Free models of enriched T-algebraic theories computed as Kan extensions

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The tensor algebra

Let k denote a commutative ring. To every k-module A is associated the tensor algebra

$$TA \quad = \quad \bigoplus_{n \in \mathbb{N}} A^{\otimes n}$$

computed as infinite sum of tensorial powers.

Furthermore, this construction is functorial

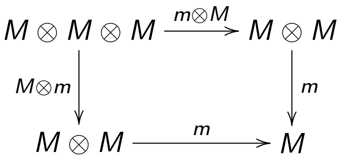
$$T \quad : \quad k\text{-Mod} \quad \longrightarrow \quad k\text{-Alg}$$

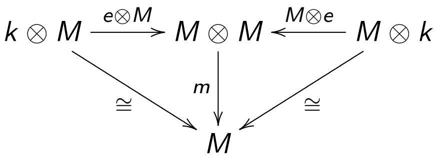
k-algebra as monoid

Recall that a k-algebra M is defined as a k-module equipped with two morphisms,

$$k \xrightarrow{\ e\ } M \xleftarrow{\ m\ } M \otimes M$$

called unit and multiplication, making the diagrams below commute:





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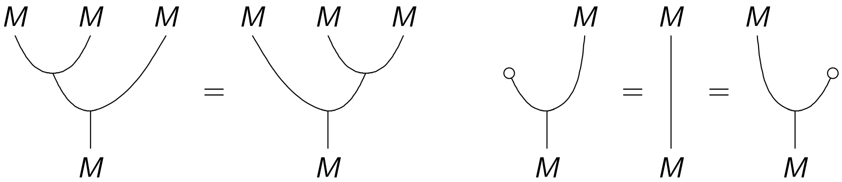


The tensor algebra as a free monoid

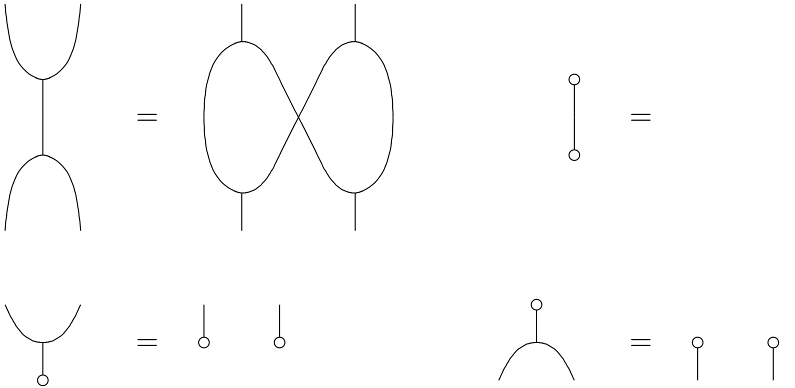
k-algebra = monoid object in the category k-Mod

(k-Mod seen as a monoidal category equipped with the familiar tensor product $\otimes$ of k-modules)

The k-algebra TA is the free monoid object in the category k-Mod

A basic problem in algebra

A k-bialgebra H is a k-module equipped with a k-algebra and a k-cogebra structure, making the bialgebra's compatibility diagrams commute:



A basic problem in algebra

There exists (in general) no free k-bialgebra for a given k-module [Loday]

That is, the forgetful functor

$$U_{\mathsf{Big}} \quad : \quad k\text{-}\mathsf{Big} \longrightarrow k\text{-}\mathsf{Mod}$$

does not have a left adjoint.

A basic problem in algebra

We want to understand more conceptually what distinguishes

- the forgetful functor U_{Alg} which has a left adjoint
- from the forgetful functor U_{Big} which does not have a left adjoint.

