Variable-basis topological systems

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Outline

Topological systems

- 1959 D. Papert and S. Papert construct an adjunction between the categories **Top** (topological spaces) and $\mathbf{Frm}^{op}$ (the dual of the category **Frm** of frames).
- 1972 J. Isbell uses the name locale for the objects of $\mathbf{Frm}^{op}$ and considers the category **Loc** (locales) as a substitute for **Top**.
- 1982 P. Johnstone gives a coherent statement to localic theory in his book "Stone Spaces".
- 1989 Using the logic of finite observations S. Vickers introduces the notion of topological system to unite both topological and localic approaches.

Fuzzy topology

- 1965 L. A. Zadeh introduces fuzzy sets. His approach is generalized by J. A. Goguen in 1967.
- 1968 C. L. Chang introduces fuzzy topological spaces. His approach is generalized by R. Lowen in 1976.
- 1983 S. E. Rodabaugh studies the category **FUZZ** of variable-basis fuzzy topological spaces. Later on he considers the category **C-Top** of variable-basis lattice-valued topological spaces.
- ... Starting from 1983 U. Höhle, S. E. Rodabaugh, A. P. Šostak *et al.* consider fixed- and variable-basis fuzzy topologies and their properties.

Fuzzy topology & Topological systems

2007 J. T. Denniston and S. E. Rodabaugh consider functorial relationships between lattice-valued topology and topological systems.

!!! Using fuzzy topological spaces and crisp topological systems they encounter some problems.

Our contribution

2008 We introduced the category of variable-basis topological spaces over an arbitrary variety of algebras generalizing the category **C-Top** of S. E. Rodabaugh.

- This talk introduces the notion of variable-basis topological system over an arbitrary variety of algebras.
- By analogy with J. T. Denniston and S. E. Rodabaugh we consider functorial relationships between variable-basis topological spaces and variable-basis topological systems.

The basic point

While considering fuzzy topological spaces, one should consider fuzzy topological systems.

Ω-algebras and Ω-homomorphisms

- Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a class of cardinal numbers.

Definition 1

- An Ω-algebra is a pair $(A, (\omega_\lambda^A)_{\lambda \in \Lambda})$ (denoted by A), where A is a set and $(\omega_\lambda^A)_{\lambda \in \Lambda}$ is a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$.
- An Ω-homomorphism $(A, (\omega_\lambda^A)_{\lambda \in \Lambda}) \xrightarrow{f} (B, (\omega_\lambda^B)_{\lambda \in \Lambda})$ is a map $A \xrightarrow{f} B$ such that $f \circ \omega_\lambda^A = \omega_\lambda^B \circ f^{n_\lambda}$ for every $\lambda \in \Lambda$.

Definition 2

- $\mathbf{Alg}(\Omega)$ is the category of Ω-algebras and Ω-homomorphisms.
- $|-|$ is the forgetful functor to the category **Set** (sets).

Varieties of algebras

Definition 3

- Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps.
- A variety of Ω-algebras is a full subcategory of $\mathbf{Alg}(\Omega)$ closed under the formation of products, $\mathcal{M}$-subobjects (subalgebras) and $\mathcal{E}$-quotients (homomorphic images).
- The objects (resp. morphisms) of a variety are called algebras (resp. homomorphisms).

Example 4

The categories **Frm**, **SFrm** and **SQuant** of frames, semiframes and semi-quantales (popular in lattice-valued topology) are varieties.

Q-powersets

- From now one fix a variety **A** and an algebra Q.

Definition 5

Given a set X, Q^X is the Q-powerset of X.

- An arbitrary element of Q^X is denoted by p (with indices).
- Q^X is an algebra with operations lifted point-wise from Q by

$$(\omega_\lambda^{Q^X}(\langle p_i\rangle_{n_\lambda}))(x) = \omega_\lambda^Q(\langle p_i(x)\rangle_{n_\lambda}).$$

Image and preimage operators

- Let $X \xrightarrow{f} Y$ be a map and let $A \xrightarrow{g} B$ be a homomorphism.
- There exist:
  - the standard image and preimage operators $\mathcal{P}(X) \xrightarrow{f^{\rightarrow}} \mathcal{P}(Y)$ and $\mathcal{P}(Y) \xrightarrow{f^{\leftarrow}} \mathcal{P}(X)$;
  - the Zadeh preimage operator $Q^Y \xrightarrow{f_Q^{\leftarrow}} Q^X$ defined by $f_Q^{\leftarrow}(p) = p \circ f$;
  - a map $A^X \xrightarrow{g_{\rightarrow}^X} B^X$ defined by $g_{\rightarrow}^X(p) = g \circ p$.

