Weak inverses for Strict n-categories

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* based on a joint work w. S. Kasangian + E.M. Vitale

Notations : $\boxed{n\text{-}\mathbf{Cat}}$

$\mathbb{C}$, $\mathbb{D}$ .... : (small and) strict n-categories

$F : \mathbb{C} \longrightarrow \mathbb{D}$ ... : strict n-functors = morphisms

$\alpha : F \Rightarrow G : \mathbb{C} \to \mathbb{D}$ ... : weak n-transform. = 2-morphisms

Given $\mathbb{C}$, cells :

0-cells : (objects) $c_0, c_0', c_0'' \ldots$

1-cells : (arrows) $c_0 \xrightarrow{c_1} c_0'$ $c_1', \ldots$

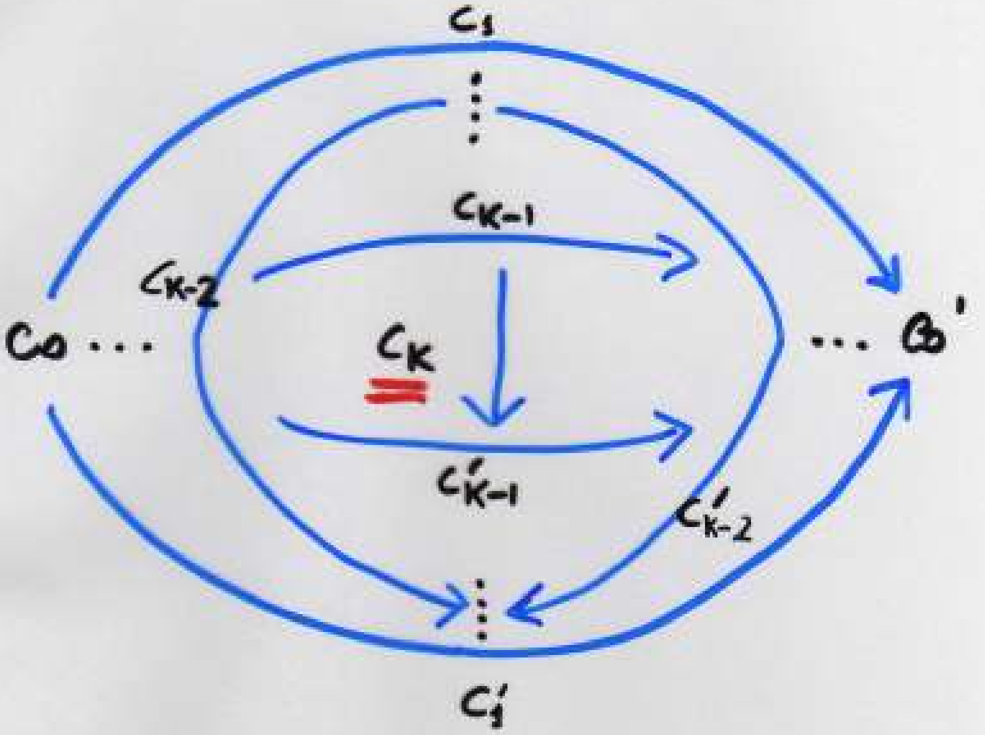
⋮

k-cells : $c_k : c_{k-1} \longrightarrow c_{k-1}'$

$\left(c_{k-1}, c_{k-1}' : c_{k-2} \longrightarrow c_{k-2}' : \ldots\ c_1 \to c_1' : c_0 \to c_0' \right)$

comp.

Question



What does it mean for c_k being *invertible* ?

Answer

There exists c_k^{-1} s.t.

$$\boxed{c_k \circ^{k-1} c_k^{-1} = 1_{c_{k-1}}} \quad \boxed{c_k^{-1} \circ^{k-1} c_k = 1_{c'_{k-1}}}$$

... end of the Talk !

If we mean weakly invertible then there are several possible answers

I) [R. Street, 1985] (in a ω-category) defines what means for two objects c_0, c_0' being n-equivalent:

$\exists$ $c_1 : c_0 \longrightarrow c_0'$, $c_1^* : c_0' \longrightarrow c_0$ s.t.