Algebraic theories

An algebraic theory is a category $\mathbb{L}$ with finite products

- objects

$$0, 1, 2, \ldots$$

- categorical product provided by

$$m_1 + \ldots + m_k.$$

An $\mathbb{L}$-model A in a Cartesian category $(\mathbb{C}, \times, \mathbf{1})$ is a finite-product preserving functor $A \quad : \quad \mathbb{L} \to \mathbb{C}$

$$A[m_1 + \ldots + m_k] \quad \longrightarrow \quad A[m_1] \times \ldots \times A[m_k]$$

Examples of algebraic theories

- trivial theory: $\mathbb{L}$, the free category with finite product generated by the category with one object

$$\mathsf{Model}(\mathbb{L}, \mathbb{C}) \cong \mathbb{C}$$

- theory of monoids: $\mathbb{M}$, the category whose n-ary operations are the finite words (of arbitrary length) built on an alphabet $[n] = \{1, \ldots, n\}$ of n letters

$$\mathsf{Model}(\mathbb{L}, \mathbb{C}) \cong \mathit{Mon}(\mathbb{C})$$

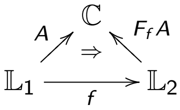
Free models as Kan extensions

Any finite-product preserving morphism $f : \mathbb{L}_1 \to \mathbb{L}_2$ defines a forgetful functor by precomposition

$$U_f \quad : \quad \text{Model}(\mathbb{L}_2, \mathbb{C}) \quad \longrightarrow \quad \text{Model}(\mathbb{L}_1, \mathbb{C}).$$

When $\mathbb{C}$ is Cartesian closed and has all small colimits (e.g. *Set*),

free model $F_f(A)$ of $A : \mathbb{L}_1 \to \mathbb{C}$ along f $\quad = \quad$ left Kan extension



Free models as Kan extensions

The construction is functorial

For example, the free monoid in *Set* is computed as

$$A^* \quad = \quad \coprod_{n \in \mathbb{N}} A^{\times n}.$$

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The analogy with the tensor algebra is striking

$\Rightarrow$ adapt algebraic theory to linear theory

Linear theory : PRO

Cartesian category	$\longrightarrow$	monoidal category
finite-product preserving functor	$\longrightarrow$	monoidal functor

Examples of PROs

- trivial PRO: $\mathbb{N}$ = the free monoidal category generated by the category with one object:

$$MonCat(\mathbb{N})(\mathbb{C}) \quad \cong \quad \mathbb{C}$$

- PRO of monoids: Δ = the category of augmented simplices

$$MonCat(\Delta)(\mathbb{C}) \quad \cong \quad Mon(\mathbb{C})$$

The tensor algebra

Let f be the unique monoidal functor from $\mathbb{N}$ to Δ that sends $1 \mapsto 1$

When $\mathbb{C} = k$-Mod, the Kan extension is

$$\mathrm{Lan}_f A \quad : \quad p \quad \mapsto \quad \bigoplus_{n \in \mathbb{N}} \Delta(n,p) \otimes A^{\otimes n}$$

where the k-module $\Delta(n,p) \otimes A^{\otimes n}$ means the direct sum of as many copies of the k-module $A^{\otimes n}$ as there are elements in the hom-set $\Delta(n,p)$.

$$\mathrm{Lan}_f A(1) \quad = \quad TA$$

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Unfortunately, the Kan extension in Cat is not always a Kan extension in *MonCat*.

When is the left Kan extension of a monoidal functor A

along a monoidal functor f, a monoidal left Kan extension?

T-algebraic theory

Given a pseudo-monad T on Cat, define the 2-category Cat^T

- T-algebraic category = pseudo-algebra of the pseudo-monad T,
- T-algebraic functor = pseudo-algebra pseudo-functor,
- T-algebraic natural transformation = pseudo-algebra natural transformation.

