Algebraic structures associated to weak Yang-Baxter operators

(joint work with J.N. Alonso , J.M. Fernández)

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Preliminaries

Let $(\mathcal{C}, \otimes, K, a, l, r)$ be a monoidal category $\mathcal{C}$ where $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is the tensor product, K is the unit,

$$a_{U,V,W} : U \otimes (V \otimes W) \rightarrow (U \otimes V) \otimes W$$

are the associativity constrains, and

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For simplicity of notation, given objects V, U, W in $\mathcal{C}$ and a morphism $f : V \rightarrow U$, we write

$W \otimes f$ for $id_W \otimes f$ and $f \otimes W$ for $f \otimes id_W$.

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The monoidal category is said to be strict if the associativity and the unit constraints a, l, r are all identities of the category.

Preliminaries

Definition.(**Joyal and Street, Adv. in Math., 1993**) Let $(\mathcal{C}, \otimes, K, a, l, r)$ be a monoidal category. Let $D \in Obj(\mathcal{C})$ and let $\tau_{D,D} : D \otimes D \to D \otimes D$ be an morphism in $\mathcal{C}$. We will say that $\tau_{D,D}$ satisfies the Yang-Baxter equation if

$$(\tau_{D,D} \otimes D) \circ a_{D,D,D} \circ (D \otimes \tau_{D,D}) \circ a^{-1}_{D,D,D} \circ (\tau_{D,D} \otimes D) =$$

$$a_{D,D,D} \circ (D \otimes \tau_{D,D}) \circ a^{-1}_{D,D,D} \circ (\tau_{D,D} \otimes D) \circ a_{D,D,D} \circ (D \otimes \tau_{D,D}) \circ a^{-1}_{D,D,D}.$$

A Yang-Baxter operator in $\mathcal{C}$ is an isomorphism $\tau_{D,D} : D \otimes D \to D \otimes D$ satisfying the Yang-Baxter equation.

Trivially, if $\mathcal{C}$ is strict the previous equality admits this formulation:

$$(\tau_{D,D} \otimes D) \circ (D \otimes \tau_{D,D}) \circ (\tau_{D,D} \otimes D) = (D \otimes \tau_{D,D}) \circ (\tau_{D,D} \otimes D) \circ (D \otimes \tau_{D,D}).$$

Weak Yang-Baxter Operators

Let $\mathcal{C}$ be an strict monoidal category equipped with a tensor product $\otimes$, unit K and such that admits split idempotents, i.e.:

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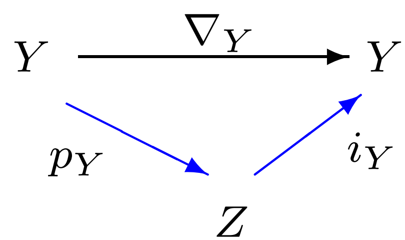

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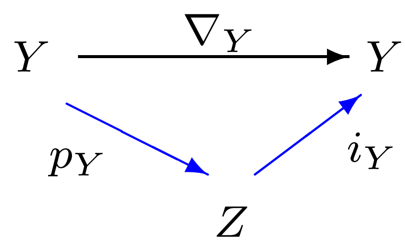

$$\nabla_Y = i_Y \circ p_Y , \quad p_Y \circ i_Y = id_Z .$$

Weak Yang-Baxter Operators

Definition. Let $D \in Obj(\mathcal{C})$. A weak Yang-Baxter operator is a morphism $t_{D,D} : D \otimes D \to D \otimes D$ in $\mathcal{C}$ such that:

(a1) $t_{D,D}$ satisfies the Yang-Baxter equation.

(a2) There exists an idempotent morphism $\nabla_{D\otimes D} : D \otimes D \to D \otimes D$ satisfying:

- (a2-1) $(\nabla_{D\otimes D} \otimes D) \circ (D \otimes \nabla_{D\otimes D}) = (D \otimes \nabla_{D\otimes D}) \circ (\nabla_{D\otimes D} \otimes D)$,
- (a2-2) $(\nabla_{D\otimes D} \otimes D) \circ (D \otimes t_{D,D}) = (D \otimes t_{D,D}) \circ (\nabla_{D\otimes D} \otimes D)$,
- (a2-3) $(t_{D,D} \otimes D) \circ (D \otimes \nabla_{D\otimes D}) = (D \otimes \nabla_{D\otimes D}) \circ (t_{D,D} \otimes D)$,
- (a2-4) $t_{D,D} \circ \nabla_{D\otimes D} = \nabla_{D\otimes D} \circ t_{D,D} = t_{D,D}$.