(i) $c_1 \circ^\circ c_1^*$ and 1_{c_0} $(n-1)$-equiv. in $\mathcal{C}(c_0, c_0)$

(ii) $c_1^* \circ^\circ c_1$ and $1_{c_0'}$ $(n-1)$-equiv. in $\mathcal{C}(c_0', c_0')$

Of course $\boxed{\text{0-equiv.} = \text{equal.}}$

We will say:

•) equivalent in a n-cat = n-equiv.

•) c_1 is S-invertible if it establishes an equivalence $c_0 \xrightarrow{\sim} c_0'$

→ s-inverses in low dimensions

$\boxed{n=1}$ (categories)

$c_0 \xrightarrow{c_1} c_0'$ **s-invertible** if

$\exists\, c_1^* : c_0' \to c_0$ s.t. $1_{c_0} = c_1 c_1^*$

$$c_1^* c_1 = 1_{c_0'}$$

$\rightsquigarrow$ iso

$\boxed{n=2}$ (2-categories)

$c_0 \xrightarrow{c_1} c_0'$ **s-invertible** if

$\exists\, c_1^* : c_0' \to c_0$, $\exists\, \eta : 1_{c_0} \to c_1 c_1^*$, $\varepsilon : c_1^* c_1 \to 1_{c_0'}$

and $\exists\, \eta^* : c_1^* c_1 \to 1_{c_0}$, $\varepsilon^* : 1_{c_0'} \to c_1^* c_1$

s.t. $\eta\eta^* = 1_{1_{c_0}}$ $\qquad \varepsilon\varepsilon^* = 1_{c_1^* c_1}$

$\eta^*\eta = 1_{c_1 c_1^*}$ $\qquad \varepsilon^*\varepsilon = 1_{1_{c_0'}}$

$\rightsquigarrow$ equiv

II) [M.M. Kapranov, V.A. Voevodsky, 1991] give a definition of ω-groupoid as n-category with weak inverses.

SLOGAN

All equations

$$\boxed{a \cdot x \cong b}$$

admit a (weak) solution.

(--- and $\boxed{x \cdot a \cong b}$ also)

THEN

a solution of $a \cdot x = 1$

gives a possible inverse of a.

<u>DEFINITION</u> (KV) $\mathbb{C}$ is a n-groupoid if $\forall\ i < k \leq n$

$\boxed{GR'_{i,k}}$ $\forall\ a_{i+1}, b_k, u_{k-1}, v_{k-1}$, (with $i < k-1$) below

$\exists\ x_k : u_{k-1} \longrightarrow v_{k-1}$, $\varphi_{k+1} : \boxed{x_k \circ^i a_{i+1} \longrightarrow b_k}$

$$\begin{tikzcd} \bullet \arrow[r, bend left, "u_{k-1}"] \arrow[r, bend right, "v_{k-1}"'] \arrow[r, phantom, "\Downarrow x_k"] & \overset{i}{\bullet} \arrow[r, "a_{i+1}"] & \bullet \end{tikzcd} \overset{\varphi_{k+1}}{\Longrightarrow} \begin{tikzcd} \bullet \arrow[r, bend left, "u_{k-1} \circ^i a_{i+1}"] \arrow[r, bend right, "v_{k-1} \circ^i a_{i+1}"'] \arrow[r, phantom, "\Downarrow b_k"] & \bullet \end{tikzcd}$$

$\boxed{GR'_{k-1,k}}$ $\forall\ a_k, b_k$ $\exists\ x_k$, φ_{k+1}

$$\begin{tikzcd} & \overset{k-1}{\bullet} \arrow[dr, "a_k"] & \\ \bullet \arrow[ur, "x_k"] \arrow[rr, bend right, "b_k"'] & \Downarrow \varphi_{k+1} & \bullet \end{tikzcd}$$

