Strict morphisms and weak morphisms

joint work with Mike Shulman

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Outline

The strict/weak dichotomy

$\mathcal{F}$-categories

Pseudo and lax aspects of $\mathcal{F}$-category theory

Variations on the $\mathcal{F}$-category theme

The context

There are many situations in which one has two types of morphism between the same object.

object	"good" morphism	"bad" morphism
monoidal category	strict monoidal functor	monoidal functor
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category	functor	profunctor
ring	homomorphism	bimodule
set	function	partial function
topos	geometric morphism	left exact functor
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- e.g. let 1 be the terminal category, and $\mathcal{I}$ the "free-living isomorphism". These are both monoidal, and the two functors

$$1 \rightrightarrows \mathcal{I}$$

are both strong monoidal, but they have no equalizer

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- It therefore has all PIE-limits (Power-Robinson): the limits that can be constructed out of Products, Inserters, and Equifiers

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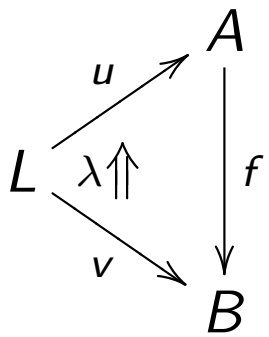

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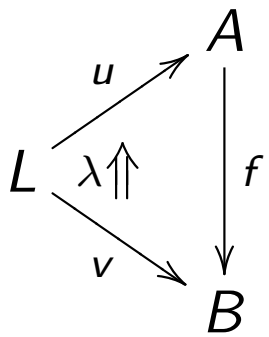

- e.g. the oplax limit of an arrow $f : A \to B$ in **Cat** is the comma category B/f

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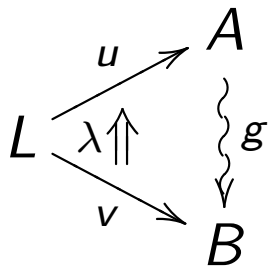

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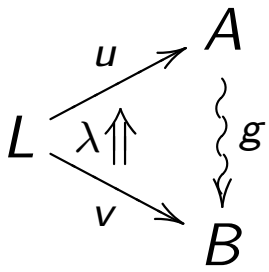

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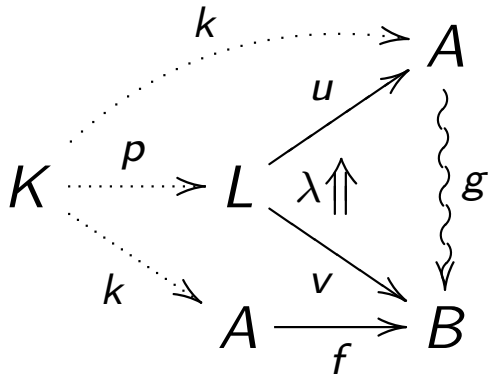

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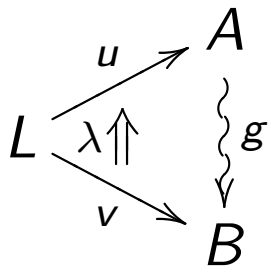

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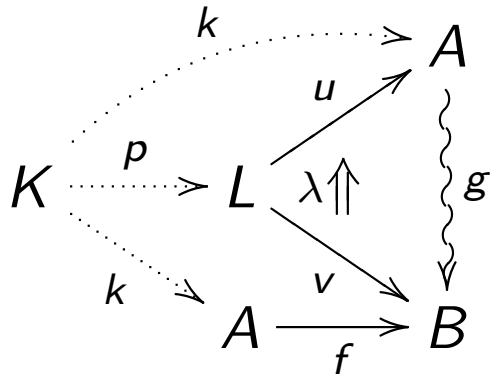

- Similar situation for equifiers

It takes all sorts...

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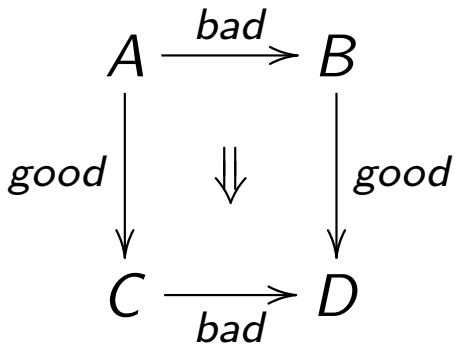
Everything remains true!

Are double categories the answer?

- Double categories have two types of morphisms: horizontal and vertical



- Gives a very good explanation of which limits should exist
- Also captures their universal property *with respect to strict maps*
- Does not (easily) capture the universal property with respect to weak maps
- You need extra structure on a double category to be able to think of strict maps as special case of weak maps

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- Example: $\mathcal{K}_t = \mathcal{K}_\ell = \mathcal{K}$, $J = 1$ (call such an $\mathcal{F}$-category *tight*)

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- satisfying associative and identity laws

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- In other words: a category enriched in $\mathcal{F}$

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- Functor categories $[\mathbb{K}, \mathbb{L}]$: objects are $\mathcal{F}$-functors, tight maps are $\mathcal{F}$-natural transformations; loose maps $F \to G$ are 2-naturals $m_\ell : F_\ell \to G_\ell$

