ON THE FUČIK SPECTRUM OF THE LAPLACIAN AND THE p-LAPLACIAN

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"2000 Seminar in Differential Equations" May-June 2000,
Kvilda (Czech Republic)

Contents

1. Introduction

Let $N \geq 1$, Ω be a bounded domain of $\mathbb{R}^N$ and $\Delta u \stackrel{\text{def}}{=} \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$ the laplacian of u defined for all $u \in C^2(\Omega)$. Let us recall that a real value λ is called an *eigenvalue of* $-\Delta$ on Ω with Dirichlet boundary conditions if the problem

$$\begin{cases} -\Delta u(x) = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases} \tag{1.1}$$

has a nontrivial solution u (in a sense that will be precised later). The set σ of all eigenvalues is called de *spectrum.*

The *Fučik spectrum of* $-\Delta$ on Ω with Dirichlet boundary conditions is defined as the set $\Sigma \subset \mathbb{R}^2$ of pairs $(\alpha, \beta) \in \mathbb{R}^2$ such that the problem

$$\begin{cases} -\Delta u(x) = \alpha u^+ - \beta u^- \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

has a nontrivial solution. Here $s^+ = \max\{s, 0\}$ and $s^- = (-s)^+$. Clearly Σ generalizes the notion of the spectrum since σ consists of all the values $\lambda \in \mathbb{R}$ such that (1.2) for $\alpha = \beta = \lambda$ has a nontrivial solution u.

The Fučik spectrum was introduced by S. Fučik (c.f. [18]) and N. Dancer [11] in the 70's, mainly motivated by the works of A. Ambrosetti and G.Prodi (see for instance [3]) in connexion with what is known in the litterature as "asymmetric problems". Briefly, this kind of problems deals with the solvability of the following BVP

$$\begin{cases} -\Delta u(x) = f(u) + h(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

when the term $f(\cdot)$ is nonlinear and the limits $\alpha := \lim_{s\to+\infty} \frac{f(s)}{s}$, $\beta := \lim_{s\to-\infty} \frac{f(s)}{s}$ are different. As a general fact, the solvability of the above BVP for any h depends strongly on the position of (α, β) with respect to Σ.

The set Σ itself has attracted an enormous interest among mathematicians and it is today a current topic of research. It can be easily verified when $N = 1$ that Σ is composed by two lines and a sequence of hyperbolic-like curves. The Fučik spectrum for $N > 1$ and $\Omega \subset \mathbb{R}^N$ an arbitrary domain have not yet been completely determined, despite the fact that Σ has been studied with at least three different approaches : topological, variational and analitical.

The Fučik spectrum, as a notion, can be extended to other differential operators not necessarily linear. We will consider here the so called "p-laplacian operator" which is defined, for any $p > 1$, as : $\Delta_p \stackrel{\text{def}}{=} div(|\nabla u|^{p-2}\nabla u)$. The Fučik spectrum of the p-laplacian is then defined as the set Σ_p of $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\begin{cases} -\Delta_p u = \alpha u^{+p-1} - \beta u^{-p-1} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{1.3}$$

has a nontrivial solution u.

In this work we present some of the results on Σ and Σ_p that have been obtained, by differents authors, using mainly variational methods. We also show some examples where the Fučik spectrum can be totally or partially computed.

2. The spectrum of the laplacian

Let $H_0^1(\Omega)$ be the usual Sobolev espace. We will denote by $||u|| \stackrel{\text{def}}{=} (\int_\Omega |\nabla u|^2\, dx)^{1/2}$ the norm of $H_0^1(\Omega)$, $|\cdot|_2$ the L^2−norm and $<,>$ the duality product on $H_0^1(\Omega)$.

Let us recall the weak formulation of the eigenvalue problem (1.1). A value $\lambda \in \mathbb{R}$ is an *eigenvalue of* $-\Delta$ *on* $H_0^1(\Omega)$ if and only if there exists $u \in H_0^1(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \nabla u \nabla \varphi dx = \lambda \int_\Omega u\varphi dx \quad \text{for all } \varphi \in H_0^1(\Omega). \tag{2.1}$$

The function u is called an eigenfunction associated to λ.

For any $f \in L^2(\Omega)$ let us denote by $G(f)$ the unique solution $u \in H_0^1(\Omega)$ of $-\Delta u = f$ and by i the compact inclusion of $H_0^1(\Omega)$ in $L^2(\Omega)$. Let $T = i \circ G$. Since

T is a selfadjoint compact operator the theory of selfadjoint compact operators on Hilbert spaces gives the existence of a sequence of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ going to $+\infty$ such that each λ_n has finite multiplicity. Moreover $H_0^1(\Omega)$ possesses a hilbertian base composed of eigenfunctions associated to the sequence λ_n.

One can use a different approach to get the existence of an infinite sequence of eigenvalues satisfying all the properties mentioned above. The following proof comes from [34] and it has the advantage of giving the Courant-Fischer minimax formulae for the eigenvalues.

First we need to reformulate (2.1) variationally. Let us consider the following C^1-functionals defined in $H_0^1(\Omega)$:

$$E(u) \stackrel{\text{def}}{=} ||u||^2$$

and

$$I(u) \stackrel{\text{def}}{=} |u|_2^2.$$

We denote the Fréchet derivative of E and I by E' and I' respectively. Thus we can write

$$(2.1) \Leftrightarrow E'(u) = \lambda I'(u).$$

Let us now put the problem of finding eigenvalues in a more abstract setting.

2.1. **Critical values of functionals restricted to manifolds.** Let X be a real Banach espace and $g \in C^1(X, \mathbb{R})$. Set $M \stackrel{\text{def}}{=} \{u \in X,\, g(x) = 1\}$ and assume that $M \neq \emptyset$. Assume further that for all $u \in M$

$$Ker g'(u) \neq X. \tag{2.2}$$

Let $J \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We say that c is a *critical value of J restricted to M* if there exists $u \in M$ such that

$$J(u) = c, \tag{2.3}$$

$$< J'(u), v >= 0 \quad \forall v \in Ker g'(u). \tag{2.4}$$

u is then called *a critical point of J restricted to M*.

Condition (2.2) is equivalent to saying that the set M is a submanifold of X of codimension 1. The tangent space to M at u is the subspace

$$T_u M = \{v \in X,\, < g'(u), v >= 0\}.$$

The norm of the derivative at $u \in M$ of the restriction of J to M is defined as $||J'(u)||_* \stackrel{\text{def}}{=} ||dJ(u)||_{(T_u M)^*}$, where $||\cdot||_{(T_u M)^*}$ denotes the norm on the dual space $(T_u M)^*$.

By the Lagrange multiplier rule, condition (2.4) is equivalent to saying that there exists $\lambda \in \mathbb{R}$ such that

$$J'(u) = \lambda g'(u). \tag{2.5}$$

The following lemma can be easily proved.

Lemma 2.1. *Assume that there exists some $e \in X$ such that*

$$\min\{J(u) : u \in M\} = J(e).$$

Then $J(e)$ is a critical value of J restricted to M and e is a critical point of J restricted to M.

We will use this lemma in the next paragraph.

2.2. **Courant-Fischer minimax formulae.** Now we can recall the "Courant-Fischer" minimax formulae for the eigenvalues. For that purpose let us take $g = I$ $J = E$ and $M = S$ in the previous abstract setting where S is now the following manifold

$$S \stackrel{\text{def}}{=} \{u \in H_0^1(\Omega) : I(u) = 1\}.$$

Theorem 2.1. (i) *Let us define for all $\mathbb{N}_*$ the value*

$$\lambda_n \stackrel{\text{def}}{=} \min_{X_n} \max_{u \in X_n \cap S} E(u) \tag{2.6}$$

where the minimum corresponds to the familly of subspaces X_n of dimension n of $H_0^1(\Omega)$. Then $\{\lambda_n\}_n$ forms a sequence of eigenvalues of $-\Delta$ on $H_0^1(\Omega)$ such that $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and $\lim\limits_{n \to +\infty} \lambda_n = +\infty$. **(ii)** *There exists a hilbertian base of $H_0^1(\Omega)$ composed of eigenfunctions associated to the sequence λ_n.*

Remark 2.1. (i) When $n = 1$ formula (2.6) gives the well known variational characterization

$$\lambda_1 = \min_{u \in S} \int_\Omega |\nabla u|^2 dx. \tag{2.7}$$

λ_1 is often called the *best Poincaré constant.*
(ii) Formula (2.6) can be rewritten as follows

$$\lambda_n = \min_{X_n} \max_{u \in X_n \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx} \tag{2.8}$$

where the minimum corresponds to subspaces X_n of dimension n of $H_0^1(\Omega)$. The quotient $\frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$ is called the *Rayleigh quotient.*

Proof. Let us start by considering the problem

$$(P_1) \qquad \min_{u \in S} E(u).$$

It is clear that every minimizing sequence u_n is uniformly bounded in $H_0^1(\Omega)$. Moreover, since the norm in a Banach space is weakly lower semicontinuous and S is weakly closed (because of the compact inbedding of $H_0^1(\Omega)$ in $L^2(\Omega)$), we can extract a subsequence u_{n_k} converging weakly in $H_0^1(\Omega)$ to some e_1 and strongly in $L^2(\Omega)$. Hence $e_1 \in S$ and $E(e_1) = \min_{u \in S} E(u)$. By lemma 2.1, $\lambda_1 := \min_{u \in S} E(u)$ is a positive eigenvalue and e_1 is an eigenfunction associated to λ_1.

Let us now assume that we have built $e_1, \dots, e_{n-1}$ eigenfunctions associated to $\lambda_1, \dots, \lambda_{n-1}$ such that $\int_\Omega e_i e_j dx = \delta_{ij}$. Using (2.1) with $u = e_i$ and $\varphi = e_j$ we deduce that

$$< E'(e_i), e_j >= \lambda_i \delta_{ij}. \tag{2.9}$$

Consider now the following problem

$$(P_n) \qquad \min\{E(u) \,:\, u \in S \text{ and } \int_\Omega u e_i dx = 0 \;\; \forall i = 1 \dots, n-1\}.$$

We associate to this problem the space $X = \{u \in H_0^1(\Omega);\, \int_\Omega u e_i dx = 0 \;\; \forall i = 1 \dots, n-1\}$ in order to apply lemma 2.1. Again the $\inf_{u \in X \cap S} E(u)$ is achieved at some point $e_n \in X \cap S$ because the constraints in the definition of X are weakly continuous. Hence $\lambda_n := E(e_n)$ is a critical value of E restricted to $X \cap S$ and we have

$$< E'(e_n), v >= \lambda_n \int_\Omega e_n v dx \qquad \forall v \in X. \tag{2.10}$$

Let v be arbitrarily choosen in $H_0^1(\Omega)$ and consider $w = v - \sum_{i=1}^n (\int_\Omega e_i v dx) e_i$. Then $w \in X$ and by (2.9) and (2.10) we have

$$< E'(e_n), v >=< E'(e_n), w >= \lambda_n \int_\Omega e_n w \, dx = \lambda_n \int_\Omega e_n v \, dx.$$

We have proved that λ_n is an eigenvalue.

We prove now the minimax formula (2.6). Let $\hat{X}_n = span\{e_1, \dots, e_n\}$ and $u = \sum_{i=1}^n \alpha_i e_i \in \hat{X}_n$. We have

$$E(u) = \sum_{i=1}^n \alpha_i^2 \lambda_i \le \lambda_n \sum_{i=1}^n \alpha_i^2 = \lambda_n I(u)$$

and then

$$\max_{u \in \hat{X}_n \cap S} E(u) = \lambda_n.$$

Besides if X_n is any subspace of dimension n there exists $v \in X_n$ such that $I(v) = 1$ and $\int_\Omega e_1 v dx = \dots = \int_\Omega e_n v dx = 0$. Hence $\lambda_n \le E(v) \le \max_{u \in X_n \cap S} E(u)$ and the result follows.