Lemma 6

For every map $X \xrightarrow{f} Y$ and every homomorphism $A \xrightarrow{g} B$, both $Q^Y \xrightarrow{f_Q^{\leftarrow}} Q^X$ and $A^X \xrightarrow{g_{\rightarrow}^X} B^X$ are homomorphisms.

Fixed-basis topological spaces

Definition 7

- Given a set X, a subset τ of Q^X is a Q-topology on X provided that τ is a subalgebra of Q^X.
- A Q-topological space (also called a Q-space) is a pair (X, τ), where X is a set and τ is a Q-topology on X.
- A map $(X, \tau) \xrightarrow{f} (Y, \sigma)$ between Q-spaces is Q-continuous provided that $(f_Q^{\leftarrow})^{\rightarrow}(\sigma) \subseteq \tau$.

Definition 8

- Q-**Top** is the category of Q-spaces and Q-continuous maps.
- $|-|$ is the forgetful functor to the category **Set**.

Notations

- From now on introduce the following notations:

- The dual of the category **A** is denoted by **LoA** (the "**Lo**" comes from "localic").
- The objects (resp. morphisms) of **LoA** are called localic algebras (resp. homomorphisms).
- The respective homomorphism of a localic homomorphism f is denoted by f^{op} and vice versa.
- To distinguish between maps and homomorphisms denote them by "f, g" and "φ, ψ" respectively.

Variable-basis preimage operator

Definition 9

Given a **Set** $\times$ **LoA**-morphism $(X, A) \xrightarrow{(f,\varphi)} (Y, B)$, there exists the Rodabaugh preimage operator $B^Y \xrightarrow{(f,\varphi)^{\leftarrow}} A^X$ defined by $(f, \varphi)^{\leftarrow}(p) = \varphi^{op} \circ p \circ f$.

Lemma 10

For every **Set** $\times$ **LoA**-*morphism* $(X, A) \xrightarrow{(f,\varphi)} (Y, B)$, *the diagram*

$$\begin{array}{ccc} B^Y & \xrightarrow{(\varphi^{op})^Y_{\rightarrow}} & A^Y \\ {\scriptstyle f_B^{\leftarrow}}\downarrow & \searrow{\scriptstyle (f,\varphi)^{\leftarrow}} & \downarrow{\scriptstyle f_A^{\leftarrow}} \\ B^X & \xrightarrow{(\varphi^{op})^X_{\rightarrow}} & A^X \end{array}$$

commutes and therefore $B^Y \xrightarrow{(f,\varphi)^{\leftarrow}} A^X$ *is a homomorphism.*

Variable-basis topological spaces

Definition 11

- Given a subcategory **C** of **LoA**, the category **C**-**Top** comprises the following data:
  - Objects: **C**-topological spaces or **C**-spaces (X, A, τ), where (X, A) is a **Set** $\times$ **C**-object and (X, τ) is an A-space.
  - Morphisms: **C**-continuous pairs $(X, A, \tau) \xrightarrow{(f,\varphi)} (Y, B, \sigma)$, where (f, φ) is a **Set** $\times$ **C**-morphism and $((f, \varphi)^{\leftarrow})^{\rightarrow}(\sigma) \subseteq \tau$.
- $| - |$ is the forgetful functor to the category **Set** $\times$ **C**.

- **C**-**Top** generalizes the respective category of S. E. Rodabaugh.
- This talk considers the case **C** $=$ **LoA**.
- Call **LoA**-spaces by spaces and **LoA**-continuity by continuity.

Satisfaction relation

Definition 12

Let X be a set and A be a frame. Then $X \xrightarrow{\models} A$ is a satisfaction relation on (X, A) if $\models$ is a binary relation from X to A satisfying the following join interchange law and meet interchange law:

- For any family $\{a_i\}_{i \in I}$ of elements of A,

$$x \models \bigvee_{i \in I} a_i \text{ iff } x \models a_i \text{ for at least one } i \in I.$$

- For any finite family $\{a_i\}_{i \in I}$ of elements of A,

$$x \models \bigwedge_{i \in I} a_i \text{ iff } x \models a_i \text{ for every } i \in I.$$

If $x \models a$, then x satisfies a.

