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Intrinsic n-stack completions over a topos * †

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†Category Theory 2008, Calais, France, June 26, 2008

OUTLINE

Our *long-term goal* is to generalize to higher dimensions the construction of the *stack completion of a groupoid* $\mathbb{G}$ in an elementary topos $\mathscr{S}$ from (Bunge 1979), where the notion of a stack is to be interpreted with respect to the *intrinsic topology* of the epis in $\mathscr{S}$.

In *dimension 1*, the main tools are the *monadicity and descent theorems* (Beck 1967, Bénabou-Roubaud 1970), and an application of the two in conjunction (Bunge-Paré 1979). It is shown therein that $\mathscr{S}$, regarded as a fibration over itself, is a stack.

As a corollary, the *stack completion* of any groupoid in $\mathscr{S}$ is constructed in (Bunge 1979) as the fibration of (essential) points of the topos $\mathscr{S}^{\mathbb{G}^{op}}$, and the *classification* $\mathbb{G}$*-torsors* (Diaconescu 1995) is obtained as a consequence. Examples of 1-stack completions abund in mathematics.

In *dimension 2*, we resort likewise to the *2-monadicity and 2-descent theorems* of (Hermida 2004), in order to prove that the 2-fibration of groupoid stacks is a 2-stack. A restriction on $\mathscr{S}$, in the form of an 'axiom of stack completions' (Lawvere 1974) is needed already for the passage from $n = 1$ to $n = 2$.

As argued in (Bunge 2002), this result leads to the *2-stack completion* of a 2-groupoid $\mathbb{G}$ in $\mathscr{S}$ and to *classifications of 2-torsors* but, unlike the case of dimension 1, a restriction on $\mathbb{G}$, to wit, that it be 'hom-by-hom' a 1-stack, is needed for it to hold, and similarly for any passage from n to (n + 1). In particular, this explains, in a more general setting, why *gerbes and bouquets* (Duskin 1989, Breen 1994) are considered as coefficients in non-abelian cohomology.

Although different in outlook, the program outlined in (Bunge 2002) has been motivated in spirit by (Duskin 1989) and (Street 1995). A *comparison* with the work of (Hirshchowitz-Simpson 2001) and others, where the emphasis is on the existence of specific *Quillen model structures*, is expected to give a conceptual simplification of the latter. *Applications* of n-stack completions (particularly in dimension 2) are envisaged.

INTRINSIC 1-STACKS

In (Lawvere 1974), it was suggested to make the notion of a *stack* (or *champ*) meaningful for any elementary topos $\mathscr{S}$, the latter regarded itself as a (big) site consisting of the class of epimorphisms. In addition, it was suggested therein that an 'axiom of stack completions' be added to those of an elementary topos, since such an axiom is satisfied and useful when the topos $\mathscr{S}$ is a Grothedieck topos, on account the existence of a set of generators.

In (Bunge-Paré 1979), motivated by Lawvere's lectures on stacks, a theory of *intrinsic stacks* (from now on simply *stacks*) was undertaken, laying down the basis for a construction of the *stack completion* of a category, or of a groupoid (Bunge 1979, 1990). Although the notion of a stack makes sense for internal categories or groupoids *in* $\mathscr{S}$, their stack completions are fibrations *over* $\mathscr{S}$, not necessarily representable. For this reason, we define this notion directly for fibrations.

Definition. (Lawvere 1974) Let $\mathscr{S}$ be an elementary topos. A fibration $\mathscr{A} \longrightarrow \mathscr{S}$ is said to be a *stack* if for every epimorphism $e : J \longrightarrow I$ in $\mathscr{S}$, the functor $e^* : \mathscr{A}^I \longrightarrow \mathscr{A}^J$ is of effective descent. This means that the canonical functor Φ_e in the diagram below, is an equivalence.

$$\begin{array}{ccc} \mathscr{A}^I & \xrightarrow{\Phi_e} & \mathsf{Des}_e(\mathscr{A}) \\ & {\scriptstyle e^*}\searrow \quad \swarrow{\scriptstyle U} & \\ & \mathscr{A}^J & \end{array}$$

BASIC FACTS ABOUT 1-STACKS

The following are taken from (Bunge-Paré 1979).

Definition Let $F : \mathscr{B} \longrightarrow \mathscr{C}$ be a functor between fibrations over $\mathscr{S}$. It is said to be a *weak equivalence* if the following conditions hold.

1. (*essentially surjective*) For each $I \in \mathscr{S}$, $F^I : |\mathscr{B}^I| \longrightarrow |\mathscr{C}^I|$, and $c \in |\mathscr{C}^I|$, there exists an epimorphism $e : J \longrightarrow I$ in $\mathscr{S}$, $b \in |\mathscr{B}^I|$, and an isomorphism $\theta : F^J(b) \longrightarrow e^*(c)$.

2. (*fully faithful*) $\forall I \in \mathscr{S}$ $\forall x, x' \in |\mathscr{B}^I|$, the functor

   $$\mathrm{Hom}_{\mathscr{B}^I}(x, x') \xrightarrow{F_{x,x'}} \mathrm{Hom}_{\mathscr{C}^I}(Fx, Fx')$$

   is an isomorphism.

