-adjoint, then $\overline{A}$ is real-valued.

The Gel'fand spectrum $\underline{\Sigma}_V$ has all the properties of a **local state space** at V.

The spectral presheaf

We now form a global object from all the local state spaces: to each $V \in \mathcal{V}(\mathcal{N})$, we assign its Gel'fand spectrum $\underline{\Sigma}_V$.

If $V' \subseteq V$, we have a morphism $i_{V'V} : V' \to V$ in the context category $\mathcal{V}(\mathcal{N})$ and define

$$\begin{aligned} \underline{\Sigma}(i_{V'V}) : \underline{\Sigma}_V &\longrightarrow \underline{\Sigma}_{V'} \\ \lambda &\longmapsto \lambda|_{V'}. \end{aligned}$$

$\underline{\Sigma}$ is a contravariant functor from the context category $\mathcal{V}(\mathcal{N})$ to the category Sets, i.e., a **presheaf over** $\mathcal{V}(\mathcal{N})$. We regard the **spectral presheaf** $\underline{\Sigma}$ as a quantum analogue of state space.

The presheaves over the context category $\mathcal{V}(\mathcal{N})$ form a topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{N})^{op}}$. *This is the topos associated to the quantum physical system, and $\underline{\Sigma}$ is the state object within this topos.*

Reformulation of the Kochen-Specker theorem

Isham, Butterfield '98: a global element of $\underline{\Sigma}$ would allow to assign values to all physical quantities at once, which is impossible due to the Kochen-Specker theorem. So we have a reformulation of the KS theorem:

Thm.: The spectral presheaf $\underline{\Sigma}$ has no global elements.

This is a 'geometric' version of the KS theorem.

Representation of physical quantities

We want to represent physical quantities as arrows in our quantum topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{N})^{op}}$. These arrows will be natural transformations from $\underline{\Sigma}$, the state object, to some quantity-value object $\underline{\mathcal{R}}$, yet to be specified.

For each self-adjoint operator $\widehat{A}$ and each context V, we have to define a function

$$\breve{\delta}(\widehat{A})_V : \underline{\Sigma}_V \longrightarrow \underline{\mathcal{R}}_V,$$

and for each $\widehat{A}$, these functions must fit together to form a natural transformation.

Since $\underline{\Sigma}_V$ is the space of algebra homomorphisms $V \to \mathbb{C}$, and these states can be evaluated on operators *in* V only, we want to approximate $\widehat{A}$ by an operator in V, for each $V \in \mathcal{V}(\mathcal{N})$. Actually, we will use one approximation from above and one from below.

Daseinisation of self-adjoint operators

Let $\widehat{A} \in \mathcal{N}_{sa}$. From the spectral family $\widehat{E}^A = (\widehat{E}^A_\lambda)_{\lambda\in\mathbb{R}}$, we obtain a new spectral family in $\mathcal{P}(V)$ by defining

$$\forall\lambda \in \mathbb{R} : \widehat{E}_\lambda^{\delta^o(\widehat{A})_V} := \bigvee\{\widehat{Q} \in \mathcal{P}(V) \mid \widehat{Q} \leq \widehat{E}^A_\lambda\}.$$

This gives a self-adjoint operator $\delta^o(\widehat{A})_V$, which is the smallest operator in V larger than $\widehat{A}$ in the so-called 'spectral order'.

Similarly, we can define

$$\forall\lambda \in \mathbb{R} : \widehat{E}_\lambda^{\delta^i(\widehat{A})_V} := \bigwedge\{\widehat{Q} \in \mathcal{P}(V) \mid \widehat{Q} \geq \widehat{E}^A_\lambda\}.$$

The corresponding operator $\delta^i(\widehat{A})_V$ approximates $\widehat{A}$ from below in the spectral order.

Constructing an arrow from $\underline{\Sigma}$ to $\underline{\mathcal{R}}$

For all $V' \subseteq V$, we have $\delta^i(\widehat{A})_{V'} \leq \delta^i(\widehat{A})_V \leq \widehat{A} \leq \delta^o(\widehat{A})_V \leq \delta^o(\widehat{A})_{V'}$.

We define

$$\begin{aligned} \breve{\delta}(\widehat{A})_V : \underline{\Sigma}_V &\longrightarrow \underline{\mathcal{R}}_V \\ \lambda &\longmapsto \{(\lambda|_{V'}(\delta^i(\widehat{A})_{V'}), \lambda|_{V'}(\delta^o(\widehat{A})_{V'})) \mid V' \subseteq V\}. \end{aligned}$$

I.e., to each $\lambda \in \underline{\Sigma}_V$, we assign a pair of functions from the set $\downarrow V = \{V' \in \mathcal{V}(\mathcal{H}) \mid V' \subseteq V\}$ to the real numbers (more precisely, to the spectrum of $\widehat{A}$).