$\boxed{GR''_{i,k}}$ (with $i < k-1$)

$$\begin{tikzcd} \bullet \arrow[r, "a_{i+1}"] & \bullet \arrow[r, bend left, "u_{k-1}"] \arrow[r, bend right, "v_{k-1}"'] \arrow[r, phantom, "\Downarrow x_k"] & \bullet \end{tikzcd} \overset{\varphi_{k+1}}{\Longrightarrow} \begin{tikzcd} \bullet \arrow[r, bend left, "a_{i+1} \circ^i u_{k-1}"] \arrow[r, bend right, "a_{i+1} \circ^i v_{k-1}"'] \arrow[r, phantom, "\Downarrow b_k"] & \bullet \end{tikzcd}$$

$\boxed{GR''_{k-1,k}}$

$$\begin{tikzcd} & \bullet \arrow[dr, "x_k"] & \\ \bullet \arrow[ur, "a_k"] \arrow[rr, bend right, "b_k"'] & \Downarrow \varphi_{k+1} & \bullet \end{tikzcd}$$

[K.V.]-n-groupoid $\Rightarrow$ [S]-n-groupoid

i.e. $\boxed{GR'_{i,k}}, \boxed{GR'_{k-1,k}}, \boxed{GR''_{i,k}}, \boxed{GR''_{k-1,k}} \Rightarrow \boxed{GR'_{k-1,k}}, \boxed{GR''_{k-1,k}}$

[K.V.] claim $\cancel{\Leftarrow}$

Their argument is:

[KV]-DEF $\Rightarrow$ "coherent" system of q.-inv.

and « there is no way to construct such a system from Street's def$^{\underline{n}}$. »

III) C. S. Q. I. (description in low dim)

$\boxed{n=1}$ $\quad c_1 : c_0 \longrightarrow c_0'$, $\exists\, c_1^* : c_0' \to c_0$ s.t.

$$c_1 c_1^* = 1_{c_0} \quad , \quad c_1^* c_1 = 1_{c_0'}$$

$\boxed{n=2}$ $\quad c_1 : c_0 \longrightarrow c_0'$, $\exists\ c_1^* : c_0' \longrightarrow c_0$

$$\rho_2 : c_1 c_1^* \longrightarrow 1_{c_0} \quad , \quad \lambda_2 : c_1^* c_1 \longrightarrow 1_{c_0'}$$

s.t.

$$\begin{tikzcd} c_1 c_1^* c_1 \arrow[r, bend left, "\rho_2 c_1"] \arrow[r, bend right, "c_1 \lambda_2"'] \arrow[r, phantom, "\Uparrow"] & c_1 \end{tikzcd} \qquad \begin{tikzcd} c_1^* c_1 c_1^* \arrow[r, bend left, "\lambda_2 c_1^*"] \arrow[r, bend right, "c_1^* \rho_2"'] \arrow[r, phantom, "\Downarrow"] & c_1^* \end{tikzcd}$$

$(+\ \rho_2^*\ \lambda_2^*)$ $\qquad \Uparrow \rho_3 \qquad\qquad \Downarrow \lambda_3$

...

<u>PROP</u> $\quad [KV] \Rightarrow$ C.S.Q.I.

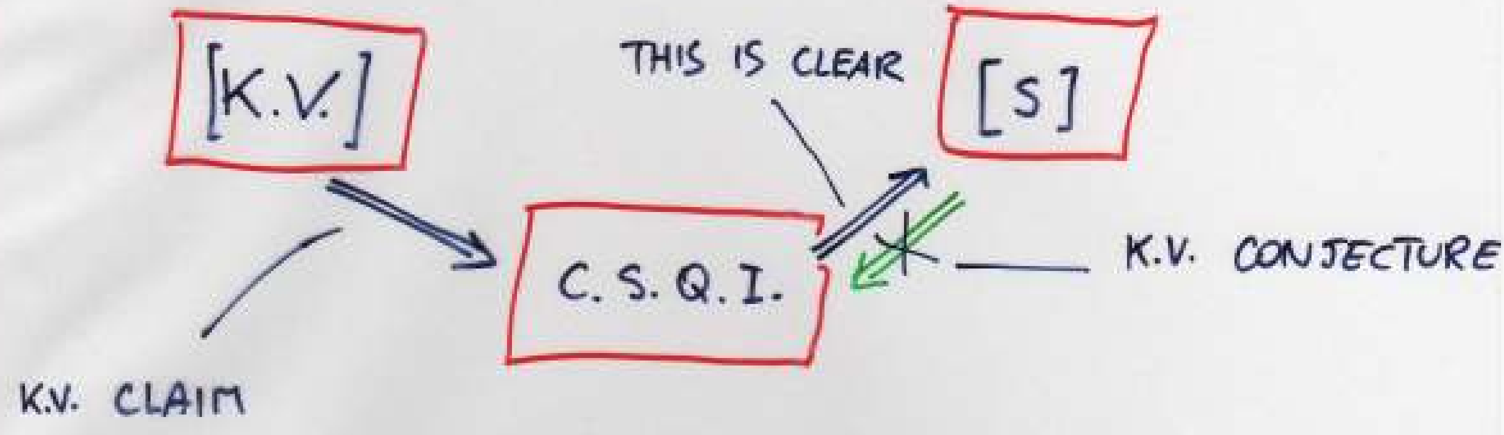
idea :

