First-Order Logical Duality

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Algebra-geometry, syntax-semantics

1. Stone duality—the fact that the ‘algebraic’ category of Boolean algebras is dual to the ‘geometric’ category of Stone spaces

$$\mathbf{BA}^{\mathrm{op}} \simeq \mathbf{Stone}$$

   has a logical interpretation as a syntax-semantics duality for classical propositional logic.

2. We present a generalization to first-order logic, which yields the propositional logical Stone duality as a special case.

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Logical interpretation - algebras

A propositional theory, $\mathbb{T}$ can be seen as a Boolean algebra.

Definition

For a propositional theory $\mathbb{T}$, the *Lindenbaum-Tarski algebra*, $L_{\mathbb{T}}$ of $\mathbb{T}$ consists of equivalence classes $[\phi]$ of formulas, where

$$\phi \sim \psi \Leftrightarrow \mathbb{T} \vdash \phi \leftrightarrow \psi,$$

ordered by provability:

$$[\phi] \leq [\psi] \Leftrightarrow \mathbb{T} \vdash \phi \rightarrow \psi.$$

The Lindebaum-Tarski (LT) algebra of a propositional theory is a Boolean algebra. Conversely, any Boolean algebra is the LT-algebra of a classical propositional theory

$$\mathcal{B} \cong L_{\mathbb{T}_{\mathcal{B}}}.$$

Logical interpretation - Stone spaces

For a propositional theory $\mathbb{T}$, a (2-valued) model is an assignment of formulas to the values 1 (true) and 0 (false) which preserves provability, and so can be considered to be a morphism of Boolean algebras

$$L_{\mathbb{T}} \longrightarrow 2.$$

Conversely, such a morphism can be seen as a model of $\mathbb{T}$. Alternatively, these morphisms can be seen as ultra-filters of $L_{\mathbb{T}}$. Therefore, the Stone space corresponding to $L_{\mathbb{T}}$ can be presented as the set of 'models'

$$X_{L_{\mathbb{T}}} := \mathrm{Hom}_{\mathbf{BA}}(L_{\mathbb{T}}, 2)$$

equipped with the 'logical' topology defined by basic opens

$$U_{\phi} = \{\mathbf{M} \vDash \mathbb{T} \mid \mathbf{M} \vDash \phi\}$$

for ϕ a formula of $\mathbb{T}$.

Representing Boolean algebras as spaces of models 1

A Boolean algebra $\mathcal{B}$ can be recovered from its Stone space of models (or ultra-filters) $X_{\mathcal{B}}$. E.g. as follows.
The map $U : \mathcal{B} \to \mathcal{O}(X_{\mathcal{B}})$ defined by $b \mapsto \{f \in X_{\mathcal{B}} \mid f(b) = 1\}$ lifts to an *isomorphism of frames* $\hat{U}$,

$$\begin{array}{ccc} \mathrm{Idl}\,(\mathcal{B}) & \xrightarrow[\cong]{\hat{U}} & \mathcal{O}(X_{\mathcal{B}}) \\ {\scriptstyle P}\uparrow & \nearrow {\scriptstyle U} & \\ \mathcal{B} & & \end{array}$$

where

- $\mathrm{Idl}\,(\mathcal{B})$ is the ideal completion of $\mathcal{B}$;
- $P : \mathcal{B} \to \mathrm{Idl}\,(\mathcal{B})$ is the principal ideal embedding.

Representing Boolean algebras as spaces of models 2

Corollary

$\mathcal{B}$ can be recovered as the compact elements of $\mathcal{O}(X_{\mathcal{B}})$, i.e. as the compact open subsets of $X_{\mathcal{B}}$.

Since $X_{\mathcal{B}}$ is Stone, in particular compact and Hausdorff, that means

Corollary

$\mathcal{B}$ can be recovered as the lattice of clopen subsets of $X_{\mathcal{B}}$.