To prove that the sequence λ_n is unbounded one applies Bessel's inequality to the orthonormal sequence $\frac{e_n}{\sqrt{\lambda_n}}$. It follows that $\frac{e_n}{\sqrt{\lambda_n}}$ converges weakly to 0 in $H_0^1(\Omega)$ and consequently $\lim_{n\to\infty} \int_\Omega \frac{e_n^2}{\lambda_n} dx = \lim_{n\to\infty} \frac{1}{\lambda_n} = 0.$

It remains to prove that $\{e_1, e_2, \dots\}$ is a hilbertian base. Let $u \in H_0^1(\Omega) \setminus \{0\}$ be such that $\int_\Omega \nabla e_n \nabla u \, dx = 0$ for all $n \in \mathbb{N}$. By (2.10) $\int_\Omega e_n u dx = 0$ for all $n \in \mathbb{N}$ and then $\frac{u}{\sqrt{I(u)}}$ is admissible in (P_n) for all $n \in \mathbb{N}$. Thus $\lambda_n \le E(\frac{u}{\sqrt{I(u)}})$. Since $\{\lambda_n\}$ is unbounded we have a contradiction. □

2.3. **Simplicity of the first eigenvalue.** The following result is known in the litterature as a "Krein-Rutman" result. The proof that we reproduce here can be found in [34].

Theorem 2.2. *λ_1 is simple, i.e., $\dim Ker(-\Delta - \lambda_1 Id) = 1$. Moreover there exists an eigenfunction ϕ_1 associated to λ_1 such that $\phi_1(x) > 0$ for all $x \in \Omega$.*

Remark 2.2. If follows from the L^p regularity theory (see [8] or [19]) that the eigenfunctions are of class $C^\infty(\Omega)$. If moreover Ω is of class C^∞ then the eigenfunctions are of class $C^\infty(\overline{\Omega})$.

Proof. Let u be an eigenfunction associated to λ_1. By choosing $\varphi = u$ in (2.1) we deduce that $\int_\Omega |\nabla u|^2 dx = \lambda_1 \int_\Omega |u|^2 dx$. By dividing u by its L^2-norm we get a function $e_1 \in S$ where the infimum on the definition of λ_1 is achieved. Since $E(e_1) = E(|e_1|)$ and $J(e_1) = J(|e_1|)$ it follows that $|e_1|$ also realizes the minimum on the definition of λ_1 and therefore $|e_1|$ is an eigenfunction for λ_1. By Remark 2.2 above, $|e_1|$ is of class C^∞ and then we have pointwise in Ω

$$-\Delta(-|e_1|) = \lambda_1(-|e_1|) \leq 0.$$

By Hopf's maximum principle (see [19]) $-|e_1|$ cannot achieve its supremum in Ω unless it is constant. But this last possibility is ruled out by the condition $e_1 \in S$. Hence $-|e_1(x)| < 0 = \sup_\Omega -|e_1|$ for all $x \in \Omega$. We have proved that any eigenfuntion u associated to λ_1 is such that either $u(x) > 0$ for all $x \in \Omega$ or $u(x) < 0$ for all $x \in \Omega$.

To prove the simplicity of λ_1 let u, v be two eigenfuctions associated to λ_1. From the previous result we can assume without loss of generality that $u > 0$ and that $v > 0$. It is also clear that we can choose $\alpha \in \mathbb{R}$ such that $u + \alpha v$ vanishes somewhere in Ω. This is in contradiction with the previous result since $u + \alpha v$ is an eigenvalue associated to λ_1. □

3. Computing the Fučik spectrum in some domains

Let us recall that the Fučik spectrum of the laplacian in $H_0^1(\Omega)$ is the set $\Sigma \subset \mathbb{R}^2$ of pairs $(\alpha, \beta) \in \mathbb{R}^2$ such that problem (1.2) has a nontrivial solution $u \in H_0^1(\Omega)$. Solutions are understood in the weak sense, i.e.

$$\int_\Omega \nabla u \nabla \varphi dx = \int_\Omega (\alpha u^+ - \beta u^-)\varphi dx \quad \text{for all } \varphi \in H_0^1(\Omega). \tag{3.1}$$

The function u is called an eigenfunction associated to the pair (α, β).

Let $\{\lambda_k\}$ be the sequence of eigenvalues of the laplacian on $H_0^1(\Omega)$. Hence the pairs (λ_k, λ_k) belong to Σ. Besides, using that the eigenfuctions associated to λ_1 have definite sign (c.f. Theorem 2.2.) one can easily prove that the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$ are contained in Σ. Furthermore, the variational characterization (2.7) of λ_1 implies that

$$\Sigma \setminus (\{\lambda_1\} \times \mathbb{R} \cup \mathbb{R} \times \{\lambda_1\}) \subset \{(\alpha, \beta) \in \mathbb{R}^2 \, : \, \alpha > \lambda_1, \beta > \lambda_1\}.$$

We call this two lines the "trivial part" of Σ.

The existence of curves of points contained in the nontrivial part of Σ has been studied by many authors. We will see in Corollary 6.1 in section 6 the proof given by O. Kavian [21] showing there exists at least a pair of curves (not necesserely different) emanating from each point (λ_k, λ_k) of Σ.

We give below some examples were the Fučik spectrum can be completely or partially computed.

Since we are dealing here with different domains Ω we will denote Σ by $\Sigma(\Omega)$.

3.1. **The Fučik spectrum in dimension 1.** Let us consider problem (1.2) when $N = 1$ and $\Omega = (0, 1)$. Then problem (1.2) becomes

$$\begin{cases} -u''(t) = \alpha u^+(t) - \beta u^-(t) & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{3.2}$$

Solving the above second order equation in the case $\alpha = \beta$ one can easily show that the eigenvalues of the laplacian in $H_0^1(0, 1)$ form the sequence : $\lambda_k = k^2\pi^2, k \in \mathbb{N}_*$. The eigenfunctions associated to λ_k are the multiples of the function $v_k(t) = \sin(k\pi t)$.

In order to compute the "nontrivial part" of $\Sigma(0, 1)$ let us assume that u is a generalized eigenfunction for the pair (α, β) and that u changes sign. Since the zeros of u are simple, the function u has a finite number of positive and negative bumps. We can estimate the length of these bumps as follows. If $u \geq 0$ in, let say, $(0, l_1)$ then

$$-u''(t) = \alpha u, \qquad u(0) = u(l_1) = 0$$

and consequently $u(t) = a \sin \sqrt{\alpha} t$ and $\sqrt{\alpha} = \frac{\pi}{l_1}$. If $u \leq 0$ in some $(0, l_2)$ then

$$-u''(t) = \beta u, \qquad u(l_2) = u(0) = 0$$

and therefore $u(t) = a \sin \sqrt{\beta} t$ and $\sqrt{\beta} = \frac{\pi}{l_2}$. Thus if u is a solution with n positive and p negative bumps then $|n - p| = 1, 0$ and the total lenght is $nl_1 + pl_2 = 1$. Thus we have find the following identity :

$$n\frac{\pi}{\sqrt{\alpha}} + p\frac{\pi}{\sqrt{\beta}} = 1.$$

By considering all the pairs $(n, p) \in (\mathbb{N}_*)^2$ such that $|n - p| = 1, 0$ we obtain that the nontrivial part of $\Sigma(0, 1)$ is composed by a sequence of curves: there is only one curve

$$C_k = \{(\alpha, \beta) \, : \, k\frac{\pi}{\sqrt{\alpha}} + k\frac{\pi}{\sqrt{\beta}} = 2\}$$

going through (λ_k, λ_k) for k even and two curves

$$C_k^1 = \{(\alpha, \beta) \, : \, (k+1)\frac{\pi}{\sqrt{\alpha}} + (k-1)\frac{\pi}{\sqrt{\beta}} = 2\}$$

$$C_k^2 = \{(\alpha, \beta) \, : \, (k+1)\frac{\pi}{\sqrt{\alpha}} + (k-1)\frac{\pi}{\sqrt{\beta}} = 2\}$$

going through (λ_k, λ_k) if k is odd.

3.2. **The Fučik spectrum in a rectangle.** Let us now consider problem (1.2) when Ω is a rectangle in $\mathbb{R}^N$, say $\Omega = \Pi_{i=1}^N [0, c_i]$, $c_i \in \mathbb{R}^+_*$. Using separation of variables one can show that the values λ_n, $n = (n_1, \dots, n_N) \in (\mathbb{N})^N_*$, defined by

$$\lambda_n = \sum_{i=1}^{N} \frac{\pi^2}{c_i^2} n_i^2, \tag{3.3}$$

are eigenvalues of $-\Delta$ in $H_0^1(\Pi_{i=1}^N [0, c_i])$. Notice that you can have $\lambda_n = \lambda_m$ for $n \neq m$. The eigenfunctions associated to λ_n are the constant multiples of

$$\phi_n(x) = \Pi_{i=1}^N \sin(\frac{n_i \pi}{c_i} x_i); \qquad x = (x_1, \dots, x_N).$$

By Fourier analysis the set of all these eigenfunctions forms a complete orthogonal set of $H_0^1(\Pi_{i=1}^N [0, c_i])$ and then there are no other eigenvalues apart from those listed in (3.3). See for instance [12].

We can also find points of the Fučik spectrum in a rectangle using separated variables. For that purpose let us introduce the set

$\Sigma^s(\Pi_{i=1}^N [0, c_i]) \stackrel{\text{def}}{=} \{(\alpha, \beta) \in \Sigma(\Pi_{i=1}^N [0, c_i]) \, : \, (\alpha, \beta)$ possesses an eigenfunction with separated variables $\}$.

Let us fix $i \in \{1, \dots, N\}$. If $(\alpha, \beta) \in \Sigma([0, c_i])$, with generalized eigenfunction $v = v(x_i)$, then one can easily show that

$$(\alpha + \sum_{j \neq i}^{N} \frac{\pi^2}{c_j^2}, \beta + \sum_{j \neq i}^{N} \frac{\pi^2}{c_j^2}) \in \Sigma^s(\Pi_{i=1}^N [0, c_i]) \tag{3.4}$$

and that the associated eigenfunction (up to a constant multiple) is

$$u(x) = v(x_i) \Pi_{j \neq i}^N \sin(\frac{\pi}{c_j} x_j).$$

Observe that, since the function u above does not change sign on the $j-$variables for $j \neq i$, you can write $u^+(x) = v^+(x_i) \Pi_{j \neq i} \sin(\frac{\pi}{c_j} x_j)$ and similarly for u^-. The right hand side of (1.2) can be then easily computed. But of course one can imagine eigenfunctions having separated variables that change sign in more than one variable so u^+ can not be splited as easily as before!.

Let us show that in the case $N = 2$ this last situation does not occur.