(a3) There exists a morphism $t'_{D,D} : D \otimes D \to D \otimes D$ such that:

- (a3-1) $t'_{D,D}$ satisfies the Yang-Baxter equation.
- (a3-2) The morphism $p_{D,D} \circ t_{D,D} \circ i_{D,D} : D \times D \to D \times D$ is an isomorphism with inverse $p_{D,D} \circ t'_{D,D} \circ i_{D,D} : D \times D \to D \times D$, where $p_{D,D}$ and $i_{D,D}$ are the morphisms such that $i_{D,D} \circ p_{D,D} = \nabla_{D\otimes D}$ and $p_{D,D} \circ i_{D,D} = id_{D\times D}$ being $D \times D$ the image of $\nabla_{D\otimes D}$.
- (a3-3) $t'_{D,D} \circ \nabla_{D\otimes D} = \nabla_{D\otimes D} \circ t'_{D,D} = t'_{D,D}$.

Weak Yang-Baxter Operators

We have the following:

- $t_{D,D}$ is a weak Yang-Baxter operator $\Longleftrightarrow t'_{D,D}$ is a weak Yang-Baxter operator.

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- $t_{D,D}$ is a weak Yang-Baxter operator $\Longleftrightarrow t'_{D,D}$ is a weak Yang-Baxter operator.
- $\nabla_{D\otimes D} = t'_{D,D} \circ t_{D,D} = t_{D,D} \circ t'_{D,D}$.
- If $\nabla_{D\otimes D} = id_{D\otimes D}$ we recover the usual definition of Yang-Baxter operator.

Weak Yang-Baxter Operators

Definition. Let $(\mathcal{D}, \times, T, a, l, r)$ be a monoidal category such that the objects and the morphisms of $\mathcal{D}$ are also in $\mathcal{C}$. The category $\mathcal{D}$ is an idemp-monoidal category of $\mathcal{C}$ if there exists a family of idempotent morphisms in $\mathcal{C}$,

$$\nabla = \{\nabla_{M\otimes N} : M \otimes N \to M \otimes N \; ; \; M, N \in Obj(\mathcal{D})\},$$

with factorization $\nabla_{M\otimes N} = i_{M,N} \circ p_{M,N}$, such that:

(1) The image of $\nabla_{M\otimes N}$ is $M \times N$.

(2) If $f : M \to N$ and $g : M' \to N'$ are morphisms in $\mathcal{D}$, $f \times g = p_{N,N'} \circ (f \otimes g) \circ i_{M,M'}$.

(3) For every triple M, N, P of objects in $\mathcal{D}$, we have

$$(i_{M,N} \otimes P) \circ \nabla_{(M\times N)\otimes P} \circ (p_{M,N} \otimes P) = (M \otimes i_{N,P}) \circ \nabla_{M\otimes(N\times P)} \circ (M \otimes p_{N,P}) =$$

$$(\nabla_{M\otimes N} \otimes P) \circ (M \otimes \nabla_{N\otimes P}) = (M \otimes \nabla_{N\otimes P}) \circ (\nabla_{M\otimes N} \otimes P).$$

(4) The associativity constraints of $\mathcal{D}$ and their inverses admit the following expressions:

$a_{M,N,P} = p_{M\times N,P} \circ (p_{M,N} \otimes P) \circ (M \otimes i_{N,P}) \circ i_{M,N\times P} : M \times (N \times P) \to (M \times N) \times P,$

$a^{-1}_{M,N,P} = p_{M,N\times P} \circ (M \otimes p_{N,P}) \circ (i_{M,N} \otimes P) \circ i_{M\times N,P} : (M \times N) \times P \to M \times (N \times P).$

Weak Yang-Baxter Operators

Theorem. Let $\mathcal{D}$ be an idemp-monoidal category of $\mathcal{C}$. Let M be an object in $\mathcal{D}$ and suppose that $\tau_{M,M} : M \times M \to M \times M$ is a Yang-Baxter operator in $\mathcal{D}$. Then, if the morphism $t_{M,M} = i_{M,M} \circ \tau_{M,M} \circ p_{M,M} : M \otimes M \to M \otimes M$ satisfies the equalities

$$(1) \quad (t_{M,M} \otimes M) \circ (M \otimes \nabla_{M \otimes M}) = (M \otimes \nabla_{M \otimes M}) \circ (t_{M,M} \otimes M),$$

$$(2) \quad (M \otimes t_{M,M}) \circ (\nabla_{M \otimes M} \otimes M) = (\nabla_{M \otimes M} \otimes M) \circ (M \otimes t_{M,M}),$$

we obtain that $t_{M,M}$ is a weak Yang-Baxter operator in $\mathcal{C}$.

Weak Yang-Baxter Operators

Definition. **(Böhm, Nill and Szlachányi, J. of Algebra, 1999)**

A weak Hopf algebra (or quantum groupoid) in a strict symmetric monoidal category $\mathcal{C}$ is by definition an algebra (H, η_H, μ_H) and coalgebra $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)$.

