

Quotient completion for preordered fibrations

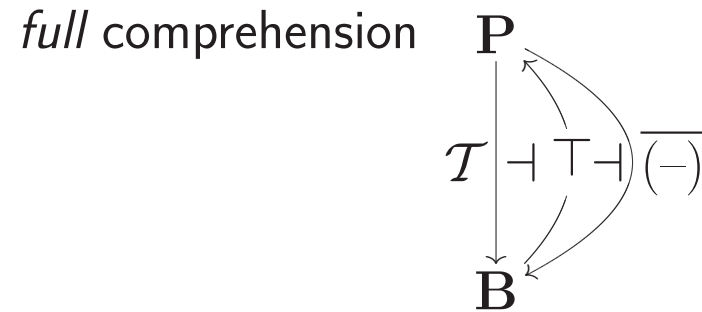
G. Rosolini

joint work with Maria Emilia Maietti

Category Theory 2008, Calais, 22-29 June 2008

A regular category from a fibration

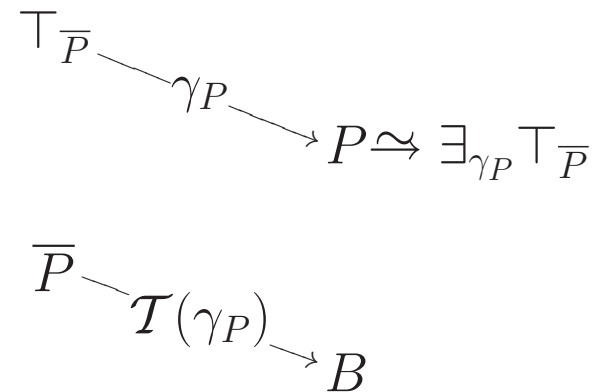
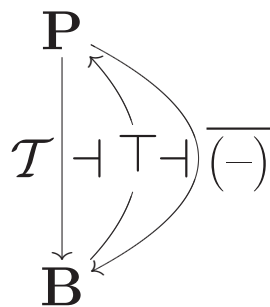
A fibration $\mathcal{T} \downarrow \mathbf{B}$ with \mathbf{P} above \mathcal{T} with finite products in \mathbf{B} , preordered fibers, and fibred finite limits \wedge, \top
 left adjoints $\exists_f \dashv f^*$ satisfying Beck-Chevalley and Frobenius



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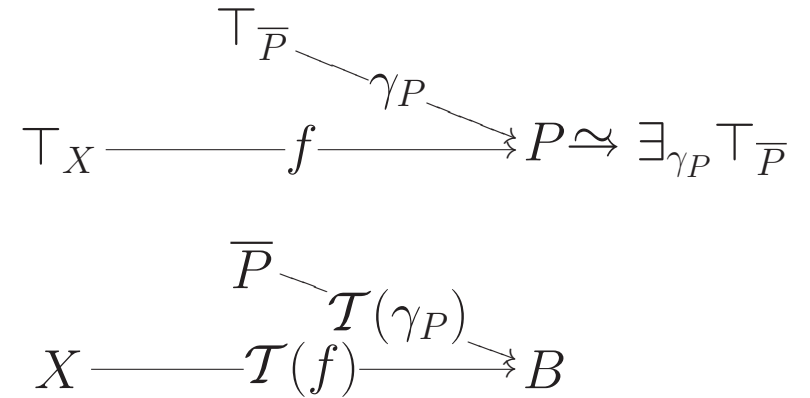
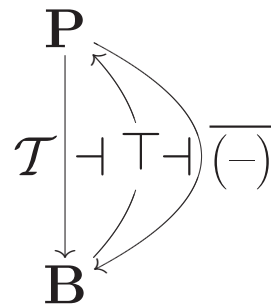
full comprehension



