The category of k-groups

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Preliminaries

- Since $T_0 \Leftrightarrow T_{3\frac{1}{2}}$ in $\mathsf{Grp}(\mathsf{Top})$, all topological groups and spaces are assumed to be at least T_2.
- $f\colon X \to Y$ is ***k*-cts** if $f_{|K}$ is cts for every compact $K \subseteq X$.
- X is a ***k*-space** if k-cts $=$ cts on X.
- kHaus is a coreflective subcategory of Haus.
- $\mathsf{k}\colon \mathsf{Haus} \longrightarrow \mathsf{kHaus}$ makes all sets $F \subseteq X$ with closed trace on each compact subset of X closed in $\mathsf{k}X$.
- If $f\colon X \to Y$ is a bijection that induces a bijection between the compact subsets, then $\mathsf{k}X \cong \mathsf{k}Y$.
- kHaus is cartesian closed (Brown, 1964).
- $[X, Y] = \mathsf{k}\mathscr{C}(X, Y)$ ($\mathscr{C}$ – compact-open topology).

Noble (1970)

- $G \in \mathsf{Grp}(\mathsf{Haus})$ is a k-***group*** if k-cts $=$ cts for group homomorphisms $\varphi\colon G \to H$.
- Not every k-group is a k-space. [$\mathrm{ev}\colon \mathscr{C}(G,\mathbb{R}) \times G \to \mathbb{R}$ is cts only if G is LC.]
- kGrp is a coreflective subcategory of $\mathsf{Grp}(\mathsf{Haus})$.
- $\mathsf{k_g}\colon \mathsf{Grp}(\mathsf{Haus}) \longrightarrow \mathsf{kGrp}$ equips G with the finest group topology coarser than $\mathsf{k}G$.
- $\mathsf{k_g}G$ has the finest group topology whose compact sets coincide with those in G.
- If $\{G_\alpha\}_{\alpha\in I}$ are k-groups, then so is $\prod\limits_{\alpha\in I}^{\mathsf{Grp}(\mathsf{Haus})} G_\alpha$.
- $\lim_{\mathsf{Grp}(\mathsf{Haus})} \neq \mathsf{k_g} \lim_{\mathsf{Grp}(\mathsf{Haus})} = \lim_{\mathsf{kGrp}}$.

Free topological groups

- Both $U\colon \mathsf{Grp}(\mathsf{Haus}) \longrightarrow \mathsf{Tych}$ and $U\colon \mathsf{Ab}(\mathsf{Haus}) \longrightarrow \mathsf{Tych}$ have left adjoints:
  - $F\colon \mathsf{Tych} \to \mathsf{Grp}(\mathsf{Haus})$;
  - $A\colon \mathsf{Tych} \to \mathsf{Ab}(\mathsf{Haus})$.
  - X generates $F(X)$ and $A(X)$ algebraically.
  - Units $X \to F(X)$ and $X \to A(X)$ are closed embs.
  - Counits $F(G) \to G$ and $A(E) \to E$ are quotients.
- $U\colon \mathsf{kGrp} \longrightarrow \mathsf{Tych}$ has no left adjoint. [No pres. of $\lim$.]

What is the “right" forgetful functor for k-groups?