Theorem 3.1. *Let $N = 2$ and $(\alpha, \beta) \in \Sigma^s([0, c_1] \times [0, c_2])$ with associated eigenfunction $u(x, y) = v(x)\xi(y)$. Then either* **(i)** *v has constant sign and $(\alpha - \frac{\pi^2}{c_1^2}, \beta - \frac{\pi^2}{c_1^2}) \in \Sigma([0, c_2])$ or* **(ii)** *ξ has constant sign and $(\alpha - \frac{\pi^2}{c_2^2}, \beta - \frac{\pi^2}{c_2^2}) \in \Sigma([0, c_1])$ or* **(iii)** *$\alpha = \beta = \lambda_n$ for some $n \in (\mathbb{N}_*)^2$.*

Proof. Let us assume that both v and ξ change sign and choose maximal intervals $[a, a+p_1], [b, b+n_1], [c, c+p_2], [d, d+n_2]$ such that

$$\begin{array}{llll} v(x) > 0 & \text{in } (a, a+p_1), & v(x) < 0 & \text{in } (b, b+n_1); \\ \xi(y) > 0 & \text{in } (c, c+p_2), & \xi(y) < 0 & \text{in } (d, d+n_2). \end{array} \tag{3.5}$$

In $[a, a+p_1] \times [c, c+p_2]$ the equation in (1.2) becomes

$$-\xi''(y)v(x) - \xi(y)v''(x) = \alpha v(x)\xi(y)$$

and hence

$$\frac{-v'' - \alpha v}{v} = \frac{\xi''}{\xi} = const.$$

From the second identity we deduce that $const = -\frac{\pi^2}{p_2^2}$ and from the first one that $\alpha + const = \frac{\pi^2}{p_1^2}$. Thus

$$\alpha = \frac{\pi^2}{p_1^2} + \frac{\pi^2}{p_2^2}.$$

Similarly considering the equation of (1.2) in $[b, b+n_1] \times [d, d+n_2]$ we deduce

$$\alpha = \frac{\pi^2}{n_1^2} + \frac{\pi^2}{n_2^2},$$

in $[a, a+p_1] \times [d, d+n_2]$,

$$\beta = \frac{\pi^2}{p_1^2} + \frac{\pi^2}{n_2^2},$$

and in $[b, b+n_1] \times [c, c+p_2]$,

$$\beta = \frac{\pi^2}{n_1^2} + \frac{\pi^2}{p_2^2}.$$

Hence $p_1 = n_1, p_2 = n_2$ and consequently $\alpha = \beta$. □

Example 3.1. Let us study the beginning of the separated Fučik spectrum of the rectangle $[0, \sqrt{2}] \times [0, 4]$ in $\mathbb{R}^2$. The first 5 eigenvalues are

$$\lambda_{(1,1)} = \frac{9\pi^2}{16},\ \lambda_{(1,2)} = \frac{3\pi^2}{4},\ \lambda_{(1,3)} = \frac{17\pi^2}{16},$$
$$\lambda_{(1,4)} = \frac{3\pi^2}{2},\ \lambda_{(1,5)} = \frac{33\pi^2}{16} = \lambda_{(2,1)}$$

The trivial part of $\Sigma([0, \sqrt{2}] \times [0, 4])$, i.e. the lines $\{\lambda_{(1,1)}\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_{(1,1)}\}$ are contained in $\Sigma^s([0, \sqrt{2}] \times [0, 4])$. The equations of the first 6 curves of the set $\Sigma^s([0, \sqrt{2}] \times [0, 4])$ are the following:

$$\text{through } \lambda_{(1,2)} : C_{(1,2)} = \{(\alpha + \frac{\pi^2}{2}, \beta + \frac{\pi^2}{2}) \ : \ \frac{\pi}{\sqrt{\alpha}} + \frac{\pi}{\sqrt{\beta}} = 4\}$$

$$\text{through } \lambda_{(1,3)} : C^1_{(1,3)} = \{(\alpha + \frac{\pi^2}{2}, \beta + \frac{\pi^2}{2}) \ : \ \frac{\pi}{\sqrt{\alpha}} + \frac{2\pi}{\sqrt{\beta}} = 4\}$$

$$\text{through } \lambda_{(1,3)} : C^2_{(1,3)} = \{(\alpha + \frac{\pi^2}{2}, \beta + \frac{\pi^2}{2}) \ : \ \frac{2\pi}{\sqrt{\alpha}} + \frac{\pi}{\sqrt{\beta}} = 4\}$$

$$\text{through } \lambda_{(1,4)} : C_{(1,4)} = \{(\alpha + \frac{\pi^2}{2}, \beta + \frac{\pi^2}{2}) \ : \ \frac{2\pi}{\sqrt{\alpha}} + \frac{2\pi}{\sqrt{\beta}} = 4\}$$

$$\text{through } \lambda_{(1,5)} : C^1_{(1,5)} = \{(\alpha + \frac{\pi^2}{2}, \beta + \frac{\pi^2}{2}) \ : \ \frac{2\pi}{\sqrt{\alpha}} + \frac{3\pi}{\sqrt{\beta}} = 4\}$$

$$\text{through } \lambda_{(1,5)} : C^2_{(1,5)} = \{(\alpha + \frac{\pi^2}{2}, \beta + \frac{\pi^2}{2}) \ : \ \frac{3\pi}{\sqrt{\alpha}} + \frac{2\pi}{\sqrt{\beta}} = 4\}$$

$$\text{through } \lambda_{(1,5)} = \lambda_{(2,1)} : C_{(2,1)} = \{(\alpha + \frac{\pi^2}{16}, \beta + \frac{\pi^2}{16}) \ : \ \frac{\pi}{\sqrt{\alpha}} + \frac{\pi}{\sqrt{\beta}} = \sqrt{2}\}.$$

Notice that there are 3 curves through $(\lambda_{(1,5)}, \lambda_{(1,5)})$ Observe also that, since the limits as $\beta \to \infty$ of $C_{(1,4)}$ and $C_{(2,1)}$ are, respectively, $\alpha = \frac{3\pi^2}{4}$ and $\alpha = \frac{9\pi^2}{16}$, there are *crossings between different curves of the Fučik spectrum.*

Remark 3.1. It was already observed by [24] that you can obtain more than two curves of Σ^s emanating from multiple eigenvalues.

Remark 3.2. One cannot expect to have $\Sigma(\Pi_{i=1}^N[0,c_i]) = \Sigma^s(\Pi_{i=1}^N[0,c_i])$ in general. In [29] the author proves the existence of a curve in $\Sigma(\Pi_{i=1}^N[0,c_i])\backslash\Sigma^s(\Pi_{i=1}^N[0,c_i])$ for some example in dimension $N = 2$. This result is obtained through a combination of the Implicit Function Theorem and numerical methods.

3.3. **The radial Fučik spectrum.** Let $\Omega = B(0,1)$ be the unit ball of $\mathbb{R}^N$. The radial solutions $u(x) = v(|x|) = v(r)$ of (1.2) satisfy

$$\left\{ \begin{array}{l} v'' + \frac{N-1}{r}v' + \alpha v^+ - \beta v^- = 0, \text{ in } (0,1) \\ v(1) = 0, \ \ \limsup_{r\to 0} v(r) < \infty. \end{array} \right. \tag{3.6}$$

Let us denote by $\Sigma^r(B(0,1))$ (resp. $\sigma^r(B(0,1))$) the set of pairs (α,β) (resp. values $\alpha = \beta = \lambda$) such that (3.6) has a nontrivial solution. We call Σ^r the *radial Fučik spectrum* (resp. σ^r the *radial spectrum*).

Let us recall some known results about $\sigma(B(0,1))$. When $N = 2$ one can decompose $H^1_0(B(0,1))$ into orthogonal linear subspaces $L_n, n \in \mathbb{Z}$, consisting of functions of the form $u(r,\theta) = v(r)e^{in\theta}$. Since the laplacian commutes with the rotations then it maps the subspace L_n into itself, and its eigenvalues can be studied in each subspace independently. The eigenfuntions are then of the form

$$u(r,\theta) = J_n(j_{(n,s)}r)e^{in\theta}$$

where $s = 1, 2, \ldots, n \in \mathbb{Z}$ and $j_{(n,s)}$ is the sth zero of the Bessel function J_n. The corresponding eigenvalue is $j^2_{(n,s)}$. The first 6 eigenvalues in increasing order are listed below. This list comes from [12].

$$
\begin{array}{ll}
\lambda_1 = j^2_{(0,1)} = 5,784 & u_1(r,\theta) = J_0(\sqrt{\lambda_1} r) \\
\lambda_2 = j^2_{(1,1)} = j^2_{(-1,1)} = 14,684 & u_2(r,\theta) = J_1(\sqrt{\lambda_3} r)e^{i\theta} \\
\lambda_3 = j^2_{(2,1)} = j^2_{(-2,1)} = 26,378 & u_3(r,\theta) = J_2(\sqrt{\lambda_5} r)e^{2i\theta} \\
\lambda_4 = j^2_{(0,2)} = 30,470 & u_4(r,\theta) = J_0(\sqrt{\lambda_6} r) \\
\lambda_5 = j^2_{(3,1)} = j^2_{(-3,1)} = 40,704 & u_5(r,\theta) = J_3(\sqrt{\lambda_2} r)e^{3i\theta} \\
\lambda_6 = j^2_{(1,2)} = j^2_{(-1,2)} = 49,224 & u_6(r,\theta) = J_1(\sqrt{\lambda_{10}} r)e^{i\theta}.
\end{array} \tag{3.7}
$$

Observe that for $n = 0$ the eigenvalues $j_{(0,s)}$ are simple and radial and that for $n \neq 0$ the eigenvalues $j^2_{(n,s)}$ are double and non radial. Thus

$$\sigma^r(B(0,1)) = \{j^2_{(0,s)} \,:\, s \in \mathbb{N}\}.$$

The situation for $N > 2$ is rather similar. The radial eigenvalues are the square of the zeros $j_{(\nu,s)}$ of the Bessel function J_ν, where $\nu = \frac{N-2}{2}$, and the corresponding eigenfunctions are constant multiples of $v(r) = r^{1-n/2} J_\nu(j_{(\nu,s)} r)$.

The *radial Fučik spectrum* in dimension $N > 1$ has been completely determined by [6] using shooting methods and Sturm comparison Theorems. We recall briefly their result. Let $v_a(r) = r^{1-N/2} J_\nu(\sqrt{a} r)$ be a solution of the following equation

$$v'' + \frac{N-1}{r} v' + av = 0, \text{ in } (0,1) \tag{3.8}$$

with boundary condition

$$\limsup_{r \to 0} v(r) < \infty, \tag{3.9}$$

where a is any real poisitive number. For $s > 0$ let $v(\cdot, a, s)$ be the solution of (3.8) satisfying the initial conditions

$$v(s,a,s) = 0, \;\; v'(s,a,s) = 1. \tag{3.10}$$

Thus if $v(\cdot,a,s)$ is bounded near 0 then $v(\cdot,a,s) = C v_a(\cdot)$ for some $C > 0$. Define further

$$\Phi_a(s) \stackrel{\text{def}}{=} \min\{t > s \,:\, v(t,a,s) = 0\}.$$

Take now a solution v of (3.6) and assume for instance that $v(0^+) > 0$ (we always have $v'(0) \neq 0$) and that v vanishes once in $(0,1)$, say at t_1. Thus v satisfies (3.8) with $a = \alpha$ on $(0, t_1)$ and with $a = \beta$ on $(t_1, 1)$. Hence $v = C_1 v_\alpha$ on $(0, t_1)$ and $v = C_2 v_\beta$ on $(t_1, 1)$ for some $C_1, C_2 \in \mathbb{R}^+$. Since v_α vanishes for the first time on $\frac{j_{(\nu,1)}}{\sqrt{\alpha}}$ it follows that $\frac{j_{(\nu,1)}}{\sqrt{\alpha}} < 1$ and hence

$$\Phi_\beta\left(\frac{j_{(\nu,1)}}{\sqrt{\alpha}}\right) = 1.$$

If v vanishes twice in $(0,1)$ then

$$\Phi_\alpha \circ \Phi_\beta\left(\frac{j_{(\nu,1)}}{\sqrt{\alpha}}\right) = 1$$

and so on. The case $v(0^+) < 0$ is similar. For instance if v vanishes once in $(0,1)$ then

$$\Phi_\alpha(\frac{j_{(\nu,1)}}{\sqrt{\beta}}) = 1$$

and if v vanishes twice in $(0,1)$ then

$$\Phi_\beta \circ \Phi_\alpha(\frac{j_{(\nu,1)}}{\sqrt{\beta}}) = 1$$

and so on. Thus $(\alpha,\beta) \in \Sigma^r(B(0,1))$ if and only if there exists $n \in \mathbb{N}$ such that

$$(3.11) \qquad \begin{cases} R^1_{2n}(\alpha,\beta) \overset{\text{def}}{=} (\Phi_\alpha \circ \Phi_\beta)^n(\frac{j_{(\nu,1)}}{\sqrt{\alpha}}) = 1 \quad \text{or} \\ R^1_{2n+1}(\alpha,\beta) \overset{\text{def}}{=} \Phi_\beta \circ (\Phi_\alpha \circ \Phi_\beta)^n(\frac{j_{(\nu,1)}}{\sqrt{\alpha}}) = 1 \quad \text{or} \\ R^2_{2n}(\alpha,\beta) \overset{\text{def}}{=} (\Phi_\beta \circ \Phi_\alpha)^n(\frac{j_{(\nu,1)}}{\sqrt{\beta}}) = 1 \quad \text{or} \\ R^2_{2n+1}(\alpha,\beta) \overset{\text{def}}{=} \Phi_\alpha \circ (\Phi_\beta \circ \Phi_\alpha)^n(\frac{j_{(\nu,1)}}{\sqrt{\beta}}) = 1. \end{cases}$$

Notice that the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$ are respectively R^1_0 and R^2_0.