(2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$

$= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H)$.

(3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$

$= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.

(4) There exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called antipode of H) satisfying:

(4-1) $id_H \wedge \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$.

(4-2) $\lambda_H \wedge id_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$.

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A weak Hopf algebra is a Hopf algebra if an only if the morphism δ_H (comultiplication) is unit-preserving (if and only if the counit is a homomorphism of algebras).

Weak Yang-Baxter Operators

If H is a weak Hopf algebra, the antipode λ_H is unique, antimultiplicative, anticomultiplicative and leaves the unit η_H and the counit ε_H invariant:

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,$$

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$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H.$$

The morphisms Π_H^L (target), Π_H^R (source), $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ defined by

$$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H) : H \to H,$$

$$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) : H \to H,$$

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are idempotent.

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In this talk we denote by H_L the image of Π_H^L and by $p_L : H \to H_L$, $i_L : H_L \to H$ the morphisms such that $i_L \circ p_L = \Pi_H^L$ and $i_L \circ p_L = id_{H_L}$.

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Let $(M, \varphi_M, \varrho_M)$, $(N, \varphi_N, \varrho_N)$ be left H-modules-comodules.

$$\varphi_{M\otimes N} = (\varphi_M \otimes \varphi_N) \circ (H \otimes c_{H,M} \otimes N) \circ (\delta_H \otimes M \otimes N) : H \otimes M \otimes N \to M \otimes N$$

$$\varrho_{M\otimes N} = (\mu_H \otimes M \otimes N) \circ (H \otimes c_{M,H} \otimes N) \circ (\varrho_M \otimes \varrho_N) : M \otimes N \to H \otimes M \otimes N.$$

Weak Yang-Baxter Operators

The morphisms

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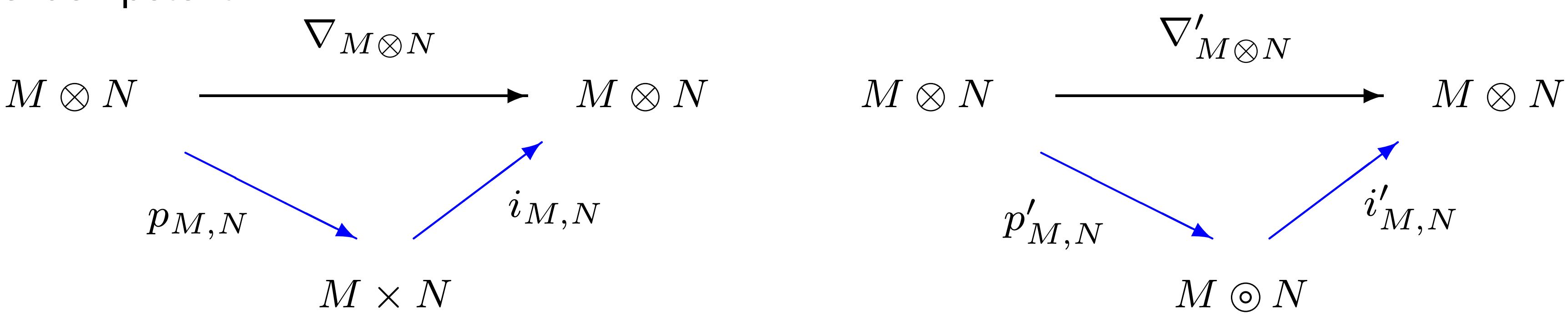

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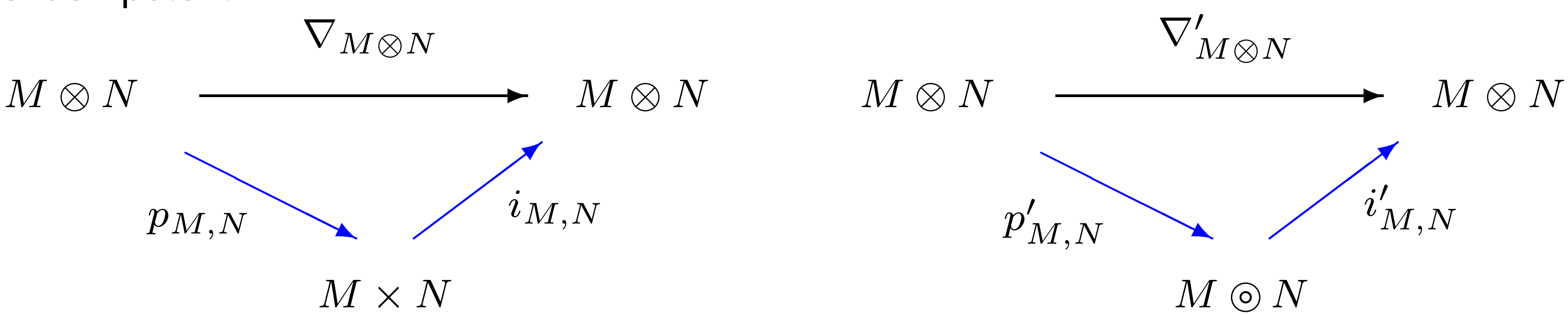
are idempotent.