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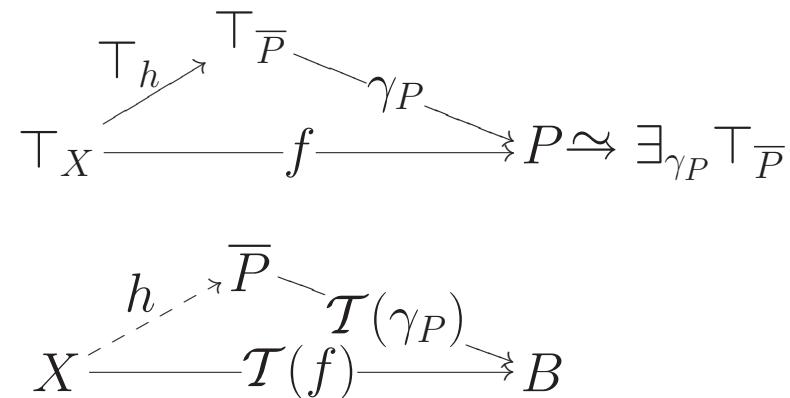
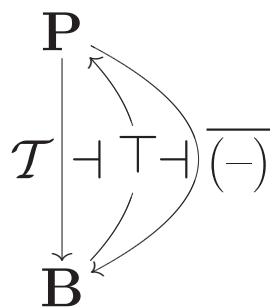
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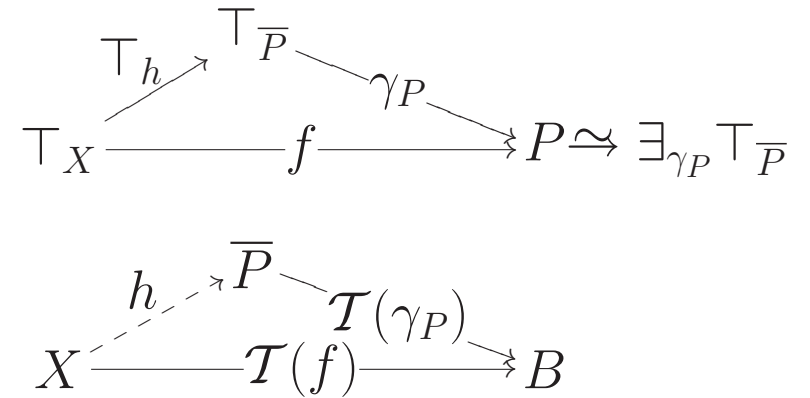
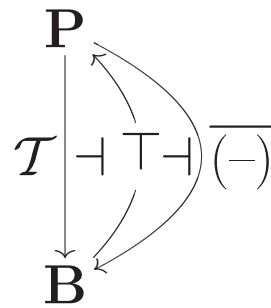
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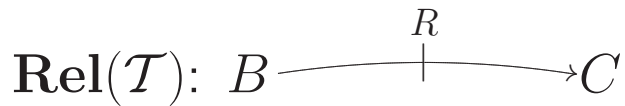
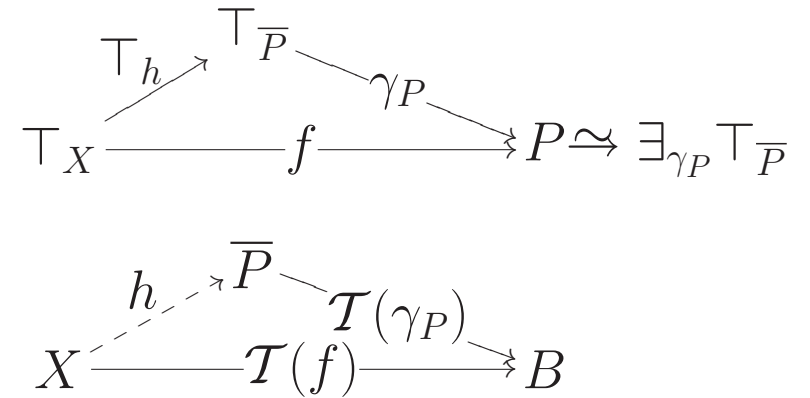
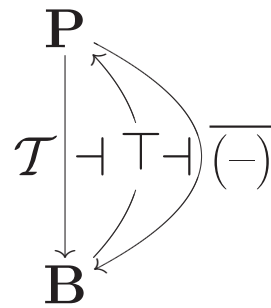


$\mathbf{Rel}(\mathcal{T})$:

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A fibration $\mathcal{T} \downarrow \mathbf{B}$ with \mathbf{P} above \mathcal{T} and \mathbf{B} below \mathcal{T} , with finite products in \mathbf{B} , preordered fibers, and fibred finite limits \wedge, \top left adjoints $\exists_f \dashv f^*$ satisfying Beck-Chevalley and Frobenius