Proposition. A fibration $\mathscr{A} \longrightarrow \mathscr{S}$ is a stack iff for every weak equivalence functor $F : \mathscr{B} \longrightarrow \mathscr{C}$ in $\mathscr{S}$, the induced

$$\mathscr{A}^F : \mathscr{A}^{\mathscr{C}} \longrightarrow \mathscr{A}^{\mathscr{B}}$$

is an equivalence of fibrations.

Corollary. Let $\mathscr{A}$ be a fibration over $\mathscr{S}$, and let $F : \mathscr{A} \longrightarrow \mathscr{B}$ be a weak equivalence functor, with $\mathscr{B}$ a stack over $\mathscr{S}$. Then, the pair $(\mathscr{B}, F)$ is the *stack completion* of $\mathscr{A}$ in the sense of satisfying the obvious universal property . Stack completions of a given $\mathscr{A}$ are unique up to equivalence.

EXAMPLE 1

Theorem. In the *Zariski topos Zar*, for $\mathscr{U}$ the generic local ring, the canonical functor

$$\alpha_{\mathscr{U}} : \mathbb{F}_{\mathscr{U}} \longrightarrow \mathbb{P}_{\mathscr{U}},$$

where $\mathbb{F}_{\mathscr{U}}$ is the internal category of free $\mathscr{U}$-modules of finite rank, and $\mathbb{P}_{\mathscr{U}}$ is the internal category of finitely generated projective $\mathscr{U}$-modules, is a *weak equivalence*.

- **Corollary 1.**(*Kaplansky's theorem.*) For a local ring L in Set, the canonical functor

  $$\alpha_L : \mathbb{F}_L \longrightarrow \mathbb{P}_L$$

  is an equivalence. This follows from the fact that $L = \varphi^*(\mathscr{U})$ for a (unique) geometric morphism $\varphi :$ Set $\longrightarrow$ Zar, that any inverse image part of a geometric morphism preserves weak equivalence functors, and that in any topos satisfying the axiom of choice, every weak equivalence is an equivalence.

- **Corollary 2.** (*Swan's theorem.*) In Sh(X), with X paracompact, and $\mathscr{C}_{\mathbb{R}}$ the sheaf of germs of $\mathbb{R}$-valued continuous functions, we have the following commutative diagram:

  $$\begin{array}{ccc} \mathbb{F}_{\mathscr{C}_{\mathbb{R}}} & \xrightarrow{\alpha_{\mathscr{C}_{\mathbb{R}}}} & \mathbb{P}_{\mathscr{C}_{\mathbb{R}}} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle P} \\ \widetilde{\mathbb{F}_{\mathscr{C}_{\mathbb{R}}}} & \xrightarrow[\widetilde{\alpha_{\mathscr{C}_{\mathbb{R}}}}]{} & \widetilde{\mathbb{P}_{\mathscr{C}_{\mathbb{R}}}} \end{array}$$

  where $\alpha_{\mathscr{C}_{\mathbb{R}}}$ is a wef (same argument as in Corollary 1), and where P and F are weak equivalence functors into the stack completions, so that also the induced $\widetilde{\alpha_{\mathscr{C}_{\mathbb{R}}}}$ is one but, between stacks, any weak equivalence is an equivalence. In view of classical theorems from Analysis, this equivalence translates in turn into the statement that there is an equivalence between the categories of *real vector bundles over* X and that of *finitely generated projective* Cont($X, \mathbb{R}$)*-modules*.

THE MAIN THEOREMS IN DIMENSION 1

Theorem A. (Bunge-Paré 1979) The fibration

$$\text{cod} : \mathscr{S}^{\rightarrow} \longrightarrow \mathscr{S}$$

is a stack.

Remarks.

- In addition to the basic facts about 1-stacks, the main tools used in the proof of Theorem A are the *monadicity and descent theorems* of (Beck 1967) and (Bénabou-Roubaud 1970).

- The category $\mathscr{S}$ plays two roles in the above. As a base for the fibration, $\mathscr{S}$ is regarded as a *topos*. As a fibration over itself, $\mathscr{S}$ need only be a *Barr-exact category*. Moreover, the motivating interpretation for future generalizations is to regard the fibration $\mathscr{S}$ over itself as the fibration of *0-stacks*, that is, that of sheaves for the intrinsic topology of $\mathscr{S}$ consisting of its epimorphisms, which just happens to be a topos.

Theorem B. (Bunge 1979) The stack completion of a groupoid $\mathbb{G}$ in $\mathscr{S}$ is identified with the first factor in the factorization of yon : $\mathbb{G} \longrightarrow \mathscr{S}^{\mathbb{G}^{\text{op}}}$ given by

$$[\mathbb{G}] \xrightarrow{\text{yon}} \text{LocRep}(\mathscr{S}^{\mathbb{G}^{\text{op}}}) \hookrightarrow \mathscr{S}^{\mathbb{G}^{\text{op}}}.$$

Theorem C. (Bunge 1990) For an *etale complete* groupoid $\mathbb{G}$ which is furthermore 'non-empty' and 'connected', there are equivalences

$$\mathsf{LocRep}(\mathscr{S}^{\mathbb{G}^{op}}) \cong \mathsf{Tors}(\mathbb{G}) \cong \mathsf{Points}(\mathscr{S}^{\mathbb{G}^{op}}).$$

Proof.