This is the 'value' the physical quantity A has at λ. (Remark: This is still a 'local' argument, since $\lambda \in \underline{\Sigma}_V$.)

Order-preserving and order-reversing functions

The first function, given by $\delta^i(\widehat{A})$, is order-preserving as a function from $\downarrow V$ to $\mathbb{R}$. The second function, given by $\delta^o(\widehat{A})$, is order-reversing.

If a context $V' \subseteq V$ contains $\widehat{A}$, then $\delta^i(\widehat{A})_{V'} = \delta^o(\widehat{A})_{V'} = \widehat{A}$ and $\lambda|_{V'}(\delta^i(\widehat{A})_{V'}) = \lambda|_{V'}(\delta^o(\widehat{A})_{V'}) =$ eigenvalue of $\widehat{A}$ in the state λ.

If a context $V' \subseteq V$ does not contain $\widehat{A}$, then $\delta^i(\widehat{A})_{V'} < \widehat{A} < \delta^o(\widehat{A})_{V'}$ and $\lambda|_{V'}(\delta^i(\widehat{A})_{V'}) < \lambda|_{V'}(\delta^o(\widehat{A})_{V'})$. In such a context V', we get a 'range' or 'unsharp value' for $\widehat{A}$ in the state λ.

The approximations of $\widehat{A}$ by $\delta^i(\widehat{A})_V$ and $\delta^o(\widehat{A})_V$ should be understood as a coarse-graining of $\widehat{A}$ to the context V. As a consequence, the 'values' that $\widehat{A}$ takes are also coarse-grained to intervals at each stage V.

The quantity-value object $\underline{\mathbb{R}^{\leftrightarrow}}$

The pairs of functions defined above form a presheaf which we denote by $\underline{\mathbb{R}^{\leftrightarrow}}$. The restriction is simply given by restriction of the order-preserving and order-reversing functions.

By construction, $\breve{\delta}(\widehat{A})$ is a natural transformation from $\underline{\Sigma}$ to $\underline{\mathbb{R}^{\leftrightarrow}}$, and $\underline{\mathbb{R}^{\leftrightarrow}}$ is the quantity-value object for quantum theory.

$\underline{\mathbb{R}^{\leftrightarrow}}$ is a monoid object in $\mathrm{Sets}^{\mathcal{V}(\mathcal{N})^{op}}$. Using the k-construction by Grothendieck, we can get an abelian-group object $k(\underline{\mathbb{R}^{\leftrightarrow}})$.

Physical quantities, i.e., self-adjoint operators $\widehat{A}$, are represented by arrows $\breve{\delta}(\widehat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}^{\leftrightarrow}}$. This is structurally similar to classical physics (but quite different from ordinary quantum theory).

Subobjects from inverse images

A proposition of the form "$A \in \Delta$" refers to the real numbers, since $\Delta \subset \mathbb{R}$. The real numbers lie *outside* the topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{N})^{op}}$ (resp. the formal language describing the system abstractly).

Now that we have defined $\underline{\mathbb{R}^{\leftrightarrow}}$, we can construct subobjects of $\underline{\Sigma}$ by taking inverse images: let $\underline{\Theta}$ be a subobject of $\underline{\mathbb{R}^{\leftrightarrow}}$, then $\breve{\delta}(\widehat{A})^{-1}(\underline{\Theta})$ is a subobject of $\underline{\Sigma}$.

In this way, we get a topos-internal construction of propositions that do not refer to the real numbers. The 'meaning' of such propositions must be discussed from 'within the topos'.

Pure states and truth objects

In classical theory, a pure state is nothing but a point of state space.

Since the spectral presheaf $\underline{\Sigma}$ has no global elements, we must use another description for (pure) states. The idea is to associate a filter of subobjects of $\underline{\Sigma}$ with a pure quantum state w_ψ. Each subobject in the filter represents a proposition that is totally true in the state w_ψ.

Let ψ be a unit vector in Hilbert space. For each $V \in \mathcal{V}(\mathcal{N})$, we define

$$\begin{aligned} \underline{\mathfrak{w}}^\psi(V) &:= \bigwedge\{S \in P_{cl}(\underline{\Sigma}_V) \mid \langle\psi|\widehat{P}_S|\psi\rangle = 1\} \\ &= \bigwedge\{S \in P_{cl}(\underline{\Sigma}_V) \mid \widehat{P}_S \geq \delta(\widehat{P}_\psi)_V\}. \end{aligned}$$

This is the smallest subobject of $\underline{\Sigma}$ representing a totally true proposition in the state w_ψ. Define the **truth object** $\mathbb{T}^\psi$ corresponding to ψ by

$$\mathbb{T}^\psi := \{\underline{S} \in \mathrm{Sub}(\underline{\Sigma}) \mid \underline{\mathfrak{w}}^\psi \subseteq \underline{S}\}.$$

The subobject classifier in $\mathbf{Sets}^{\mathcal{V}(\mathcal{N})^{op}}$

As is well known, the subobject classifier $\underline{\Omega}$ in a topos of presheaves is the presheaf of **sieves**.