A T-algebraic theory is then a small T-algebraic category

Examples of T-algebraic theories

T-algebraic theories	$T\mathbb{A}$
algebraic theories	free category with finite products
linear theories	free monoidal category
symmetric theories	free symmetric monoidal category
braided theories	free braided monoidal category
projective sketches	free category with finite limits

Algebraic distributors at work [Benabou]

The bicategory of distributors consists in

- Categories as 0-cells
- Functors from

$$\mathbb{A} \times \mathbb{B}^{\mathsf{op}} \longrightarrow \mathit{Set}$$

  as 1-cells, noted

$$\mathbb{A} \longrightarrow\!\!\!\!\!\!\mapsto \mathbb{B}$$

- Natural transformations as 2-cells

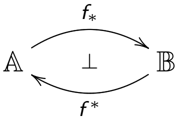
Right adjoint and Kan extension

Every functor $f : \mathbb{A} \longrightarrow \mathbb{B}$ gives rise to a distributor

$$f_* : \mathbb{A} \longrightarrow\!\!\!\!\!\!\!\!\;\;+\;\; \mathbb{B}$$

which as a right adjoint

$$f^* : \mathbb{B} \longrightarrow\!\!\!\!\!\!\!\!\;\;+\;\; \mathbb{A}$$

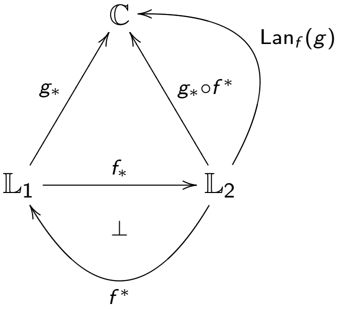


Right adjoint and Kan extension

The Kan extension of a functor f along a functor j is obtained by

- first composing g_* and f^*
- then taking the representative $\mathsf{Lan}_f(g)$ of $g_* \circ f^*$

$$\mathsf{Dist}(g_* \circ f^*, h_*) \quad \cong \quad \mathsf{Cat}(\mathsf{Lan}_f(g), h)$$



The two ingredients of the recipe

Ingredient n°1:

the adjunction
$f_* \dashv f^*$
is T-algebraic

Ingredient n°2:

the T-algebraic distributor
$g_* \circ f^* : \mathbb{A} \quad \longrightarrow\!\!\!\!\!\!\mapsto \quad \mathbb{C}$
is represented by a T-algebraic functor

The two ingredients of the recipe

Ingredient n°1:

the adjunction $f_* \dashv f^*$ is T-algebraic	$\Longrightarrow$	operadicity

Ingredient n°2:

the T-algebraic distributor $g_* \circ f^* : \mathbb{A} \longrightarrow\!\!\!\!\!\!+\;\; \mathbb{C}$ is represented by a T-algebraic functor	$\Longrightarrow$	as the required algebraic colimits

Proarrow equipment [Wood]

A proarrow equipment is a formalisation of the homomorphism of bicategories between Cat and Dist. It consists in a homomorphism of bicategories

$$(-)_* : \mathcal{K} \to \mathcal{M}$$

satisfying the three axioms:

1. The object of $\mathcal{M}$ are those of $\mathcal{K}$ and $(-)_*$ is the identity on objects.
2. $(-)_*$ is locally fully faithful, ie.

$$\mathcal{K}(f, g) \cong \mathcal{M}(f_*, g_*)$$

3. For every arrow f in $\mathcal{K}$, f_* has a right adjoint f^*.

Representative of $\mathcal{M}$ in $\mathcal{K}$

an arrow $g : B \longrightarrow C$ of $\mathcal{K}$ represents an arrow $f : B \longrightarrow C$ of $\mathcal{M}$

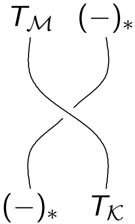
when

$$\mathcal{M}(f, (-)_*) \quad \cong \quad \mathcal{K}(g, -)$$

Pseudomonad in a proarrow equipment

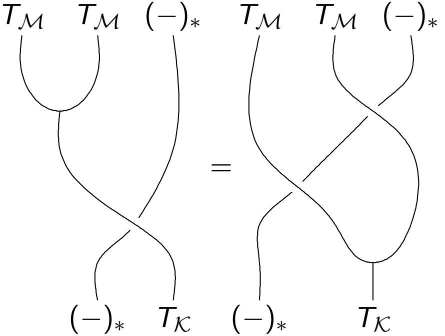
A pseudomonad T in a proarrow equipment $(-)_* : \mathcal{K} \to \mathcal{M}$ is given by