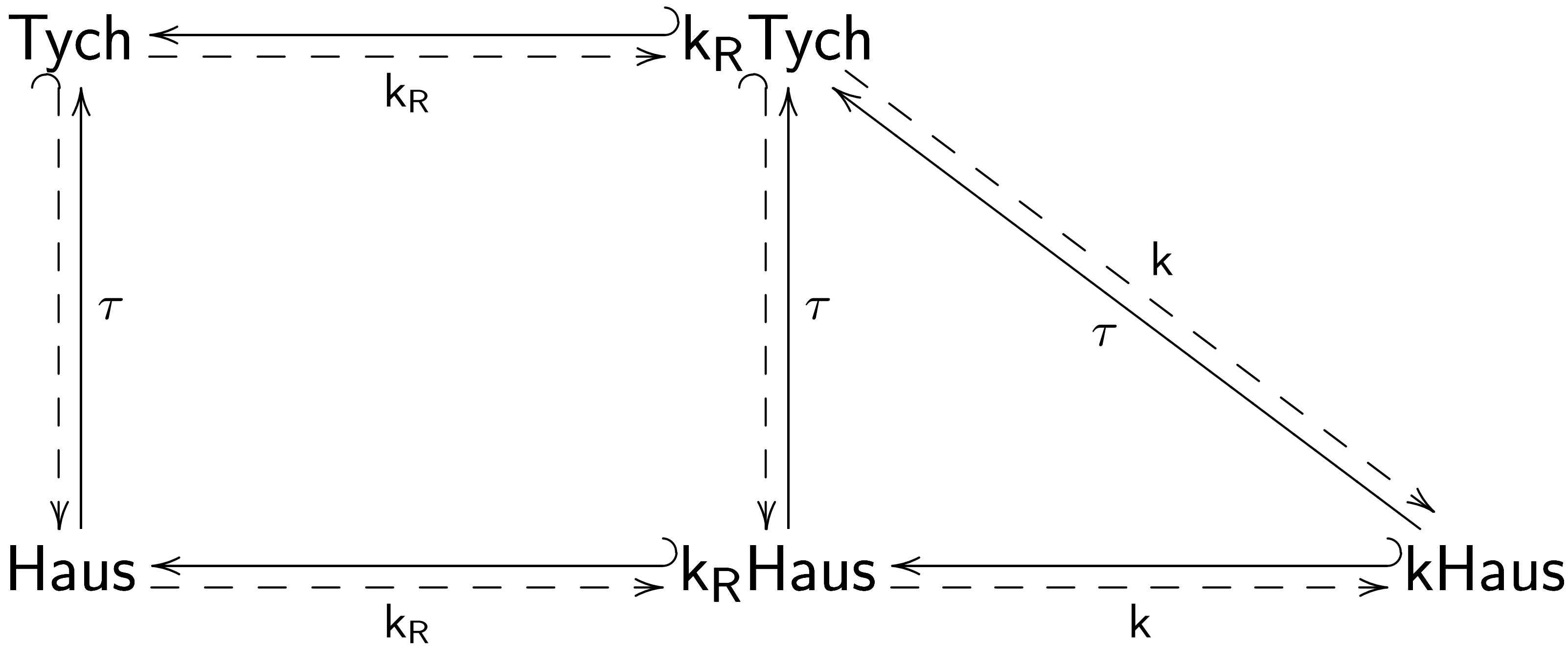
k_R-spaces

- X is a k_R***-space*** if k-cts $=$ cts for $f\colon X \to \mathbb{R}$.
- X is a k_R-space $\Leftrightarrow$ k-cts $=$ cts for $f\colon X \to Z$ with $Z \in \mathsf{Tych}$.

GL (2002):

- $\mathsf{k_R Haus}$ is coreflective in Haus.
- $\mathsf{k_R} Z \in \mathsf{Tych}$ for $Z \in \mathsf{Tych}$.
- $\mathsf{k_R Tych}$ is coreflective in Tych.
- $\mathsf{k_R Tych}$ is cartesian closed.
- $[X, Y] = \mathsf{k_R}\mathscr{C}(X, Y)$.
- $\mathsf{k_R Tych}$ is equivalent to a (full) epireflective subcategory of kHaus.

k_R-spaces



(The dashed arrows are right adjoints.)

