

Injective spaces via adjunction

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Basic definitions

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 $\underline{a : TX \dashrightarrow X}$ (V-matrix)

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Example: $\mathbb{V} = (V, \text{hom}_\xi)$ where $\text{hom}_\xi(v, v) = \text{hom}(\xi(v), v)$.

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We have $\mathcal{T}\text{-Cat} \begin{array}{c} \xleftarrow{A} \\ \xrightarrow[\perp]{S} \\ \xrightarrow{\quad} \end{array} V\text{-Cat}$ and $\text{Set}^{\mathbb{T}} \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow[\perp]{\quad} \\ \xrightarrow{\quad} \end{array} \mathcal{T}\text{-Cat}$.

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Hence $PX = \{\psi : X \dashrightarrow E\}$

Theorem

The following are equivalent for $\varphi : X \dashrightarrow Y$.

- (i)** $\varphi : X \dashrightarrow Y$ is a \mathcal{T} -module.
- (ii)** $\varphi : |X| \otimes Y \longrightarrow V$, $\varphi : X^{\text{op}} \otimes Y \longrightarrow V$ are \mathcal{T} -functors.

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Hence $PX = \{\psi : X \dashrightarrow E\} \hookrightarrow V^{|X|}$

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Here $- \circ \varphi \dashv - \circ \varphi$,

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 \quad \psi \circ - \varphi(z, y) = \llbracket T \lceil \varphi^{-1}(z), \lceil \psi^{-1}(y) \rrbracket$$

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Here $\varphi \circ \downarrow \dashv \uparrow \circ \varphi$, and $\ulcorner \varphi \urcorner : Y \rightarrow PX$ and $\ulcorner \psi \urcorner : Z \rightarrow PX$.

The Yoneda functor

Put $y_X := \ulcorner a \urcorner : X \rightarrow PX$, for a \mathcal{T} -category $X = (X, a)$.

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For distributors φ, ψ :

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Put $y_X := \ulcorner a^{-1} : X \rightarrow PX$, for a \mathcal{T} -category $X = (X, a)$.

Corollary

For $x \in TX$, $\psi \in PX$, we have $\psi(x) = \llbracket T y_X(x), \psi \rrbracket$

Theorem

For distributors φ, ψ :

$$\begin{array}{ccc}
 X & \xrightarrow{\psi} & Y \\
 \varphi \downarrow & \uparrow & \nearrow \\
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Here $\varphi \circ \psi = \varphi \circ \psi$, and $\ulcorner \varphi^{-1} : Y \rightarrow PX$ and $\ulcorner \psi^{-1} : Z \rightarrow PX$.

The Yoneda functor

Put $y_X := \ulcorner a^{-1} : X \rightarrow PX$, for a \mathcal{T} -category $X = (X, a)$.

Corollary

For $x \in TX$, $\psi \in PX$, we have $\psi(x) = \llbracket T y_X(x), \psi \rrbracket = (y_X)_*(x, \psi)$.

Colimits

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ \downarrow \varphi & & \\ B & & \end{array}$$

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The diagram illustrates the relationship between a morphism $h: A \rightarrow X$ and its pushout along a morphism $\varphi: A \rightarrow B$. The left side shows the original morphism h and the morphism φ pointing from A to B . The right side shows the pushout construction, where the morphism h is replaced by h_* (the pushout of h along φ), and a new morphism $h_* \circ \varphi$ is introduced, pointing from B to X . The pushout is indicated by a dashed line and a double arrow \uparrow pointing to the new morphism h_* .

Cocomplete \mathcal{T} -categories

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The diagram illustrates the universal property of a colimit. On the left, a morphism $h: A \rightarrow X$ is shown with a morphism $\varphi: A \rightarrow B$ pointing to it. On the right, the same morphism h is shown with a new morphism $h_*: A \rightarrow X$ and a new morphism $g_*: B \rightarrow X$. A dashed arrow points from g_* to h_* , and a solid arrow points from h_* to h . The text "(then $g = \text{colim}(\varphi, h)$)" indicates that g is the colimit of the pair (φ, h) .

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The diagram shows the relationship between a morphism $h: A \rightarrow X$ and its colimit $g_*: A \rightarrow X$. A morphism $\varphi: A \rightarrow B$ is shown on the left. The right diagram shows a commutative square with a diagonal arrow $g_*: B \rightarrow X$ and a curved arrow $\uparrow \circlearrowleft$ from B to A . The equation $h_* \circ \varphi = g_*$ is written below the square. The text "(then $g = \text{colim}(\varphi, h)$)" is to the right.

Kan extension (X cocomplete)

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ i \downarrow & \Downarrow & \nearrow \\ B & \xrightarrow{g} & X \end{array}$$

The diagram shows a commutative square with a diagonal arrow $g: B \rightarrow X$ and a curved arrow \Downarrow from A to X . The text "Kan extension (X cocomplete)" is above the diagram.

