Quantales and Toposes

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Topics

- Groupoids (mostly étale)
- Quantales
- Toposes

Sketches of a talk


A TOPOS IS A CATEGORY OF EQUIVARIANT SHEAVES


A TOPOS IS A GENERALIZED UNIVERSE OF SETS


A TOPOS IS A GENERALIZED SPACE

Part I — Étale groupoids and quantales

A quick recap

Definition

A *unital involutive quantale* Q is an involutive monoid...

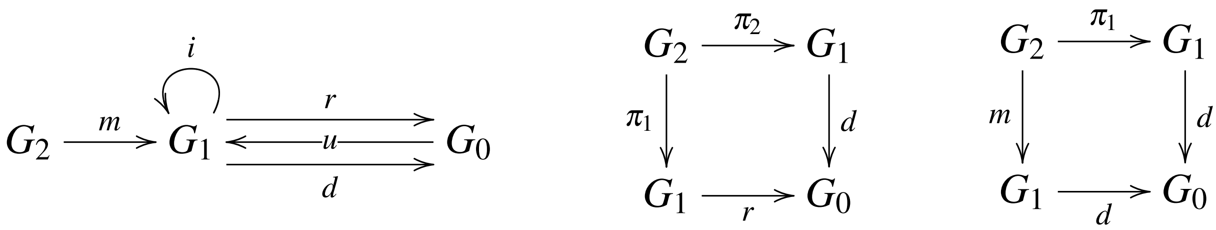
$$
\begin{aligned}
(ab)c &= a(bc) \\
ae &= a \\
ea &= a
\end{aligned}
\qquad
\begin{aligned}
a^{**} &= a \\
(ab)^* &= b^* a^*
\end{aligned}
$$

...in the monoidal category of sup-lattices $\mathbf{SL}$:

$$
\begin{aligned}
(\bigvee a_i)b &= \bigvee a_i b \\
b(\bigvee a_i) &= \bigvee b a_i
\end{aligned}
\qquad
(\bigvee a_i)^* = \bigvee a_i^*
$$

Part I — Étale groupoids and quantales

A quick recap



in **SL** : $\quad G_1 \otimes G_1 \xrightarrow{q} G_2 \xrightarrow{m_!} G_1 \qquad G_1 \xrightarrow[\cong]{i_!} G_1$

Proposition

This defines a unital involutive quantale $\mathscr{O}(G)$:

1. $ab = m_!(q(a \otimes b))$
2. $a^* = i_!(a)$
3. $e = u_!(1_{G_0})$

Part I — Étale groupoids and quantales

A quick recap

Definition

The uiqs of this type are the *inverse quantal frames*. In particular, $\downarrow(e) \cong G_0$. Let us denote this locale by B (the base locale).

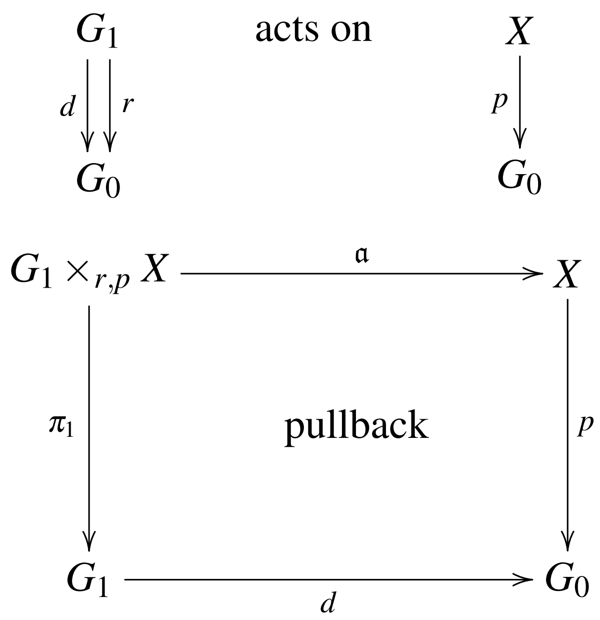
Characterization of open groupoids is trickier... but almost done!

Theorem

Inverse quantal frames are the "same" as étale groupoids via "non-functorial equivalence" of categories:

$$Q \cong \mathscr{O}(\mathscr{G}(Q))$$
$$G \cong \mathscr{G}(\mathscr{O}(G))$$

$\mathscr{G}(Q)$ is the "germ groupoid" of Q.



d open $\Rightarrow$ $\mathfrak{a}$ open

X is a left $\mathscr{O}(G)$-module: we denote it by $\mathscr{O}(X)$.