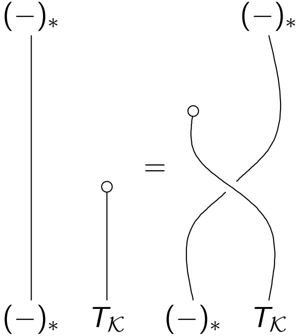
- a pseudomonad $T_{\mathcal{K}}$ on $\mathcal{K}$
- a pseudomonad $T_{\mathcal{M}}$ on $\mathcal{M}$
- a pseudo natural transformation $h : T_{\mathcal{M}} \circ (-)_* \to (-)_* \circ T_{\mathcal{K}}$ noted



making $((-)_*, h)$ be a map of pseudomonads from $T_{\mathcal{K}}$ to $T_{\mathcal{M}}$,

Pseudomonad in a proarrow equipment





Operadicity

A $T_{\mathcal{K}}$-algebraic morphism f of $\mathcal{K}$ is operadic

when its right adjoint f^* in $\mathcal{M}$ is $T_{\mathcal{M}}$-algebraic

f_* is $T_{\mathcal{M}}$-algebraic

$$\mathbb{A} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathbb{B} \quad (\perp)$$

f^* is $T_{\mathcal{M}}$-algebraic

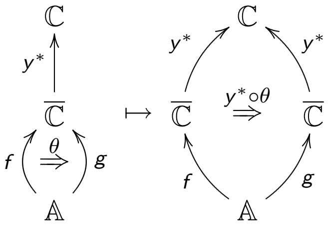
Recall that f^* is always a lax $T_{\mathcal{M}}$-algebraic morphism.

Algebraic colimits

An object $\mathbb{C}$ of $\mathcal{K}$ is algebraically cocomplete
(wrt. the object $\overline{\mathbb{C}}$) when there is an adjunction in $\mathcal{K}$

$$\text{colim} : \overline{\mathbb{C}} \rightleftarrows \mathbb{C} : y$$

- y is full and faithful
- colim,y and y^* are algebraic
- y^* creates isomorphisms, ie.



$y^* \circ \theta$ iso $\Rightarrow \theta$ iso

Main result

Hypotheses:

- $f : \mathbb{L}_1 \to \mathbb{L}_2$ is operadic,
- $\mathbb{C}$ is algebraically cocomplete via the adjunction

$$\text{colim} : \overline{\mathbb{C}} \rightleftarrows \mathbb{C} : y$$

- for all morphism $g : \mathbb{L}_1 \to \mathbb{C}$ in $\mathcal{K}$, $g_* \circ f^*$ factorises through y^*.

Main result

Then, the forgetful functor

$$U_f \quad : \quad \mathsf{Model}(\mathbb{L}_2, \mathbb{C}) \to \mathsf{Model}(\mathbb{L}_1, \mathbb{C})$$

has a left adjoint computed by left Kan extension :

$$\mathsf{Lan}_f \quad : \quad \mathsf{Model}(\mathbb{L}_1, \mathbb{C}) \to \mathsf{Model}(\mathbb{L}_2, \mathbb{C}).$$

When the proarrow equipment is $(-)_* : \mathsf{Cat} \to \mathsf{Dist}$, this left Kan extension is computed by

$$\boxed{\mathsf{Lan}_f A \quad = \quad \int^{m \in \mathbb{L}_1} \mathbb{L}_2(fm, n) \otimes A^{\otimes m}}$$