It is easy to show directly that the canonical morphism

$$[\mathbb{G}] \xrightarrow{\text{triv}} \mathsf{Tors}^1(\mathbb{G})$$

(defined by regarding $\mathbb{G}$ as the trivial $\mathbb{G}$-1-torsor) is a weak equivalence of 1-fibrations, and that $\mathsf{Tors}^1(\mathbb{G})$ is a 1-stack, hence 'the' 1-stack completion of $\mathbb{G}$.

On the other hand, Theorem B applies to $\mathbb{G}$. That is, we have

$$[\mathbb{G}] \xrightarrow{\text{yon}} \mathsf{LocRep}(\mathscr{S}^{\mathbb{G}^{op}})$$

is a weak equivalence, and $\mathsf{LocRep}(\mathscr{S}^{\mathbb{G}^{op}})$ is a 1-stack, hence 'the' 1-stack completion of $\mathbb{G}$.

There is a direct identification of $\mathsf{LocRep}(\mathscr{S}^{\mathbb{G}^{op}})$ with the fibration if *essential points* of the topos $\mathscr{S}^{\mathbb{G}^{op}}$ (Bunge 1979), and yet another (Bunge 1990) with the fibration of (localic – in this case discrete) points of $\mathscr{S}^{\mathbb{G}^{op}}$, hence all the various versions of stack completions of $\mathbb{G}$ must be equivalent.

Remark.

We conclude (as shown directly by Diaconescu 1995) that the topos $\mathscr{S}^{\mathbb{G}^{op}}$ *classifies* $\mathbb{G}$-torsors in the usual meaning of this terminology in the case of a topos.

Recall that *1-dimensional cohomology* of $\mathscr{S}$ with coefficients in an etale complete groupoid $\mathbb{G}$ is given by the formula

$$\mathsf{H}^1(\mathscr{S};\mathbb{G}) = \Pi_0(\mathsf{Tors}^1(\mathbb{G}))$$

where Π_0 denotes 'isomorphism classes'.

EXAMPLE 2

In "La longue marche..."), Grothendieck defines the *fundamental groupoid* $\Pi_1(\mathscr{G})$ of a Galois topos $\mathscr{G}$ (bounded over Set) to be the fibration Points($\mathscr{G}$) over Set.

We have argued (coincidentally) elsewhere (Bunge 2001) that this is the correct choice also over an arbitrary base topos $\mathscr{S}$, whereas a natural candidate for the *Galois groupoid* of $\mathscr{G}$ is the groupoid $\mathbb{G}_U = \mathrm{Iso}(p_U)$ where U is a universal cover in $\mathscr{G}$. To be noted is that the former is the stack completion of the latter so that, for $\mathscr{S} =$ Set, they are equivalent, a reason why this distinction is not made in the case of Grothendieck toposes.

One important difference between Galois and fundamental groupoids can be noted in the case of an arbitrary (for simplicity, a locally connected) topos $e : \mathscr{E} \longrightarrow \mathscr{S}$ $(e_! \dashv e^* \dashv e_*)$, where a generating system of covers is needed in order to define the fundamental groupoid $\Pi_1(\mathscr{E})$ as a limit (Bunge 2001).

Let $U \leq V$ in $\mathscr{S}$ be covers with corresponding Galois groupoids $\mathbb{G}_U$ and $\mathbb{G}_V$, with $\mathscr{G}_U$, $\mathscr{G}_V$ their classifying (Galois) toposes, and with $p_U : \mathscr{S}/e_!U \longrightarrow \mathscr{G}_U$, $p_V : \mathscr{S}/e_!V \longrightarrow \mathscr{G}_V$ the canonical (bags of) points. There is an equivalence of categories:

$$\mathrm{Hom}(\mathbb{G}_U, \mathbb{G}_V) \simeq \mathbf{Top}_{\mathscr{S}}(\mathscr{G}_U, \mathscr{G}_V)_+,$$

where the symbol $+$ indicates commutation with the canonical (bags of) points.

By contrast, letting $F_U : \mathbb{G}_U \longrightarrow \widetilde{\mathbb{G}_U}$ and $F_V : \mathbb{G}_V \longrightarrow \widetilde{\mathbb{G}_V}$ be the stack completions and weak equivalence functors, hence inducing equivalences of the classifying (Galois) toposes, there is an equivalence of categories:

$$\mathrm{Hom}(\widetilde{\mathbb{G}_U}, \widetilde{\mathbb{G}_V}) \simeq \mathbf{Top}_{\mathscr{S}}(\mathscr{G}_U, \mathscr{G}_V).$$

By adding multiplicity (in the stack completions) one eliminates the dependence on the chosen points. This difference is of importance when taking a limit in order to define $\Pi_1(\mathscr{E})$.

n-FIBRATIONS and n-STACKS

In view of our ultimate goal, we shall now introduce the notion of an intrinsic n-stack and state some basic facts by analogy with the case $n = 1$. As before, $\mathscr{S}$ is an elementary topos.