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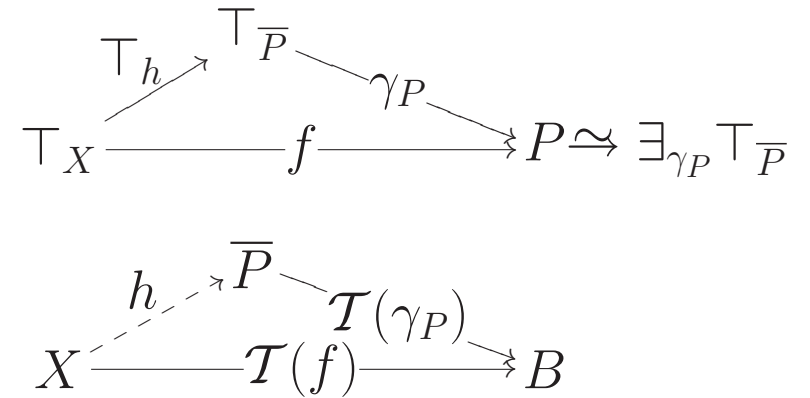
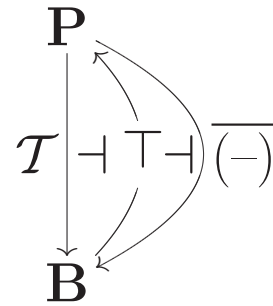


with $\mathcal{T}(R) = B \times C$

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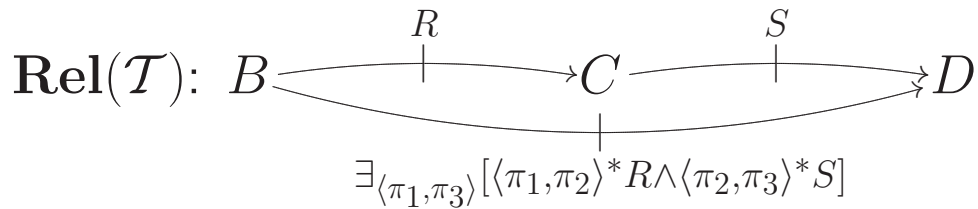
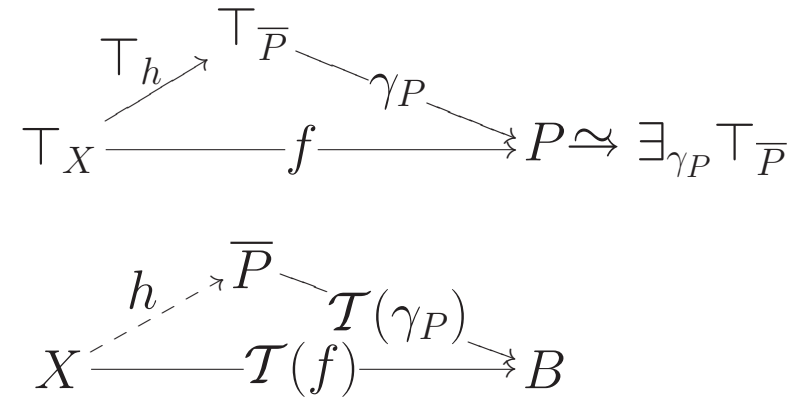
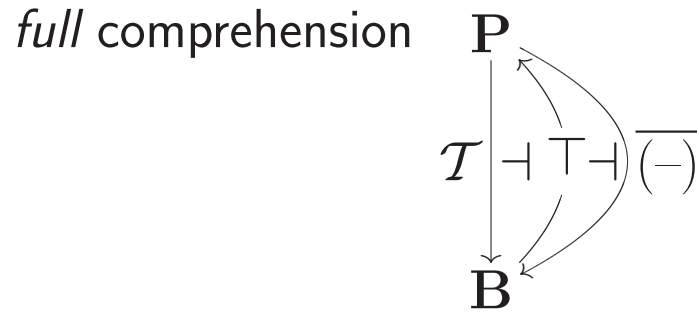
full comprehension



with $\mathcal{T}(R) = B \times C$, $\mathcal{T}(S) = C \times D$.