(c_1^*, ρ_2) solution of $c_1 \circ x \cong 1_{c_0}$

(λ_2, ρ_3) solution of $c_1 \circ x \cong \rho_2 \circ c_1$

....

The situation so far :



Need another notion of "inverse"

IV (variation on) [C. SIMPSON, 1998]

→ relies on a notion of equivalence of n-categories :

DEFINITION $F : \mathbb{C} \longrightarrow \mathbb{D}$ n-equivalence

$\boxed{n=0}$ isomorphism of sets

$\boxed{n>0}$ $\boxed{(E1)}$ $\forall\, c_0, c_0' \in \mathbb{C}_0$

$F_1^{c_0,c_0'} : \mathbb{C}_1(c_0,c_0') \longrightarrow \mathbb{D}_1(Fc_0, Fc_0')$ $(n-1)$-equiv.

$\boxed{(E2)}$ $\forall\, d_0 \in \mathbb{D}_0 \ \exists\, c_0 \in \mathbb{C}_0$, $d_1 : d_0 \longrightarrow Fc_0$

s.t. $d_1 \circ^{\circ} -$, $- \circ^{\circ} d_1$ $(n-1)$-equiv.

small and...

$\sqrt{\text{Strict n-Categories}}$

inductive definition:

$$\mathbb{C} = \langle \mathbb{C}_0, \{\mathbb{C}_1(c_0, c_0')\}_{c_0, c_0' \in \mathbb{C}_0} \rangle$$

Set of objects

(n-1)-categories

+ compositions and identities

$\forall c_0, c_0', c_0'' \in \mathbb{C}_0$, $-\circ-: \mathbb{C}_1(c_0, c_0') \times \mathbb{C}_1(c_0', c_0'') \longrightarrow \mathbb{C}_1(c_0, c_0'')$

$\forall c_0 \in \mathbb{C}_0$, $U(c_0): \mathbb{1} \longrightarrow \mathbb{C}_1(c_0, c_0)$

+ axioms:
(n-1)- functors satisfying
associativity, **left/right** unit
axioms.

Strict n- functors ...

$\boxed{\text{DEFINITION}}$ $c_1 : c_0 \longrightarrow c_0'$

is ω-invertible if $\forall\, c_0'' \in \mathcal{C}$

$\boxed{(i)}$ $c_1 \circ - \; : \; \mathcal{C}_1(c_0', c_0'') \longrightarrow \mathcal{C}_1(c_0, c_0'')$

$\boxed{(ii)}$ $- \circ c_1 \; : \; \mathcal{C}_1(c_0'', c_0) \longrightarrow \mathcal{C}_1(c_0'', c_0')$

are equivalences of $(n-1)$-categories

$\left(? \quad (i) \Longleftrightarrow (ii) \quad ? \right)$

$\boxed{\text{DEFINITION}}$

$\mathcal{C}$ is a n-groupoid: $\boxed{n=0}$ $\mathcal{C}$ is a set

$\boxed{n>0}$

- $\forall\, c_0, c_0'$ $\mathcal{C}_1(c_0, c_0')$ is a $(n-1)$-groupoid
- $\forall\, c_1 : c_0 \to c_0'$ is ω-invertible.