The latter can be identified with the continuous functions from $X_{\mathcal{B}}$ into the discrete (Stone) space 2,

$$CL(X_{\mathcal{B}}) \cong \mathrm{Hom}_{\mathbf{Stone}}(X_{\mathcal{B}}, 2)$$

Stone duality

Sending a Boolean algebra to its Stone space of ‘models’ is (contravariantly) functorial, as is recovering a Boolean algebra as the clopens of a Stone space, and we get the familiar Stone duality:

$$\mathbf{BA}^{\mathrm{op}} \underset{\mathrm{Hom}_{\mathbf{BA}}(-,2)}{\overset{\mathrm{Hom}_{\mathbf{Stone}}(-,2)}{\rightleftarrows}} \mathbf{Stone} \qquad (\simeq)$$

Logical Duality - Table

	SYNTAX	Intermediate	SEMANTICS
Class. Prop. Logic	*Boolean algebras* $\mathcal{B} \cong L_{\mathbb{T}}$ algebraic object built from syntax	*Frames* $\mathsf{Idl}(\mathcal{B}) \cong \mathcal{O}(X_{\mathcal{B}})$	*Stone spaces* $X_{\mathcal{B}} \cong \mathrm{Hom}_{\mathbf{BA}}(\mathcal{B}, 2)$ space of models
FOL	*Bool. coh. cats* $\mathcal{B} \simeq \mathcal{C}_{\mathbb{T}}$ algebraic object built from syntax	*Topoi* $\mathsf{Sh}(\mathcal{B}) \simeq \mathsf{Sh}_{G_{\mathcal{B}}}(X_{\mathcal{B}})$	*Top. gpds* $G_{\mathcal{B}} \rightrightarrows X_{\mathcal{B}}$ top. grpd of models and isomorphisms

Syntactical categories - $\mathcal{C}_{\mathbb{T}}$

For a first-order theory $\mathbb{T}$, the *syntactical category* $\mathcal{C}_{\mathbb{T}}$ of $\mathbb{T}$ has as objects formulas-in-context

$$[\vec{x} \mid \phi]$$

of $\mathbb{T}$, with arrows classes of $\mathbb{T}$-provably equivalent formulas-in-context

$$|\,[\vec{x}, \vec{y} \mid \sigma]\,| : [\vec{x} \mid \phi] \longrightarrow [\vec{y} \mid \psi]$$

such that σ is $\mathbb{T}$-provably a functional relation from ϕ to ψ. With $\mathbb{T}$ a classical f.o. theory, $\mathcal{C}_{\mathbb{T}}$ is a Boolean (coherent) category (BC). Moreover, every BC is, up to equivalence, the syntactic category of a classical f.o. theory, so that BCs represent first-order logical theories.

Models

- Ordinary set-models of $\mathbb{T}$ correspond to coherent functors $\mathcal{C}_{\mathbb{T}} \longrightarrow \textbf{Sets}$,

$$\mathrm{Mod}_{\mathbb{T}}(\textbf{Sets}) \simeq \mathrm{Hom}_{\textbf{Coh}}(\mathcal{C}_{\mathbb{T}}, \textbf{Sets})$$

  $\mathbb{T}$-model isomorphisms correspond to invertible natural transformations between these coherent functors. Accordingly, the *groupoid* (category with all arrows invertible) of $\mathbb{T}$-models and isomorphisms between them can be represented as the groupoid of coherent set-valued functors from $\mathcal{C}_{\mathbb{T}}$ with invertible natural transformations between them:
- In order to have *sets* of models and isomorphisms, lets say $\mathbb{T}$ (and $\mathcal{C}_{\mathbb{T}}$) is countable, and we only consider the countable models, i.e. those functors that take values in countable sets.

Semantical groupoids

- For a countable Boolean coherent category $\mathcal{B}$, then, we consider the groupoid

$$G_{\mathcal{B}} \times_{X_{\mathcal{B}}} G_{\mathcal{B}} \xrightarrow{\ \mathrm{c}\ } \overset{\mathrm{i}\ \circlearrowleft}{G_{\mathcal{B}}} \;\begin{array}{c} \xrightarrow{\ \mathrm{s}\ } \\ \xleftarrow{\ \mathrm{Id}\ } \\ \xrightarrow[\ \mathrm{t}\ ]{} \end{array}\; X_{\mathcal{B}}$$

  of countable 'models' (coherent functors) and isomorphisms between them.
- We equip the sets $X_{\mathcal{B}}$ and $G_{\mathcal{B}}$ with topologies to make this a topological groupoid.

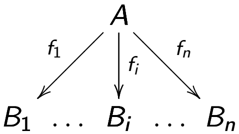
The topology on $X_{\mathcal{B}}$

Definition

The *coherent topology* on $X_{\mathcal{B}}$ is the coarsest containing all sets of the form

$$\{M \in X_{\mathcal{B}} \mid \exists x \in M(A).\, M(f_1)(x) = b_1 \wedge \ldots M(f_n)(x) = b_n\}$$

given by a finite span in $\mathcal{B}$,



and a list $b_1, \ldots, b_n \in \mathbf{Sets}_c$.

