

A topological model for studying branching and merging homologies of time flows

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Organization of the talk

1. The combinatorial model category of flows
2. The topological version: the combinatorial model category of multipointed d -spaces
3. The link between the two combinatorial model categories
4. The question

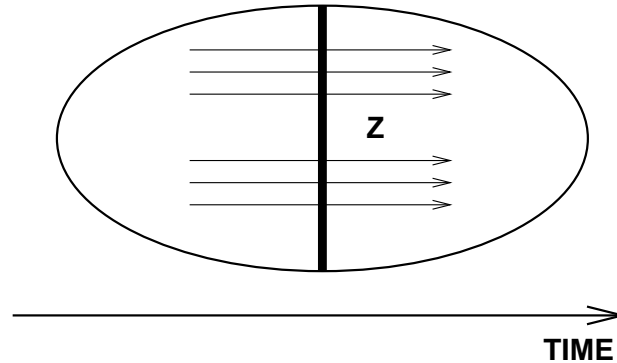
Time flow

- **Flow** X = small category **without identity maps** enriched over Δ -generated spaces (colimit of simplices)
- Set of **objects** X^0 modelling the **states of the concurrent system**
- Space of **morphisms** $\mathbb{P}X$ modelling the **non-constant execution paths of the concurrent system**
- $f : X \rightarrow Y$ **weak S-homotopy equivalence** if $f^0 : X^0 \rightarrow Y^0$ bijection and $\mathbb{P}f : \mathbb{P}X \rightarrow \mathbb{P}Y$ weak homotopy equivalence

Categorical structure of Flow

- Locally presentable
- Tensored and cotensored over Δ -generated spaces with $\mathbf{Flow}(X \otimes K, Y) \cong \mathbf{Top}(K, \mathbf{FLOW}(X, Y)) \cong \mathbf{Flow}(X, Y^K)$
- Combinatorial proper simplicial model category with class of weak equivalences the weak S-homotopy equivalences
- $X \mapsto X^0 \sqcup \mathbb{P}X$ is not topological
- $X \mapsto \mathbb{P}X$ is **functorial**: key point for studying branching and merging homologies

Globe of a topological space



- The **globe** $\text{Glob}(Z)$ of the topological space Z
 - $\text{Glob}(Z)^0 = \{\hat{0}, \hat{1}\}$
 - $\mathbb{P}\text{Glob}(Z) = Z$
 - $s = \hat{0}$
 - $t = \hat{1}$
 - no composable non-constant execution paths
- The **directed segment** $\text{Glob}(\{*\}) = \vec{I}$

Weak S-homotopy model structure

- A set S can be viewed as flow with $S^0 = S$ and $\mathbb{P}S = \emptyset$
- **Generating cofibrations:**

$$I_+^{gl} = \{\text{Glob}(\mathbf{S}^{n-1}) \rightarrow \text{Glob}(\mathbf{D}^n), n \geq 0\} \cup \{C, R\}$$

with $C : \emptyset \rightarrow \{0\}$, $R : \{0, 1\} \rightarrow \{0\}$

- **Generating trivial cofibrations:**

$$J^{gl} = \{\text{Glob}(\mathbf{D}^n \times \{0\}) \rightarrow \text{Glob}(\mathbf{D}^n \times [0, 1]), n \geq 0\}$$

- **Fib = $\{f : X \rightarrow Y \text{ s.t. } \mathbb{P}f \text{ Serre fibration}\}$**
- **Every flow is fibrant**

A topological version of flows ?

- **Advantage** of a topological version of the category of flows: **the full subcategory of colimits of cubes** have nice properties (topological, locally presentable, and also complete, cocomplete, etc...)
- The n -cubes model the concurrent execution of n actions: possibility of getting rid of meaningless geometric shapes
- See [Fajstrup-Rosický's paper](#) for an example of such a category convenient for dealing with some problems in directed algebraic topology