A sieve in a poset like $\mathcal{V}(\mathcal{N})$ is particularly simple: let $V \in \mathcal{V}(\mathcal{N})$. A sieve σ on V is a collection of subalgebras $V' \subseteq V$ such that, whenever $V' \in \sigma$ and $V'' \subset V'$, then $V'' \in \sigma$ (so σ is a downward closed set).

A truth-value is a global element of the presheaf $\underline{\Omega}$.

The global element consisting entirely of maximal sieves is interpreted as 'totally true', the global element consisting of empty sieves as 'totally false'. There are many other global elements of $\underline{\Omega}$, interpreted as truth-values between 'totally false' and 'totally true'.

Truth values from truth objects

We saw that subobjects of $\underline{\Sigma}$ represent propositions about the physical system under consideration, and that states are represented by truth objects.

Let $\underline{S} \in \mathrm{Sub}(\underline{\Sigma})$ be such a subobject, and let $\mathbb{T}^{\psi}$ be a truth object.

Let

$$\nu(\underline{S} \in \mathbb{T}^{\psi})_V := \{V' \subseteq V \mid \underline{S}(V') \in \mathbb{T}^{\psi}(V')\}.$$

One can show that this is a sieve on V. Moreover, for varying V, these sieves form a global element

$$\nu(\underline{S} \in \mathbb{T}^{\psi}) : \underline{1} \to \underline{\Omega}.$$

This is the truth-value of the proposition represented by $\underline{S}$, given by the truth object $\mathbb{T}^{\psi}$.

Summary so far

We have arrived at a **neo-realist** formulation of quantum theory. By using the topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{N})^{op}}$ that is directly motivated from the Kochen-Specker theorem, we can

- identify the state object $\underline{\Sigma}$, the quantity-value object $\underline{\mathbb{R}^{\leftrightarrow}}$, and write physical quantities as arrows $\breve{\delta}(\widehat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}^{\leftrightarrow}}$,
- systematically incorporate contextuality in the description,
- define truth objects corresponding to pure states,
- assign truth-values to all propositions once, without any reference to measurement, observers etc.,
- use a powerful logical structure, which is fixed by the topos,
- overcome the direct dependence on the continuum: $\underline{\mathbb{R}^{\leftrightarrow}}$ is *not* the real-number object in the topos,
- achieve a structural similarity between classical and quantum physics, previously not given.

Formal languages

There is a very elegant way of describing what we are doing: to construct a theory of a physical system S is equivalent to finding a representation in a topos of a certain formal language, $\mathcal{L}(S)$, that is attached to S.

- The language $\mathcal{L}(S)$ will depend on the physical system S, but not on the theory type (classical, quantum, ...).
- The representation will depend on the theory type.
- We allow for a logic that is not Boolean, but still is a deductive system. We choose *intuitionistic* axioms for the language.

For quantum theory, we choose a representation in the topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{N})^{op}}$. *For classical theory, one uses* Sets. *Most importantly, the whole topos scheme allows for major generalisations; in future theories other topoi will play a rôle.*

Related work and outlook

Recently, Landsman, Spitters and Heunen realised that by changing from presheaves to functors over $\mathcal{V}(\mathcal{N})$, and more generally over $\mathcal{V}(\mathcal{A})$ for a C^*-algebra $\mathcal{A}$, one can obtain an internal *abelian* C^*-algebra $\underline{\overline{\mathcal{A}}}$. Results by Banaschewski and Mulvey then show that this internal algebra has an internal Gel'fand spectrum $\overline{\underline{\Sigma}}$, which is a compact, completely regular locale.

Since in the covariant picture the physically important concept of coarse-graining is missing, it remains to be seen whether there is a useful physical interpretation of this (otherwise closely related) scheme.

Spitters and Coquand also developed a certain kind of constructive integration and measure theory recently. Applied to our situation, this will be useful to recover the expectation values and probabilities of ordinary quantum theory.

Open problems and goals

There are many interesting open questions in the topos programme. Some of the things we are working on are:

- description of time evolution
- action of the unitary group, 'geometry' of $\underline{\Sigma}$
- topos formulation of uncertainty relations
- composite systems and entanglement
- internal vs. external formulations
- abstract characterisation of Σ and $\mathcal{R}$
- space-time concepts
- ...

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