The topology on $G_{\mathcal{B}}$

Definition

The *coherent topology* on $G_{\mathcal{B}}$ is the coarsest such that the source and target maps $G_{\mathcal{B}} \rightrightarrows X_{\mathcal{B}}$ are both continuous, and containing all sets of the form

$$U_{A,a\mapsto b} = \{f : M \to N \mid a \in M(A) \wedge f_A(a) = b\}$$

given by an object A in $\mathcal{B}$ and $a, b \in \mathbf{Sets}_c$.

Sheaves: $\mathrm{Sh}(X)$

For a space X, the topos of sheaves on X

$$\mathrm{Sh}(X)$$

consists of local homeomorphisms over X

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & {\scriptstyle a}\searrow \quad \swarrow{\scriptstyle b} & \\ & X & \end{array}$$

If X is the space of objects of a topological groupoid:

$$G \underset{t}{\overset{s}{\rightrightarrows}} X$$

the *topos of equivariant sheaves*, $\mathrm{Sh}_G(X)$, is constructed by equipping sheaves on X with an action by G.

Equivariant sheaves: $\mathrm{Sh}_G(X)$

$\mathrm{Sh}_G(X)$ has as objects pairs $\langle a : A \to X, \alpha \rangle$ where the first component is an element of $\mathrm{Sh}(X)$ and the second component is a continuous action

$$G \times_X A \xrightarrow{\alpha} A$$

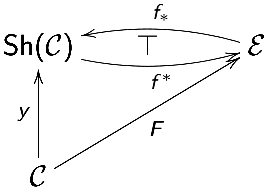
$$\langle g : y \to z, d \rangle \quad \mapsto \quad \alpha(g, d)$$

An arrow between objects $\langle a : A \to X, \alpha \rangle$ and $\langle b : B \to X, \beta \rangle$ is an arrow $f : A \to B$ of $\mathrm{Sh}(X)$ which commutes with the actions:

$$\begin{array}{ccc} G \times_X A & \xrightarrow{1_G \times f} & G \times_X B \\ \alpha \downarrow & & \downarrow \beta \\ A & \xrightarrow[f]{} & B \end{array}$$

The topos of coherent sheaves

For a coherent category $\mathcal{C}$, the *topos of coherent sheaves*—i.e. sheaves for the coherent, or finite epimorphic families, coverage—$\mathrm{Sh}(\mathcal{C})$ is the 'free topos on $\mathcal{C}$', in the sense that coherent functors from $\mathcal{C}$ into a topos $\mathcal{E}$ correspond to geometric morphisms from $\mathcal{E}$ to $\mathrm{Sh}(\mathcal{C})$:



$\mathcal{C}$ can be recovered, up to pretopos completion, from $\mathrm{Sh}(\mathcal{C})$ as the *coherent* objects, or, if $\mathcal{C}$ is Boolean, as the compact decidable objects.

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Stone representation theorem

1. The Stone representation theorem says that a Boolean algebra can be embedded in the lattice of subsets of a set

$$\mathcal{B} \hookrightarrow \mathcal{P}(X_{\mathcal{B}})$$

   By equipping that set with a topology, on can recover $\mathcal{B}$ as the compact open sets.

2. Generalizing, we show that a (countable) Boolean category can be 'embedded' in the topos of sets over a set

$$\mathcal{B} \hookrightarrow \mathbf{Sets}/X_{\mathcal{B}}$$

   By equipping that set with a topology and introducing continuous actions, on can recover $\mathcal{B}$ as the compact decidable objects.

Analogue to the Stone representation THM

For an object A in $\mathcal{B}$ we have the set E_A over $X_{\mathcal{B}}$ whose fiber over $M \in X_{\mathcal{B}}$ is $M(A)$:

$$E_A = \{\langle M, d\rangle \mid M \in X_{\mathcal{B}} \wedge d \in M(A)\} \xrightarrow{\pi_1} X_{\mathcal{B}}$$

Which gives the assignment:

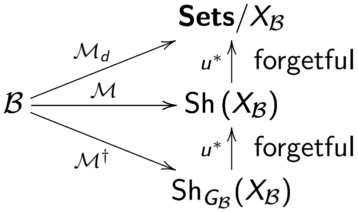
$\mathcal{B}$	$\mathbf{Sets}/X_{\mathcal{B}}$	Fiber over $M \in X_{\mathcal{B}}$
$A \xrightarrow{f} B$	$E_A \xrightarrow{E_f} E_B$, $E_A \to X_{\mathcal{B}}$, $E_B \to X_{\mathcal{B}}$	$M(A) \xrightarrow{M(f)} M(B)$, $M(A) \to M$, $M(B) \to M$