It is proved in [6] that if you set $C^j_i \overset{\text{def}}{=} \{(\alpha,\beta) \in \mathbb{R}^+ \times \mathbb{R}^+ \ : \ R^j_i(\alpha,\beta) = 1\}$, then for all $i \geq 1$ and all $j = 1,2$ there exists a strictly decreasing analytic homeomorphism $c^j_i : (a^j_i, +\infty) \to (b^j_i, +\infty)$ such that $C^j_i = \{(\alpha, c^j_i(\alpha)) \ : \ \alpha \in (a^j_i, +\infty)\}$. Moreover $(j^2_{(\nu,i+1)}, j^2_{(\nu,i+1)}) \in C^j_i$ and, when $\alpha \to +\infty$,

$$(3.12) \qquad \begin{array}{l} c^1_{2n}(\alpha) \to j^2_{(\nu,n)}, \\ c^2_{2n}(\alpha) \to j^2_{(\nu,n+1)}, \\ c^1_{2n+1}(\alpha) \to j^2_{(\nu,n+1)}, \\ c^1_{2n+1}(\alpha) \to j^2_{(\nu,n+1)}. \end{array}$$

When $N = 3$ the radial Fučik spectrum can be easily computed by making the change of variables $w(r) = rv(r)$. Problem (3.6) becomes

$$-w'' = \alpha w^+ - \beta w^- \text{ in } (0,1),\ w(0) = w(1) = 0$$

which is exactly the O.D.E. of paragraph 3.1.

4. The spectrum of the p-laplacian

Let $p > 1$ be a real number, Ω a domain in $\mathbb{R}^N$, $N \geq 1$ and $W^{1,p}_0(\Omega)$ the usual Sobolev space endowed with the norm

$$||u||_p \overset{\text{def}}{=} (\int_\Omega |\nabla u|^p \, dx)^{1/p}.$$

If $p' \overset{\text{def}}{=} \frac{p}{p-1}$ and $W^{-1,p'}(\Omega)$ denotes the dual space of $W^{1,p}_0(\Omega)$, the $p-$laplacian operator $\Delta_p : W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega)$ is defined as $\Delta_p u \overset{\text{def}}{=} div(|\nabla u|^{p-2}\nabla u)$. Thus for $p = 2$ the p-laplacian is the usual laplacian.

The *spectrum of the p-laplacian on* $W_0^{1,p}(\Omega)$ is defined as the set σ_p of $\lambda \in \mathbb{R}$ such that

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{4.1}$$

has a nontrivial solution $u \in W_0^{1,p}(\Omega)$. λ is called an *eigenvalue* and u an *eigenfuction associated to* λ.

When $N = 1$ the spectrum of the $p-$laplacian can be easily computed using a shooting technique (see paragraph 4.1 below) but there does not exist a complete description of the spectrum of the p-laplacian when $N > 1$. It is possible, using variational methods, to define two sequences of eigenvalues going to ∞ but it is not known, for neither of those two sequences, if they form the whole spectrum of the $p-$laplacian. It is not even known if all the eigenvalues are isolated or if the spectrum (which is a closed set in $\mathbb{R}$) contains open sets.

In this section we will summarize the properties on the *first eigenvalue* of the p-laplacian and we will also recall the definiton of the sequence of eigenvalues of Ljusternik-Schnirelman type. The construction of a second sequence of eigenvalues will be done in section 5.

Let us start by computing the spectrum of the p-laplacian when $N = 1$.

4.1. **The spectrum in dimension 1.** Let $\Omega = (0, 1)$. The eigenvalue problem (4.1) reads in this case

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda |u|^{p-2}u \text{ in } (0,1) \\ u(0) = u(1) = 0. \end{cases} \tag{4.2}$$

Set $p' \stackrel{\text{def}}{=} \frac{p}{p-1}$. After multiplying (4.2) by u' we find

$$\frac{d}{dt}((p-1)|u'|^{p'} + \lambda |u|^p) = 0.$$

Assume that λ is an eigenvalue and u an eigenfunction associated to λ such that $u'(0) = 1$. We can always assume this last condition by dividing u by $u'(0)$ ($u'(0) \neq 0$ by the uniqueness of the Cauchy problem for the equation in (4.2), see [27]). Then we have

$$((p-1)|u'|^{p'} + \lambda |u|^p) = p - 1 \tag{4.3}$$

and we can write

$$|u'| = (1 - \lambda \frac{|u|^p}{p-1})^{1/p'}. \tag{4.4}$$

Assume that u does not vanish in $(0, 1)$. By uniqueness, $u(t) = u(1-t)$ for all $t \in (0, 1)$ so then $u'(1/2) = 0$ and $t = 1/2$ is the unique value where u' is zero. Using (4.3) we find $u(1/2) = (\frac{p-1}{\lambda})^{1/p}$. Hence we get from (4.4) after integration over $(0,1/2)$,

$$1/2 = \int_0^{(\frac{p-1}{\lambda})^{1/p}} \frac{du}{(1-\lambda\frac{|u|^p}{p-1})^{1/p'}} = \lambda^{-1/p}\int_0^1 \frac{dz}{(1-\frac{|z|^p}{p-1})^{1/p'}},$$

$$\lambda^{1/p} = \pi_p \stackrel{\text{def}}{=} 2\int_0^1 \frac{dz}{(1-\frac{|z|^p}{p-1})^{1/p'}}.$$

If u vanishes twice in $(0,1)$ then by symmetry, the points where u vanishes are $1/3$ and $2/3$ and consequently $u'(1/6) = u'(1/2) = 0$. Proceeding as before we will find that $\lambda^{1/p} = 2\pi_p$. Proceeding in this manner we find that the eigenvalues are

$$\lambda_{n,p} \stackrel{\text{def}}{=} (\pi_p n)^p, \tag{4.5}$$

$n \in \mathbb{N}^*$. The eigenfunctions associated to $\lambda_{n,p}$ are the constant multiples of

$$u(t) = \sin_p(\lambda_{n,p}^{1/p} t).$$

where the function $y = \sin_p(s)$ is defined in $[0, \dfrac{\pi_p}{2}]$ implicitely by the equation

$$s = \int_0^y \frac{d\tau}{(1-\frac{\tau^p}{p-1})^{1/p}},$$

and extended by symmetry on $[\dfrac{\pi_p}{2}, \pi_p]$.

4.2. **Around the first eigenvalue.** We recall the weak formulation of (4.1). A value $\lambda \in \mathbb{R}$ is an *eigenvalue* of problem (4.1) if and only if there exists $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx = \lambda \int_\Omega |u|^{p-2}u\varphi dx \tag{4.6}$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

Let us introduce the functionals $E_p, I_p : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$E_p(u) \stackrel{\text{def}}{=} \int_\Omega |\nabla u|^p\, dx,$$

$$I_p(u) \stackrel{\text{def}}{=} \int_\Omega |u|^p\, dx.$$

E_p, I_p are C^1-functionals satisfying

$$< E_p'(u), v >= p\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx$$

$$< I_p'(u), v >= p\int_\Omega |u|^{p-2}uv\, dx.$$

Let us introduce the manifold

$$S_p \stackrel{\text{def}}{=} \{u \in W_0^{1,p}(\Omega) \,:\, I_p(u) = 1\}.$$

Notice that S_p is a manifold of class C^1 if $1 < p < 2$.

According to Definition 2.1 and (4.6), the critical values of E_p restricted to S_p correspond to the eigenvalues of (4.1).

Remark 4.1. The regularity theory for quasilinear elliptic equations implies that solutions to (1.3) belong to $L^\infty(\Omega)$ and are of class $C^{1,\alpha}_{loc}(\Omega)$ for some $0 < \alpha < 1$. If moreover we assume that Ω is a bounded domain with a $C^{1,\alpha}$-boundary $\partial\Omega$, for some $\alpha \in (0,1)$, then there exists a constant $0 < \beta < 1$, such that $u \in C^{1,\beta}(\overline{\Omega})$. The reader is refered to Serrin [28] (for the local boundedness regularity up to C^α); to Anane [4], Guedda-Véron [20] and De Thélin [17] (for the global boundeness in Ω); to DiBenedetto [13], Guedda-Véron [20] and Tolksdorf [32] for C^1 local regularity) and to Lieberman [22] (for the C^1 regularity up to the boundary).

Let us define

$$\lambda_{1,p} \stackrel{\text{def}}{=} \min\{\int_\Omega |\nabla u|^p \ : \ u \in S_p\}. \tag{4.7}$$

The following properties for a general bounded domain $\Omega \subset \mathbb{R}^N$ can be found for instance in [23].

Proposition 4.1. (i) *$\lambda_{1,p}$ is the least eigenvalue of* (4.1) *and, for any $v \in S_p$, v achieves the infimum in* (4.7) *if and only if v is an eigenfunction associated to $\lambda_{1,p}$.* **(ii)** *The eigenfunctions associated to $\lambda_{1,p}$ are either positive or negative in Ω.* **(iii)** *$\lambda_{1,p}$ is simple in the sense that the eigenfunctions associated to $\lambda_{1,p}$ are merely constant multiples of one other.*

Proof. (i) Using that E_p is a weakly lower semicontinuous functional and that S_p is a weakly closed subset (this last fact follows from the compact imbedding of $W^{1,p}_0(\Omega)$ in $L^p(\Omega)$) one can prove that the infimum is achieved at some $u \in S_p$. Hence, by lemma 2.1, $\lambda_{1,p} = \int_\Omega |\nabla u|^p\, dx > 0$ is a critical value of E_p under the constraint $I_p(u) = 1$.It is trivial that $\lambda_{1,p}$ is the least eigenvalue.

(ii) Let $u \in S_p$ be an eigenfunction associated to $\lambda_{1,p}$. Then u is a function where the infimum in the definition of $\lambda_{1,p}$ is achieved. Since $||\nabla |u|\ ||_p = ||\nabla u||_p$ it follows that $|u|$ is also an eigenfunction for $\lambda_{1,p}$. Using the Strong Maximum Principle (SMP for short) of [33] we conclude that $|u(x)| > 0 \quad \forall x \in \Omega$ and consequently u is either positive or negative in Ω.

(iii) We will use the so-called "Picone's identity" proved in [2]. We recall it here for completness.