$$\nabla_{M\otimes N} = i_{M,N} \circ p_{M,N}, \quad id_{M\times N} = p_{M,N} \circ i_{M,N}.$$
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Weak Yang-Baxter Operators

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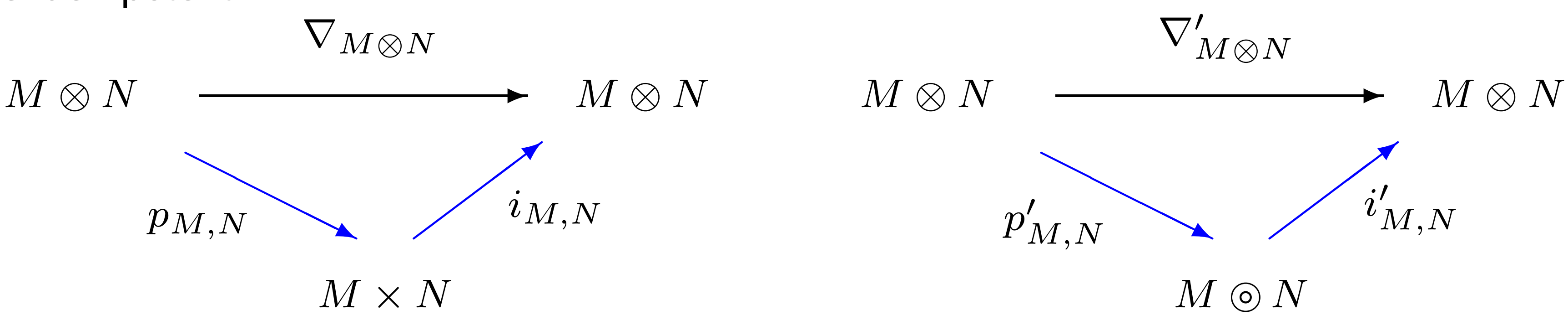

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If H is a Hopf algebra then $\nabla_{M\otimes N} = id_{M\otimes N} = \nabla'_{M\otimes N}$ and $M \times N = M \otimes N = M \odot N$.

Weak Yang-Baxter Operators

Definition. **(Böhm, Comm. in Algebra, 2000)**
Let H be a weak Hopf algebra. We shall denote by ${}^H_H\mathcal{YD}$ the category of left-left Yetter-Drinfeld modules over H. That is, $M = (M, \varphi_M, \varrho_M)$ is an object in ${}^H_H\mathcal{YD}$ if (M, φ_M) is a left H-module, (M, ϱ_M) is a left H-comodule and

(1) $$(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ ((\varrho_M \circ \varphi_M) \otimes H) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M)$$
$$= (\mu_H \otimes \varphi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes \varrho_M),$$

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$$f \circ \varphi_M = \varphi_N \circ (H \otimes f), \quad (H \otimes f) \circ \varrho_M = \varrho_N \circ f.$$

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Weak Yang-Baxter Operators

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${}^H_H\mathcal{YD}$ is a non-strict braided monoidal category

Weak Yang-Baxter Operators

For two left-left Yetter-Drinfeld modules $M = (M, \varphi_M, \varrho_M)$, $N = (N, \varphi_N, \varrho_N)$ we have $\nabla_{M\otimes N} = \nabla'_{M\otimes N}$ and then the tensor product is defined as object by

$$Im(\nabla_{M\otimes N}) = M \times N = M \circledcirc N = Im(\nabla'_{M\otimes N}).$$

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$M \times N$ is a left-left Yetter-Drinfeld module with the following action and coaction:

$$\varphi_{M\times N} = p_{M,N} \circ \varphi_{M\otimes N} \circ (H \otimes i_{M,N}),$$
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The unit object is

$$H_L = Im(\Pi_H^L).$$

The structure of left-left Yetter-Drinfeld module for H_L is

$$\varphi_{H_L} = p_L \circ \mu_H \circ (H \otimes i_L), \quad \varrho_{H_L} = (H \otimes p_L) \circ \delta_H \circ i_L.$$

Weak Yang-Baxter Operators

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The unit constrains are:

$$l_M = \varphi_M \circ (i_L \otimes M) \circ i_{H_L,M} : H_L \times M \to M,$$
$$r_M = \varphi_M \circ c_{M,H} \circ (M \otimes (\overline{\Pi}^L_H \circ i_L)) \circ i_{M,H_L} : M \times H_L \to M.$$

