Hopf modules and centres of autonomous pseudomonoids

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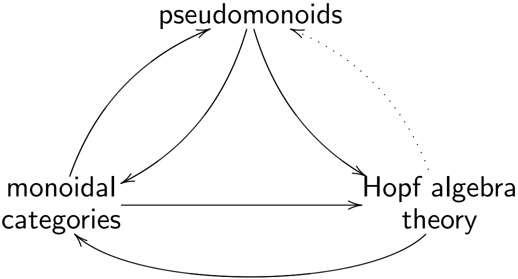
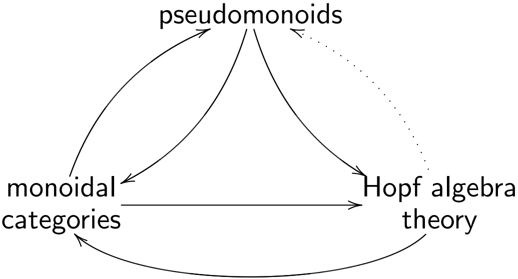


Recall:

- a *Hopf algebra H* is a bialgebra (= algebra + coalgebra + compatibility) with an *antipode* $S : H \to H$.
- If H is a Hopf algebra then $\mathrm{Comod}_f(H)$ is left autonomous monoidal.



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Basic results we generalise:

Thm (Fundamental theorem of Hopf modules)

Let H be a bialgebra. Then

$$\begin{aligned}\mathbf{Comod}(k, H) &\to \mathbf{Comod}(H, H)^{H\otimes -} \quad = \textit{category of Hopf modules}\\ M &\mapsto H \otimes M\end{aligned}$$

is a monoidal equivalence if H is a Hopf algebra

H f.d. Hopf algebra $\Rightarrow$ $D(H)$ *Drinfel'd* or *quantum* double.

Thm

$Z(\mathbf{Comod}(k, H)) \simeq \mathbf{Comod}(H, H)^{(H\otimes - \otimes H)} \simeq \mathbf{Comod}(k, D(H))$

So, $D(H)$ is some sort of *centre* of H.

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Pseudomonoids

Let $\mathcal{M}$ be a monoidal bicategory.
A *pseudomonoid* in $\mathcal{M}$ is an object A with multiplication $p : A \otimes A \to A$, unit $j : I \to A$ and isomorphisms

$$p(p \otimes A) \cong p(A \otimes p) \qquad p(j \otimes A) \cong 1_A \cong p(A \otimes j)$$

satisfying two axioms.

Ex

Pseudomonoid in:

- $\mathcal{V}$-**Cat** are monoidal enriched categories.
- $\mathcal{V}$-**Mod** are promonoidal enriched categories.

$$\mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{co}} \to \mathcal{V}\text{-}\mathbf{Mod}$$

$$(A \xrightarrow{F} B) \mapsto (A \xrightarrow{F_*} B) \qquad F_* \dashv F^*$$

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Ex

- (Normal) pseudomonoid in **Comon** (bicategory of comonoids in **Vect**) = coquasi bialgebra.
- **Comod** = bicategory of bicomodules.

$$\mathbf{Comon} \to \mathbf{Comod}$$

$$(C \xrightarrow{f} D) \mapsto (C \xrightarrow{f_*} D) \qquad f_* \dashv f^*$$

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(Coquasi) Hopf algebras live in **Comod**.

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Dualizations

Assume $\mathcal{M}$ is *right autonomous*.

Def (Day-McCrudden-Street)

A *left dualization* for the pseudomonoid A is (d, α, β) where $d : A^\circ \to A$ and

$$
\begin{array}{ccc}
A^\circ \otimes A & \xrightarrow{d \otimes A} & A \otimes A \\
{\scriptstyle n}\uparrow & \Downarrow\alpha & \downarrow{\scriptstyle p} \\
I & \xrightarrow[j]{} & A
\end{array}
\qquad
\begin{array}{ccc}
I & \xrightarrow{j} & A \\
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Dualizations: examples

Ex (Duals)

A monoidal category. Let $D : A^{op} \to A$ be a functor. D_* is a left dualization in $\mathcal{V}$-**Mod** *iff* each $x \in A$ has a left dual x^* and $D \cong (-)^*$.

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H coquasi bialgebra, $s : H^{cop} \to H$ comonoid morphism. s_* is a left dualization for H in **Comod** *iff* s is an antipode for H.

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Hopf modules

Fix $A \in \mathcal{M}$ a map pseudomonoid.
$\mathcal{M}(A, A)$ has a *convolution* monoidal structure:

$$f * g = p(f \otimes g)p^* \qquad \text{unit } jj^*$$

Obs
$1_A \in \mathcal{M}(A, A)$ is a monoid with

$$\text{multiplication: } 1 * 1 = pp^* \to 1 \qquad \text{unit: } jj^* \to 1$$

Define
$$\mathcal{M}(A \otimes X, A) \xrightarrow{\theta_X} \mathcal{M}(A \otimes X, A)$$
$$f \mapsto (A \otimes X \xrightarrow{p^* \otimes A} A^{\otimes 2} \otimes X \xrightarrow{A \otimes f} A^{\otimes 2} \xrightarrow{p} A)$$

This is an "action of 1_A" $\Rightarrow \theta_X$ is a monad.