Definition. An n-functor $P : \mathscr{E} \longrightarrow \mathscr{B}$ is a n-*fibration* if:

- for any object X in $\mathscr{E}$ and any 1-cell $u : I \longrightarrow PX$ in $\mathscr{B}$, there is an n-cartesian 1-cell $\bar{u} : u^*(X) \longrightarrow X$ with $P\bar{u} = u$.

- For any pair of objects X, Y in $\mathscr{E}$, the induced $(n-1)$-functor $P_{X,Y} : \mathscr{E}(X,Y) \longrightarrow \mathscr{B}(PX, PY)$ is an (n-1)- fibration, stable under precomposition: for every 1-cell $h : Z \longrightarrow X$ in $\mathscr{E}$, the $(n-1)$-functor $\mathscr{E}(h, Y) : \mathscr{E}(X,Y) \longrightarrow \mathscr{E}(Z,Y)$ preserves n-cartesian morphisms (from $P_{X,Y}$ to $P_{Z.Y}$).

Definition.

- Let $\mathscr{S}$ be an elementary topos. An n-fibration $\mathscr{A} \longrightarrow \mathscr{S}$ is said to be a *n-stack* if for every epimorphism $e : J \longrightarrow I$ in $\mathscr{S}$, the n-functor $e^* : \mathscr{A}^I \longrightarrow \mathscr{A}^J$ is of effective n-descent.

- Let $\mathscr{A} \longrightarrow \mathscr{S}$ be an n-fibration. Then, the n-*stack completion* of $\mathscr{A}$ (if it exists) is given a pair $(\tilde{\mathscr{A}}, F)$, where $\tilde{\mathscr{A}} \longrightarrow \mathscr{S}$ is an n-stack, and $F : \mathscr{A} \longrightarrow \tilde{\mathscr{A}}$ is a morphism of n-fibrations, satisfying the obvious universal property among such pairs.

WEAK n-EQUIVALENCES

Definition *

The notion of a *weak n-equivalence morphism* $F : \mathscr{A} \longrightarrow \mathscr{B}$ of n-fibrations over $\mathscr{S}$ is given by induction as follows.

- $(n = 0)$ A morphism of 0-fibrations is a morphism $f : A \longrightarrow B$ of $\mathscr{S}$. It is said to be a *weak 0-equivalence 0-functor* if it is an isomorphism.

- $(n > 0)$ Let $F : \mathscr{A} \longrightarrow \mathscr{B}$ be an n-functor between n-fibrations over $\mathscr{S}$. It is said to be a *weak n-equivalence* if the following conditions hold.

  1. For each $I \in \mathscr{S}$, $F^I : |\mathscr{A}^I| \longrightarrow |\mathscr{B}^I|$, and $b \in |\mathscr{B}^I|$, there exists an epimorphism $e : J \longrightarrow I$ in $\mathscr{S}$, $a \in |\mathscr{A}^I|$, and an isomorphism $\theta : F^J(e) \longrightarrow e^*(b)$.

  2. $\forall I \in \mathscr{S}\ \forall x, x' \in |\mathscr{A}^I|$, the morphism
  $$\mathrm{Hom}_{\mathscr{A}^I}(x, x') \xrightarrow{F_{x,x'}} \mathrm{Hom}_{\mathscr{B}^I}(Fx, Fx')$$
  of $(n-1)$-fibrations is a weak $(n-1)$-equivalence.

Proposition.

1. An n-fibration $\mathscr{A}$ over $\mathscr{S}$ is an n-stack iff for every weak n-equivalence $F : \mathscr{B} \longrightarrow \mathscr{C}$, the induced $\mathscr{A}^F : \mathscr{A}^{\mathscr{C}} \longrightarrow \mathscr{A}^{\mathscr{B}}$ is an equivalence of n-fibrations.

2. If $F : \mathscr{A} \longrightarrow \mathscr{B}$ is a morphism of n-fibrations that is a weak n-equivalence, and $\mathscr{B}$ is an n-stack, then the pair $(\mathscr{B}, F)$ is the n-stack completion of $\mathscr{A}$.

*For internal n-categories, the definition of a weak n-equivalence n-fucntor was given in (Bourn 1980).

n-KERNELS

Any morphism $e : J \longrightarrow I$ in $\mathscr{S}$ induces an n-functor $F_e^n : \mathbb{J}_e^{(n)} \longrightarrow \mathbb{I}_{\mathsf{dis}}^n$, the n-kernel of e.