THEOREM (Simpson)

$\mathbb{C}$ (K.V.)- n-groupoid $\iff$ (Simpson)- n-groupoid

in particular,

$c_1 : c_0 \to c_0'$ [KV]-invertible $\iff$ ω-invertible

$\boxed{n=1}$ c_1 ω-invert. $\overset{\text{def}}{\iff}$ $\begin{cases} c_1 \circ - \\ - \circ c_1 \end{cases}$ isos of sets

$\iff$ c_1 is an iso of $\mathbb{C}$

$\boxed{n=2}$ c_1 ω-invert. $\overset{\text{def}}{\iff}$ $\begin{cases} c_1 \circ - \\ - \circ c_1 \end{cases}$ equiv. of cats.

i.e.

(i) $\forall\, x \in \mathbb{C}_0$ $\quad c_1 \circ - : \mathbb{C}(c_0', x) \longrightarrow \mathbb{C}(c_0, x)$

is an equiv. of cats when

•) ess. surjective :

$$\begin{tikzcd} & c_0' \arrow[dr, "\exists"] \arrow[d, phantom, "\Downarrow\exists"] & \\ c_0 \arrow[ur, "c_1"] \arrow[rr, "\forall"'] & {} & x \end{tikzcd}$$

•) full

$$\begin{tikzcd} c_0 \arrow[r, "c_1"] & c_0' \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\exists"] & x \end{tikzcd} \quad \overset{\exists}{\underset{\text{"="}}{\Longrightarrow}} \quad \begin{tikzcd} c_0 \arrow[r, bend left, "u \circ c_1"] \arrow[r, bend right, "v \circ c_1"'] \arrow[r, phantom, "\Downarrow\forall"] & x \end{tikzcd}$$

•) faithful

$$\begin{tikzcd} c_0 \arrow[r, "c_1"] & c_0' \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "a \Downarrow = \Downarrow b"] & x \end{tikzcd}$$

$\forall$ "$=_1$" $: c_1 a \longrightarrow c_1 b$

$\exists$ "$=_2$" $a \to b$, "$=_1$" $\Rightarrow$ "$=$".

+ duals (ii)

$$
\begin{tikzcd}
 & {[Si]} \arrow[dr, Leftrightarrow, "\text{THM}"] & \\
{[K.V.]} \arrow[ur, Leftrightarrow, "\text{Simpson's THM}"] \arrow[dr, Rightarrow] & & {[S]} \\
 & {[C.S.Q.I.]} \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

THEOREM (Kasangian, M, Vitale)

A 1-cell $c_1 : c_0 \longrightarrow c_0'$ of a n-cat $\mathbb{C}$ is $\boxed{\text{w-invertible}}$ iff $\boxed{\text{s-invertible}}$

COROLLARY

The corresponding notions of n-groupoid coincide.

The proof is not straightforward
(... still looking for a better one!)

Key-idea:

$$\boxed{\text{equivalence of } n\text{-categories}} = \boxed{n+1\text{-levels of surjectivity}}$$

for istance, $n = 1$ (1-cat = cat)
a functor is an equivalence if

ess. surjective = "surjective" on objects

full = "surjective" on arrows

faithful = "surjectiv" on equations

LOCALIZE

Main tool : $\boxed{\text{inductive def}}$

Consider n-Cat as a sesquicategory

obj : n-categories
mor : n-functors
2-mor : weak n-transformations

Defn

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbb{D} \end{tikzcd} \qquad \alpha = \langle \alpha_0, \alpha_1^{c_0,c_0'} \rangle_{c_0 c_0' \in \mathbb{C}_0}$$

$$\alpha_0 : \underset{\in \mathbb{C}_0}{c_0} \longmapsto F c_0 \xrightarrow{\alpha_0(c_0)} G c_0 \in \mathbb{D}_1$$