Topological systems

Definition 13

- A topological system is a triple $(X, A, \models)$, where (X, A) is a **Set** $\times$ **Loc**-object and $\models$ is a satisfaction relation on (X, A).
- Elements of X are points and elements of A are opens.
- The category **TopSys** comprises the following data:
  - Objects: topological systems $(X, A, \models)$.
  - Morphisms: continuous maps

    $$(X, A, \models_1) \xrightarrow{f=(\mathrm{pt}\, f, (\Omega f)^{op})} (Y, B, \models_2),$$

    where f is a **Set** $\times$ **Loc**-morphism and for every $x \in X$, $b \in B$,

    $$\mathrm{pt}\, f(x) \models_2 b \text{ iff } x \models_1 \Omega f(b).$$

Variable-basis topological systems

Definition 14

- Given a subcategory **C** of **LoA**, the category **C-TopSys** comprises the following data:
  - Objects: **C**-topological systems or **C**-systems $(X, A, B, \models)$, where (X, A, B) is a $\mathbf{Set} \times \mathbf{C} \times \mathbf{C}$-object and $X \times B \xrightarrow{\models} A$ is a map (satisfaction relation) such that for every $x \in X$,
  $$B \xrightarrow{\models(x,-)} A \text{ is a homomorphism.}$$
  - Morphisms: **C**-continuous maps
  $$(X, A, B, \models_1) \xrightarrow{f=(\mathrm{pt}\, f, (\Sigma f)^{op}, (\Omega f)^{op})} (Y, C, D, \models_2),$$
  where f is a $\mathbf{Set} \times \mathbf{C} \times \mathbf{C}$-morphism and for every $x \in X$, $d \in D$,
  $$\Sigma f(\models_2(\mathrm{pt}\, f(x), d)) = \models_1(x, \Omega f(d)).$$
- $| - |$ is the forgetful functor to the category $\mathbf{Set} \times \mathbf{C} \times \mathbf{C}$.

From variable-basis to fixed-basis

Definition 15

- For a **C**-object Q, Q-**TopSys** is the subcategory of **C**-**TopSys** of all **C**-systems $(X, Q, B, \models)$ with basis Q and all continuous f such that $\Sigma f = 1_Q$.
- $| - |$ is the forgetful functor to the category **Set** $\times$ **C**.

Lemma 16

- *The subcategory Q-**TopSys** is full iff $\mathbf{C}(Q, Q) = \{1_Q\}$.*
- *If Q is an initial (terminal) object in **A**, then Q-**TopSys** is full.*

Examples

Example 17

$2 = \{\bot, \top\}$ is initial in **Frm**. The full subcategory 2-**TopSys** of **Loc**-**TopSys** is isomorphic to the category **TopSys** of S. Vickers.

Example 18

Given a set K, the subcategory K-**TopSys** of **LoSet**-**TopSys** is isomorphic to the category $\mathtt{Chu}(\mathbf{Set}, K)$ of Chu spaces over K. K-**TopSys** is full iff K is the empty set or a singleton.

- The following considers the category **LoA**-**TopSys**.
- Call **LoA**-systems by systems and **LoA**-continuity by continuity.

From spaces to systems

Lemma 19

There exists a full embedding **LoA-Top** $\xhookrightarrow{E_T}$ **LoA-TopSys** *with*

$$E_T((X,A,\tau) \xrightarrow{(f,\varphi)} (Y,B,\sigma)) =$$
$$(X,A,\tau,\models_1) \xrightarrow{(f,\varphi,((f,\varphi)^{\leftarrow})^{op})} (Y,B,\sigma,\models_2)$$

where $\models_i(z,p) = p(z)$.

Proof.