We depict $\mathbb{J}_e^{(2)}$ and $F_e^{(2)} : \mathbb{J}_e^{(2)} \longrightarrow \mathbb{I}_{\mathsf{dis}}^2$ in simplified form.

$$J \times_I J \times_I J \; \underset{\pi_{12}}{\overset{\pi_{01}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xrightarrow{\pi_{02}}}}}} \; J \times_I J \; \underset{\pi_1}{\overset{\pi_0}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{i}}}}} \; J$$

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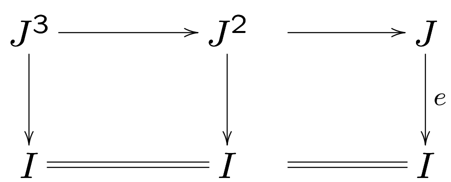


Proposition. For any epimorphism $e : J \longrightarrow I$ in $\mathscr{S}$, and any $n > 0$, $F_e^n : \mathbb{J}_e^{(n)} \longrightarrow \mathbb{I}_{\mathsf{dis}}^n$ is a weak n-equivalence n-functor.

Proof. In this case, the internal definition of a weak n-equivalence (Bourn 1980) is more useful. The proof is by induction on n.

THE 2-FIBRATION STACK

Let cod : **Stack** $\longrightarrow \mathscr{S}$ be the 2-fibration of groupoid stacks in $\mathscr{S}$, defined specifically as follows.

The fiber **Stack**^{I} above an object I of $\mathscr{S}$ is given by the 2-category **Stack**$/\mathbb{I}_{dis}$, whose objects are pairs $\langle \mathbb{G}, P \rangle$, where $\mathbb{G}$ is a groupoid stack in $\mathscr{S}$ and $P : \mathbb{G} \longrightarrow \mathbb{I}_{dis}$ a functor (necessarily both a fibration and a cofibration since $\mathbb{I}_{dis}$ is discrete), whose morphisms are functors F which fit into a commutative diagram

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{F} & \mathbb{H} \\ & {\scriptstyle P}\searrow \quad \swarrow{\scriptstyle Q} & \\ & \mathbb{I}_{\mathsf{dis}} & \end{array}$$

and whose 2-cells are natural transformations $\alpha : F \Longrightarrow F' : \mathbb{G} \longrightarrow \mathbb{H}$ (necessarily natural isomorphisms).

Change of base along a morphism $u : J \longrightarrow I$ in $\mathscr{S}$ is given by the 2-functor $u^* :$ **Stack**$/\mathbb{I}_{dis} \longrightarrow$ **Stack**$/\mathbb{J}_{dis}$, pullback in **Cat** along the functor $u_{dis} : \mathbb{J}_{dis} \longrightarrow \mathbb{I}_{dis}$.

AXIOM OF STACK COMPLETIONS

Denote by **Stack** the 2-category whose objects are 1-groupoid 1-stacks in $\mathscr{S}$, whose morphisms are functors $F : \mathbb{G} \longrightarrow \mathbb{H}$, and whose 2-cells are natural transformations $\alpha : F \longrightarrow F' : \mathbb{G} \longrightarrow \mathbb{H}$ (necessarily natural isomorphisms).

(ASC). We say that $\mathscr{S}$ satisfies *(ASC)* if the 2-inclusion

$$i : \textbf{Stack} \hookrightarrow \textbf{Gpd}$$

admits a left biadjoint $a \dashv i$, such that the unit $\eta : \mathrm{id} \longrightarrow i \cdot a$, evaluated at any groupoid $\mathbb{G}$, is a weak equivalence functor

$$\eta_{\mathbb{G}} : \mathbb{G} \longrightarrow \widetilde{\mathbb{G}} = (i \cdot a)(\mathbb{G}).$$

Remark. In view of the (already shown) existence of 1-stack completions of 1-groupoids in $\mathscr{S}$, as fibrations, the axiom says, equivalently, that for any 1-groupoid $\mathbb{G}$ in $\mathscr{S}$, the 1-fibration $\mathsf{LocRep}(\mathscr{S}^{\mathbb{G}^{\mathrm{op}}})$ is representable: there is an equivalence $\Phi : \mathsf{LocRep}(\mathscr{S}^{\mathbb{G}^{\mathrm{op}}}) \longrightarrow \widetilde{\mathbb{G}}$ of 1-fibrations (where we denote by $\mathbb{G}$ (and $\widetilde{\mathbb{G}}$) both the 1-groupoid and its associated representable 1- fibration, fitting into a commutative diagram

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{\ \mathsf{yon}\ } & \mathsf{LocRep}(\mathscr{S}^{\mathbb{G}^{\mathrm{op}}}) \\ & {\scriptstyle \eta_{\mathbb{G}}} \searrow \quad \swarrow {\scriptstyle \Phi} & \\ & \widetilde{\mathbb{G}} & \end{array}$$

Remark. Denote by $(ASC)^n$ the analogue, for arbitrary $n > 0$, of (ASC). It makes sense regardless of whether a specific construction of the n-stack completion of an n-groupoid ((n -1)-stack) is available as a (possibly non-representable) n-fibration, in the form we shall discuss.