Theorem 4.1. [2] *Let $v > 0, u \geq 0$ be two continuous functions in Ω differentiable a.e. Denote*

$$L(u,v) = |\nabla u|^p + (p-1)\tfrac{u^p}{v^p}|\nabla v|^p - p\tfrac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla v \nabla u\,,$$

$$R(u,v) = |\nabla u|^p - |\nabla v|^{p-2}\nabla(\tfrac{u^p}{v^{p-1}})\nabla v.$$

Then **(i)** *$L(u,v) = R(u,v)$,* **(ii)** *$L(u,v) \geq 0$ a.e. and* **(iii)** *$L(u,v) = 0$ a.e. in Ω if and only if $u = kv$ for some $k \in \mathbb{R}$.*

Let u, v be two eigenfunctions associated to $\lambda_{1,p}$. We can assume without restriction that u and v are positive in Ω. Let $\epsilon > 0$. From Picone's identity we have

$$0 \le \int_\Omega L(u, v+\epsilon)\,dx = \int_\Omega R(u, v+\epsilon)\,dx =$$
$$\lambda_{1,p} \int_\Omega u^p\,dx - \int_\Omega |\nabla v|^{p-2}\nabla\left(\tfrac{u^p}{(v+\epsilon)^{p-1}}\right)\nabla v\,dx.$$

The function $\frac{u^p}{(v+\epsilon)^{p-1}}$ belongs to $W_0^{1,p}(\Omega)$ (we use here that $\frac{u}{(v+\epsilon)} \in L^{+\infty}$ because of the regularity results from Remark 4.1) and then it is admissible for the weak formulation of $-\Delta_p v = \lambda_{1,p}|v|^{p-2}v$. Hence

$$0 \le \int_\Omega L(u, v+\epsilon)\,dx = \lambda_{1,p}\int_\Omega u^p\left(1 - \frac{v^{p-1}}{(v+\epsilon)^{p-1}}\right)dx.$$

Letting $\epsilon \to 0$ it follows that $L(u,v) = 0$. Then, by Theorem 4.1, there exists $k \in \mathbb{R}$ such that $u = kv$. □

In the next proposition we give an estimate of the measure of the nodal domains of an eigenfunction u. We recall that a *nodal domain of* u is a connected component of $\Omega \setminus \{x \in \Omega\colon u(x) = 0\}$.

A result similar to the next proposition can be found in [1]. Our exponent γ is slightly different.

Proposition 4.2. *Any eigenfunction u associated to a positive eigenvalue $0 < \lambda \neq \lambda_{1,p}$ changes sign. Moreover if $\mathcal{N}$ is a nodal domain of u then*

$$|\mathcal{N}| \ge (C\lambda)^{-\gamma} \tag{4.8}$$

where $\gamma = \frac{N}{p}$ and C is some constant depending only on N and p.

Proof. Assume by contradiction that $u \ge 0$. The case $u \le 0$ is completely analogous and the proof in this case is left to the reader. By the SMP of [33] it follows that $u(x) > 0$ for all $x \in \Omega$. Let $\varphi > 0$ be an eigenfunction associated to λ_1. For any $\epsilon > 0$ we apply Picone's identity to the pair $\varphi, v+\epsilon$. We have

$$\begin{aligned}&0 \le \int_\Omega L(\varphi, v+\epsilon)\,dx = \int_\Omega R(\varphi, v+\epsilon)\,dx =\\ &\lambda_{1,p}\int_\Omega \varphi^p\,dx - \int_\Omega |\nabla v|^{p-2}\nabla\left(\tfrac{\varphi^p}{(v+\epsilon)^{p-1}}\right)\nabla v\,dx.\end{aligned} \tag{4.9}$$

Notice that $\frac{\varphi^p}{(v+\epsilon)^{p-1}}$ belongs to $W_0^{1,p}(\Omega)$ and then it is admissible in the weak formulation of $-\Delta_p v = \lambda|v|^{p-2}v$. Then if follows from (4.9) that

$$0 \le \int_\Omega \varphi^p\left(\lambda_{1,p} - \lambda\frac{v^{p-1}}{(v+\epsilon)^{p-1}}\right)dx.$$

Letting $\epsilon \to 0$ it comes that $0 \le \int_\Omega \varphi^p(\lambda_{1,p} - \lambda)\,dx$ which is imposible because $\lambda > \lambda_{1,p}$ and $\int_\Omega \varphi^p\,dx > 0$. Hence we have proved that v must change sign.

Next we prove estimate (4.8). Assume that $v > 0$ in $\mathcal{N}$, the case $v < 0$ is completely analogous. Since $v \in W_0^{1,p}(\Omega) \cap C(\Omega)$ (c.f. Remark 4.1) it follows that $v_{|\mathcal{N}} \in W_0^{1,p}(\mathcal{N})$. Hence the function w defined as $w(x) = v(x)$ if $x \in \mathcal{N}$ and $w(x) = 0$ if $x \in \Omega \setminus \mathcal{N}$ belongs to $W_0^{1,p}(\Omega)$.

Let us start with the case $1 < p < N$. Using w as a test function in the weak equation satisfied by v we find

$$\int_{\mathcal{N}} |\nabla v|^p \, dx = \lambda \int_{\mathcal{N}} |v|^p \, dx \leq \lambda ||u||^p_{p^*,\mathcal{N}} \, |\mathcal{N}|^{\frac{p^*-p}{p^*}}$$

by Hölder inequality. Here $p^* \stackrel{\text{def}}{=} \dfrac{pN}{N-p}$ if $1 < p < N$ and $p^* \stackrel{\text{def}}{=} +\infty$ if $N \geq p$.

On the other hand using Sobolev imbeddings we have that $\int_{\mathcal{N}} |\nabla v|^p \, dx \geq C||v||^p_{p^*,\mathcal{N}}$ for some constant $C = C(N,p)$. Hence

$$C \leq \lambda |\mathcal{N}|^{\frac{p^*-p}{p^*}}$$

and the proposition follows.

In the case $p = N$ we proceed similarly using Sobolev's inclusion $W^{1,N}_0(\mathcal{N}) \subset L^N(\mathcal{N})$ and then apply Hölder inequality. We find

$$C||v||^N_{N,\mathcal{N}} |\mathcal{N}|^{-1} \leq \int_{\mathcal{N}} |\nabla v|^N \, dx \leq \lambda ||v||^N_{N,\mathcal{N}}$$

for some $C = C(N)$ and then inequality (4.8) follows.

In the case $p > N$ we have, on the one hand

$$\int_{\mathcal{N}} |\nabla v|^p \, dx \leq \lambda ||v||^p_{\infty,\mathcal{N}} \, ,$$

and on the other hand, from Morrey's lemma,

$$C||v||_{\infty,\mathcal{N}} \leq |\mathcal{N}|^{-1/p+1/N} ||\nabla v||_{p,\mathcal{N}}$$

for some $C = C(N,p)$. Then inequality (4.8) holds. □

Corollary 4.1. *Each eigenfunction has a finite number of nodal domains.*

Proof. Let $\mathcal{N}_j$ be a nodal domain of an eigenfunction associated to some positive eigenvalue λ. Then $|\mathcal{N}_j| \geq (C\lambda)^{-\gamma}$,

$$|\Omega| \geq \sum_j |\mathcal{N}_j| \geq (C\lambda)^{-\gamma} \sum_j 1$$

and the claim follows. □

Proposition 4.3. *$\lambda_{1,p}$ is isolated, that is, there exists $\delta > 0$ such that $(\lambda_{1,p}, \lambda_{1,p} + \delta) \cap \sigma_p = \emptyset$.*

Proof. The result follows from the estimate (4.8). Assume by contradiction that there exists a sequence λ_n of eigenvalues of (4.1) such that $0 < \lambda_n \searrow \lambda_{1,p}$. Let u_n be an eigenfunction associated to λ_n. Since $0 < \int_\Omega |\nabla u_n|^p \, dx = \lambda_n \int_\Omega |u_n|^p \, dx$ we can define

$$v_n := \frac{u_n}{(\int_\Omega |u_n|^p \, dx)^{1/p}}.$$

v_n is bounded in $W_0^{1,p}(\Omega)$ so there exist a subsequence (still denoted v_n) and $v \in W_0^{1,p}(\Omega)$ such that $v_n \to v$ in $L^p(\Omega)$ and weakly in $W_0^{1,p}(\Omega)$. Moreover $\int_\Omega |v|^p\, dx = 1$. On the other hand

$$\int_\Omega |\nabla v|^p\, dx \le \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^p\, dx = \lambda_{1,p}$$

and then $\int_\Omega |\nabla v|^p\, dx = \lambda_{1,p}$ by (4.7). Using Proposition 4.1 we get that v is an eigenfunction associated to $\lambda_{1,p}$. Then it follows from the SMP of [33] that either $v > 0$ or $v < 0$. In the case $v > 0$ (the other case is analogous) we conclude from the convergence in measure of the sequence v_n towards v that

$$|\Omega_n^-| \to 0 \tag{4.10}$$

where Ω_n^- denotes the negative set of u_n. But (4.10) clearly contradicts the estimate (4.8). □

Proposition 4.4. *Let Ω_1 be a proper subset of Ω_2. Then $\lambda_{1,p}(\Omega_2) < \lambda_{1,p}(\Omega_1)$.*

Proof. Let $u \in W_0^{1,p}(\Omega_1)$ be a positive eigenfunction associated to $\lambda_{1,p}(\Omega_1)$ and put $\tilde u$ the function obtained by extending u by zero in $\Omega_2 \setminus \Omega_1$. Then $\tilde u \in W_0^{1,p}(\Omega_2)$ and $\int_{\Omega_2} \tilde u^p\, dx = \int_{\Omega_1} u^p\, dx > 0$. Using $\dfrac{\tilde u}{(\int_{\Omega_2} \tilde u^p\, dx)^{1/p}}$ as an admissible function for $\lambda_{1,p}(\Omega_2)$ we get

$$\lambda_{1,p}(\Omega_2) \le \frac{\int_{\Omega_2} |\nabla \tilde u|^p\, dx}{\int_{\Omega_2} \tilde u^p\, dx} = \frac{\int_{\Omega_1} |\nabla u|^p\, dx}{\int_{\Omega_1} u^p\, dx} = \lambda_{1,p}(\Omega_1).$$

The equality holds only if $\tilde u$ is an eigenfunction associated to $\lambda_{1,p}(\Omega_2)$ but this is impossible because $\tilde u$ vanishes in Ω_2. □

4.3. **A sequence of eigenvalues of Ljusternik-Schnirelman type.** We have shown in the previous section that the eigenvalues of the p-laplacian can be seen as critical values of the energy functional $E_p(u)$ on the manifold S_p. In the case $p = 2$ we have also shown that the eigenvalues correspond to the minimax values of the energy functional $E = E_2$ along some family of subsets of the manifold $S = S_2$.

The idea of trying to find critical values for a functional J defined on a manifold M (including the case of a Banach space itself) by minimazing along families of subsets has been largely exploited after the pionering work of Ljusternik-Schinerlman. A standard approach to prove this kind of results requiresa deformation lemma on the manifold M. Classically the deformation lemma is constructed with the help of integral lines of a pseudo-gradient vector field of J on M. Since this construction requires the vector field to be locally Lipschitz continuous, it seems necessary to assume that M is at least $C^{1,1}$. However, in some applications the manifold M is merely of class C^1 and then one has either to construct de deformation more carefully or to use a different approach to prove a minimax principle. For instance, in the case of the manifold S_p one has at most C^1 regularity if $1 < p < 2$.

The first minimax principle for C^1 manifolds was obtained by [30] who proves the existence of a sequence of eigenvalues of the p-laplacian using families of odd subsets of M with different genus. This type of sequences are said to be of *Ljusternik-Schnirelman type.* Let us explain the definition of this sequence in an abstract setting.