$\forall c_0 c_0' \in \mathbb{C}_0,$

$\alpha_1^{c_0,c_0'}$

$$\begin{tikzcd}[column sep=small] & \mathbb{C}_1(c_0,c_0') \arrow[dl, "F_1^{c_0c_0'}"'] \arrow[dr, "G_1^{c_0c_0'}"] & \\ \mathbb{D}_1(Fc_0,Fc_0') \arrow[dr, "-\circ^{\circ}\alpha_{c_0'}"'] & \overset{\alpha_1^{c_0,c_0'}}{\Longleftarrow} & \mathbb{D}_1(Gc_0,Gc_0') \arrow[dl, "\alpha_{c_0}\circ^{\circ}-"] \\ & \mathbb{D}_1(Fc_0,Gc_0') & \end{tikzcd}$$

+ axioms

(idea of the) proof:

We prove that $\boxed{c, \circ^{\bullet} -}$

is an equivalence of $(n-1)$-categories

by double induction over

- the dimension
- the "level" of localization

Case Study : direct

Proof for $\boxed{n=2}$

Let c_1 be S-invertible, i.e.

$$\exists\, c_1^*\ ,\ \eta : 1_{c_0} \longrightarrow c_1 c_1^*\ ,\ \eta^* \text{ s.t. } \eta\eta^* = 1 \quad \eta^*\eta = 1$$

$$\varepsilon : c_1^* c_1 \longrightarrow 1_{c_0'}\ ,\ \varepsilon^* \text{ s.t. } \varepsilon^*\varepsilon = 1 \quad \varepsilon\varepsilon^* = 1$$

we want to show $c_1 \circ -\ ,\ - \circ c_1$ equiv.

(1) fix c_0'' and consider

$$c_1 \circ - : \mathbb{C}(c_0', c_0'') \longrightarrow \mathbb{C}(c_0, c_0'')$$

- <u>ess. surjective</u>

$\forall\, b : c_0 \longrightarrow c_0''$, $\exists\, a : c_0' \longrightarrow c_0''$ s.t. $c_1 a = b$?

let $a = c_1^* b$

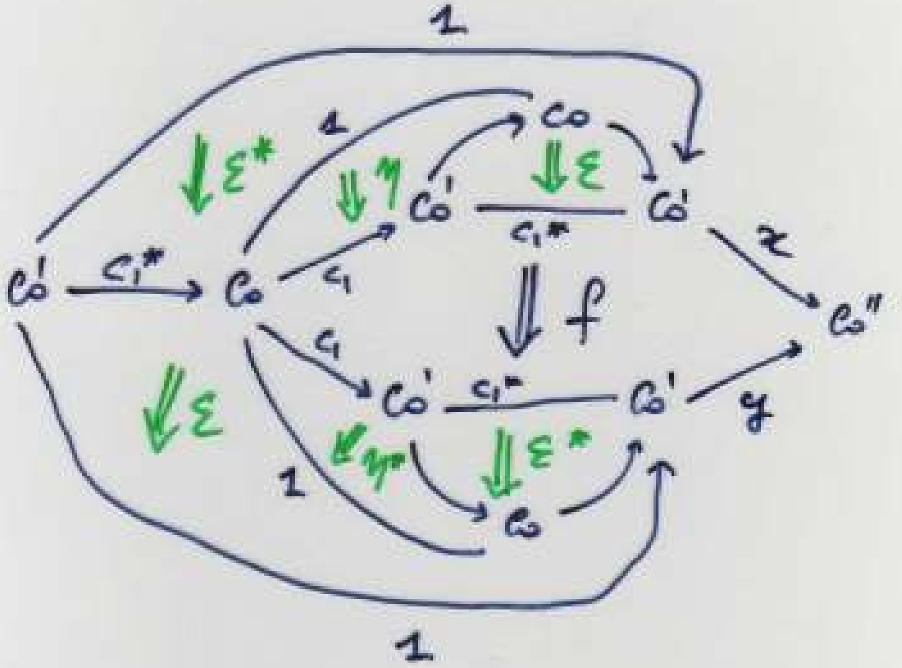
then

$$\begin{tikzcd}
 & c_0' \arrow[r, "c_1^*"] \arrow[rr, bend left, "a"] \arrow[d, phantom, "\Downarrow \eta^*"] & c_0 \arrow[dr, "b"] \arrow[d, phantom, "\cong"] & \\
c_0 \arrow[ur, "c_1"] \arrow[urr] \arrow[rrr, "b"'] & & {} & c_0''
\end{tikzcd}$$