Weak Yang-Baxter Operators

These morphisms are isomorphisms with inverses:

$$l_M^{-1} = p_{H_L,M} \circ (p_L \otimes \varphi_M) \circ ((\delta_H \circ \eta_H) \otimes M) : M \to H_L \times M,$$
$$r_M^{-1} = p_{M,H_L} \circ (\varphi_M \otimes p_L) \circ (H \otimes c_{H,M}) \circ ((\delta_H \circ \eta_H) \otimes M) : M \to M \times H_L.$$

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If M, N, P are objects in the category ${}^H_H\mathcal{YD}$, the associativity constrains are defined by

$$a_{M,N,P} : M \times (N \times P) \to (M \times N) \times P,$$
$$a_{M,N,P} = p_{M \times N,P} \circ (p_{M,N} \otimes P) \circ (M \otimes i_{N,P}) \circ i_{M,N \times P}$$

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where the inverse are the morphisms:

$$a^{-1}_{M,N,P} : (M \times N) \times P \to M \times (N \times P).$$
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If $\gamma : M \to M'$ and $\phi : N \to N'$ are morphisms in the category, then

$$\gamma \times \phi = p_{M',N'} \circ (\gamma \otimes \phi) \circ i_{M,N} : M \times N \to M' \times N'$$

is a morphism in ${}_H^H\mathcal{YD}$.

Weak Yang-Baxter Operators

Finally, the braiding is

$$\tau_{M,N} = p_{N,M} \circ t_{M,N} \circ i_{M,N} : M \times N \longrightarrow N \times M,$$

where

$$t_{M,N} = (\varphi_N \otimes M) \circ (H \otimes c_{M,N}) \circ (\varrho_M \otimes N) : M \otimes N \longrightarrow N \otimes M.$$

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The morphism $\tau_{M,N}$ is a natural isomorphism with inverse:

$$\tau_{M,N}^{-1} = p_{M,N} \circ t'_{M,N} \circ i_{N,M} : N \times M \longrightarrow M \times N$$

where

$$t'_{M,N} = c_{N,M} \circ (\varphi_N \otimes M) \circ (c_{N,H} \otimes M) \circ (N \otimes \lambda_H^{-1} \otimes M) \circ (N \otimes \varrho_M).$$

Weak Yang-Baxter Operators

Then, if M, N and P are left-left Yetter-Drinfeld modules, we have the following:

- The image of $\nabla_{M \otimes N}$ is $M \times N$(the tensor product in ${}^{H}_{H}\mathcal{YD}$).

Weak Yang-Baxter Operators

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- The image of $\nabla_{M\otimes N}$ is $M \times N$ (the tensor product in ${}_H^H\mathcal{YD}$).
- If $f : M \to N$ and $g : M' \to N'$ are morphisms in ${}_H^H\mathcal{YD}$ then

$$f \times g = p_{N,N'} \circ (f \otimes g) \circ i_{M,M'}.$$

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$$f \times g = p_{N,N'} \circ (f \otimes g) \circ i_{M,M'}.$$

- We have

$$(i_{M,N} \otimes P) \circ \nabla_{(M\times N)\otimes P} \circ (p_{M,N} \otimes P) = (M \otimes i_{N,P}) \circ \nabla_{M\otimes(N\times P)} \circ (M \otimes p_{N,P}) =$$

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- If $f : M \to N$ and $g : M' \to N'$ are morphisms in ${}^H_H\mathcal{YD}$ then

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$$(\nabla_{M\otimes N} \otimes P) \circ (M \otimes \nabla_{N\otimes P}) = (M \otimes \nabla_{N\otimes P}) \circ (\nabla_{M\otimes N} \otimes P).$$

- The associativity constraints of ${}^H_H\mathcal{YD}$ and their inverses admit the following expressions:

$$a_{M,N,P} = p_{M\times N,P} \circ (p_{M,N} \otimes P) \circ (M \otimes i_{N,P}) \circ i_{M,N\times P} : M \times (N \times P) \to (M \times N) \times P,$$
$$a^{-1}_{M,N,P} = p_{M,N\times P} \circ (M \otimes p_{N,P}) \circ (i_{M,N} \otimes P) \circ i_{M\times N,P} : (M \times N) \times P \to M \times (N \times P).$$

Weak Yang-Baxter Operators

Moreover:

- $t_{M,N} = i_{M,N} \circ \tau_{M,N} \circ p_{M,N}.$

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$$(t_{M,M} \otimes M) \circ (M \otimes \nabla_{M\otimes M}) = (M \otimes \nabla_{M\otimes M}) \circ (t_{M,M} \otimes M),$$

$$(M \otimes t_{M,M}) \circ (\nabla_{M\otimes M} \otimes M) = (\nabla_{M\otimes M} \otimes M) \circ (M \otimes t_{M,M}),$$