Let X be a real Banach space, $g \in C^1(X, \mathbb{R})$ and $M = \{u \in X,\ g(x) = 1\}$ as in paragraph 2.1. Assume that $M \neq \emptyset$ and that (2.2) holds. Let us assume further that g is an *even functional* so then $0 \notin M$. A subset $A \subset M$ is said to be *symmetric* if $A = -A$. Call $\mathcal{F}$ the familly of closed, symmetric not empty subset A of M. For any $A \in \mathcal{F}$ set $\gamma(A) = 0$ if $A = \emptyset$ and

$$\gamma(A) \stackrel{\text{def}}{=} \inf\{m \in \mathbb{N}_* \,:\, \exists h \in C(A, \mathbb{R}^m \setminus \{0\}), h \text{ is odd}\}$$

and $\gamma(A) = \infty$ if there is no $m \in \mathbb{N}$ satisfying the definition above. $\gamma(A)$ is called the *Krasnoselskii's genus* of A and generalizes the notion of dimension of a linear space. The krasnoselskii's genus satisfies the following properties :

Property 1 :*Monotonicity.* Let $A, B \in F$ and let $g : A \to B$ be a continuous map. Then $\gamma(A) \leq \gamma(B)$.

Property 2 :*Sub-additivity.* Let $A, B \in F$. Then $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$. If moreover $\gamma(B) < +\infty$ then $\overline{A \backslash B} \in F$ and $\gamma(\overline{A/B}) \geq \gamma(A) - \gamma(B)$.

Property 3 :*Continuity.* Let $K \in F$ be a compact set. Then $\gamma(K) < \infty$ and there exists $\delta > 0$ such that $\gamma(N_\delta(K)) = \gamma(K)$ where $N_\delta(K) = \{u \in X \,:\, \text{dist}(u, K) \leq \delta\}$.

Property 4 : Assume that $X = E_1 \oplus E_2$ with $\dim E_1 < \infty$. Let $A \in F$ be such that $A \cap E_2 = \emptyset$. Then $\gamma(A) \leq \dim E_1$. If moreover $\dim E_1 < +\infty$ then for all $A \in F$ such that $\gamma(A) \geq \dim E_1 + 1$ it holds that $A \cap E_2 \neq \emptyset$.

Property 5 :*Borsuk-Ulam* Let E be an invariant subspace of finite dimension and consider the set $B = \{u \in E \,:\, ||u|| = 1\}$. Then $B \in F$ and $\gamma(B) = \dim E$.

Property 6 : For any $k \in \mathbb{N}$ we denote by S^k the unit sphere of $\mathbb{R}^{k+1}$. Let us introduce the set

$$\mathcal{O}(S^k, S_p) \stackrel{\text{def}}{=} \{h \in C(S^k, S_p) \,:\, h \text{ is odd }\}.$$

Then for any $h \in \mathcal{O}(S^k, S_p)$, $\gamma(h(S^k)) \geq k + 1$.

Definition 4.1. We recall that J is said to satisfy the *Palais-Smale condition on* M at level c ($(P.S)_{c,M}$ for short) if any sequence $u_n \in M$ such that

$$\lim_{n\to\infty} J(u_n) = c \tag{4.11}$$

and

$$\lim_{n\to\infty} ||J'(u_n)||_* = 0 \tag{4.12}$$

possesses a convergent subsequence. J satisfies the $(P.S)$ *condition on* M if J satisfies $(P.S)_{c,M}$ for all $c \in \mathbb{R}$.

The following result was proved by [30].

Theorem 4.2. *Suppose J is even, bounded below and that J satisfies the $(P.S)$ condition on M. Define*

$$c_j \stackrel{\text{def}}{=} \inf_{A\in\Gamma_j} \sup_{u\in A} J(u),$$

where $\Gamma_j \stackrel{\text{def}}{=} \{A \subset M;\ A \text{ is compact, symmetric, } \gamma(A) \geq j\}$. If $c_k = c_{k+1} = \cdots = c_{k+l} = c \in \mathbb{R}$ for some $k \geq 1$ and $l \geq 0$ then

$$\gamma(K_c) \geq l + 1. \tag{4.13}$$

In particular c_k is a critical value of J restricted to M.

As an application of this theorem when $X = W_0^1(\Omega)$, $J = E_p$ and $g = I_p$ we obtain the following result :

Proposition 4.5. *Let us define for all $k \in \mathbb{N}_*$ the value*

$$\mu_{k,p} \stackrel{\text{def}}{=} \inf_{A\in\Gamma_{k,p}} \sup_{u\in A} E_p(u), \tag{4.14}$$

where $\Gamma_{k,p} \stackrel{\text{def}}{=} \{A \subset S_p;\ A \text{ is compact, symmetric, } \gamma(A) \geq k\}$ and γ stands here for the Krasnoselskii's genus on $W_0^{1,p}(\Omega)$. Then $\mu_{k,p}$ is an eigenvalue of $-\Delta_p$ in $W_0^1(\Omega)$. Moreover $0 < \mu_{1,p} \leq \mu_{2,p} \leq \ldots$ and $\lim_{n\to\infty} \mu_{n,p} = \infty$. The eigenvalue $\mu_{1,p}$ is equal to the eigenvalue $\lambda_{1,p}$ defined in (4.7).

Proof. We will prove below in lemma 4.1 that E_p satisfies the $(P.S)$ condition on S_p. Set $\mu_{k,p} \stackrel{\text{def}}{=} c_k$. Since S_p has genus equal to ∞, all the values $\mu_{k,p}$ are real numbers. Hence the values $\mu_{k,p}$ are eigenvalues for the p-laplacian and the first part of the proposition is proved. To prove the second assertion let us assume by contradiction that the sequence $\mu_{k,p}$ is stationnary. Hence $\gamma(K_{\mu_{k,p}}) = \infty$. This is impossible because the $(P.S)$ condition implies that $K_{\mu_{k,p}}$ is a compact set. □

Lemma 4.1. *E_p satisfies the $(P.S)$ condition on S_p.*

Proof. Let $u_n \in S_p$ satisfy (4.11) and (4.12). From (4.11) it follows that u_n remains uniformly bounded in $W_0^{1,p}(\Omega)$. Then there exists a subsequence, still denoted u_n, and $u_0 \in W_0^{1,p}(\Omega)$ such that u_n converges to u_0 weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$. For any $u \in S_p$ and any $v \in W_0^{1,p}(\Omega)$ let us define $P_u(v) := v - \dfrac{1}{p}\langle I_p'(u), v\rangle u$. Notice that $P_u(v) \in T_u S_p$. The computation of $d\,E_p(u)(v)$ with $v = P_{u_n}(u_n - u_0)$ gives

$$\left| \int_\Omega (|\nabla u_n|^{p-2}\nabla u_n \nabla u_0 - t_n |\nabla u_n|^p)\,dx \right| \leq ||d\,\tilde{E}_p(u_n)||_* ||P_{u_n}(u_n - u_0)|| \tag{4.15}$$

where $t_n = \int_\Omega |u_n|^{p-2}u_n u_0\, dx$. Since $t_n \to 1$ and the sequence $||P_{u_n}(u_n - u_0)||$ is bounded (because $||u_n||$ is uniformily bounded), it follows from (4.12) and (4.15) that

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u_0)\, dx \to 0.$$

Then the $(S)_+$ property of $-\Delta_p$ (see [26]) implies that $||u_n - u_0|| \to 0$. □

Remark 4.2. Clearly $\lambda_{1,p} = \mu_{1,p}$ for all $p > 1$ and we will see in section 6 that $\mu_{2,p}$ is the *second eigenvalue* of the p-laplacian.

In the case $N = 1$ the Ljusternik-Schnirelmann sequence gives all the eigenvalues of the p-laplacian. We prove this in the next theorem (see also [16]).

Proposition 4.6. *Let us consider* $\Omega = (0, 1)$. *Let* $\lambda_{k,p}$ *be defined in* (4.5) *and let* $\mu_{k,p}$ *be defined in* (4.14). *Then for all* $k \in \mathbb{N}_*$,

$$\lambda_{k,p} = \mu_{k,p}.$$

Proof. Since we know that all the eigenvalues are simple, it follows from (4.13) (recall that $\mu_{k,p} = c_k$ from Theorem 4.2) that the values $\mu_{k,p}$ are all different. Hence $\lambda_{k,p} \le \mu_{k,p}$. It remains to prove that $\mu_{k,p} \le \lambda_{k,p}$. In order to prove this inequality let $v \in S_p$ be an eigenfunction associated to $\lambda_{k,p}$. Then v posseses k nodal sets, $I_1, \dots, I_k$, each of length $1/k$ (c.f. paragraph 4.1.). Set $v_i(t) = v(t)$ if $t \in I_i$ and $v_i(t) = 0$ otherwise. We have that $|v_i|_p^p = 1/k|v|_p^p$ and $|v_i'|_p^p = \lambda_{k,p}|v_i|_p^p$. Let us consider the map $h : S^{k-1} \to S_p$ defined by

$$h(z_1, \dots, z_k) = k \sum_{i=1}^{k} |z_i|^{2/p-1} z_i v_i.$$

Clearly $h(S^{k-1})$ is an admissible set for the infimum on the definition of $\mu_{k,p}$ (by Property 6 on this section) and hence

$$\mu_{k,p} \le \max_{z\in S^{k-1}} E_p(h(z)) = \lambda_{k,p}.$$

□

When $N > 1$ the previous result is still true if $p = 2$. Let us show it.

Proposition 4.7. *Denote* $\mu_n = \mu_{n,2}$ *the sequence defined in Proposition 4.5 and let* λ_n *be the sequence given by* (2.6). *Then* $\lambda_n = \mu_n$.

Proof. Notice that, for any subspace X_j of dimension j, $\gamma(X_j \cap S) = j$ so then

$$\mu_j \le \lambda_j \quad \forall j \ge 0. \tag{4.16}$$

Assume that you have proved that $\mu_n = \lambda_n$ up to some $n \ge 1$ and assume by contradiction that

$$\mu_{n+1} < \lambda_{n+1}.$$

From (4.16) we deduce that $\mu_n = \mu_{n+1}$ and then, using Theorem 4.2, $\gamma(K_{\mu_n}) \ge 2$. Then you can find at least two orthogonal eigenfunctions e_n, e_n' associated to

$\mu_n = \lambda_n$. Taking the subspace F of dimension $n+1$ generated by $e_1, e_2, \ldots, e_n, e'_n$ you get, from the definition of λ_{n+1}, that

$$\lambda_{n+1} \leq \max_{u \in F} E(u) = \lambda_n = \mu_n,$$

a contradiction. □

5. Minimax principles on C^1 manifolds

5.1. **Two minimax principles on C^1 manifolds.** In this section we are going to state two minimax principles of mountain pass type for C^1 functionals defined on general C^1 manifolds of codimension one. The proofs can be found in [10].

We will apply these minimax principles in paragraph 5.2 to get a second sequence of eigenvalues for the p-laplacian and again in section 6 to prove the existence of curves on Σ_p.

Let us state the minimax theorems in an abstract setting. Let X be a Banach space with norm $||\cdot||$. We will assume throughout this section that X is uniformly convex (see [14]), which is already the case if $X = W_0^{1,p}(\Omega)$ for some $p > 1$.

Let $g : X \to \mathbb{R}$ be given and assume that $g \in C^1(X, \mathbb{R})$ and also that (2.2) is satisfied. Consider the C^1 manifold $M = \{u \in X \,:\, g(u) = 1\}$.

In what follows K is a given compact metric space and $K_0 \subset K$ a closed subset. Observe that, when $K = [-1, 1]$ and $k_0 = \{-1, 1\}$, Theorem 5.1 below is the classical Mountain Pass Thorem of Rabinowitz.

Theorem 5.1. *Let $J \in C^1(X, \mathbb{R})$ and $h_0 \in C(K_0, M)$ be fixed. Consider the family $\Gamma = \{h \in C(K, M) \,:\, h_{|K_0} = h_0\}$ and assume that $\Gamma \neq \emptyset$. Assume further that the following condition holds*

$$\max_{z \in K_0} J(h_0(z)) \,<\, \max_{z \in K} J(h(z)) \tag{5.1}$$

for all $h \in \Gamma$. Let us define $c \overset{\text{def}}{=} \inf_{h \in \Gamma} \max_{z \in K} J(h(z))$. Let $\epsilon > 0$ and $h \in \Gamma$ be such that

$$\max_{z \in K} J(h(z)) \,<\, c + \frac{\epsilon^2}{2}. \tag{5.2}$$

Then there exists $u \in M$ such that

$$\begin{cases} c \leq J(u) \leq c + \frac{\epsilon^2}{2}, \\ dist(u, h(K)) \leq \epsilon, \\ ||J'(u)||_* \leq \epsilon. \end{cases} \tag{5.3}$$

The following propositions follow directly from Theorem 5.1.