As an example show that $E_T(f,\varphi)$ is in **LoA-TopSys**:

$$\models_1(x,(f,\varphi)^{\leftarrow}(p)) = \models_1(x,\varphi^{op} \circ p \circ f) =$$
$$\varphi^{op} \circ p \circ f(x) = \varphi^{op}(\models_2(f(x),p)).$$

From systems to spaces: spatialization

Lemma 20

There exists a functor **LoA-TopSys** $\xrightarrow{\text{Spat}}$ **LoA-Top** *defined by*

$$\text{Spat}((X, A, B, \models_1) \xrightarrow{f} (Y, C, D, \models_2)) =$$
$$(X, A, \tau) \xrightarrow{(\text{pt}\, f, (\Sigma f)^{op})} (Y, C, \sigma)$$

where $\tau = \{\models_1(-, b) \mid b \in B\}$ $(\models_1(-, b)$ *is the* *extent* *of* $b)$.

Proof.

As an example show that $\text{Spat}(f)$ is in **LoA-Top**:

$$((\text{pt}\, f, (\Sigma f)^{op})^{\leftarrow}(\models_2(-, d)))(x) = \Sigma f \circ \models_2(-, d) \circ \text{pt}\, f(x) =$$
$$\Sigma f(\models_2(\text{pt}\, f(x), d)) = \models_1(x, \Omega f(d)) = (\models_1(-, \Omega f(d)))(x).$$

E_T and Spat form an adjoint pair

Theorem 21

Spat *is a right-adjoint-left-inverse of* E_T.

Proof.

- Given a system $(X, A, B, \models)$,
$$E_T \operatorname{Spat}(X, A, B, \models) \xrightarrow{(1_X, 1_A, \Phi^{op})} (X, A, B, \models)$$
with $\Phi(b) = \models(-, b)$ provides an E_T-(co-universal) map.
- Straightforward computations show that Spat $E_T = 1_{\mathbf{LoA\text{-}Top}}$.

Corollary 22

LoA-Top *is isomorphic to a full (regular mono)-coreflective subcategory of* **LoA-TopSys**.

From localic algebras to systems

Lemma 23

- *There exists an embedding* **LoA** $\xrightarrow{E_L^Q}$ **LoA-TopSys** *defined by*

$$E_L^Q(B \xrightarrow{\varphi} C) = (\mathtt{Pt}_Q(B), Q, B, \models_1) \xrightarrow{((\varphi^{op})_Q^{\leftarrow}, 1_Q, \varphi)} (\mathtt{Pt}_Q(C), Q, C, \models_2)$$

  where $\mathtt{Pt}_Q(B) = \mathbf{A}(B, Q)$ *and* $\models_i(p, d) = p(d)$.
- E_L^Q *is full iff* $\mathbf{A}(Q, Q) = \{1_Q\}$.
- *If* Q *is an initial (terminal) object in* $\mathbf{A}$, *then* E_L^Q *is full.*

From systems to localic algebras: localification

Lemma 24

There exists a functor **LoA-TopSys** $\xrightarrow{\text{Loc}}$ **LoA** *defined by* $\text{Loc}((X, A, B, \models_1) \xrightarrow{f} (Y, C, D, \models_2)) = B \xrightarrow{(\Omega f)^{op}} D$.

Lemma 25

In general E_L^Q *does not have a left adjoint and therefore* Loc *is not a left adjoint of* E_L^Q.

From fixed-basis topologies to systems

Lemma 26

- *Every homomorphism* $A \xrightarrow{\varphi} B$ *provides an embedding* $A\text{-}\mathbf{Top} \xhookrightarrow{E_\varphi} \mathbf{LoA\text{-}TopSys}$ *defined by*

$$E_\varphi((X,\tau) \xrightarrow{f} (Y,\sigma)) =$$
$$(X,B,\tau,\models_1^\varphi) \xrightarrow{(f,1_B,(f_A^{\leftarrow})^{op})} (Y,B,\sigma,\models_2^\varphi)$$

where $\models_i^\varphi(z,p) = \varphi \circ p(z)$.

