Injective spaces via adjunction

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Basic definitions

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  $\underline{a : TX \,{-\!\!\!\mapsto}\, X}$ (V-matrix)

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We have $\mathcal{T}\text{-Cat} \underset{\mathsf{S}}{\overset{\mathsf{A}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{V}\text{-Cat}$ and $\mathsf{Set}^{\mathbb{T}} \underset{\longrightarrow}{\overset{\longleftarrow}{\bot}} \mathcal{T}\text{-Cat}$.

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Some functors

- $(-)^* : \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathcal{T}\text{-}\mathsf{Dist}$ where $f^*(\mathfrak{y}, x) = b(\mathfrak{y}, f(x))$.

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The following are equivalent for $\varphi : X \rightharpoonup\!\!\!\!\!+ \; Y$.

1. $\varphi : X \multimapdotinv Y$ *is a* $\mathcal{T}$*-module.*
2. $\varphi : |X| \otimes Y \longrightarrow \mathsf{V}$, $\varphi : X^{\mathrm{op}} \otimes Y \longrightarrow \mathsf{V}$ *are* $\mathcal{T}$*-functors.*

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For distributors φ, ψ*:*

$$\begin{array}{ccc} X & \xrightarrow{\psi\ \circ} & Y \\ {\scriptstyle \varphi\,\circ}\downarrow & & \\ Z & & \end{array}$$

Yoneda

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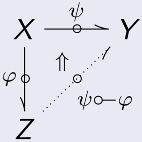

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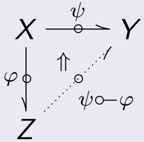

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Put $y_X := \ulcorner a \urcorner : X \longrightarrow PX$, for a $\mathcal{T}$-category $X = (X, a)$.

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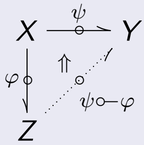

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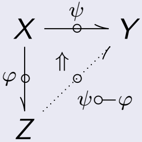

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Cocomplete $\mathcal{T}$-categories

Colimits

$$\begin{array}{ccc} A & \xrightarrow{\;h\;} & X \\ {\scriptstyle \varphi}\downarrow\!\circ & & \\ B & & \end{array}$$

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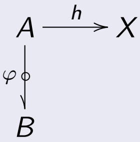
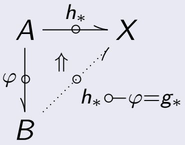


$\rightsquigarrow$



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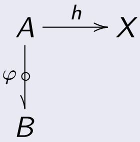

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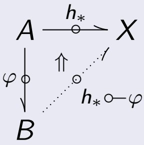

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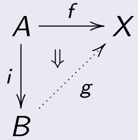
Kan extension (X cocomplete)



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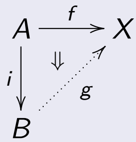

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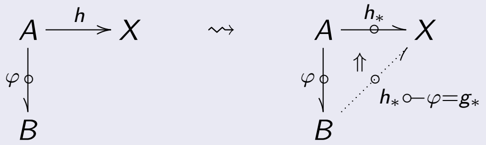

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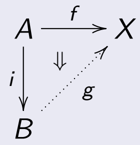

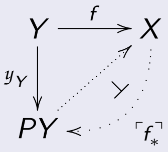

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3. *The functor $\mathcal{T}\text{-}\mathrm{Cocont}_{\mathrm{sep}} \longrightarrow \mathcal{T}\text{-}\mathrm{Cat}$ is monadic and the induced monad (P, y, y^{-1}) on $\mathcal{T}\text{-}\mathrm{Cat}$ is of Kock-Zöberlein typ.*

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Monadicity over Set

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3. $\mathcal{T}\text{-Cocont}_{\mathrm{sep}}$ has and G preserves coequaliser of G-equivalence relations.

About (3):

Proof.

Let $\pi_1, \pi_2 : R \rightrightarrows X$ in $\mathcal{T}$-Cocont$_{\mathrm{sep}}$ be an equivalence relation in Set.

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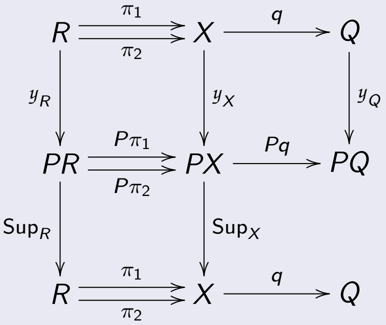

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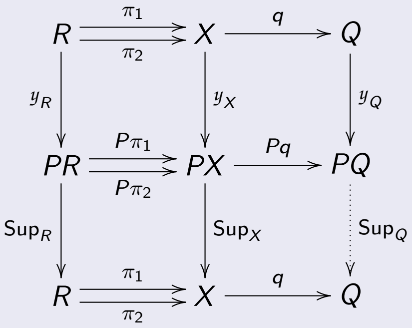

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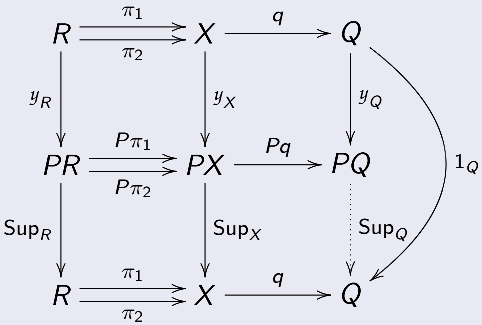

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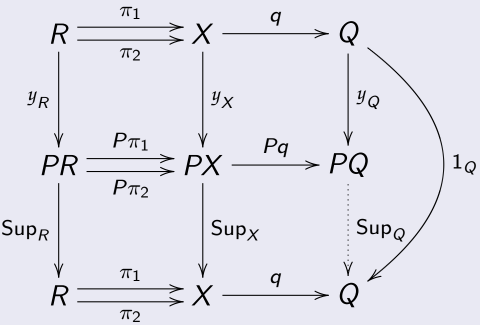

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