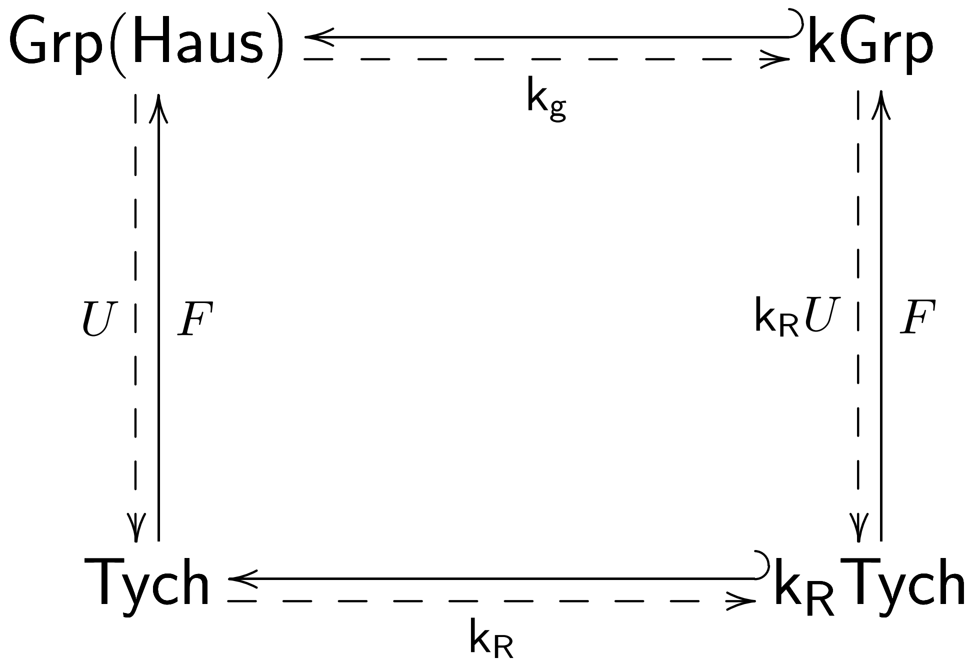
Free k-groups

- $\mathsf{k_R k_g} = \mathsf{k_R}$.
- $\mathsf{k_R}U \colon \mathsf{kGrp} \longrightarrow \mathsf{k_R Tych}$ preserves limits.

Theorem. (GL, 2004) If $X \in \mathsf{k_R Tych}$, then $F(X), A(X) \in \mathsf{kGrp}$.

- $F_{|\mathsf{k_R Tych}}$ is left adjoint to $\mathsf{k_R}U \colon \mathsf{kGrp} \longrightarrow \mathsf{k_R Tych}$.
- $A_{|\mathsf{k_R Tych}}$ is left adjoint to $\mathsf{k_R}U \colon \mathsf{kAb} \longrightarrow \mathsf{k_R Tych}$.
- G is an [abelian] k-group if and only if G is a quotient of $F(X)$ [$A(X)$], where X is a Tychonoff k_R-space.

Free k-groups



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Tensor product of abelian k-groups

Let $B, C \in \mathsf{kAb}$, and consider the following subgroup of $A(\mathsf{k_R}(B \times C))$:

$$R(B,C) = \langle (b_1+b_2, c)-(b_1,c)-(b_2,c), (b, c_1+c_2)-(b,c_1)-(b,c_2) \rangle$$

We put $B \otimes_{\mathsf{k}} C \stackrel{def}{=} A(\mathsf{k_R}(B \times C))/\overline{R(B,C)}$.

Theorem. (GL, 2004/8) There are bijections

$$\mathsf{kAb}(B \otimes_{\mathsf{k}} C, D) \longleftrightarrow \mathsf{kBil}(B \times C, D) \longleftrightarrow \mathsf{kAb}(B, \mathsf{k_g}\mathscr{H}(C,D))$$

that are natural in $B, C, D \in \mathsf{kAb}$. [$\mathscr{H}$ = cts homo. $\subseteq \mathscr{C}$.]

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$\mathsf{k_g}\mathscr{H}(B, \mathsf{k_g}\mathscr{H}(C, D)) \cong \mathsf{k_g}\mathscr{B}(B \times_{\mathsf{k_R}} C, D) \cong \mathsf{k_g}\mathscr{H}(C, \mathsf{k_g}\mathscr{H}(B, D))$
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Question. (Easy?) Is this enough to conclude that $\otimes_{\mathsf{k}}$ is: Associative? Coherent? Makes kAb monoidal closed?

Question. Is $\mathsf{k_g}\mathscr{H}(B \otimes_{\mathsf{k}} C, D) \cong \mathsf{k_g}\mathscr{H}(B, \mathsf{k_g}\mathscr{H}(C, D))$?

Pros and cons

Cons:

- k-groups are not closed under the formation of closed subgroups.
- Put $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and consider the dual $G' = \mathsf{k}_{\mathrm{g}}\mathscr{H}(G, \mathbb{T})$. Although the evaluation $G \to G''$ is continuous, it need not be a topological isomorphism.
- Thus, kAb is not $*$-autonomous with respect to this structure. (Michael Barr's proposed structure is!)

Pros and cons

Pros:

- kAb contains all metrizable abelian and LCA groups as well as their arbitrary products.
- kAb is closed under the formation of open subgroups, quotients, and coproducts (in $\mathsf{Ab}(\mathsf{Haus})$).
- $\otimes_{\mathsf{k}}$ is defined without any reference to the group of continuous characters. Both Barr's and Garling's (1974) constructions require the groups to be determined by their continuous characters.
- If B and C are LCA, then k-cts bilinear maps $B \times C \to D$ coincide with the cts bilinear ones.
- G'' is precisely the k_g-ification of the Binz-Butzmann dual of G (cf. convergence groups).