Moreover:

- $t_{M,N} = i_{M,N} \circ \tau_{M,N} \circ p_{M,N}$.
- $$(t_{M,M} \otimes M) \circ (M \otimes \nabla_{M\otimes M}) = (M \otimes \nabla_{M\otimes M}) \circ (t_{M,M} \otimes M),$$
  $$(M \otimes t_{M,M}) \circ (\nabla_{M\otimes M} \otimes M) = (\nabla_{M\otimes M} \otimes M) \circ (M \otimes t_{M,M}),$$

Theorem. Let H be a weak Hopf algebra in a strict symmetric monoidal category $\mathcal{C}$ with split idempotens such that the antipode λ_H is an isomorphism. The category ${}^H_H\mathcal{YD}$ is an idemp-monoidal category of $\mathcal{C}$ and if $(M, \varphi_M, \varrho_M)$ is an object in ${}^H_H\mathcal{YD}$, the morphism

$$t_{M,M} = (\varphi_M \otimes M) \circ (H \otimes c_{M,M}) \circ (\varrho_M \otimes M) : M \otimes M \to M \otimes M$$

is a weak Yang-Baxter operator in $\mathcal{C}$ where

$$\nabla_{M\otimes M} = \varphi_{M\otimes M} \circ (\eta_H \otimes M \otimes M) : M \otimes M \to M \otimes M.$$

Weak Yang-Baxter Operators

Lemma. Let H be a weak Hopf algebra. The morphisms

(1) $\Omega_H^1 = \mu_{H\otimes H} \circ ((c_{H,H} \circ \delta_H \circ \eta_H) \otimes H \otimes H) : H \otimes H \to H \otimes H$,

(2) $\Omega_H^2 = \mu_{H\otimes H} \circ (H \otimes H \otimes (\delta_H \circ \eta_H)) : H \otimes H \to H \otimes H$,

(3) $\Omega_H^3 = \mu_{H\otimes H} \circ (H \otimes H \otimes (c_{H,H} \circ \delta_H \circ \eta_H)) : H \otimes H \to H \otimes H$,

(4) $\Omega_H^4 = \mu_{H\otimes H} \circ ((\delta_H \circ \eta_H) \otimes H \otimes H) : H \otimes H \to H \otimes H$,

are idempotent and $\Omega_H^1 \circ \Omega_H^2 = \Omega_H^2 \circ \Omega_H^1$, $\Omega_H^3 \circ \Omega_H^4 = \Omega_H^4 \circ \Omega_H^3$.

Weak Yang-Baxter Operators

Lemma. Let H be a weak Hopf algebra. The morphisms

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Let $\Omega_H = \Omega_H^2 \circ \Omega_H^1$ and $\Omega'_H = \Omega_H^4 \circ \Omega_H^3$. Then, Ω_H and Ω'_H are idempotent morphisms and if $\sigma : A \to H \otimes H$ is a morphism in $\mathcal{C}$,

$$\Omega_H \circ \sigma = \sigma \quad \Leftrightarrow \quad \Omega_H^1 \circ \sigma = \sigma \text{ e } \Omega_H^2 \circ \sigma = \sigma,$$

$$\Omega'_H \circ \sigma = \sigma \quad \Leftrightarrow \quad \Omega_H^3 \circ \sigma = \sigma \text{ e } \Omega_H^4 \circ \sigma = \sigma.$$

Weak Yang-Baxter Operators

Definition. (Nikshych, Turaev and Vainerman , Topology Appl., 2003)
Let Ω_H and Ω'_H be the idempotent morphisms defined previously. A quasitriangular weak Hopf algebra is a pair (H, σ) where H is a weak Hopf algebra and $\sigma : K \longrightarrow H \otimes H$ is a morphism in $\mathcal{C}$ satisfying the following conditions:

(1) $\Omega_H \circ \sigma = \sigma$.

(2) $(\delta_H \otimes H) \circ \sigma = (H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\sigma \otimes \sigma)$.

(3) $(H \otimes \delta_H) \circ \sigma = (\mu_H \otimes c_{H,H}) \circ (H \otimes c_{H,H} \otimes H) \circ (\sigma \otimes \sigma)$.

(4) $\mu_{H\otimes H} \circ (\sigma \otimes \delta_H) = \mu_{H\otimes H} \circ ((c_{H,H} \circ \delta_H) \otimes \sigma)$.

(5) There exists a morphism $\overline{\sigma} : K \longrightarrow H \otimes H$ such that:

(5-1) $\Omega'_H \circ \overline{\sigma} = \overline{\sigma}$.

(5-2) $\sigma \wedge \overline{\sigma} = c_{H,H} \circ \delta_H \circ \eta_H$.