Proposition 5.1. *Let J, Γ and c be as in Theorem 5.1 and assume that* (5.1) *holds. If J satisfies $(P.S)_{c,M}$ then there exists a critical point $u \in M$ of J restricted to M such that $J(u) = c$.*

Next we state a second minimax principle that will give "almost critical points" of J restricted to M when we minimize along continuous odd maps defined on spheres of finite dimension. To that effect, let us assume that the map g is *even* so that in particular $-M = M$. Let us introduce the set

$$\mathcal{O}(S^k, M) \stackrel{\text{def}}{=} \{h \in C(S^k, M) \,:\, h \text{ is odd }\}.$$

Theorem 5.2. *Let $J \in C^1(X, \mathbb{R})$ be a even function and let us fix $k \in \mathbb{N}_*$. Let us define*

$$d_k \stackrel{\text{def}}{=} \inf_{h \in \mathcal{O}(S^{k-1}, M)} \max_{z \in S^{k-1}} J(h(z))$$

and assume that $d_k \in \mathbb{R}$. Let $\epsilon > 0$ and $h \in \mathcal{O}(S^{k-1}, M)$ be such that

$$\max_{z \in S^{k-1}} J(h(z)) \,<\, d_k + \frac{\epsilon^2}{2}. \tag{5.4}$$

Then there exists $u \in M$ such that

$$\begin{cases} d_k \le J(u) \le d_k + \frac{\epsilon^2}{2}, \\ dist(u, h(S^{k-1})) \le \epsilon, \\ \|J'(u)\|_* \le \epsilon. \end{cases} \tag{5.5}$$

The following proposition follows directly from Theorem 5.2.

Proposition 5.2. *Let J, Γ and d_k be as in Theorem 5.2. If J satisfies $(P.S)_{d_k,M}$ then there exists a critical point $u \in M$ of J restricted to M such that $J(u) = d_k$.*

5.2. **A second sequence of eigenvalues of the p-laplacian.** Let us apply Theorem 5.2 when $X = W_0^{1,p}(\Omega)$, $J = E_p$ and $g = I_p$. We get the following result.

Proposition 5.3. *For all $k \in \mathbb{N}_*$ the value*

$$\nu_{k,p} \stackrel{\text{def}}{=} \inf_{h \in \mathcal{O}(S^{k-1}, S_p)} \max_{z \in S^{k-1}} E_p(h(z)) \tag{5.6}$$

is an eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$. Moreover $\nu_{k,p} \ge \mu_{k,p}$ and $\lim_{k\to\infty} \nu_{k,p} = +\infty$.

*Proof.*Let $\nu_{k,p} = d_k$ where d_k is given by Theorem 5.2. By Proposition 5.2 and Lemma 4.1, $\nu_{k,p}$ is a critical point of E_p restricted to S_p and hence $\nu_{k,p}$ is an eigenvalue of the p-laplacian. The inequality $\nu_{k,p} \ge \mu_{k,p}$ follows from the fact that for any $h \in \mathcal{O}(S^{k-1}, S_p)$, $h(S^{k-1}) \subset \Gamma_k$. □

When the dimension $N = 1$ or $p = 2$ we can prove the following. As before we have written $\nu_n = \nu_{n,2}$.

Proposition 5.4. *For $p = 2$ or $N = 1$, $\lambda_{n,p} = \nu_{n,p}$ for all $n \in \mathbb{N}_*$.*

Proof. Assume that $p = 2$. From Proposition 4.7 and Proposition 5.3 we have $\lambda_k = \mu_k \le \nu_k$. To prove that $\nu_k \le \lambda_k$ let us consider the map $h : S^{k-1} \to S$ defined by $h(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i e_i$ where the $e_i' s$ are the hilbertian base of eigenfunctions associated to the sequence λ_k. Clearly $\max_{u \in h(S^{k-1})} E(u) = \lambda_k$. Since $h \in \mathcal{O}(S^k, S)$ the claim follows now from the definition of ν_k.

Assume now that $N = 1$ and $p > 1$. From Proposition 4.6 and Proposition 5.3 we get $\lambda_{k,p} \le \nu_{k,p}$. To prove the reverse inequality consider the map h defined in the proof of Proposition 4.6 and adapt the same arguments to the present situation. □.

Remark 5.1. Proposition 5.4 was proved by [16]. Proposition 5.3 was proved by [7] for a class of more general elliptic operators.

6. The Fučik spectrum of the p-laplacian

The *Fučik spectrum of the p-laplacian* on $W_0^{1,p}(\Omega)$ is defined as the set Σ_p of pairs $(\alpha, \beta) \in \mathbb{R}^2$ such that problem (1.3) has a nontrivial solution $u \in W_0^{1,p}(\Omega)$. Solutions u of (1.3) are understood in the weak sense which means here

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \int_\Omega (\alpha (u^+)^{p-1} - \beta (u^-)^{p-1}) \varphi dx \quad \text{for all } \varphi \in W_0^1(\Omega). \tag{6.1}$$

A pair (λ, λ) belongs to Σ_p if and only if λ is an eigenvalue of $-\Delta_p$ on $W_0^{1,p}(\Omega)$. Besides, using that the eigenfunctions associated to $\lambda_{1,p}$ have definite sign, one can easily prove that the lines $\{\lambda_{1,p}\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_{1,p}\}$ are contained in Σ_p. Furthermore the variational characterization (4.7) implies that

$$\Sigma \setminus (\{\lambda_{1,p}\} \times \mathbb{R} \cup \mathbb{R} \times \{\lambda_{1,p}\}) \subset \{(\alpha, \beta) \in \mathbb{R}^2 \, : \, \alpha > \lambda_{1,p}, \; \beta > \lambda_{1,p}\}.$$

In this section we will compute Σ_p when $N = 1$ and we will obtain branches on Σ_p (in any dimension) using the minimax principles of section 5.

6.1. The Fučik spectrum in dimension 1.

Let $\Omega = (0, 1)$. We compute $\Sigma_p(0, 1)$ using the same technique as in the case $p = 2$ (i.e. considering the number of nodal domains of the solutions). The result is that $\Sigma_p(0, 1)$ is composed of the two lines $\mathbb{R} \times \{\lambda_{1,p}\}$ et $\{\lambda_{1,p}\} \times \mathbb{R}$ and the sequence of curves

$$C_k = \{(\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \, : \, \alpha^{-1/p} + \beta^{-1/p} = \frac{1}{\pi_p k}\}$$

if k is even and

$$C_k^1 = \{(\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \, : \, (\frac{k-1}{2}) \alpha^{-1/p} + (\frac{k+1}{2}) \beta^{-1/p} = \frac{1}{\pi_p}\}$$

$$C_k^2 = \{(\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \, : \, (\frac{k+1}{2}) \alpha^{-1/p} + (\frac{k-1}{2}) \beta^{-1/p} = \frac{1}{\pi_p}\}$$

if k is odd.

6.2. **The beginning of the Fučik spectrum.** Let Ω be any domain in $\mathbb{R}^N$, $N \geq 1$. We summarize in this paragraph the results of [9] concerning the existence of a *first curve in* Σ_p. We refere to [9] for the proofs.

Let us fix $s \in \mathbb{R}$ and let us draw in the (α, β) plane a line parallel to the diagonal and passing through $(s, 0)$. Since Σ_p is symmetric with respect to the diagonal we will always assume in this paragraph that $s \geq 0$. We will first see that the points of Σ_p on that line correspond exactly to the critical values of some constrained functional.

For that purpose let us consider the functional

$$E_p^s(u) \stackrel{\text{def}}{=} \int_\Omega |\nabla u|^p - s \int_\Omega u^{+p}.$$

E_p^s is a C^1 functional on $W_0^{1,p}(\Omega)$. We are interested in the critical points of the restriction of E_p^s to S_p. By Lagrange multipliers rule, $u \in S_p$ is a critical point of E_p^s restricted to S_p if and only if there exists $t \in \mathbb{R}$ such that $(E_p^s)'(u) = t(I_p^s)'(u)$, i.e.

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v - s \int_\Omega u^{+p-1} v = t \int_\Omega |u|^{p-2} uv \tag{6.2}$$

for all $v \in W_0^{1,p}(\Omega)$. This means that

$$\begin{cases} -\Delta_p u = (s+t) u^{+p-1} - t u^{-p-1} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

holds in the weak sense, i.e. that $(s + t, t) \in \Sigma_p$. Taking $v = u$ in (6.2), one also see that the Lagrange multiplier t is equal to the corresponding critical value $E_p^s(u)$. We have then that the points in Σ_p on the parallel to the diagonal passing through $(s, 0)$ are exactly of the form $(s + E_p^s(u), E_p^s(u))$ with u a critical point of E_p^s on S_p.

From now on we assume $s \geq 0$, which is no restriction since Σ_p is clearly symmetric with respect to the diagonal. Let us denote by $\varphi_{1,p}$ the positive eigenfunction in S_p associated to $\lambda_{1,p}$.

A first critical point of E_p^s restricted to S_p comes from global minimization. Indeed

$$E_p^s(u) \geq \lambda_{1,p} \int_\Omega |u|^p - s \int_\Omega u^{+p} \geq \lambda_{1,p} - s$$

for all $u \in S_p$, and one has $E_p^s(u) = \lambda_{1,p} - s$ for $u = \varphi_{1,p}$ by Proposition 4.1. Consequently we have that the corresponding point in Σ_p is $(\lambda_{1,p}, \lambda_{1,p} - s)$, which lies on the vertical line through $(\lambda_{1,p}, \lambda_{1,p})$.

A second critical point of E_p^s restricted to S_p comes from the following proposition.

Proposition 6.1. *$-\varphi_{1,p}$ is a strict local minimum of E_p^s, and $E_p^s(-\varphi_{1,p}) = \lambda_{1,p}$. The corresponding point in Σ_p is $(\lambda_{1,p}+s, \lambda_{1,p})$, which lies on the horizontal line through $(\lambda_{1,p}, \lambda_{1,p})$.*

Of course, when $s = 0$, the two critical values $E_p^s(\varphi_{1,p})$ and $E_p^s(-\varphi_{1,p})$ coincide as well as the corresponding points in Σ_p.

To get a third critical point, we can use our version of the mountains pass theorem on a C^1 manifold, c.f. Proposition 5.1. First we need the following

Lemma 6.1. *E_p^s satisfies the $(P.S)$ condition on S_p.*

The proof is similar to the one of Lemma 4.1 and it is left to the reader.

The geometric condition (5.1) is a consequence of the following result.

Lemma 6.2. *Let $\varepsilon_0 > 0$ be such that*

$$E_p^s(u) > E_p^s(-\varphi_{1,p}) \tag{6.3}$$

for all $u \in B(-\varphi_{1,p}, \varepsilon_0) \cap S_p$ with $u \neq -\varphi_{1,p}$, where the ball B is taken in $W_0^{1,p}(\Omega)$ and ϵ is given by Proposition 6.1. Then, for any $0 < \varepsilon < \varepsilon_0$,

$$\inf\{E_p^s(u) \,:\, u \in S_p \text{ and } ||u - (-\varphi_{1,p})||_{1,p} = \varepsilon\} > E_p^s(-\varphi_{1,p}). \tag{6.4}$$

We are now in a position to apply the mountain pass theorem of Proposition 5.1. Clearly

$$\Gamma = \{\gamma \in C([-1,+1], S_p) \,:\, \gamma(-1) = -\varphi_{1,p} \text{ and } \gamma(1) = \varphi_{1,p}\}$$

is nonempty (take e.g. $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \notin \mathbb{R}\varphi_{1,p}$, consider the path $t\varphi_{1,p} + (1-|t|)\varphi$ and normalize it). Consequently

$$c(s) \overset{\text{def}}{=} \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,+1]} E_p^s(u) \tag{6.5}$$

is a critical value of E_p^s. Moreover

$$c(s) > \max\{E_p^s(-\varphi_{1,p}), E_p^s(\varphi_{1,p})\} = \lambda_{1,p}. \tag{6.6}$$

In this way we have been able to obtain a third critical point of E_p^s.