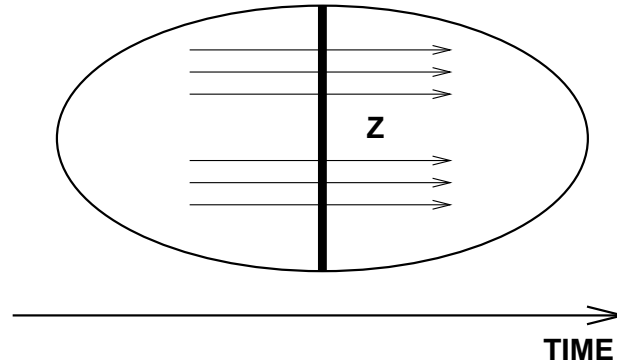
Multipointed d -space

- **Multipointed d -space** $(|X|, X^0, \mathbb{P}^{top} X)$
 - Δ -generated space $|X|$ together with a subset $X^0 \subset |X|$
 - $\mathbb{P}^{top} X$ set of continuous paths closed under strictly increasing reparametrization and composition such that $\gamma : [0, 1] \rightarrow X$ implies $\gamma(0), \gamma(1) \in X^0$
- $f : X \rightarrow Y$ map of multipointed d -spaces
 - A continuous map $|f| : |X| \rightarrow |Y|$ with $f(X^0) \subset Y^0$
 - $\phi \in \mathbb{P}^{top} X$ implies $\mathbb{P}^{top} f(\phi) := |f| \circ \phi \in \mathbb{P}^{top} Y$
- $f : X \rightarrow Y$ **weak S-homotopy equivalence** if $f^0 : X^0 \rightarrow Y^0$ bijection and $\mathbb{P}^{top} f$ weak homotopy equivalence

Categorical structure of MdTop

- Locally presentable
- Tensored and cotensored over Δ -generated spaces with $\text{MdTop}(X \otimes K, Y) \cong \text{Top}(K, \text{MDTOP}(X, Y)) \cong \text{MdTop}(X, Y^K)$
- Combinatorial **right proper** simplicial model category with class of weak equivalences the weak S-homotopy equivalences
- $(X \mapsto \text{underlying set of } |X|)$ is topological
- And of course $X \mapsto \mathbb{P}^{\text{top}} X$ is functorial
- **Left properness is still a conjecture**

Topological globe of a topological space



- The **topological globe** $\text{Glob}^{top}(Z)$ of the topological space Z
- $\text{Glob}^{top}(Z)^0 = \{\hat{0}, \hat{1}\}$
- $|\text{Glob}^{top}(Z)| = \{\hat{0}, \hat{1}\} \sqcup (Z \times [0, 1]) / \left((z, 0) = (z', 0) = \hat{0}, (z, 1) = (z', 1) = \hat{1} \right)$
- $\mathbb{P}^{top} \text{Glob}^{top}(Z)$ closure by strict increasing reparametrization of $\{t \mapsto (z, t), z \in Z\}$

Weak S-homotopy model structure

- Discrete space S viewed as multipointed d -space with $S^0 = S$ and $\mathbb{P}^{top} S = \emptyset$

- **Generating cofibrations:**

$$I_+^{gl,top} = \{ \text{Glob}^{top}(\mathbf{S}^{n-1}) \rightarrow \text{Glob}^{top}(\mathbf{D}^n), n \geq 0 \} \cup \{C, R\}$$

with $C : \emptyset \rightarrow \{0\}$, $R : \{0, 1\} \rightarrow \{0\}$

- **Generating trivial cofibrations:**

$$J^{gl,top} = \{ \text{Glob}^{top}(\mathbf{D}^n \times \{0\}) \rightarrow \text{Glob}^{top}(\mathbf{D}^n \times [0, 1]), n \geq 0 \}$$

- **Fib = $\{ f : X \rightarrow Y \text{ s.t. } \mathbb{P}^{top} f \text{ Serre fibration} \}$**
- **Every multipointed d -space is fibrant**

From multipointed d -spaces to flows (I)

- Multipointed d -space $X = (|X|, X^0, \mathbb{P}^{top} X)$
- Flow $cat(X)$ defined as follows:
- $cat(X)^0 := X^0$
- $\mathbb{P}cat(X)$ defined by

$$\mathbb{P}^{top} X$$

strictly increasing reparametrization with fixed extremities

- $cat : \mathbf{MdTop} \rightarrow \mathbf{Flow}$ well-defined functor
- Example : $cat(\mathbf{Glob}^{top}(Z)) \cong \mathbf{Glob}(Z)$

From multipointed d -spaces to flows (II)

- The composite functor

$$\mathbf{MdTop} \xrightarrow{(-)^{cof}} \mathbf{MdTop} \xrightarrow{cat} \mathbf{Flow}$$

induces an equivalence of categories

$$\mathbf{Ho}(\mathbf{MdTop}) \simeq \mathbf{Ho}(\mathbf{Flow}).$$

- The functor $cat : \mathbf{MdTop} \rightarrow \mathbf{Flow}$ preserves cofibrations, trivial cofibrations and weak S-homotopy equivalences between cofibrant objects.
- The functor $cat : \mathbf{MdTop} \rightarrow \mathbf{Flow}$ is not colimit-preserving

Question

- IF: $F : \mathcal{M} \rightarrow \mathcal{N}$ functor between two combinatorial model categories preserving cofibrations, trivial cofibrations, weak equivalences between cofibrant objects, not colimit-preserving, \mathcal{M} topological over \mathbf{Set} and $F \circ (-)^{cof} : \mathbf{Ho}(\mathcal{M}) \rightarrow \mathbf{Ho}(\mathcal{N})$ equivalence of categories
- THEN: are \mathcal{M} and \mathcal{N} Quillen equivalent ?