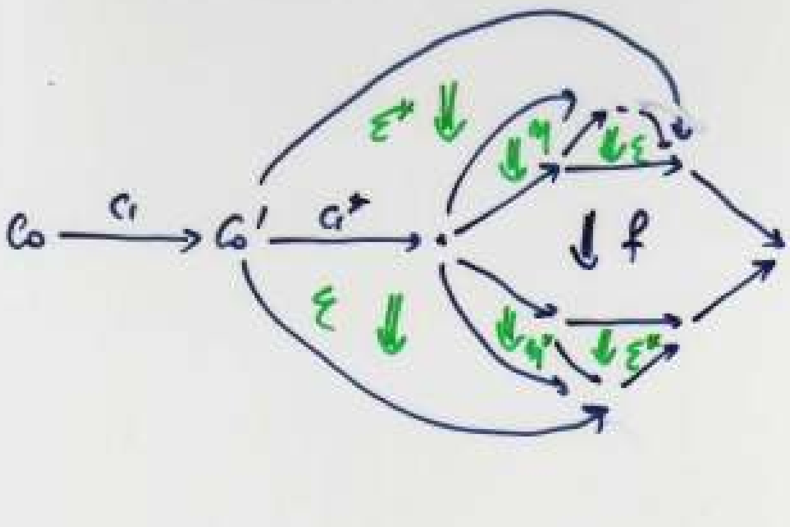
• <u>full</u> : let $x, y : c_0' \longrightarrow c_0''$;

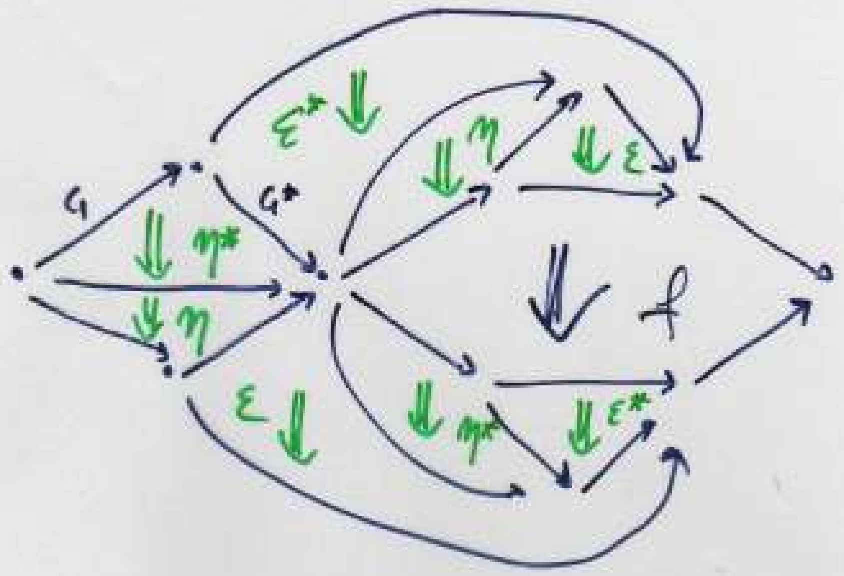
$\forall$ $\begin{tikzcd} c_0 \arrow[r, bend left, "c_1 x"] \arrow[r, bend right, "c_1 y"'] \arrow[r, phantom, "\Downarrow f"] & c_0'' \end{tikzcd}$ $\exists$ $\begin{tikzcd} c_0' \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow g"] & c_0'' \end{tikzcd}$ s.t. $c_1 g = f$?