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The diagram shows the relationship between a morphism $h: A \rightarrow X$ and its colimit $g: B \rightarrow X$. On the left, a morphism $\varphi: B \rightarrow A$ is shown as a vertical arrow with a circle at its top, indicating it is part of a colimit. On the right, the colimit $g: B \rightarrow X$ is shown, with a vertical arrow $\varphi: B \rightarrow A$ and a horizontal arrow $h_*: A \rightarrow X$. A dashed arrow $g_*: B \rightarrow X$ is shown, and a dashed arrow $h_* \circ \varphi = g_*$ is shown. The text "(then $g = \text{colim}(\varphi, h)$)" is written to the right.

Kan extension (X cocomplete)

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The diagram shows a Kan extension. A morphism $f: A \rightarrow X$ is shown. A morphism $i: B \rightarrow A$ is shown as a vertical arrow with a circle at its top. A dashed arrow $g: B \rightarrow X$ is shown. A dashed arrow $f \circ i = g$ is shown. The text " $g \cdot i \cong f$ if i is fully faithful" is written below the diagram.

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$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ y_Y \downarrow & \Downarrow & \nearrow \\ PY & \xrightarrow{f_*} & X \end{array}$$

The diagram shows a Kan extension. A morphism $f: Y \rightarrow X$ is shown. A morphism $y_Y: PY \rightarrow Y$ is shown as a vertical arrow with a circle at its top. A dashed arrow $f_*: PY \rightarrow X$ is shown. A dashed arrow $f \circ y_Y = f_*$ is shown. The text " f_* " is written below the diagram.

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② *For $f : X \rightarrow Y$ between cocomplete \mathcal{T} -categories:*

$$f \text{ is left adjoint} \iff \begin{array}{ccc} PX & \xrightarrow{Pf} & PY \\ \text{Sup}_X \downarrow & \cong & \downarrow \text{Sup}_Y \\ X & \xrightarrow{f} & Y \end{array}$$

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③ The functor $\mathcal{T}\text{-Cocont}_{\text{sep}} \longrightarrow \mathcal{T}\text{-Cat}$ is monadic and the induced monad (P, y, y^{-1}) on $\mathcal{T}\text{-Cat}$ is of Kock-Zöberlein typ.

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Coequaliser in $\mathcal{T}\text{-Cat}$:
$$R \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{\pi_2} \end{array} X \xrightarrow{q} Q$$

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Coequaliser in $\mathcal{T}\text{-Cat}$:

$$\begin{array}{ccccc} R & \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{\pi_2} \end{array} & X & \xrightarrow{q} & Q \\ y_R \downarrow & & y_X \downarrow & & y_Q \downarrow \\ PR & \begin{array}{c} \xrightarrow{P\pi_1} \\ \xrightarrow{P\pi_2} \end{array} & PX & \xrightarrow{Pq} & PQ \end{array}$$

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Let $\pi_1, \pi_2 : R \rightrightarrows X$ in $\mathcal{T}\text{-Cocont}_{\text{sep}}$ be an equivalence relation in Set . Let $q : X \rightarrow Q$ be its coequaliser in $\mathcal{T}\text{-Cat}$.

Coequaliser in $\mathcal{T}\text{-Cat}$:

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The diagram consists of three rows of objects and arrows connecting them:

- Top row:** $R \xrightarrow{\pi_1} X \xrightarrow{q} Q$. A second arrow π_2 also points from R to X .
- Middle row:** $PR \xrightarrow{P\pi_1} PX \xrightarrow{Pq} PQ$. A second arrow $P\pi_2$ also points from PR to PX .
- Bottom row:** $R \xrightarrow{\pi_1} X \xrightarrow{q} Q$. A second arrow π_2 also points from R to X .

Vertical arrows connect the rows:

- $y_R : R \rightarrow PR$ (downward)
- $y_X : X \rightarrow PX$ (downward)
- $y_Q : Q \rightarrow PQ$ (downward)
- $\text{Sup}_R : PR \rightarrow R$ (downward)
- $\text{Sup}_X : PX \rightarrow X$ (downward)
- $\text{Sup}_Q : PQ \rightarrow Q$ (dotted downward arrow)

A curved arrow labeled 1_Q points from Q in the top row to Q in the bottom row.

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Theorem (D. Scott)

The algebras for the proper filter monad on Top_0 are precisely the T_0 -spaces which are injective with respect to dense embeddings.

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- A \mathcal{T} -functor $f : X \rightarrow Y$ is called **dense** if f_* is inhabited.
- A \mathcal{T} -category X is **inhabited-cocomplete** if X has all φ -weighted colimits where φ is inhabited.