(5-3) $\overline{\sigma} \wedge \sigma = \delta_H \circ \eta_H$.

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(2) $(\delta_H \otimes H) \circ \sigma = (H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\sigma \otimes \sigma)$.

(3) $(H \otimes \delta_H) \circ \sigma = (\mu_H \otimes c_{H,H}) \circ (H \otimes c_{H,H} \otimes H) \circ (\sigma \otimes \sigma)$.

(4) $\mu_{H\otimes H} \circ (\sigma \otimes \delta_H) = \mu_{H\otimes H} \circ ((c_{H,H} \circ \delta_H) \otimes \sigma)$.

(5) There exists a morphism $\overline{\sigma} : K \to H \otimes H$ such that:

(5-1) $\Omega'_H \circ \overline{\sigma} = \overline{\sigma}$.

(5-2) $\sigma \wedge \overline{\sigma} = c_{H,H} \circ \delta_H \circ \eta_H$.

(5-3) $\overline{\sigma} \wedge \sigma = \delta_H \circ \eta_H$.

If $\upsilon : K \to H \otimes H$ is a morphism satisfying (5-1), (5-2) and (5-3) we have

$$\upsilon = \Omega^3_H \circ \upsilon = \upsilon \wedge (\sigma \wedge \overline{\sigma}) = (\upsilon \wedge \sigma) \wedge \overline{\sigma} = \Omega^4_H \circ \overline{\sigma} = \overline{\sigma}.$$

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The unit and the associative constrains are the ones defined in the Yetter-Drinfeld setting.
If $\gamma : M \to M'$ and $\phi : N \to N'$ are morphisms in the category, then

$$\gamma \times \phi = p_{M',N'} \circ (\gamma \otimes \phi) \circ i_{M,N} : M \times N \to M' \times N'$$

is a morphism in ${}_H\mathcal{C}$.

Weak Yang-Baxter Operators

The braiding is

$$\chi_{M,N} = p_{N,M} \circ \vartheta_{M,N} \circ i_{M,N} : M \times N \longrightarrow N \times M$$

where

$$\vartheta_{M,N} = c_{M,N} \circ (\varphi_M \otimes \varphi_N) \circ (H \otimes c_{H,M} \otimes N) \circ (\sigma \otimes M \otimes N).$$

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The morphism $\chi_{M,N}$ is a natural isomorphism with inverse:

$$\chi_{M,N}^{-1} = p_{M,N} \circ \vartheta'_{M,N} \circ i_{N,M} : N \times M \to M \times N$$

where

$$\vartheta'_{M,N} = (\varphi_M \otimes \varphi_N) \circ (H \otimes c_{H,M} \otimes N) \circ (\overline{\sigma} \otimes c_{N,M}).$$

Weak Yang-Baxter Operators

Lemma. Let (H, σ) be a quasitriangular weak Hopf algebra in a strict symmetric monoidal category $\mathcal{C}$ with split idempotens. There is a monoidal functor

$$F: \ {}_H\mathcal{C} \rightarrow \ {}_H^H\mathcal{YD}$$

defined by $F((M, \varphi_M)) = (M, \varphi_M, \varrho_M = (H \otimes \varphi_M) \circ ((c_{H,H} \circ \sigma) \otimes M))$ for the objects and by the identity for the morphisms. Moreover, for all M, N in ${}_H\mathcal{C}$ we have $t_{M,N} = \vartheta_{M,N}$.

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Theorem. Let (H, σ) be a quasitriangular weak Hopf algebra in a strict symmetric monoidal category $\mathcal{C}$ with split idempotens. The category ${}_H\mathcal{C}$ is an idemp-monoidal category of $\mathcal{C}$ and if (M, φ_M) is an object in ${}_H\mathcal{C}$, the morphism

$$\vartheta_{M,M} = c_{M,M} \circ (\varphi_M \otimes \varphi_M) \circ (H \otimes c_{H,M} \otimes M) \circ (\sigma \otimes M \otimes M) : M \otimes M \to M \otimes M$$

is a weak Yang-Baxter operator in $\mathcal{C}$ where

$$\nabla_{M \otimes M} = \varphi_{M \otimes M} \circ (\eta_H \otimes M \otimes M) : M \otimes M \to M \otimes M.$$

Weak braided Hopf algebras

Definition. A weak braided Hopf algebra (**WBHA**) D is an object in $\mathcal{C}$ with an algebra structure (D, η_D, μ_D) and a coalgebra structure $(D, \varepsilon_D, \delta_D)$ such that there exists a weak Yang-Baxter operator $t_{D,D} : D \otimes D \to D \otimes D$ with associated idempotent $\nabla_{D\otimes D}$ satisfying the following conditions:

(1)

(1-1) $\mu_D \circ \nabla_{D\otimes D} = \mu_D$,

(1-2) $\nabla_{D\otimes D} \circ (\mu_D \otimes D) = (\mu_D \otimes D) \circ (D \otimes \nabla_{D\otimes D})$,

(1-3) $\nabla_{D\otimes D} \circ (D \otimes \mu_D) = (D \otimes \mu_D) \circ (\nabla_{D\otimes D} \otimes D)$.