A regular category from a fibration

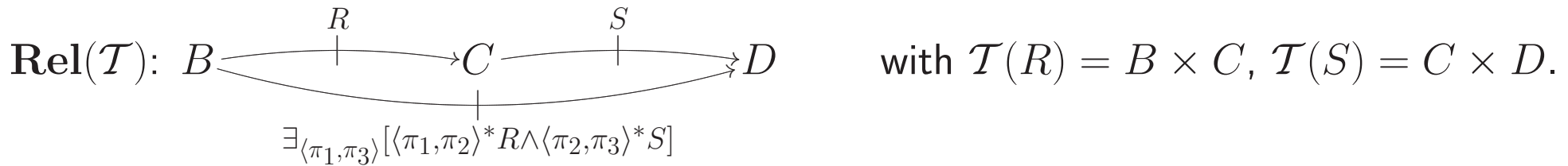
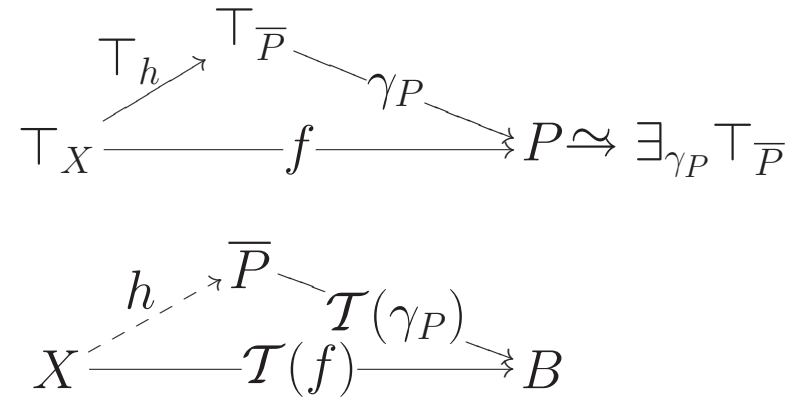
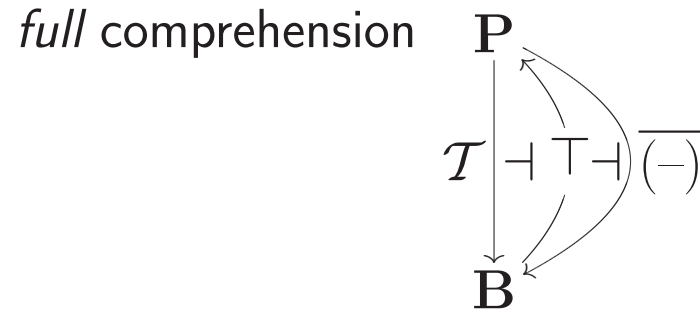
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$\mathbf{Rel}(\mathcal{T})$ is a preordered cartesian bicategory. All objects are discrete and Frobenius.

Moreover $\mathbf{Map}(\mathbf{Rel}(\mathcal{T}))$ is regular and $\mathcal{T} \downarrow$ is a change of base.

$$\begin{array}{ccc} \mathbf{P} & \longrightarrow & \mathbf{Mono}(\mathbf{Map}(\mathbf{Rel}(\mathcal{T}))) \\ \mathcal{T} \downarrow & & \downarrow \text{cod} \\ \mathbf{B} & \longrightarrow & \mathbf{Map}(\mathbf{Rel}(\mathcal{T})) \end{array}$$

An example

\mathbf{B} with finite limits and a stable factorization $(\mathcal{E}, \mathcal{M})$

Take

$$\begin{array}{c} \mathbf{P} = \mathcal{M}^{\rightarrow} \\ \downarrow \\ \mathcal{T} = \text{cod} \\ \downarrow \\ \mathbf{B} \end{array}$$

Then $\mathbf{Rel}(\text{cod})$ is $\mathbf{Rel}(\mathbf{B}, \mathcal{E}, \mathcal{M})$ as in

G.M. Kelly, *A note on relations relative to a factorization system*, CT'90

In particular,

$$\mathbf{B} \longrightarrow \mathbf{Map}(\mathbf{Rel}(\mathbf{B}, \mathcal{E}, \mathcal{M}))$$

is the reflection from the 2-category of categories with a stable factorization system to that of regular categories, inverting precisely the monos in \mathcal{E} .

The only ~~An~~ example that comes to mind

\mathbf{B} with finite limits and a stable factorization $(\mathcal{E}, \mathcal{M})$