- *If* φ *is an* $\mathbf{A}$*-monomorphism and* $\mathbf{A}(B,B) = \{1_B\}$, E_φ *is full.*

From systems to spaces again

Lemma 27

There exists a functor **LoA-TopSys** $\xrightarrow{F_{\models}}$ **LoA-Top** *defined by*

$$F_{\models}((X, A, B, \models_1) \xrightarrow{f} (Y, C, D, \models_2)) =$$
$$(X, B, \tau) \xrightarrow{(\mathrm{pt}\, f, (\Omega f)^{op})} (Y, D, \sigma)$$

where

$$\tau = \{p \in B^X \mid \models_1(x, p(x)) = \models_1(x', p(x')) \text{ for every } x, x' \in X\}.$$

- $F_{\models}$ is related to stratified topological spaces of R. Lowen.
- In contrast to Spat, $F_{\models}$ forgets the basis.

LoA-Top is topological

Definition 28

Given an algebra A and a subset $S \subseteq A$, $\langle S \rangle$ denotes the smallest subalgebra of A which contains S. Given a Q-space (X, τ), a subset $S \subseteq Q^X$ is a subbasis of τ provided that $\tau = \langle S \rangle$.

Lemma 29

The concrete category $(\mathbf{LoA}\text{-}\mathbf{Top}, |-|)$ *is topological.*

Proof.

Given a $|-|$-structured source $\mathcal{S} = ((X, A) \xrightarrow{(f_i, \varphi_i)} |(X_i, A_i, \tau_i)|)_{i \in I}$, the initial structure on (X, A) can be defined by $\tau = \langle \bigcup_{i \in I} S_i \rangle$ with $S_i = ((f_i, \varphi_i)^{\leftarrow})^{\rightarrow}(\tau_i)$.

LoA-TopSys is not topological

Theorem 30

The concrete category $(\mathbf{LoA\text{-}TopSys}, |-|)$ *is topological iff* $|-|$ *is an isomorphism.*

Proof.

Since the sufficiency is clear show the necessity. Let $A \xrightarrow{\varphi} B$ be a homomorphism. Show that φ is an isomorphism.

φ is an isomorphism

φ is injective

- For the singleton $\imath = \{\star\}$ define $\imath \times A \xrightarrow{\models_2} A$ by $\models_2(\star, a) = a$.
- Let $(\imath, A, B, \models_1) \xrightarrow{(1_\imath, 1_A, \varphi^{op})} (\imath, A, A, \models_2)$ be an initial lift of $(\imath, A, B) \xrightarrow{(1_\imath, 1_A, \varphi^{op})} |(\imath, A, A, \models_2)|$.
- Given $a \in A$, $a = \models_2(\star, a) = \models_1(\star, \varphi(a)) = \models_1(\star, -) \circ \varphi(a)$.

φ is surjective

- Define $\imath \times B \xrightarrow{\models_1} B$ by $\models_1(\star, b) = b$.
- Let $(\imath, B, B, \models_1) \xrightarrow{(1_\imath, \varphi^{op}, 1_B)} (\imath, A, B, \models_2)$ be a final lift of $|(\imath, B, B, \models_1)| \xrightarrow{(1_\imath, \varphi^{op}, 1_B)} (\imath, A, B)$.
- Given $b \in B$, $b = \models_1(\star, b) = \varphi(\models_2(\star, b)) = \varphi \circ \models_2(\star, -)(b)$.

The structure of **A**

Proof.

- If **A** has the empty algebra A, then for every algebra B, the projection $A \times B \xrightarrow{\pi_B} B$ is an isomorphism and then B is the empty algebra. Thus $\mathcal{O}\mathrm{b}(\mathbf{A}) = \{\text{the empty algebra}\}$.
- If **A** has a non-empty algebra A, then for every algebra B, the projection $A \times B \xrightarrow{\pi_A} A$ is an isomorphism and then B is a singleton algebra. Thus $\mathcal{O}\mathrm{b}(\mathbf{A}) = \{\text{singleton algebras}\}$.
- In both cases **A** is a thin, connected category.
- It follows that the forgetful functor

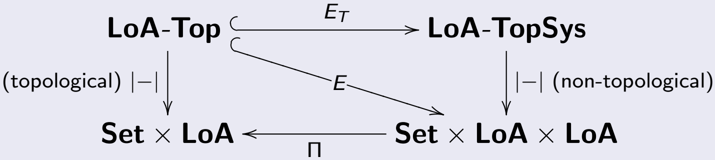
$$\mathbf{LoA\text{-}TopSys} \xrightarrow{|-|} \mathbf{Set} \times \mathbf{LoA} \times \mathbf{LoA}$$

is an isomorphism. Q.E.D.