We have thus proved that for each $s \geq 0$, the point $(s + c(s), c(s))$, where $c(s) > \lambda_{1,p}$ is defined by the minimax formula (6.5), belongs to Σ_p.

Proceeding in this manner for each $s \geq 0$, we get *a non trivial curve $s \in \mathbb{R}^+ \to (s + c(s), c(s)) \in \mathbb{R}^2$ in Σ_p.* Of course the symmetric points with respect to the diagonal also belong to Σ_p. The whole curve will be denoted by $\mathcal{C}$.

Several properties of $\mathcal{C}$ are proved in [9]. The main property is that $\mathcal{C}$ is the *first nontrivial curve in* Σ_p, in the following sense :

Theorem 6.1. *Let $s \geq 0$. The point $(s + c(s), c(s))$ is the first nontrivial point of Σ_p on the parallel to the diagonal through $(s, 0)$.*

Remark 6.1. Since $\lambda_{1,p}$ is isolated in the spectrum (c.f. Proposition 4.3) it makes sense to define

$$\overline{\lambda}_{2,p} \stackrel{\text{def}}{=} \inf\{\lambda \in \sigma_p \,:\, \lambda > \lambda_{1,p}\} \tag{6.7}$$

In particular, for $s = 0$, Theorem 6.1 tell us that the curve $\mathcal{C}$ passes through $(\overline{\lambda}_{2,p}, \overline{\lambda}_{2,p})$, which provides the following variational characterization of $\overline{\lambda}_{2,p}$:

$$\overline{\lambda}_{2,p} = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,+1]} \int_\Omega |\nabla u|^p, \tag{6.8}$$

where Γ is the family of all continuous paths in p going from $-\varphi_{1,p}$ to $+\varphi_{1,p}$.

This characterization of $\overline{\lambda}_{2,p}$ is slightly different from the one obtained recently in [5]. We recall that it was shown in [5] that $\overline{\lambda}_{2,p}$ is equal to the second Ljusternik-Schnirelman eigenvalue :

$$\overline{\lambda}_{2,p} = \mu_{2,p} \tag{6.9}$$

It is easily seen that the inf max in the definition of $\mu_{2,p}$ is $\leq$ the inf max in (6.8). Indeed starting with a path $\gamma \in \Gamma$ and joining it with its symmetric counterpart $-\gamma$, one gets a compact symmetric set of genus ≥ 2 on which E_p has not increased its values. The characterization (6.8) shows that in order to get $\overline{\lambda}_{2,p}$, it suffices to restrict oneself to this particular class of sets A of genus 2.

6.3. **Existence of infinitely many curves.** We will prove in Theorem 6.2 below a result that generalizes partially for $p \neq 2$ a result of [21], c.f. Corollary 6.1. Theorem 6.2 was announced in [21] and has been recently proved by [25] using a different technique.

Let $s \geq 0$ and consider the Fučik problem in its form (6.2). In order to construct branches in Σ_p we are going to apply Proposition 5.1 and Proposition 5.2 to the functional E_p^s restricted to the manifold S_p.

First let us give a *new variational characterization* of the values $\lambda_{k,p}$ that where defined in (4.14). For that purpose let us introduce, for any $k \in \mathbb{N}_*$, the set

$$S_+^{k-1} \stackrel{\text{def}}{=} \{x = (x_1, x_2, \ldots, x_k) \,:\, x \in S^{k-1}, x_k \geq 0\}.$$

In what follows we write $E_p := E_p^0$.

Proposition 6.2. *For all $k \geq 2$*

$$\lambda_{k,p} = \inf_{h\in\Upsilon_{k,p}} \max_{u\in h(S_+^{k-1})} E_p(u) \tag{6.10}$$

where $\Upsilon_{k,p} \stackrel{\text{def}}{=} \{h \in C(S_+^{k-1}, S_p) \,:\, h(S^{k-2})$ is odd $\}$.

Proof. Extend the maps of $\Upsilon_{k,p}$ over S^{k-1} by oddness and use that E_p is an even functional. ☐

Theorem 6.2. *Let $k \in \mathbb{N}_*$ be such that $\lambda_{k,p} < \lambda_{k+1,p}$ and fix $0 < \epsilon < \lambda_{k+1,p} - \lambda_{k,p}$. Then there exists a decreasing continuous map $c : [0, \lambda_{k+1,p} - \lambda_{k,p} - \epsilon] \to]\lambda_{k,p}, +\infty]$ such that the curve*

$$C_{k+1} \stackrel{\text{def}}{=} \{(c(s) + s, c(s)) \,:\, 0 \le s \le \lambda_{k+1,p} - \lambda_{k,p} - \epsilon\}$$

belong to Σ_p. Moreover $\lambda_{k+1,p} \le c(0)$.

Proof. Clearly

$$E_p(u) - s \le E_p^s(u) \le E_p(u) \tag{6.11}$$

for all $u \in S_p$ and all $s \ge 0$. Taking the "minimax" along the familly of maps $\mathcal{O}(S^{k-1}, S_p)$ and using the characterization of $\lambda_{k,p}$ given in Proposition 4.5 we have, for all $s \ge 0$,

$$\lambda_{k,p} - s \le a_{k,p} \stackrel{\text{def}}{=} \inf_{h \in \mathcal{O}(S^{k-1}, S_p)} \max_{u \in h(S^{k-1})} E_p^s(u) \le \lambda_{k,p}. \tag{6.12}$$

Similarly we have, using Proposition 6.2,

$$\lambda_{k+1,p} - s \le b_{k+1,p} \stackrel{\text{def}}{=} \inf_{h \in \Upsilon_{k+1,p}} \max_{u \in h(S_+^k)} E_p^s(u) \le \lambda_{k+1,p}. \tag{6.13}$$

Let $h_0 \in \mathcal{O}(S^{k-1}, S_p)$ be such that

$$\max_{u \in h_0(S^{k-1})} E_p(u) < \lambda_{k,p} + \epsilon \tag{6.14}$$

and let us apply Theorem 5.1 to $J = E_p^s$, $M = S_p$, $K = S_+^k$, $K_0 = S^{k-1}$ and h_0 satisfying (6.14). The family Γ defined in Theorem 5.1 is formed, in this context, by the continuous maps from $K = S_+^k$ to S_p that are equal to h_0 on $K_0 = S^{k-1}$. Hence

$$\Gamma \subset \Upsilon_{k+1,p}. \tag{6.15}$$

Let us prove that $\Gamma \neq \emptyset$. Since $h_0(S^{k-1})$ is a compact set and S_p is not, there exists $e \in S_p$ such that $e \notin h_0(S^{k-1})$. Then the map $h : S_+^k \to S_p$ defined by

$$h(z_1, \ldots, z_{k+1}) = \frac{(1 - z_{k+1}^2)^{1/2} h_0(z_1, \ldots, z_k) + z_{k+1} e}{|(1 - z_{k+1}^2)^{1/2} h_0(z_1, \ldots, z_k) + z_{k+1} e|_p}, \tag{6.16}$$

belong to Γ.

The geometric conditon (5.1) follows from (6.11), (6.14) and (6.15). Indeed we have, for all $s \in [0, \lambda_{k+1,p} - \lambda_{k,p} - \epsilon]$,

(6.17)

$$\max_{u \in h_0(S^{k-1})} E_p^s(u) \le \max_{u \in h_0(S^{k-1})} E_p(u) < \lambda_{k,p} + \epsilon \le \lambda_{k+1,p} - s \le b_{k+1,p} \le \max_{u \in h(S_+^k)} E_p^s(u)$$

for all $h \in \Gamma$. From Theorem 5.1 we conclude that the value

$$c(s) \stackrel{\text{def}}{=} \inf_{h \in \Gamma} \max_{u \in h(S_+^k)} E_p^s(u)$$

is a critical value of E_p^s restricted to S_p. Finally from (6.17), (6.12) and (6.13) we have, for all $s \in [0, \lambda_{k+1,p} - \lambda_{k,p} - \epsilon]$,

$$\lambda_{k,p} + \epsilon \leq \lambda_{k+1,p} - s \leq b_{k+1,p} \leq c(s)$$

so in particular $c(0) \geq \lambda_{k+1,p}$.

To prove that c is continuous and decreasing, let $0 < \delta$ small enough and $t \leq s \leq t + \delta$. For all $u \in S_p$ we have

$$E_p^t(u) - \delta \leq E_p^s(u) \leq E_p^t(u).$$

Hence taking the infmax along Γ (notice that h_0 *does not depends on s or t*) we find

$$c(t) - \delta \leq c(s) \leq c(t)$$

for all $t \leq s \leq t + \delta$ and consequently c is decreasing and continuous. □

In a similar way we can define another function, say $d(s)$, using the functional $F_p^s(u) \stackrel{\text{def}}{=} E_p^s(u) - sI_p(u^-)$ providing a new point $(d(s), s + d(s))$ on Σ_p.

Corollary 6.1. [21] *If $p = 2$, $c(0) = \lambda_{k+1}$.*

Proof. Let us onsider the map h defined in (6.16) where $e \in S$ is an eigenfunction associated to λ_{k+1}. Then for all $z \in S_+^k$ we have

$$E(h(z)) = \frac{\int_\Omega((1 - z_{k+1}^2)|\nabla h_0|^2 + 2(1 - z_{k+1}^2)^{1/2} z_{k+1} \nabla h_0 \nabla e + z_{k+1}^2 |\nabla e|^2) dx}{(1 - z_{k+1}^2)|h_0|_2^2 + 2(1 - z_{k+1}^2)^{1/2} z_{k+1} \int_\Omega h_0 e \, dx + z_{k+1}^2 |e|_2^2} =$$

$$\frac{\int_\Omega((1 - z_{k+1}^2)|\nabla h_0|^2 + 2\lambda_{k+1}(1 - z_{k+1}^2)^{1/2} z_{k+1} h_0 e + \lambda_{k+1} z_{k+1}^2 |e|^2) dx}{(1 - z_{k+1}^2)|h_0|_2^2 + 2(1 - z_{k+1}^2)^{1/2} z_{k+1} \int_\Omega h_0 e \, dx + z_{k+1}^2 |e|_2^2} \leq \lambda_{k+1}$$

and $E(e) = \lambda_{k+1}$. Thus

$$\max_{u \in h(S_+^k)} E(u) \leq \lambda_{k+1}.$$

Then the inequality $c(0) \leq \lambda_{k+1}$ follows from the definition of $c(0)$. □

Remark 6.2. Under the hypothesis of Theorem 6.2 assume further that there exists $h_0 \in \mathcal{O}(S^{k-1}, S_p)$ such that, instead of (6.14), we have

$$\max_{u \in h(S^{k-1})} E_p(u) = \lambda_{k,p}.$$

Then the value $c(s)$ on the proof of the theorem can be defined *for all* $s \in [0, \lambda_{k+1,p} - \lambda_{k,p}[$.

Remark 6.3. Since $\lim_{k \to +\infty} \lambda_{k,p} = +\infty$ there exist infinitely many k's such that $\lambda_{k,p} < \lambda_{k+1,p}$ and consequently there exist infinitely many branches of points contained in Σ_p.

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