Let g be the following pasting:



compute $c_1 \circ g$:

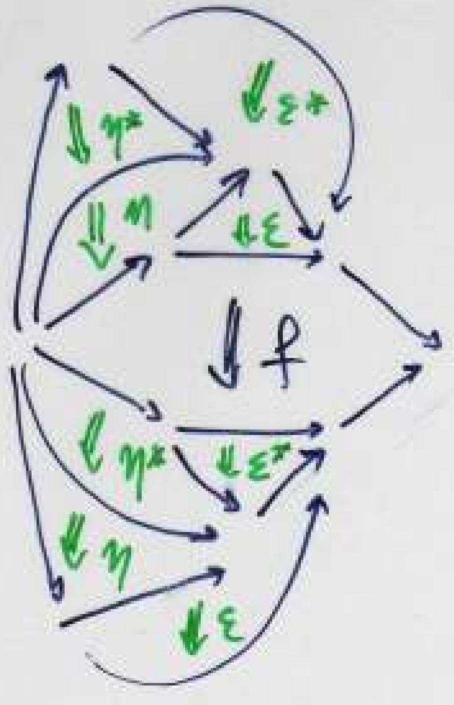




But

$$
\begin{tikzcd}[column sep=small]
 & C_0' \arrow[dr, "C_1^*"] & & & \\
C_0 \arrow[ur, "C_1"] \arrow[rr, no head] \arrow[rr, phantom, shift left=4, "\Downarrow\eta^*"] & & C_0 \arrow[rr, no head] \arrow[rr, phantom, shift right=4, "\Downarrow\eta"] \arrow[dr, "C_1"'] & & C_0 \\
 & & & C_0' \arrow[ur, "C_1^*"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[column sep=small]
 & C_0' \arrow[dr] & \\
C_0 \arrow[ur, "C_1"] \arrow[rr, no head] \arrow[rr, phantom, shift left=4, "\Downarrow\eta^*"] \arrow[rr, phantom, shift right=4, "\Downarrow\eta"] \arrow[dr] & & C_0 \\
 & C_0' \arrow[ur] &
\end{tikzcd}
$$

hence

 $= f$

• faithful: Let $x, y : c_0' \longrightarrow c_0''$

$$\forall \quad \begin{tikzcd} c_0' \arrow[r, bend left=50, "x"] \arrow[r, bend right=50, "y"'] \arrow[r, phantom, "f\Downarrow\ \Downarrow g"] & c_0'' \end{tikzcd} \qquad c_1 f = c_1 g \implies f = g \, ?$$

consider

$$f = \begin{tikzcd} c_0' \arrow[r, "c_1^*"] \arrow[rr, bend left=60, "1"] \arrow[rr, bend right=60, "1"'] & c_0 \arrow[r, "c_1"] \arrow[u, phantom, "\Downarrow \varepsilon"] \arrow[d, phantom, "\Downarrow \varepsilon^*"] & c_0' \arrow[r, bend left=50, "x"] \arrow[r, bend right=50, "y"'] \arrow[r, phantom, "\Downarrow f"] & c_0'' \end{tikzcd} =$$

$$= \begin{array}{c} \begin{tikzcd} \cdot \arrow[r] \arrow[rr, bend left=60] & \cdot \arrow[r] \arrow[u, phantom, "\Downarrow \varepsilon"] & \cdot \arrow[r] & \cdot \end{tikzcd} \\ \begin{tikzcd} \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow f"] & \cdot \end{tikzcd} \\ \begin{tikzcd} \cdot \arrow[r] \arrow[rr, bend right=60] & \cdot \arrow[r] \arrow[d, phantom, "\Downarrow \varepsilon^*"] & \cdot \arrow[r] & \cdot \end{tikzcd} \end{array} =$$

$$= \begin{array}{c} \begin{tikzcd} \cdot \arrow[r] \arrow[rr, bend left=60] & \cdot \arrow[r] \arrow[u, phantom, "\Downarrow \varepsilon"] & \cdot \arrow[r] & \cdot \end{tikzcd} \\ \begin{tikzcd} \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow g"] & \cdot \end{tikzcd} \\ \begin{tikzcd} \cdot \arrow[r] \arrow[rr, bend right=60] & \cdot \arrow[r] \arrow[d, phantom, "\Downarrow \varepsilon^*"] & \cdot \arrow[r] & \cdot \end{tikzcd} \end{array} = g$$