Basic functorial relations

Lemma 31

- *There is an embedding* **LoA-Top** $\overset{E}{\hookrightarrow}$ **Set** $\times$ **LoA** $\times$ **LoA** *with* $E((X,A,\tau) \xrightarrow{(f,\varphi)} (Y,B,\sigma)) = (X,A,\tau) \xrightarrow{(f,\varphi,((f,\varphi)^{\leftarrow})^{op})} (Y,B,\sigma)$.
- *There is the projection* **Set** $\times$ **LoA** $\times$ **LoA** $\xrightarrow{\Pi}$ **Set** $\times$ **LoA** *with* $\Pi((X,A,B) \xrightarrow{(f,\varphi,\psi)} (Y,C,D)) = (X,A) \xrightarrow{(f,\varphi)} (Y,C)$.
- *The following diagram commutes:*



Products of systems

- Suppose **A** has coproducts.

- Let $((X_i, A_i, B_i, \models_i))_{i\in I}$ be a set-indexed family of systems.
- Let $(X, (\pi_i)_{i\in I})$ be a product of $(X_i)_{i\in I}$ in **Set**.
  Let $((\mu_i)_{i\in I}, A)$, $((\rho_i)_{i\in I}, B)$ be coproducts of $(A_i)_{i\in I}$, $(B_i)_{i\in I}$ in **A**.
- Define $X \times B \xrightarrow{\models} A$ with $\models(x, -) = \coprod_{i\in I} \models_i(\pi_i(x), -)$ given by the commutativity for every $i \in I$ of the following diagram:

$$\begin{array}{ccc} B_i & \xrightarrow{\rho_i} & B \\ {\scriptstyle \models_i(\pi_i(x),-)}\downarrow & & \vdots\, {\scriptstyle \coprod_{i\in I} \models_i(\pi_i(x),-)} \\ A_i & \xrightarrow[\mu_i]{} & A \end{array}$$

- $((X, A, B, \models), ((\pi_i, \mu_i^{op}, \rho_i^{op}))_{i\in I})$ is a (concrete) product of $((X_i, A_i, B_i, \models_i))_{i\in I}$ in **LoA-TopSys**.

Coproducts of systems

- In general **LoA-TopSys** does not have concrete coproducts.
- It is still unclear whether **LoA-TopSys** has coproducts.
- Q-**TopSys** has concrete coproducts constructed as follows:

- Let $((X_i, Q, B_i, \models_i))_{i \in I}$ be a set-indexed family of systems.
- Let $((\mu_i)_{i \in I}, X)$ be a coproduct of $(X_i)_{i \in I}$ in **Set**.
  Let $(B, (\pi_i)_{i \in I})$ be a product of $(B_i)_{i \in I}$ in **A**.
- Define $X \times B \xrightarrow{\models} Q$ by $\models(x, b) = \models_i(x_i, b_i)$ for $x = \mu_i(x_i)$.
- $(((\mu_i, 1_Q, \pi_i^{op}))_{i \in I}, (X, Q, B, \models))$ is a (concrete) coproduct of $((X_i, Q, B_i, \models_i))_{i \in I}$ in Q-**TopSys**.

Algebraic properties of **LoA-TopSys**

Lemma 32

LoA-**TopSys** $\xrightarrow{|-|}$ **Set** $\times$ **LoA** $\times$ **LoA** *creates isomorphisms.*

- Lemma 32 and topological properties of **LoA**-**TopSys** suggest the following questions:

Problem 33

- *Is the category* **LoA**-**TopSys** *algebraic?*
- *Does* **LoA**-**TopSys** $\xrightarrow{|-|}$ **Set** $\times$ **LoA** $\times$ **LoA** *have a left adjoint?*

What about the category **LoA** $\times$ **LoA**?

- By Lemma 23 **LoA** can be embedded into **LoA**-**TopSys**.

Problem 34

Is it possible to embed **LoA** $\times$ **LoA** *fully into* **LoA**-**TopSys***?*

A possible answer

- It is possible to change the definition of **LoA**-**TopSys** in such a way that **A** $\times$ **LoA** can be fully embedded into it.
- The embedding has a left-adjoint-left-inverse Loc defined appropriately.

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Thank you for your attention!