THEOREM

An $\mathscr{O}(G)$-module M is of the form $\mathscr{O}(X)$ iff it is an $\mathscr{O}(G)$-locale; that is, if and only if the following conditions hold:

1. *M is a locale*
2. *$bx = b1 \wedge x$ for all $b \in B$ and $x \in M$ (bundle condition).*

In other words: if M is an $\mathscr{O}(G)$-module and it is a B-bundle at all then it is a G-bundle.

PROOF.

The existence of the groupoid action $\mathfrak{a}$ depends on the following formulas for the right adjoint of the action $\alpha : \mathcal{O}(G) \otimes_B M \to M$ (the elements s satisfy $\{ss^*, s^*s\} \subset B$ and form both an inverse monoid and a locale basis):

$$\begin{aligned} \alpha_*(x) &= \bigvee_{ay \le x} a \otimes y = \bigvee_{sy \le x} s \otimes y \le \bigvee_{s^*sy \le s^*x} s \otimes y \\ &= \bigvee_{s^*sy \le s^*x} ss^*s \otimes y = \bigvee_{s^*sy \le s^*x} s \otimes s^*sy \\ &\le \bigvee_s s \otimes s^*x \le \alpha_*(x) \\ \Rightarrow \quad & \alpha_* \text{ preserves joins} \\ \Rightarrow \quad & \text{define } \mathfrak{a} \text{ by } \mathfrak{a}^* = \alpha_* \end{aligned}$$

∎

PART I — G-BUNDLES
= CONTINUOUS G-ACTIONS

Let $\mathcal{O}(G)$-**Loc** be the category of $\mathcal{O}(G)$-locales whose morphisms are the maps of locales f such that f^* is a homomorphism of $\mathcal{O}(G)$-modules.

THEOREM

*$\mathcal{O}(G)$-**Loc** is isomorphic to the category of G-bundles G-**Loc**.*

PROOF.

∃ faithful functor $\mathcal{O} : G\text{-}\mathbf{Loc} \to \mathcal{O}(G)\text{-}\mathbf{Loc}$ due to Beck–Chevalley (equivariance diagrams are pullbacks).

$\mathcal{O}$ is full: use again formula for α_*. ∎

Part I — Bibundles

If G and H are groupoids, a *bibundle* from G to H is a locale X equipped with a left G-bundle structure and a right H-bundle structure that are compatible in a natural way:

$$\begin{array}{ccc} G_1 \times_{G_0} X & \xrightarrow{\mathfrak{a}} & X \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[q]{} & H_0 \end{array} \qquad \begin{array}{ccc} X \times_{H_0} H_1 & \xrightarrow{\mathfrak{b}} & X \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[p]{} & G_0 \end{array}$$

$$\begin{array}{ccc} G_1 \times_{G_0} X \times_{H_0} H_1 & \xrightarrow{\mathfrak{a}\times 1} & X \times_{H_0} H_1 \\ {\scriptstyle 1\times\mathfrak{b}}\downarrow & & \downarrow{\scriptstyle \mathfrak{b}} \\ G_1 \times_{G_0} X & \xrightarrow[\mathfrak{a}]{} & X \end{array}$$

(In particular, bibundles are spans.)

Part I — Bibundles

Definition

By an $\mathscr{O}(G)$-$\mathscr{O}(H)$-*bilocale* is meant an $\mathscr{O}(G)$-$\mathscr{O}(H)$-bimodule M that is also a locale and satisfies the bundle condition wrt both actions.

Theorem

1. *The category of G-H-bibundles is isomorphic to the category of $\mathscr{O}(G)$-$\mathscr{O}(H)$-bilocales.*
2. *The bicategory of étale groupoids is equivalent to the bicategory of inverse quantal frames.*

Part I — Sheaves

The $\mathscr{O}(G)$-locales that correspond to groupoid sheaves (p is a local homeomorphism) have two simple characterizations.
Here's one of them (don't even need to assume it's a locale):

Theorem

An $\mathscr{O}(G)$-module M corresponds to a G-sheaf if and only if it has a "sesquilinear" form

$$\langle -,- \rangle : M \times M \to \mathscr{O}(G)$$

and a "basis" $\Gamma \subset M$ such that $\bigvee \Gamma = 1$ and for all $x \in M$

$$x = \bigvee_{s \in \Gamma} \langle x, s \rangle s$$

Corollary

$$BG \cong \mathscr{O}(G)\text{-}\mathbf{Sh}$$

Part II — What does G look like in BG?

