Diads and their Application to Topoi

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The full subcategory of fixed points of a finite-limit preserving idempotent endofunctor on a topos is again a topos.

Diads

A *distributive diad* on a category $\mathcal{C}$ is a functor $T : \mathcal{C} \longrightarrow \mathcal{C}$, equipped with natural transformations $\alpha : T \longrightarrow T^2$ and $\beta : T^2 \longrightarrow T$ such that the following diagrams commute:

$$\begin{array}{ccc} T^2 & \xrightarrow{\beta} & T \\ {\scriptstyle \alpha}\uparrow & \nearrow{\scriptstyle 1} & \\ T & & \end{array} \qquad \begin{array}{ccc} T & \xrightarrow{\alpha} & T^2 \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle T\alpha} \\ T^2 & \xrightarrow[\alpha_T]{} & T^3 \end{array}$$

$$\begin{array}{ccc} T^3 & \xrightarrow{\beta_T} & T^2 \\ {\scriptstyle T\beta}\downarrow & & \downarrow{\scriptstyle \beta} \\ T^2 & \xrightarrow[\beta]{} & T \end{array} \qquad \begin{array}{ccc} T^2 & \xrightarrow{\alpha_T} & T^3 \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle T\beta} \\ T & \xrightarrow[\alpha]{} & T^2 \end{array}$$

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- For a monad (T, η, μ), (T, η_T, μ) is a distributive diad if and only if the monad is idempotent.
- Any idempotent functor is a distributive diad.
- For a monad (T, η, μ), $(T, T\eta, \mu)$ is a distributive diad.

Dialgebras

A *distributive dialgebra* for the distributive diad (T, α, β) is an object X with morphisms $X \underset{\theta}{\overset{\phi}{\rightleftarrows}} TX$ such that the following diagrams commute:

$$\begin{array}{ccc} TX & \xrightarrow{\theta} & X \\ {\scriptstyle\phi}\uparrow & \nearrow{\scriptstyle 1_X} & \\ X & & \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\phi} & TX \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle T\phi} \\ TX & \xrightarrow[\alpha_X]{} & T^2X \end{array}$$

$$\begin{array}{ccc} T^2X & \xrightarrow{\beta_X} & TX \\ {\scriptstyle T\theta}\downarrow & & \downarrow{\scriptstyle\theta} \\ TX & \xrightarrow[\theta]{} & X \end{array} \qquad \begin{array}{ccc} TX & \xrightarrow{\alpha_X} & T^2X \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle T\theta} \\ X & \xrightarrow[\phi]{} & TX \end{array}$$

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- For a monad (T, η, μ), where T is faithful, distributive dialgebras for $(T, T\eta, \mu)$ are coalgebras for the comonad induced on the category of algebras for (T, η, μ).
- For any distributive diad (T, α, β) and any object X, there is a free dialgebra (TX, α_X, β_X).

Dialgebra Homomorphisms

A dialgebra homomorphism from (X, ϕ, θ) to (Y, π, ρ) is the obvious thing – namely a morphism $X \xrightarrow{f} Y$ such that

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ \theta\downarrow & & \downarrow\rho \\ X & \xrightarrow[f]{} & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ \phi\uparrow & & \uparrow\pi \\ X & \xrightarrow[f]{} & Y \end{array}$$

commute.

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- For the diad (T, η_T, μ) from a monad, dialgebra homomorphisms are exactly algebra homomorphisms.
- For any objects X and Y, and any morphism $X \xrightarrow{f} Y$, Tf is a dialgebra homomorphism between the free dialgebras on X and Y.
- For any distributive dialgebra (X, ϕ, θ), ϕ is a dialgebra homomorphism from (X, ϕ, θ) to the free dialgebra (TX, α_X, β_X).

Main Theorem

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The category of distributive dialgebras for a finite-limit preserving distributive diad on a topos is again a topos.

Limits

- The terminal object is just the dialgebra $(1, 1, 1)$.

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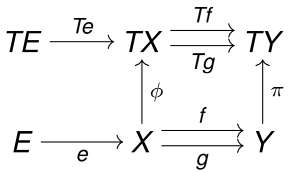
- The product of (X, ϕ, θ) and (Y, π, ρ) is $(X \times Y, \phi \times \pi, \theta \times \rho)$.
- The equaliser of dialgebra maps f and g can be given a distributive dialgebra structure using the universal property of equalisers and the fact that T preserves equalisers:

Equalisers

$$TE \xrightarrow{Te} TX \underset{Tg}{\overset{Tf}{\rightrightarrows}} TY$$

$$E \xrightarrow[e]{} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

Equalisers



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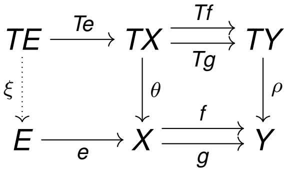

Exponentials

The exponential in the category of dialgebras is a subobject of $T(X^Y)$.

- Given $Z \xrightarrow{f} T(X^Y)$, we define $Z \times Y \xrightarrow{g} X$ by

$$g = Z \times Y \xrightarrow{f \times \pi} T(X^Y) \times TY \xrightarrow{T(\mathrm{ev})} TX \xrightarrow{\theta} X$$

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- Given a dialgebra homomorphism $Z \times Y \xrightarrow{g} X$, we define $Z \xrightarrow{f} T(X^Y)$ by

$$f = Z \xrightarrow{\psi} TZ \xrightarrow{T(\overline{g})} T(X^Y)$$

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We form dialgebra homomorphisms $T(X^Y) \xrightarrow{T(\phi^Y)} T(TX^Y)$ and $T(X^Y) \xrightarrow{\alpha_{X^Y}} T^2(X^Y) \xrightarrow{T(\epsilon)} T(TX^{TY}) \xrightarrow{T(TX^\pi)} T(TX^Y)$, where ϵ is the exponential comparison map $T(X^Y) \longrightarrow TX^{TY}$.

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The equaliser of these two homomorphisms is the exponential in the category of dialgebras.

Subobject Classifier

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The subobject classifier in the category of distributive dialgebras is the equaliser of $(T\tau)\alpha_\Omega$ and the identity on $T\Omega$.

Given a monomorphism $Y \overset{m}{\rightarrowtail} X$ in the category of dialgebras, its classifying map is the factorisation of $T(\chi_m)\pi$ through e.