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Analogue of Hopf modules: $\mathcal{M}(A \otimes -, A)^\theta \in [\mathcal{M}^{op}, \mathbf{Cat}]$.

Def

The theorem of Hopf modules holds for A when

$$\lambda : \mathcal{M}(-, A) \xrightarrow{\mathcal{M}(j^* \otimes -, A)} \mathcal{M}(A \otimes -, A) \to \mathcal{M}(A \otimes -, A)^\theta$$
$$(X \xrightarrow{f} A) \mapsto (A \otimes X \xrightarrow{A \otimes f} A^{\otimes 2} \xrightarrow{p} A)$$

is a (monoidal) equivalence.

Ex

When $\mathcal{M} = \mathbf{Comod}$, $X = k$ and A is a bialgebra: 1_A is A and $\mathcal{M}(A, A)^{\theta_k} = A$-modules in $\mathbf{Comod}(A, A) =$ Hopf modules.
λ_k equivalence = classical theorem of Hopf modules.

Thm (Fundamental theorem of Hopf modules)

Let A be a map pseudomonoid. Then, A is left autonomous iff *the theorm of Hopf modules holds for A.*

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$t : A^e \to A^e$ monad

Prop

If A is left autonomous, then t has en E-M construction given by

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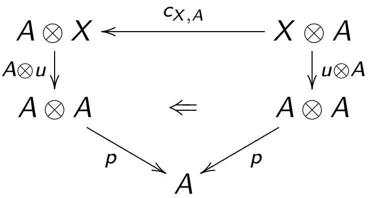
Lax centres

Now assume $\mathcal{M}$ is a braided Gray monoid, with braiding c.

Def (Street)

Let A be a pseudomonoid. A *lax centre piece* is $u : X \to A$ with



satisfying axioms.

We have a category $CP_\ell(X, A)$ and a 2-functor $CP_\ell(-, A) : \mathcal{M}^{op} \to \mathbf{Cat}$.

- A *lax centre* of A is a birepresentaion $Z_\ell A \to A$ of $CP_\ell(-, A)$.
- Requiring invertible 2-cells, we obtain a *centre* $ZA \to A$.

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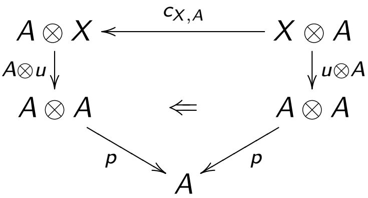

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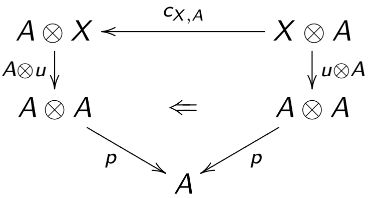

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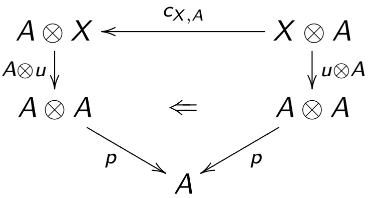

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Lax centres of automomous pseudomonoids

Recall: $\theta : \mathcal{M}(A \otimes -, A) \to \mathcal{M}(A \otimes -, A)$ "action of 1_A on the left".

Consider $\sigma : \mathcal{M}(A \otimes -, A) \to \mathcal{M}(A \otimes -, A)$ "action of 1_A on the right" (use the braiding)

σ is represented by a monad $s : A^e \to A^e$.

Prop

- *t and s are opmonoidal monads ($A^e = A^\circ \otimes A$ has endo-hom monoidal structure).*
- *There is a distributive law $ts \cong st$ in* $\mathbf{Opmon}(\mathcal{M})$.

When A is left autonomous, denote by $\hat{s} : A \to A$ the lifting of s to the E-M object of t.

$$A^{\hat{s}} \simeq (A^e)^{st} = \text{ "two-sided Hopf modules"}$$

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Thm

A left autonomous map pseudomonoid.

- $Z_\ell A \longrightarrow A$ *exits* iff $A^{\hat{s}} \longrightarrow A$ *exits; in this case they coincide.*
- *If A is left and right autonomous* $\Rightarrow$ *lax centre* $=$ *centre*

Consequence (in $\mathcal{V}$-modules)

Any left autonomous map pseudomonoid in $\mathcal{V}$-**Mod** (*e.g.*, a monoidal category) has a lax centre.

Consequence (in comodules)

H f.d. (coquasi) Hopf algebra $\Rightarrow$ $Z_\ell A$ exists in **Comod** and $Z_\ell A \simeq ZA$.

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Consequence (in comodules)

H f.d. (coquasi) Hopf algebra $\Rightarrow$ $Z_\ell A$ exists in **Comod** and $Z_\ell A \simeq ZA$.

Thm

A left autonomous map pseudomonoid.

- $Z_\ell A \to A$ *exits* iff $A^{\hat{s}} \to A$ *exits; in this case they coincide.*
- *If* A *is left and right autonomous* $\Rightarrow$ *lax centre* $=$ *centre*

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Using

$$\mathbf{Comod}_f(k, ZA) \simeq CP(k, A) = Z(\mathbf{Comod}_f(k, A)) \simeq \mathbf{Comod}_f(k, D(A))$$

Cor
$Z(A)$ is isomorphic to $D(A)$ as a coalgebra
equivalent to $D(A)$ as a coquasi bialgebra.

So $D(A)$ *is* a centre construction for A.

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