Operadicity

When the proarrow equipment is $(-)_* : \mathrm{Cat} \to \mathrm{Dist}$, operadicity means that

$$\int^{h \in T(\mathbb{L}_1)} \mathbb{L}_1(m, [h]) \otimes T(\mathbb{L}_2)(Tf(h), n) \quad \longrightarrow \quad \mathbb{L}_2(fm, [n])$$

is an isomorphism

Operadicity

operadicity	$=$	tree decomposition property

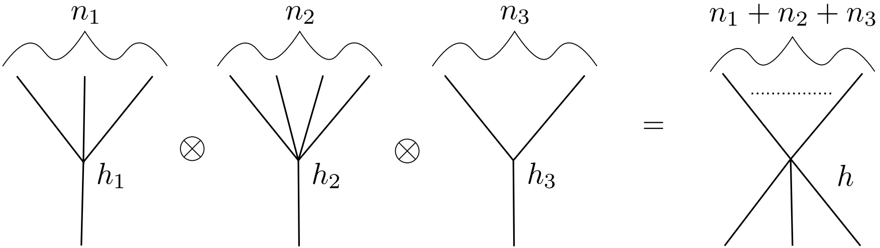
Operadicity for linear theories

When T is the pseudomonad for monoidal category, the isomorphism becomes

$$\int^{h_1 \in \mathbb{L}_1} \cdots \int^{h_k \in \mathbb{L}_1} \mathbb{L}_1(h, h_1 + \cdots + h_k) \times \mathbb{L}_2(h_1, n_1) \times \cdots \times \mathbb{L}_2(h_k, n_k)$$

$$\longrightarrow \quad \mathbb{L}_2(h, n_1 + \cdots + n_k)$$

Operadicity for linear theories



Operadicity for linear theories

This terminology "operadic" is justifies by the fact:

Every map of operads f between two operads $\mathbb{L}_1$ and $\mathbb{L}_2$ (seen as monoidal categories) is operadic

Factorisation system of Cat [Street, Walters]

- $\mathcal{E}$: the classe of final functors
- $\mathcal{M}$: the classe of discrete fibrations

Any diagram $F : J \to \mathbb{C}$ may be seen as the presheaf φ given by the decomposition

$$J \xrightarrow{F} \mathbb{C} \quad = \quad J \xrightarrow{F_1} \mathsf{Elt}\varphi \xrightarrow{F_2} \mathbb{C}$$

where F_1 is a final functor and F_2 is a discrete fibration.

Algebraically cocomplete

When the proarrow equipment is $(-)_* : \mathsf{Cat} \to \mathsf{Dist}$,

algebraic cocompleteness = colimits under some class $\mathcal{F}$ commute with the T-algebraic structure

Algebraically cocomplete : linear theories

When T is the pseudo-monad for monoidal categories, one chooses a subcategory of the category of presheaves

$$\overline{\mathbb{C}} \hookrightarrow \widehat{\mathbb{C}}$$

closed under the Day's tensor product

$$\varphi_1 \otimes_{\overline{\mathbb{C}}} \varphi_2 : b \mapsto \int^{a_1, a_2 \in \mathbb{C}} \mathbb{C}(b, a_1 \otimes_{\mathbb{C}} a_2) \otimes \varphi_1(a_1) \otimes \varphi_2(a_2)$$

Algebraically cocomplete : linear theories

This is the case for example when the class $\mathcal{F}$ is closed under product

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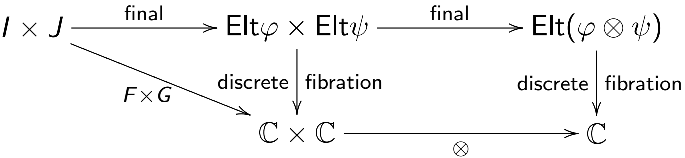


Algebraically cocomplete : linear theories

$\mathbb{C}^\bullet$ is the restriction of the category of presheaves to presheaves having a colimit in $\mathbb{C}$

$$\mathbb{C} \underset{\text{colim}}{\overset{y}{\rightleftarrows}} \mathbb{C}^\bullet \hookrightarrow \widehat{\mathbb{C}} \qquad (\text{colim} \dashv y)$$