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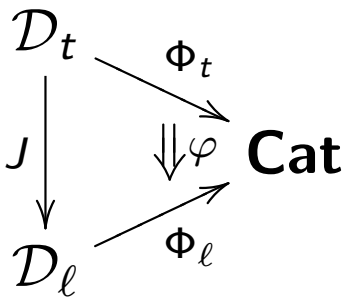

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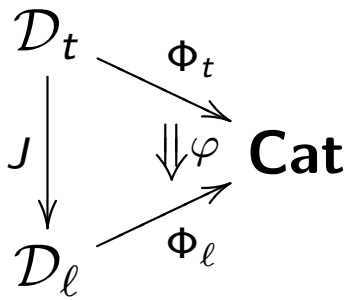

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- Also have analogues $\mathrm{Lax}(\mathbb{K}, \mathbb{L})$ and $\mathrm{Oplax}(\mathbb{K}, \mathbb{L})$

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- Question: which limits lift from $\mathbb{K}$ to $\mathbb{ALG}(\mathbb{T})$?

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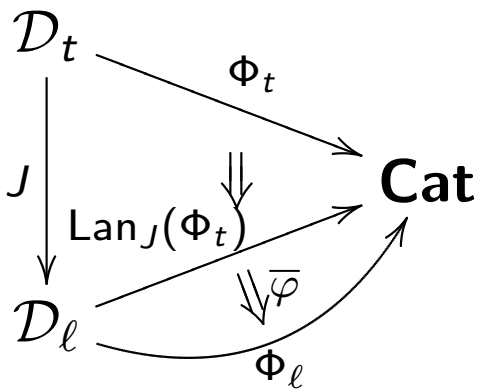

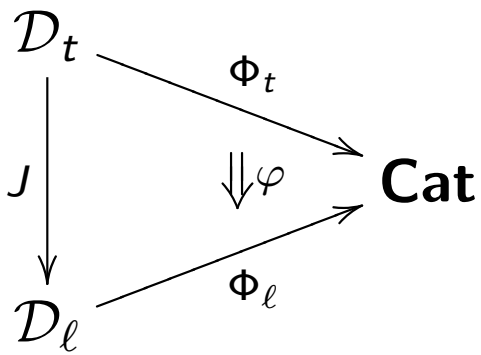

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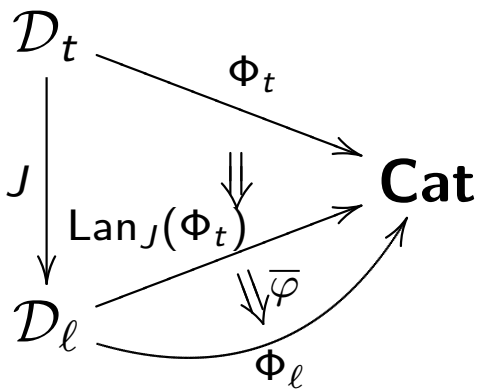

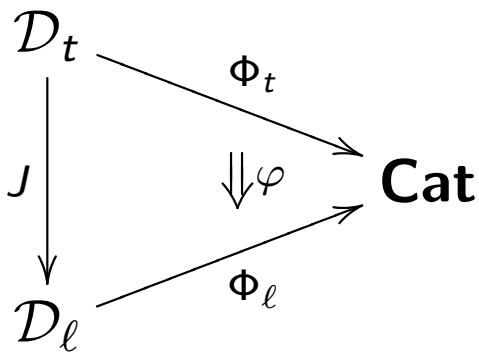

- We'll be interested in the functors which are components of $\overline{\varphi}$

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$TK \xrightarrow{k} K$ and (K, k) is the limit in $\mathbb{ALG}(\mathbb{T})^{\mathrm{ps}}$. □

A sufficient condition

There is also a useful sufficient condition which covers the most important cases:

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Theorem

The weights in the previous theorem are precisely those that can be constructed from tight limits, pseudo limits of loose morphisms, inserters, and equifiers.

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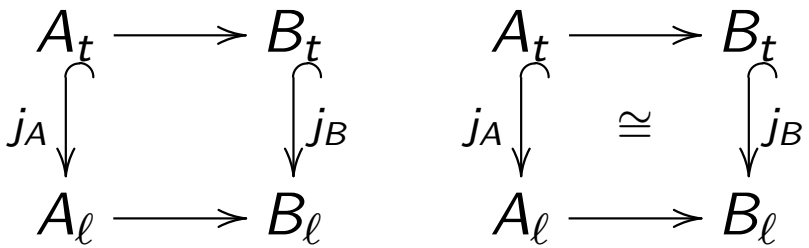

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- For example, one could take functors as tight morphisms and profunctors as loose ones

$\mathcal{F}_2$-categories

- An object of $\mathcal{F}$ consists of a single fully faithful functor
- Instead we could take a composable pair of fully faithful functors
- These form the objects of a cartesian closed category $\mathcal{F}_2$
- An $\mathcal{F}_2$-category consists of a composable pair of 2-functors which are the identity on objects and locally fully faithful
- This would allow strict, pseudo, and lax morphisms to be treated altogether

$\mathbf{Set}^2$-categories

- $\mathbf{Set}^2$ is the cartesian closed category of morphisms in $\mathbf{Set}$
- A $\mathbf{Set}^2$-category is a *functor* which is the identity on objects
- Could also restrict $\mathbf{Set}^2$ to the full subcategory of injective functions, to describe functors which are the identity on objects and faithful

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- Also have enriched variants of all of these structures