HERMIDA'S THEOREMS (2004)

A 2-fibration $P : \mathscr{A} \longrightarrow \mathbf{Cat}$ is said *to have Σ with the BCC for bicomma squares* if for every $u : \mathbb{B} \longrightarrow \mathbb{A}$ in **Cat**, the change of base $u^* : \mathscr{A}^{\mathbb{A}} \longrightarrow \mathscr{A}^{\mathbb{B}}$ admits a left 2-adjoint $\Sigma_u \dashv u^*$ such that, for every bicomma square

$$\begin{array}{ccc} (v \downarrow u) & \xrightarrow{q} & \mathbb{B} \\ {\scriptstyle p}\downarrow & \Longrightarrow \lambda & \downarrow{\scriptstyle u} \\ \mathbb{K} & \xrightarrow[v]{} & \mathbb{A} \end{array}$$

in **Cat**, the induced

$$\widetilde{\lambda} : \Sigma_p \cdot q^* \Longrightarrow v^* \cdot \Sigma_u$$

is an equivalence.

Theorem 1. If a 2-fibration $P : \mathscr{A} \longrightarrow \mathbf{Cat}$ has Σ subject to the BCC for bicomma squares in **Cat** along a cofibration, then, given $q : \mathbb{T} \longrightarrow \mathbb{Q}$ in **Cat**, there is a canonical biequivalence

$$\mathsf{Des}^2{}_q(\mathscr{A}) \longrightarrow \mathsf{Ps} - (q^*\Sigma_q)\mathsf{Alg}$$

Definition. A functor $q : O \longrightarrow Q$ in $\mathbf{Cat}$ is said to be a *2-regular eso* ('eso' for 'essentially surjective on objects') if the bicomma object

$$\begin{array}{ccc} (q \downarrow q) & \xrightarrow{d} & O \\ {\scriptstyle c}\downarrow & \Longrightarrow \lambda & \downarrow{\scriptstyle q} \\ O & \xrightarrow[q]{} & Q \end{array}$$

exhibits (q, λ) as the lax colimit of the span $O \rightharpoonup O$.

Theorem 2. Any 2-regular eso in **Cat** is of effective 2-descent for the basic fibration

$$cod : \mathbf{Fib} \longrightarrow \mathbf{Cat}.$$

Proof. Essentially the same argument as in exact categories.

STACK IS A 2-STACK

Theorem 2A. Let $\mathscr{S}$ be an elementary topos satisfying (ASC). Then, the 2-fibration

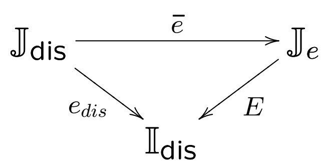
$$\text{cod} : \textbf{Stack} \longrightarrow \mathscr{S}$$

is a 2-stack.

Proof. The 2-category **Stack** (the fiber at 1 of the 2-fibration cod : **Stack** $\longrightarrow \mathscr{S}$), as a reflective subcategory of **Gpd** by (ASC), has all colimits that exist in **Gpd**. The same is true then of the fiber of **Stack** at any I and, for any $u : J \longrightarrow I$, $u^* :$ **Stack**$/\mathbb{I}_{dis} \longrightarrow$ **Stack**$/\mathbb{J}_{dis}$ preserves such colimits (in particular, pseudo-coequalizers). Moreover, since the 2-fibration cod : **Fib** $\longrightarrow$ **Cat** has Σ with the BCC for bi-comma squares, the same is true (and in an even simpler form since only groupoids are involved) for cod : **Stack** $\longrightarrow \mathscr{S}$.

By Theorem 1 (Hermida 2004), in order to prove that **Stack** is a 2-stack, it is enough to prove that, for any epimorphism $e : J \longrightarrow I$ in $\mathscr{S}$, $e^* :$ **Stack**$/\mathbb{I}_{dis} \longrightarrow$ **Stack**$/\mathbb{J}_{dis}$ reflects equivalences.

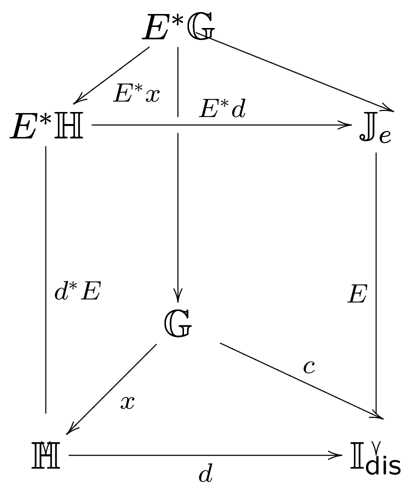
Consider the factorization



where $E : \mathbb{J}_e \longrightarrow \mathbb{I}_{dis}$ is the 1-kernel of e. Observe that $\bar{e} : \mathbb{J}_{dis} \longrightarrow \mathbb{J}_e$ is a strong (hence 2-regular) eso.

Assume that $e^*x = (\bar{e}^*E^*)x$ is an equivalence. Since $\bar{e}$ is a regular eso, it is of effective 2-descent for $cod : \mathbf{Fib} \longrightarrow \mathbf{Cat}$ by Theorem 2 (Hermida 2004). It follows then that E^*x is an equivalence. We now show that this is enough to show that x is an equivalence.

