Constructing noncommutative topology

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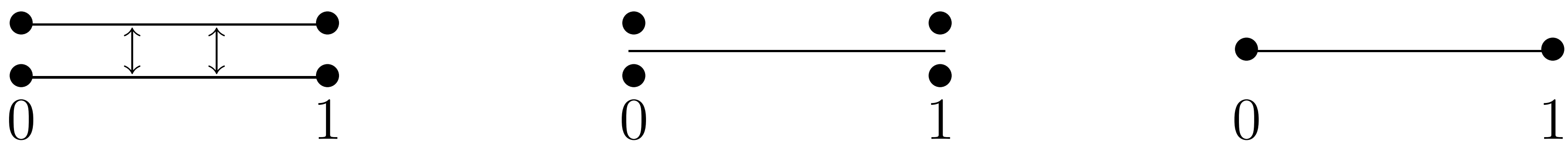
Noncommutative topology

Ex.:

X_1 : X_2 : X_3 :



(Connes 1994)

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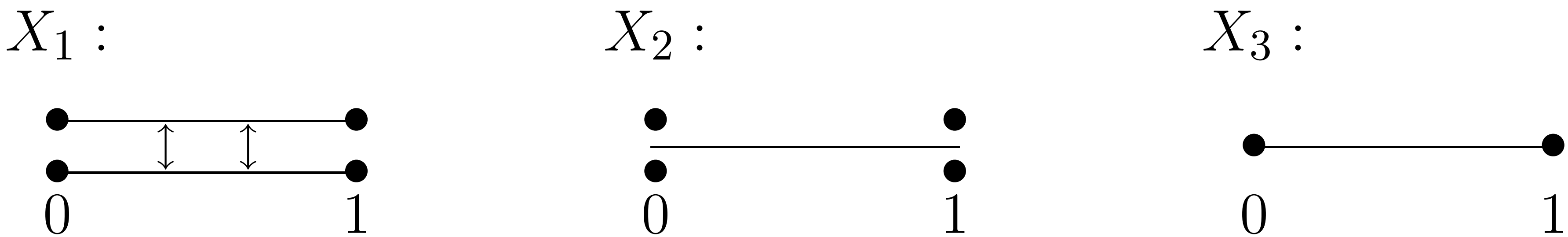

(Connes 1994)

$$A_1 = C(X_2) \cong C(X_3)$$

$$A_2 = \{\text{continuous } f : [0,1] \to M_2(\mathbb{C}) \mid f(0), f(1) \text{ diagonal}\}$$

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Construct a quantale with a given structure of right- and left-sided elements, or a C*-algebra from its spectrum of q-open sets (Akemann 1970, Giles and Kummer 1971).

Generalize the idea of quantale couples (Egger and Kruml 2008, CT 2007).

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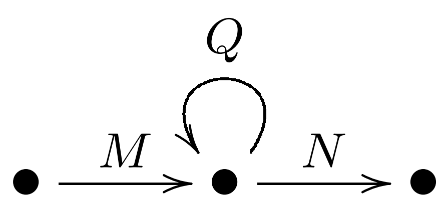
Saying that a graph



is enriched over $\mathcal{S}up$ we mean that Q is a quantale, M is a left, and N a right Q-module.

Ideals of a ring

Let A be a ring.

- T . . . two-sided ideals
- L . . . left ideals
- R . . . right ideals
- Q . . . additive subgroups (or only those which are modules of the center)

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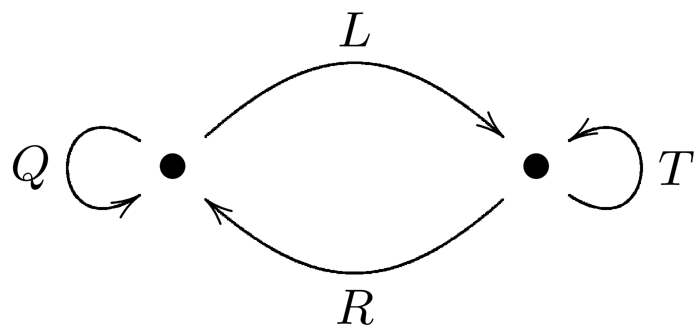
(Van den Bossche 1995)

They are all quantales, some of them also modules, bimorphisms $L \times R \to T, R \times L \to Q$.