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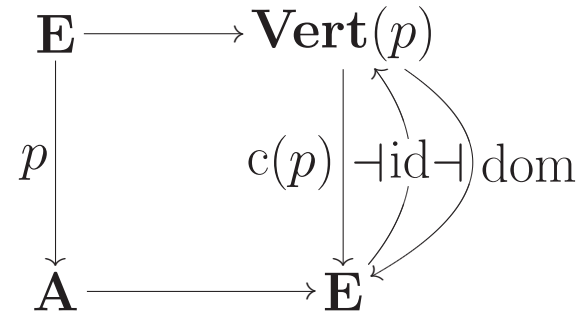
Manufacturing other examples

fibration with
fibred fin.lim's

$$\begin{array}{c} \mathbf{E} \\ \downarrow p \\ \mathbf{A} \end{array}$$

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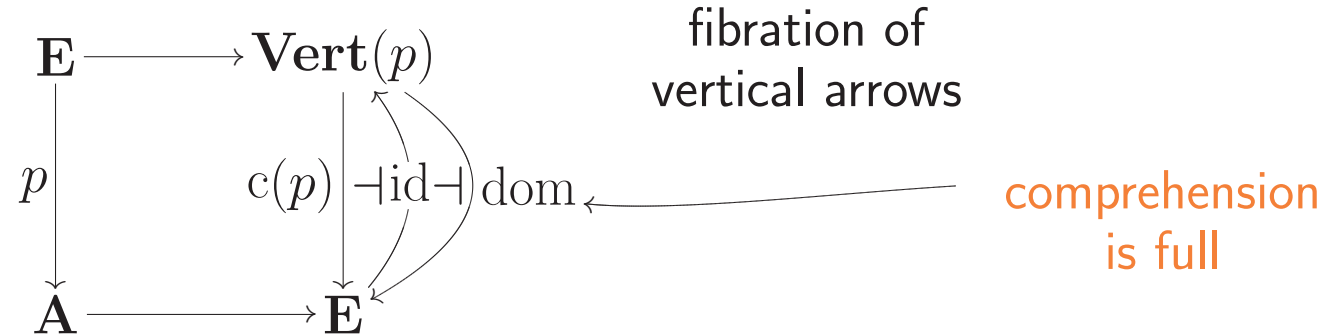


fibration of
vertical arrows

is the left biadjoint to forgetting comprehension in fibrations with fibred finite limits.

Manufacturing other examples

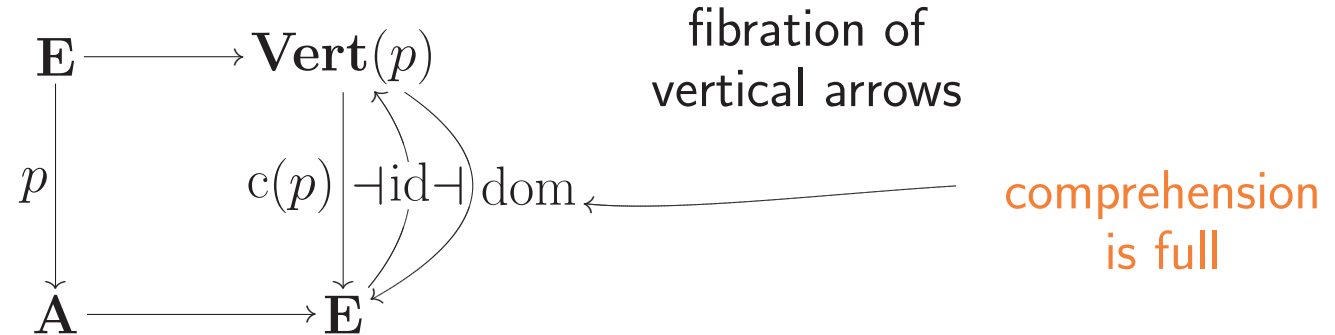
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Moreover

- if \mathbf{A} has finite products, then \mathbf{E} has finite products

- if $p \downarrow$ has left adjoints to reindexing with BCC and FR, then so does $c(p) \downarrow$

- if $p \downarrow$ is preordered, then $c(p) \downarrow$ is preordered

Manufacturing other examples, 2

- a tripos \mathbf{T} as in
$$\begin{array}{c} \mathbf{T} \\ \downarrow p \\ \mathbf{Set} \end{array}$$

J. Hyland, P. Johnstone, A. Pitts, *Tripos Theory*, Math.Proc.Camb.Phil.Soc. 88 (1980)

- for a geometric theory T , the fibration of formulas \mathbf{WFF}_T as in
$$\begin{array}{c} \mathbf{WFF}_T \\ \downarrow p \\ \mathbf{Sort}_T \end{array}$$