Consider the prism diagram



where the front and back square faces are pullbacks, and where E^*x is an equivalence. The morphisms $c^*E : E^*\mathbb{G} \longrightarrow \mathbb{G}$ and $d^*E : E^*\mathbb{H} \longrightarrow \mathbb{H}$ are weak equivalence functors, since E is one.

Therefore, $d^*E \cdot E^*x = x \cdot c^*E$ (the lhs commuting square) is a weak equivalence functor and therefore so is x (by basic properties of wef (Bunge-Paré 1979)). But $\mathbb{G}$ is a stack, hence x is an equivalence. This concludes the proof.

Remark. See also (Mauri-Tierney 1999) for an alternative proof of this result in the case of the fibers at 1, where they use (instead of our abstract (ASC)), the Quillen model structure on **Cat** (or on **Gpd**) whose fibrant objects are the strong stacks (Joyal-Tierney 1991).

2-GROUPOIDS

We shall adopt the notion of internal n-groupoid made explicit in (Bourn 1987). It is available in any exact category, in particular, in any topos $\mathscr{S}$. Moreover, each fibration

$$(-)_{n-1} : n\mathbf{Gpd}(\mathscr{S}) \longrightarrow (n-1)\mathbf{Gpd}(\mathscr{S})$$

is exact.

We shall restrict our attention first to 2-groupoids. We omit the mention of $\mathscr{S}$. The 1-cells in a 2-groupoid are equivalences, and the 2-cells are isomorphisms.

Definition. A 2-groupoid $\mathbb{G}$ in $\mathscr{S}$ is said to be a *2-groupoid 1-stack* if $(\mathbb{G})_1$ is a 1-groupoid 1-stack in $\mathscr{S}$. Denote by $\widetilde{2\mathbf{Gpd}}$ the full subcategory of $2\mathbf{Gpd}$ whose objects are the 2-groupoids 1-stacks, and by

$$\widetilde{(-)_1} : \widetilde{2\mathbf{Gpd}} \longrightarrow \mathbf{Stack}$$

the lifting to $\mathbf{Stack} \longrightarrow \mathbf{Gpd}$ of the restriction of $(-)_1 : 2\mathbf{Gpd} \longrightarrow \mathbf{Gpd}$ to $\widetilde{2\mathbf{Gpd}}$.

Remark. Any 2-groupoid 2-stack is a 2-groupoid 1-stack. More generally, any n-groupoid n-stack is an n-groupoid k-stack for any $0 < k \leq (n-1)$.

2-STACK COMPLETIONS OF 2-GROUPOIDS 1-STACKS

Proposition. A 2-fibration $\mathscr{A} \longrightarrow \mathscr{S}$ is a 2-stack iff for every weak equivalence 2-functor $F : \mathbb{B} \longrightarrow \mathbb{A}$ in $\mathscr{S}$, the induced

$$\mathscr{A}^F : \mathscr{A}^{\mathbb{A}} \longrightarrow \mathscr{A}^{\mathbb{B}}$$

is an equivalence of 2-fibrations.

Theorem 2B. Let $\mathscr{S}$ be a topos satisfying (ASC). Let $\mathbb{G}$ be a 2-groupoid 1-stack in $\mathscr{S}$. Then the 2-stack completion of $\mathbb{G}$ can be identified with the pair

$$(\mathrm{LocRep}(\mathbf{Stack}^{\mathbb{G}^{\mathrm{op}}}), F)$$

where F is

$$\mathbb{G} \xrightarrow{\text{yon}} \mathrm{LocRep}(\mathbf{Stack}^{\mathbb{G}^{\mathrm{op}}})$$

Proof.

- Since $\mathbf{Stack}$ is a 2-stack (over $\mathscr{S}$), so is $\mathbf{Stack}^{\mathbb{G}^{\mathrm{op}}}$.

- Factor the yoneda embedding as

  $$\mathbb{G} \xrightarrow{yon} \mathrm{LocRep}(\mathbf{Stack}^{\mathbb{G}^{\mathrm{op}}}) \hookrightarrow \mathbf{Stack}^{\mathbb{G}^{\mathrm{op}}}$$

  and prove (just as in the case $n = 1$ from (Bunge-Paré 1979) that, since $\mathbf{Stack}^{\mathbb{G}^{\mathrm{op}}}$ is a 2-stack, so is $\mathrm{LocRep}(\mathbf{Stack}^{\mathbb{G}^{\mathrm{op}}})$. Furthermore, the first factor is a weak 2-equivalence.

