Cauchy completeness results motivated by Myhill's characterization of combinatorial functions

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Because A can be given the structure of a free algebra.

The main example

The Schanuel topos

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- So free algebras are "essentially the same" as Myhill's *combinatorial functions*.
- CLAIM: Myhill's main result says that Kl_M is Cauchy complete (in a very strong sense).

A very simple counting problem

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It is enough to find a section for the canonical presentation $a : (M\mathbf{T}, \mathbf{m}) \to (\mathbf{T}, a)$.

Functions II

- The assignment

$$(f : T \to U) \mapsto (fT \subseteq U, f : T \to fT)$$

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- Which is the object of generators?
- It is the equalizer $\mathbf{T}_s \to \mathbf{T}$ of $s, \mathbf{u} : \mathbf{T} \to M\mathbf{T}$.

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- More explicitly, $\mathbf{T}_s U = \{$surjections $T \to U\}$.
- Numerical version: $n^{\sharp T} = \sum_{i=0}^{n} su(i, \sharp T)\binom{n}{i}$.

Strongly Cauchy monads

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Definition 2. Let $s : (A, a) \to (MA, \mathbf{m})$ be a morphism. The *subobject of generic elements* associated to s is the equalizer (in $\mathcal{C}$)

$$A_s \xrightarrow{\bar{s}} A \underset{s}{\overset{\mathbf{u}_A}{\rightrightarrows}} MA$$

of $s, \mathbf{u}_A : A \to MA$.

Strongly Cauchy monads

Definition 3. A monad $(M, \mathbf{u}, \mathbf{m})$ is *strongly Cauchy* if for every algebra (A, a) with a section $s : (A, a) \longrightarrow (MA, \mathbf{m})$ for $a : (MA, \mathbf{m}) \longrightarrow (A, a)$ the map

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Proposition 1. *Every strongly Cauchy monad is Cauchy.*

A more interesting example: Möbius categories

Möbius categories

Definition 3. A category $\mathcal{C}$ has *finite decompositions of degree 2* if for every map f the set $\{(f', f'') \mid f'f'' = f\}$ is finite.

Fix $\mathcal{C}$ with finite decompositions and A a ring.

Definition 3. The *incidence algebra* $A\mathcal{C}$ of $\mathcal{C}$ is:

1. the set of functions $\mathrm{Arr}\mathcal{C} \to A$ with
2. pointwise addition and multiplication by scalar
3. multiplication $* : A\mathcal{C} \times A\mathcal{C} \to A\mathcal{C}$ defined by

$$(\alpha * \beta)f = \sum_{f'f''=f} (\alpha f')(\beta f'')$$

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Theorem 1 (Content, Lemay et Leroux; 1980). *If $\mathcal{C}$ has finite decompositions of degree 2 then t.f.a.e.:*

1. $\alpha \in A\mathcal{C}$ *is invertible iff for all identities* i *in* $\mathcal{C}$, αi *is invertible in* A.
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Definition 4. A *Möbius category* is one satisfying the conditions of the theorem above.

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- (The general Möbius inversion principle) For every $\alpha, \beta \in A\mathcal{C}$

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On a more combinatorial proof . . .

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Lemma 1. *The monad* $(\mathtt{Ev}, \mathbf{u}, \mathbf{m})$ *is strongly Cauchy.*

- Decompositions of odd length induce a functor $\mathtt{Od} : \mathbf{Set}^{\mathrm{Arr}\mathcal{C}} \to \mathbf{Set}^{\mathrm{Arr}\mathcal{C}}$.

Combinatorial Möbius inversion

Proposition 2. *For every* β *in* $\mathbf{Set}^{\mathrm{Arr}\mathcal{C}}$*,*

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4. Observe that the subobject of generic elements is $\beta * \zeta$.

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Strongly Cauchy monads on Heyting categories

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Definition 6. The *subobject of minimal elements* associated with an M-algebra (A, a) is the subobject $a_\star : \lfloor A, a \rfloor \longrightarrow A$ of A where

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Definition 6. An algebra (A, a) is *well-founded* if the morphism

$$M\lfloor A, a \rfloor \xrightarrow{Ma_\star} MA \xrightarrow{a} A$$

is regular epi.

Quasi-exact algebras

Proposition 3. *Let $(M, \mathbf{m}, \mathbf{u})$ be a monad on a Heyting category such that M maps pullbacks to quasi-pullbacks. For an M-algebra (A, a) the following are equivalent:*

1. $a(Ma_\star) : M\lfloor A, a \rfloor \to A$ *is an iso.*
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Example 2. If $(M, \mathbf{m}, \mathbf{u})$ is the monad on $\mathbf{Set}^{\mathbb{B}}$ which induces the Schanuel topos then quasi-exact means pullback-preserving.

Quasi-exact algebras

- For any algebra (A, a) consider the pullback

$$\begin{array}{ccc} \mathsf{T}(A,a) & \xrightarrow{\pi_0} & MA \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle a} \\ MA & \xrightarrow[a]{} & A \end{array}$$

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Definition 7. An algebra (A, a) is *quasi-exact* if T applied to $a : (MA, \mathbf{m}) \to (A, a)$ is regular epi.

Basic facts and example

1. Free algebras are quasi-exact.

2. Retracts of quasi-exact algebras are quasi-exact.

$$\mathrm{Kl} \to \overline{\mathrm{Kl}} \to \mathrm{qeAlg} \to \mathrm{Alg}$$

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Example 3. For the Myhill-Schanuel monad $\mathrm{Kl} \to \mathrm{qeAlg}$ is an equivalence.