M. Makkai, G. Reyes, *First Order Categorical Logic*, LNM 611, 1977

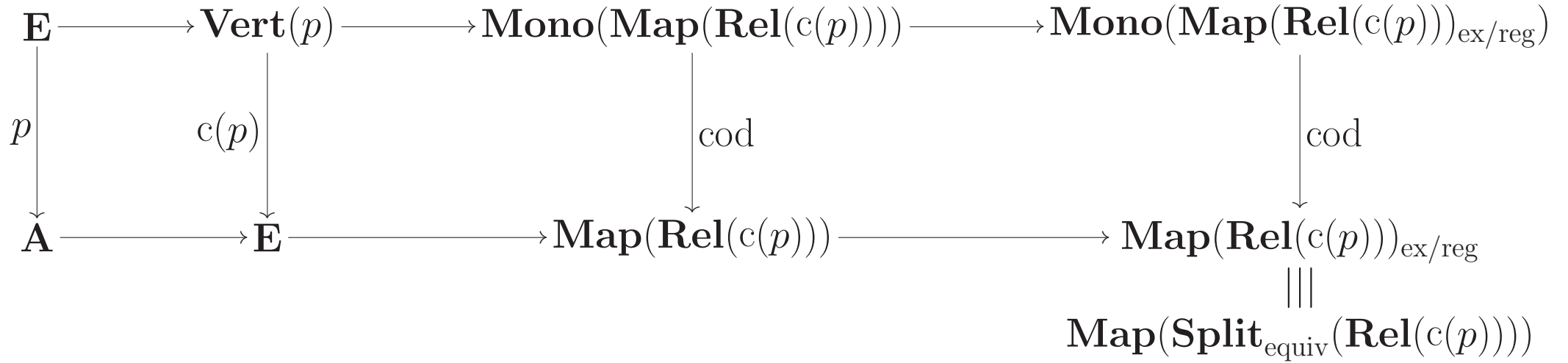
- for a category \mathbf{A} with fin.products and weak equalizers, the fibred preordered reflection $(\mathbf{A}^{\rightarrow})_{po}$ as in
$$\begin{array}{c} (\mathbf{A}^{\rightarrow})_{po} \\ \downarrow \text{cod} \\ \mathbf{A} \end{array}$$
- essentially as in

A. Carboni, E. Vitale, *Regular and exact completions*, J.Pure Appl.Alg. 125 (1998)

- for a left exact functor $F: \mathbf{A} \rightarrow \mathbf{C}$, a suitable fibred preordered reflection $(\mathbf{pb}_{\mathbf{C}}(F))_{po}$ as in
$$\begin{array}{c} (\mathbf{pb}_{\mathbf{C}}(F))_{po} \\ \downarrow \text{cod} \\ \mathbf{A} \end{array}$$

P. Hofstra, *Relative completions*, J.Pure Appl.Alg. 192 (2004)

Quotient completion



Quotient completion

$$\begin{array}{ccccccc}
 \mathbf{E} & \longrightarrow & \mathbf{Vert}(p) & \longrightarrow & \mathbf{Mono}(\mathbf{Map}(\mathbf{Rel}(c(p)))) & \longrightarrow & \mathbf{Mono}(\mathbf{Map}(\mathbf{Rel}(c(p)))_{\text{ex/reg}}) \\
 \downarrow p & & \downarrow c(p) & & \downarrow \text{cod} & & \downarrow \text{cod} \\
 \mathbf{A} & \longrightarrow & \mathbf{E} & \longrightarrow & \mathbf{Map}(\mathbf{Rel}(c(p))) & \longrightarrow & \mathbf{Map}(\mathbf{Rel}(c(p)))_{\text{ex/reg}} \\
 & & & & & & \parallel \\
 & & & & & & \mathbf{Map}(\mathbf{Split}_{\text{equiv}}(\mathbf{Rel}(c(p))))
 \end{array}$$

For \mathbf{A} with finite products and weak equalizers

$$\begin{array}{ccccccc}
 (\mathbf{A}^{\rightarrow})_{\text{po}} & \longrightarrow & \mathcal{M}(\mathbf{Fr}(\mathbf{A})) & \longrightarrow & \mathbf{Mono}(\mathbf{A}_{\text{reg}}) & \longrightarrow & \mathbf{Mono}(\mathbf{A}_{\text{ex}}) \\
 \downarrow \text{cod} & & \downarrow & & \downarrow & & \downarrow \\
 \mathbf{A} & \longrightarrow & \mathbf{Fr}(\mathbf{A}) & \longrightarrow & \mathbf{A}_{\text{reg}} & \longrightarrow & \mathbf{A}_{\text{ex}}
 \end{array}$$