Embedding $\mathcal{B}$

This defines a coherent functor

$$\mathcal{M}_d : \mathcal{B} \longrightarrow \mathbf{Sets}/X_{\mathcal{B}}$$

which is **faithful** and **cover reflecting**. By equipping $X_{\mathcal{B}}$ with the coherent topology, and then introducing continuous $G_{\mathcal{B}}$-actions, we make the objects in the image of $\mathcal{M}_d$ compact and generating, and the embedding full. That is, we factor $\mathcal{M}_d$:

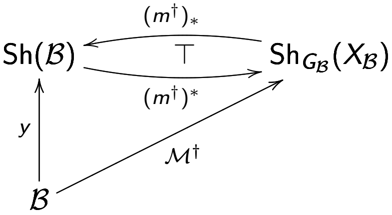


$\mathcal{M}^\dagger : \mathcal{B} \longrightarrow \mathrm{Sh}_{G_\mathcal{B}}(X_\mathcal{B})$ continued

Verifying that

1. the set $\{\mathcal{M}^\dagger(A) \mid A \in \mathcal{B}\}$ is a generating set for $\mathrm{Sh}_{G_\mathcal{B}}(X_\mathcal{B})$;
2. $\mathcal{M}^\dagger$ if full and faithful; and
3. $\mathcal{M}^\dagger$ reflects covers.

we get that $\mathcal{B} \simeq \mathcal{M}^\dagger(\mathcal{B})$ is a site for $\mathrm{Sh}_{G_\mathcal{B}}(X_\mathcal{B})$, and thus that the induced geometric morphism



is an equivalence.

Representation theorem

Theorem

For any (countable) Boolean coherent category $\mathcal{B}$,

$$\mathsf{Sh}(\mathcal{B}) \simeq \mathsf{Sh}_{G_{\mathcal{B}}}(X_{\mathcal{B}})$$

where $G_{\mathcal{B}} \rightrightarrows X_{\mathcal{B}}$ is the groupoid of countable models and isomorphisms, equipped with the coherent topologies.

Corollary

A (countable) Boolean coherent category, $\mathcal{B}$, is equivalent to the full subcategory of compact decidable objects in $\mathsf{Sh}_{G_{\mathcal{B}}}(X_{\mathcal{B}})$ up to pretopos completion. So that if $\mathcal{B}$ is a pretopos, then it is equivalent to the subcategory of compact decidable objects.

Syntax-semantics adjunction

1. Sending a BC to its semantical groupoid is functorial

$$\mathcal{G} : \mathbf{BC_c}^{\mathrm{op}} \longrightarrow \mathbf{Gpd}$$

2. By restricting to a subcategory of the category **Gpd** of topological groupoids, we can find an adjoint.
3. One way of doing this is to restrict to the category $\mathbf{BoolGpd} \hookrightarrow \mathbf{Gpd}$ of topological groupoids $G \rightrightarrows X$ such that $\mathrm{Sh}_G(X)$ has a Boolean coherent site, and morphisms between them that preserve compact (decidable) objects. Then taking the compact decidable objects in $\mathrm{Sh}_G(X)$ extracts a Boolean coherent category,

$$\mathcal{B}_{G \rightrightarrows X} \hookrightarrow \mathrm{Sh}_G(X)$$

Syntax-Semantics adjunction

There is a groupoid $\mathbb{S}$—it's the groupoid of models of the theory of equality—such that morphisms from $G \rightrightarrows X$ to $\mathbb{S}$ in **BoolGpd** corresponds to compact decidable objects in $\mathrm{Sh}_G(X)$. So 'homming into $\mathbb{S}$' gives a 'syntactical' functor extracting Boolean coherent categories from groupoids:

Theorem

The 'semantical' functor is (right) adjoint to the 'syntactical' functor ,

$$\mathbf{BC_c}^{\mathrm{op}} \underset{\mathcal{G}=\mathrm{Hom}_{\mathbf{BC_c}}(-,\mathbf{Sets}_c)}{\overset{\Theta=\mathrm{Hom}_{\mathbf{BoolGpd}}(-,\mathbb{S})}{\rightleftarrows}} \mathbf{BoolGpd} \qquad (\Theta \dashv \mathcal{G})$$

Counit components are equivalences at pretopoi.