Lax and Pseudo Presheaves and Exponentiability

Susan Niefield, Union College

Categories whose objects are sets:

morphisms

Set	$f\colon X \to Y$	functions	
Rel	$R\colon X \mathrel{-\!\!\!+\!\!\!\longrightarrow} Y$	relations $R \subseteq X \times Y$	2−*category*
Span	$A\colon X \mathrel{-\!\!\!\circ\!\!\!\longrightarrow} Y$	spans $\begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ X & & Y \end{array}$	

$$B \circ A = A \times_Y B$$

bicategory

IDEA

Consider the topos $\mathbf{Set}^{B^{op}}$ of $\mathbf{Set}$-valued presheaves on B

Replace $\mathbf{Set}$ by a bicategory $\mathcal{S}$ and consider lax or pseudo-functors

$$B^{op} \to \mathcal{S}$$

When $\mathcal{S}$ is $\mathbf{Rel}$ or $\mathbf{Span}$, and morphisms are map-valued op-lax transformation, the result is a category

When is it a topos?

Lax(B^{op}, $\mathbf{Span}$)

OBJECTS: lax functors $X\colon B^{op} \to \mathbf{Span}$

- a set X_b, for every object b
- a span $X_\beta\colon X_{b'} \multimap X_b$, for every $\beta\colon b \to b'$

with $\eta\colon X_b \to X_{id_b}$ and $\mu\colon X_{\beta'} \times_{X_{b'}} X_\beta \to X_{\beta'\beta}$ s.t. ...

MORPHSIMS: families $f_b\colon X_b \to Y_b$ with $\begin{array}{ccc} X_{b'} & \xrightarrow{f_{b'}} & Y_{b'} \\ {\scriptstyle X_\beta}\downarrow\circ & \xrightarrow[f_\beta]{} & \circ\downarrow{\scriptstyle Y_\beta} \\ X_b & \xrightarrow[f_b]{} & Y_b \end{array}$ s.t. ...

Lax(B^{op}, $\mathbf{Span}$) $\simeq$ Lax$_N$(B^{op}, $\mathbf{Prof}$), normal lax functors

$X\colon B^{op} \to \mathbf{Span}$ exponentiable iff $\hat{X}\colon B^{op} \to \mathbf{Prof}$ pseudo-functor [S]

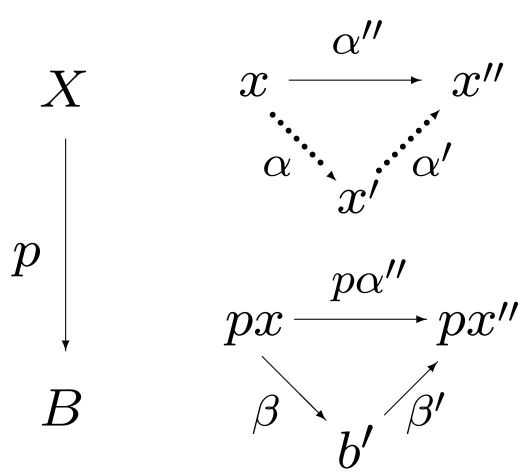
ANOTHER VIEW OF Lax($B^{op}, \mathbf{Span}$)

Lax($B^{op}, \mathbf{Span}$) $\simeq \mathbf{Cat}/B$

$p\colon X \to B$ is exponentiable iff factorization lifting (FL) holds

where (FL) = Giraud-Conduché condition [G],[C] =

(WFL)



+ connectivity condition

$$\mathsf{Lax}(B^{op}, \mathbf{Span}) \underset{pt}{\overset{\Phi}{\rightleftarrows}} \mathbf{Cat}/B, \qquad X \;\mapsto\; \begin{array}{c} \sqcup X_b \\ \downarrow \\ B \end{array} \qquad \begin{array}{c} x \xrightarrow{\alpha} x' \\ \vdots \\ b \xrightarrow{\beta} b' \end{array} \quad \alpha \in X_\beta$$

subequivalences | *objects* $p\colon X \to B$

$\Phi\colon \mathsf{Lax}(B^{op}, \mathbf{Rel}) \simeq \mathbf{Cat}_f/B$ | faithful

exponentiables : X pres $\circ \leftrightarrow$ WFL [N]

$\Phi\colon \mathsf{Lax}(B^{op}, \mathbf{Set}) \simeq \mathbf{DF}/B$ | discrete fibrations

exponentiables: all

$\Phi\colon \mathsf{Pseudo}(B^{op}, \mathbf{Span}) \simeq \mathbf{UFL}/B$ | unique factorization lifting

exponentiables: ?

$\Phi\colon \mathsf{Pseudo}(B^{op}, \mathbf{Rel}) \simeq \mathsf{DWFL}(\mathbf{Cat}_f/B)$ | discrete WFL

exponentiables: ?

IS $\mathbf{UFL}/B$ A TOPOS?