Van den Bossche quantaloid



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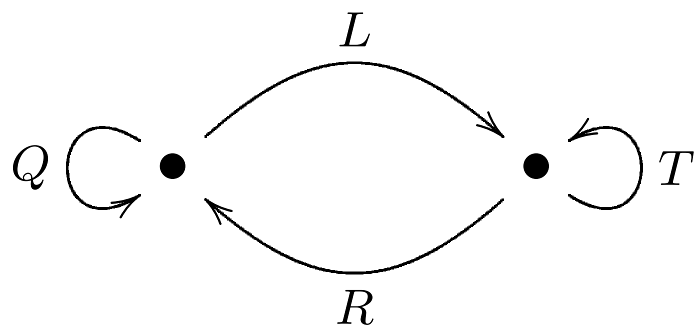

$$T \otimes T \to T \qquad Q \otimes Q \to Q$$

$$R \otimes T \to R \qquad Q \otimes R \to R$$

$$T \otimes L \to L \qquad L \otimes Q \to L$$

$$L \otimes R \to T \qquad R \otimes L \to Q$$

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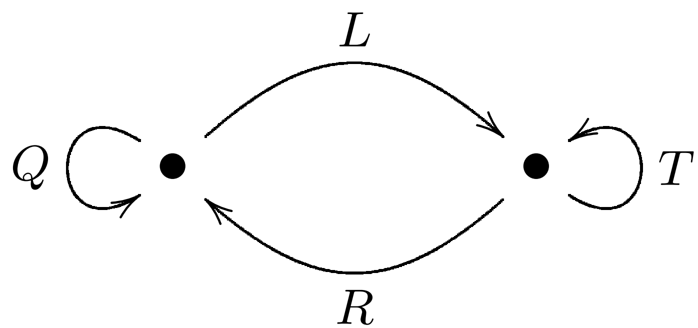

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16 pentagonal coherence axioms
+ some of the 6 triangular axioms for unital objects

Triads and solutions

Given a triad (L, T, R), is there some solution Q?

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$$\begin{array}{ccccc}
Q_1 & \longrightarrow & L \multimap L & \rightrightarrows & (T \otimes L) \multimap L \\
\downarrow & & \downarrow & & \\
R \multimap R & \longrightarrow & (L \otimes R) \multimap T & & \\
\downarrow\downarrow & & & & \\
(R \otimes T) \multimap R & & & &
\end{array}$$

Category of solutions

A quantale Q is a solution of (L, T, R) iff both diagrams commute for Q, i.e. there is a unique factorization $Q_0 \to Q \to Q_1$. The actions are given via

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In particular, $Q_0 \to Q_1$ is a unital couple of quantales.

Special instances

For any sup-lattice S, triad $(S^*, \mathbf{2}, S)$ provides a Girard couple $(S \otimes S^*) \to (S \multimap S)$. More generally, $Q_0 \to Q_1$ is a Girard couple whenever T is a Girard quantale and $L^* \cong R$ as T-modules.

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For suplattices S, T, every Galois connection between them determines a unique map $S \otimes T \to \mathbf{2}$. Then $(S, \mathbf{2}, T)$ form a triad and Q_1 is the Galois quantale (Resende 2004).

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When $L \cong R \cong T$ is unital then also $Q_0 \cong Q_1 \cong T$.

Triads from semiquantales

Assume that L is a right semiquantale, i.e. a sup-lattice with a (non-assoc.) right distributive multiplication, T a commutative quantale, $T \to L$ an open embedding, the images of elements of T are central in L, and the left adjoint $| \ | : L \to T$ satisfies $|xy| = |yx|$ for all $x, y \in L$.
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Ex.: Sasaki projection $x \dot{\wedge} y = (x \vee y') \wedge y$ and the central cover $|\ |$ in a complete orthomodular lattice.

Applications

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- L a frame, T an open subframe . . . supported quantales (e.g. Penrose tilings of Mulvey, Resende 2005).
- L a complete orthomodular lattice, T its centre . . . dynamics of quantum logics.
- Quantum frames (Rosický 1989).
- MV-algebras.

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