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2. Retracts of quasi-exact algebras are quasi-exact.

$$\mathrm{Kl} \to \overline{\mathrm{Kl}} \to \mathrm{qeAlg} \to \mathrm{Alg}$$

Example 3. For the Myhill-Schanuel monad $\mathrm{Kl} \to \mathrm{qeAlg}$ is an equivalence.

RECALL: An M-algebra (A, a) is *well-founded* if the map $M\lfloor A, a\rfloor \to A$ is regular epi.

Well-founded implies Strong Cauchy

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   - *If such a section $s : (A, a) \to (MA, \mathbf{m})$ exists then $\lfloor A, a \rfloor = A_s$.*

Concrete manifestations

Schanuel-Kleisli categories

Proposition 5. *Let* $\mathcal{M}$ *be a category with pullbacks and every map mono. Let* $\mathcal{I}$ *be the subcategory of isos and* $(M, \mathbf{u}, \mathbf{m})$ *be the monad on* $\mathbf{Set}^{\mathcal{I}}$ *induced by* $\mathcal{I} \to \mathcal{M}$*. Then the following are equivalent*

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Proof. Use previous result and (Fiore and Menni; 2005). □

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- The equivalence $\mathrm{Kl} \to \mathrm{qeAlg}$ is a combinatorial form of *the exponential principle*.
- Generalizes from $\mathbb{B} = !1$ to the symmetric monoidal completion $!G$ of any groupoid G.

Another simple example: permutations

Permutations I

1. Define $\mathfrak{G}$ in $\mathbf{Set}^{\mathbb{B}}$ by

$$\mathfrak{G}U = \{\sigma : U \to U \mid \sigma \text{ is bijective}\}$$

2. There is an M-algebra $a : M\mathfrak{G} \to \mathfrak{G}$ which, at stage U, takes a subset V of U together with a permutation π on V and produces the permutation on U which is the extension of π by leaving the elements in U/V fixed.

Is $(\mathfrak{G}, a)$ a free M-algebra?

Permutations II

1. There is a section $s : \mathfrak{G} \to M\mathfrak{G}$ which, at stage U, takes a permutation π of U and produces the subset V of U given by the elements of U that are not fixed by π, together with the restriction of π to V. It follows that $\mathfrak{G}$ is a free M-algebra.
2. What are the "generators"?
3. The equalizer of $\mathbf{u}$ and s is the object $\mathfrak{D}$ of *derangements*.
4. The numerical reflection of this is that $n! = \sum_{i=0}^{n} d_i \binom{n}{i}$.

Origin of the ideas

Myhill's combinatorial functions

- For every function $f : \mathbb{N} \to \mathbb{N}$ there exists a unique function $c : \mathbb{N} \to \mathbb{Z}$ such that for every $n \in \mathbb{N}$, $fn = \sum_{i=0}^{n} c_i \binom{n}{i}$.

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- Let Q be the set of finite sets of natural numbers and define an *operator* to be a function $Q \to Q$.
- An operator ϕ is called *numerical* if for a and b in Q of the same cardinality, ϕa and ϕb have the same cardinality.
- Clearly, every numerical operator induces a function $\mathbb{N} \to \mathbb{N}$.

Myhill and Schanuel

1. For a numerical operator ϕ let $\phi^{\varepsilon} = \bigcup\{\phi a \mid a \in Q\}$.
2. A numerical operator ϕ is *combinatorial* if there exists a $\phi^{-1} : \phi^{\varepsilon} \to Q$ such that $x \in \phi a$ if and only if $\phi^{-1}x \subseteq a$.
3. A function $f : \mathbb{N} \to \mathbb{N}$ is then called *combinatorial* if it is induced by a combinatorial operator.

Theorem 1 (Myhill)**.** *Let $f : \mathbb{N} \to \mathbb{N}$ be a function with $fn = \sum_{i=0}^{n} c_i \binom{n}{i}$. Then f is combinatorial if and only if $c_i \geq 0$ for every $i \geq 0$.*

Myhill and Schanuel

Corollary 0. *If f and g are combinatorial functions then so are the functions $n \mapsto (fn) \cdot (gn)$ and $n \mapsto f(gn)$.*

Proof. This may be shown without introducing combinatorial operators, but combinatorial operators allow “to prove these closure conditions without the algebraic complications which arise from substitution involving expressions” of the form $\sum_{i=0}^{n} c_i \binom{n}{i}$. (Dekker 1990) $\square$

The other way to look at this is ...

The Schanuel topos

1. Continuous actions for subgroup of bijections of $\mathbb{N}^{\mathbb{N}}$.
2. Classifier of infinite decidable objects.
3. Sheaves for the atomic topology on $\mathbb{I}^{\mathrm{op}}$.
4. Pullback-preserving functors $\mathbb{I} \to \mathbf{Set}$.
5. Kl_M, where $(M, \mathbf{u}, \mathbf{m})$ on $\mathbf{Set}^{\mathbb{B}}$ is induced by $\mathbb{B} \to \mathbb{I}$.

Myhill and Schanuel

We claim that

1. Myhill's theorem is essentially saying that Kl_M is Cauchy complete.
2. A numerical operator ϕ should be thought of as an M-algebra (A, a) with $a : MA \to A$.
3. The ϕ^{-1} witnessing that ϕ is combinatorial should be thought of as a section of the canonical quotient $a : (MA, \mathbf{m}) \to (A, a)$.
4. Put differently: combinatorial operators are a device to recognice free M-algebras.