- Functoriality understood
- Lack of functoriality wrt groupoid functors is useful:
  - Adjunction between inverse semigroups and quantales
  - Representation of G by "global sections" (????):
- $d : G \to G_0$ is an object of G, call it $\boldsymbol{G}$

Theorem

$$\mathscr{O}(G) \cong \hom_{BG}(1, P(\boldsymbol{G} \times \boldsymbol{G}))$$

- Suggested by Funk's work on inverse semigroups and toposes
- For open groupoids should have ($\boldsymbol{G}$ is an internal *locale*):

$$\mathscr{O}(G) \cong \hom_{BG}(1, \mathscr{O}(\boldsymbol{G} \times \boldsymbol{G}))$$

Part II — Noncommutative algebras

Let G be a finite group:

- BG is the category of G-sets; $\boldsymbol{G} \in BG$ is G with left regular action; and $\boldsymbol{C}$ is $\mathbb{C}$ with trivial action.
- The object $M_{\boldsymbol{G}}(\boldsymbol{C}) = \boldsymbol{C}^{\boldsymbol{G}\times\boldsymbol{G}} \in BG$ of "$\boldsymbol{G}$-indexed" matrices has global sections the invariant matrices:

$$\begin{aligned} A &\in \hom(1, M_{\boldsymbol{G}}(\boldsymbol{C})) \cong \hom(\boldsymbol{G}\times\boldsymbol{G}, \boldsymbol{C}) \\ a_{gh} &= a_{kg,kh} \end{aligned}$$

- Isomorphism of algebras $\hom(1, M_{\boldsymbol{G}}(\boldsymbol{C})) \to \mathbb{C}G$:

$$A \mapsto \sum_{g\in G} a_{1g} g$$

$$(AB)_{1g} = \sum_h a_{1h} b_{hg} = \sum_g a_{1h} b_{1,h^{-1}g}$$

Part II — Noncommutative algebras

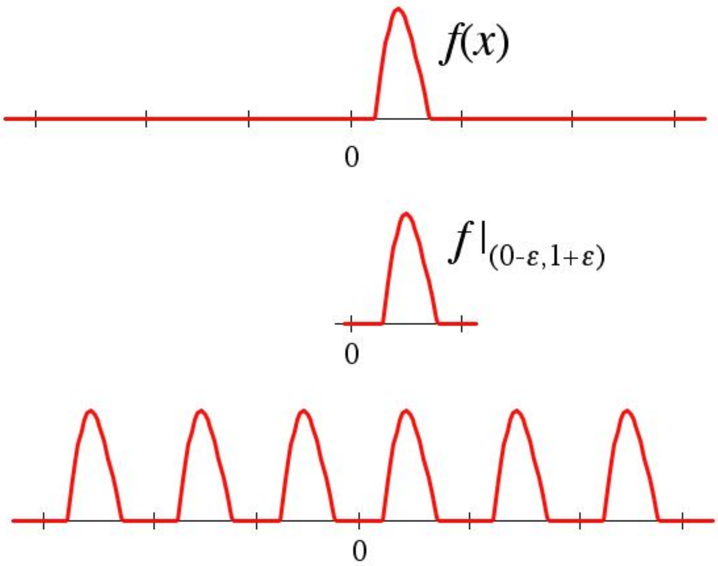
Let G be an étale groupoid:

- Bijection $C(G;\mathbb{C}) \cong \hom_{BG}(1, M_{\boldsymbol{G}}(\boldsymbol{C}))$
- Replace $\mathbb{C}$ by Sierpiński space $\Rightarrow \mathscr{O}(G) \cong \hom_{BG}(1, P(\boldsymbol{G}\times\boldsymbol{G}))$
- Back to $\mathbb{C}$: in order to have algebra need to restrict to finitely supported matrices (for some internal notion of finiteness).

Part III — Compactly supported functions

Let G be a locally compact (preferrably Hausdorff) étale groupoid.

- $C_c(G)$ is the convolution algebra of G:
$$(f*g)(x) = \sum_{x=yz} f(x)g(y)$$
- But there is no subobject $S \subset M_{\mathbf{G}}(\mathbf{C})$ such that $C_c(G) \cong \hom(1,S)$!



Part III — Toposes as (noncommutative?) spaces

- $C_c(G)$ is not internalizable! (Same applies if we think of internal locales instead of discrete internal sets.)
- Nevertheless can obtain $C_c(G)$ from $(BG,\mathbf{G})$! (And then $C_r^*(G)$, etc.)
- Direct definitions available for $HH^*(\mathscr{E},\mathbf{G})$, $HC^*(\mathscr{E},\mathbf{G})$, $HP^*(\mathscr{E},\mathbf{G})$, $C_r^*(\mathscr{E},\mathbf{G})$, $K_0(\mathscr{E},\mathbf{G})$?...
- Applications #1 — Noncommutative geometry tools applied to whatever toposes apply to...
- Applications #2 — Topos theory applied to operator algebras...
- Applications #3 — Logical handle on noncommutative geometry...
- Applications #4 — Physics...