Lamarch (1996): conjectured $\mathbf{UFL}/B$ is a topos

Bunge/Niefield (1998): $\mathbf{UFL}/B$ is a topos, for B with (IG), and coreflective in Cat/B, using "model-generated" categories [BN]

Johnstone (1998): $\mathbf{UFL}/B$ is not cartesian closed when B is a square and is a topos when B has (CFI), using sheaves [J]

Bunge/Fiori (1998): (IG) iff (CFI), and $\mathbf{UFL}/B$ is a topos for B with (IG), using sheaves [BF]

Theorem 1. $\mathbf{UFL}/B$ coreflective in $\mathbf{Cat}/B \Rightarrow \mathbf{UFL}/B$ is a topos

Proof. $\mathbf{UFL}/B(X \times_B Y, Z) \cong \mathbf{Cat}/B(X \times_B Y, Z) \cong \mathbf{Cat}/B(X, Z^Y)$

$\cong \mathbf{UFL}/B(X, \widehat{Z^Y})$ and $\mathsf{Sub}_{\mathbf{UFL}/B}(X \to B) \cong \mathsf{Sub}_{\mathbf{UFL}}(X) \cong$

$\mathbf{Cat}(X, \Omega) \cong \mathbf{Cat}/B(X, \Omega \times B) \cong \mathbf{UFL}/B(X, \widehat{\mathbf{\Omega} \times \mathbf{B}})$, where Ω

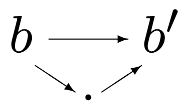
is the UFL subobject classifier in $\mathbf{Cat}$.

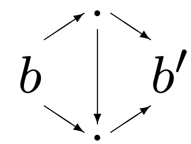
Corollary. Pseudo($B^{op}, \mathbf{Span}$) coreflective in Lax($B^{op}, \mathbf{Span}$) $\Rightarrow$ Pseudo($B^{op}, \mathbf{Span}$) is a topos

Theorem 2. $\mathbf{UFL}/B$ is coreflective in $\mathbf{Cat}/B \iff$ (IG)

Proof. later

Given $\beta\colon b \to b'$, consider the category $[\![\beta]\!]$

Objects: 

Morphisms: 

The interval glueing condition (IG)

$$\begin{array}{ccc} [\![1_{b'}]\!] & \longrightarrow & [\![\beta']\!] \\ \downarrow & & \downarrow \\ [\![\beta]\!] & \longrightarrow & [\![\beta'\beta]\!] \end{array} \qquad (*)$$

is a pushout in $\mathbf{Cat}$, for all $b \xrightarrow{\beta} b' \xrightarrow{\beta'} b''$

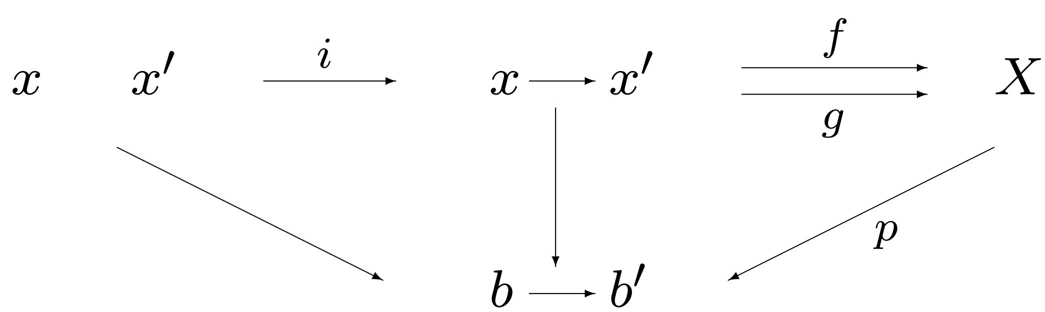
Theorem 2. $\mathbf{UFL}/B$ is coreflective in $\mathbf{Cat}/B \iff$ (IG)

Proof. Suppose $\mathbf{UFL}/B$ is coreflective in $\mathbf{Cat}/B$. Then (*) is a pushout in $\mathbf{UFL}/B \Rightarrow$ (*) is a pushout in $\mathbf{Cat}/B \Rightarrow$ (*) is a pushout in $\mathbf{Cat}$. The converse was proved in [BN].

Corollary. Pseudo($B^{op}, \mathbf{Span}$) is coreflective in Lax($B^{op}, \mathbf{Span}$) $\iff$ (IG), and in this case, Pseudo($B^{op}, \mathbf{Span}$) is a topos

Is Pseudo$(B^{op}, \mathbf{Rel})$ a topos?

No, i is mono and epi ($fi = gi \Rightarrow f = g$ since p is faithful), but i is not iso in DWFL$(\mathbf{Cat}_f/B) \simeq$ Pseudo$(B^{op}, \mathbf{Rel})$



Is Pseudo$(B^{op}, \mathbf{Rel})$ cartesian closed? Sometimes

A variation of the proof of Theorem 1 shows it is cartesian closed if Pseudo$(B^{op}, \mathbf{Rel})$ is coreflective in Lax$_N(B^{op}, \mathbf{Rel})$

REFERENCES

[BF] Bunge and Fiori, Unique factorization lifting and categories of processes, Math. Str. Comp. Sci. 10 (2000) 137–163

[BN] Bunge and Niefield, Exponentiability and single universes, JPAA 148 (2000), 217–250

[C] Conduché, Au sujet de l'existance d'adjoints à droite aux foncteurs "image réciproque" dans la catégorie des catégories, C. R. Acad. Sci. Paris 275 (1972), 891–894.

[G] Giraud, Méthode de la descent, Bull. Math. Soc. France, Mem. 2 (1964)

[J] Johnstone, A note on discrete Conduché fibrations, TAC 5 (1999), 1–11

[N] Niefield, Change of base for relational variable sets, TAC 12 (2004), 248–261

[S] Street, Powerful functors, expository note (2001)

[W] Worytkiewicz, Synchronization from a categorical perspective, arXiv:cs/0411001v1