2-GERBES AND 2-TORSORS

A *particular case* of the notion of a 2-groupoid 1-stack is that of a 2-gerbe. A 2-groupoid $\mathbb{G}$ in $\mathscr{S}$ is said to be a *2-gerbe* if for some ('non-empty' and 'connected') 1-groupoid 1-stack $\mathbb{A}$ (a *bouquet*) , there is a 2-equivalence

$$\mathbb{G} \simeq \mathsf{Equ}(\mathbb{A}).$$

We recall the notion of a *2-torsor* for a 2-groupoid $\mathbb{G}$ (Mauri-Tierney 2001). The 2-groupoid $\mathsf{Tors}^2(\mathbb{G})$ has:

- as *objects* the $\mathbb{G}$-2-torsors, where a 2-torsor is a groupoid stack $\mathbb{T}$ such that $T_0 \longrightarrow 1$ is an epimorphism, andit is equipped with an action $a : \mathbb{T} \times \mathbb{G} \longrightarrow \mathbb{T}$ of $\mathbb{G}$ on $\mathbb{T}$, such that

  $$\mathbb{T} \times \mathbb{G} \xrightarrow{<\pi_1, a>} \mathbb{T} \times \mathbb{T}$$

  is an isomorphism.

- as *1-cells* the $\mathbb{G}$-equivariant functors, as in the diagram

  $$\begin{array}{ccc} \mathbb{T} \times \mathbb{G} & \xrightarrow{h \times \mathsf{id}} & \mathbb{R} \times \mathbb{G} \\ {\scriptstyle <\pi_1, a>} \downarrow & & \downarrow {\scriptstyle <\pi_1, b>} \\ \mathbb{T} \times \mathbb{T} & \xrightarrow[h \times h]{} & \mathbb{R} \times \mathbb{R} \end{array}$$

- as *2-cells*, natural isomorphisms $h \Longrightarrow k : \mathbb{T} \longrightarrow \mathbb{R}$.

CLASSIFICATION OF 2-TORSORS

Theorem C. Let $\mathscr{S}$ satisfy (ASC). Let $\mathbb{G}$ be a *2-gerbe*. Then there is a biequivalence.

$$\mathsf{LocRep}(\mathbf{Stack}^{\mathbb{G}^{op}}) \cong \mathsf{Tor}^2(\mathbb{G}).$$

Proof.

It is easy to show directly that the canonical morphism

$$\mathbb{G} \xrightarrow{\text{triv}} \mathsf{Tors}^2(\mathbb{G})$$

(defined by regarding $\mathbb{G}$ as the trivial $\mathbb{G}$-2-torsor) is a weak 2-equivalence of 2-fibrations, and that $\mathsf{Tors}^2(\mathbb{G})$ is a 2-stack, hence 'the' 2-stack completion of $\mathbb{G}$.

On the other hand, any 2-gerbe is a 2-groupoid 1-stack, hence Theorem 2B applies to it. That is, we have

$$\mathbb{G} \xrightarrow{\text{yon}} \mathsf{LocRep}(\mathbf{Stack}^{\mathbb{G}^{op}})$$

is a weak 2-equivalence and $\mathsf{LocRep}(\mathbf{Stack}^{\mathbb{G}^{op}})$ is a 2-stack, hence 'the' 2-stack completion of $\mathbb{G}$.

Remark. We interpret this to say that, for any 2-gerbe $\mathbb{G}$, the 2-category **Stack**$^{\mathbb{G}^{op}}$ *classifies* $\mathbb{G}$-2-torsors. Recall that 2-dimensional cohomology of $\mathscr{S}$ with coefficients in a 2-gerbe $\mathbb{G}$ is given by the formula

$$\mathsf{H}^2(\mathscr{S}; \mathbb{G}) = \mathbf{\Pi_0}(\mathsf{Tors}^2(\mathbb{G}))$$

where $\mathbf{\Pi_0}$ in this case denotes 'equivalence classes'.

A PROGRAM

$\mathscr{S}$ is a 1-stack $\mapsto$ 1-stack completion of 1-groupoids (0-stacks) $\mapsto$ classification of 1-torsors.

$\rightsquigarrow$

(under the assumption (ASC))

Stack is a 2-stack $\mapsto$ 2-stack completions of 2-groupoids 1-stacks $\mapsto$ classification of 2-torsors.

...

$\rightsquigarrow$

(under the assumption (ASC)^n)

Stack^n is an $(n+1)$-stack $\mapsto$ $(n+1)$-stack completions of $(n+1)$-groupoids n-stacks $\mapsto$ classification of (n + 1)-torsors.

Conjecture. We conjecture the following theorem, which we regard as feasible, with the possible exception of a direct comparison between n-monadicity and n-descent beyond a remark of the sort 'the coherence conditions correspond to each other'. All other ingredients for such a proof seem to be in place.

'Theorem' . Let $n > 0$. Let $\mathscr{S}$ be an elementary topos satisfying (ASC)^n. Let $\mathbb{G}$ be an (n + 1)-groupoid n-stack (e.g. an (n+1)-gerbe). Then the (n+1)-stack completion of $\mathbb{G}$ exists (as an (n+1)-fibration) and is given by the weak (n+1)-equivalence

$$\mathbb{G} \xrightarrow{yon} \text{LocRep}((\textbf{Stack}^n)^{\mathbb{G}^{\text{op}}})$$

morphism of (n+1)-fibrations.

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