$\overline{\mathbb{C}}$ is the restriction of the category of presheaves to presheaves having an algebraic colimit in $\mathbb{C}$

$$\mathbb{C} \underset{\text{colim}}{\overset{y}{\rightleftarrows}} \overline{\mathbb{C}} \hookrightarrow \mathbb{C}^\bullet \hookrightarrow \widehat{\mathbb{C}} \qquad (\text{colim} \dashv y)$$

Observe that $\overline{\mathbb{C}}$ and $\widehat{\mathbb{C}}$ are equipped with $\otimes_{\mathsf{Day}}$ but not necessarily $\mathbb{C}^\bullet$.

Free monoid: the Dubuc construction

$\mathbb{C}$ is an monoidal category with colimits for which

- coequalisers commute with the tensor product
- sequential colimits commute with the tensor product

Then we can compute the free monoid on pointed object

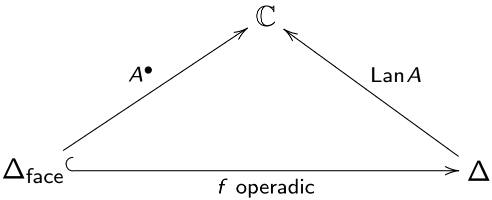
Free monoid: the Dubuc construction

- $\mathbb{L}_1 = \Delta_{\mathsf{face}}$: the category of augmented simplices and injective maps

  theory of pointed objects

- $\mathbb{L}_2 = \Delta$: the category of augmented simplices

  theory of monoids



Free monoid: the Dubuc construction

In practice, we have to show that

all the diagrams defining the Kan extension in Dist

live in $\overline{\mathbb{C}}$

Free monoid: the Dubuc construction

Coequalisers commute with the tensor product in $\mathbb{C}$.

Thus, the presheaf φ_n associated to the diagram

$$1 \longrightarrow A \rightrightarrows A^{\otimes 2} \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; A^{\otimes n}$$

lives in $\overline{\mathbb{C}}$ for every n.

Free monoid: the Dubuc construction

As sequential colimits commute with the tensor product in $\mathbb{C}$, the sequential colimit of the presheaves φ_n

$$1 \longrightarrow A \rightrightarrows A^{\otimes 2} \cdots A^{\otimes n} \cdots$$

$$=$$

$$\Delta_{\text{face}} \xrightarrow{A^{\bullet}} \mathbb{C}$$

lives in $\overline{\mathbb{C}}$

Free monoid: the Vallette/Lack construction

$\mathbb{C}$ is an monoidal category with colimits for which

- reflexive coequalisers commute with the tensor product
- sequential colimits commute with the tensor product

Then we can compute the free monoid on pointed object

Free monoid: the Vallette/Lack construction

Recipe : replace the pair

$$A \overset{f}{\underset{g}{\rightrightarrows}} A^{\otimes 2}$$

with the reflexive pair (having the same coequaliser)

$$A \oplus A^{\otimes 2} \overset{f \oplus A^{\otimes 2}}{\underset{g \oplus A^{\otimes 2}}{\rightrightarrows}} A^{\otimes 2} \qquad i_2 : A^{\otimes 2} \to A \oplus A^{\otimes 2}$$

and apply the same construction.

Free commutative monoid

$\mathbb{C}$ is an symmetric monoidal category with colimits for which

- coequalisers commute with the tensor product
- coproducts commute with the tensor product

Then we can compute the free commutative monoid

Free commutative monoid

First, we coequalise the permutation on $A^{\otimes n}$

Then we take the coproduct of the coequalisers

$$TA = \bigoplus_{n \in \mathbb{N}} A^{\otimes n} / \sim$$

Future work

Applying this construction to games
,where we have almost no colimit,
to compute the free commutative comonoid.

TA is the game where Opponent can open
as many copies of the game A as he wants.

Enable to construct a Cartesian closed category of games
starting from a symmetric monoidal one.