(2)

(2-1) $\nabla_{D\otimes D} \circ \delta_D = \delta_D$,

(2-2) $(\delta_D \otimes D) \circ \nabla_{D\otimes D} = (D \otimes \nabla_{D\otimes D}) \circ (\delta_D \otimes D)$,

(2-3) $(D \otimes \delta_D) \circ \nabla_{D\otimes D} = (\nabla_{D\otimes D} \otimes D) \circ (D \otimes \delta_D)$.

Weak braided Hopf algebras

(3) The morphisms η_D, μ_D, ε_D and δ_D commute with $t_{D,D}$, i.e.,

(3-1) $t_{D,D} \circ (\eta_D \otimes D) = \nabla_{D\otimes D} \circ (D \otimes \eta_D)$,

(3-2) $t_{D,D} \circ (D \otimes \eta_D) = \nabla_{D\otimes D} \circ (\eta_D \otimes D)$,

(3-3) $t_{D,D} \circ (\mu_D \otimes D) = (D \otimes \mu_D) \circ (t_{D,D} \otimes D) \circ (D \otimes t_{D,D})$,

(3-4) $t_{D,D} \circ (D \otimes \mu_D) = (\mu_D \otimes D) \circ (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D)$,

(3-5) $(\varepsilon_D \otimes D) \circ t_{D,D} = (D \otimes \varepsilon_D) \circ \nabla_{D\otimes D}$,

(3-6) $(D \otimes \varepsilon_D) \circ t_{D,D} = (\varepsilon_D \otimes D) \circ \nabla_{D\otimes D}$,

(3-7) $(\delta_D \otimes D) \circ t_{D,D} = (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D) \circ (D \otimes \delta_D)$,

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Weak braided Hopf algebras

(3) The morphisms μ_D and δ_D commute with $t_{D,D}$, i.e.,

(3-1) $t_{D,D} \circ (\mu_D \otimes D) = (D \otimes \mu_D) \circ (t_{D,D} \otimes D) \circ (D \otimes t_{D,D})$,

(3-2) $t_{D,D} \circ (D \otimes \mu_D) = (\mu_D \otimes D) \circ (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D)$,

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Weak braided Hopf algebras

(4) $\delta_D \circ \mu_D = (\mu_D \otimes \mu_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D)$.

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$= (\varepsilon_D \otimes \varepsilon_D) \circ (\mu_D \otimes \mu_D) \circ (D \otimes (t'_{D,D} \circ \delta_D) \otimes D)$.

(6) $(\delta_D \otimes D) \circ \delta_D \circ \eta_D = (D \otimes \mu_D \otimes D) \circ (\delta_D \otimes \delta_D) \circ (\eta_D \otimes \eta_D)$

$= (D \otimes (\mu_D \circ t'_{D,D}) \otimes D) \circ (\delta_D \otimes \delta_D) \circ (\eta_D \otimes \eta_D)$.

(7) There exists a morphism $\lambda_D : D \rightarrow D$ in $\mathcal{C}$ (called the antipode of D) satisfying:

(7-1) $id_D \wedge \lambda_D = ((\varepsilon_D \circ \mu_D) \otimes D) \circ (D \otimes t_{D,D}) \circ ((\delta_D \circ \eta_D) \otimes D)$,

(7-2) $\lambda_D \wedge id_D = (D \otimes (\varepsilon_D \circ \mu_D)) \circ (t_{D,D} \otimes D) \circ (D \otimes (\delta_D \circ \eta_D))$,

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Weak braided Hopf algebras

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If D satisfies (1) to (6), D is a weak braided bialgebra in $\mathcal{C}$.

Weak braided Hopf algebras

- If $\mathcal{C}$ is symmetric, $t_{D,D} = c_{D,D} = t'_{D,D}$, i.e. the weak Yang-Baxter operator is the twist of the symmetric category $\mathcal{C}$, then $\nabla_{D\otimes D} = id_{D\otimes D}$ and the last definition is the usual definition of weak Hopf algebra (**Böhm, Nill and Szlachányi**).

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- If $\mathcal{C}$ is braided, $t_{D,D} = c_{D,D}$, $t'_{D,D} = c^{-1}_{D,D}$ and $\eta_D \otimes \eta_D = \delta_D \circ \eta_D$, D is a Hopf algebra in a braided category (**Majid**).

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- If $\mathcal{C}$ is braided, $t_{D,D} = c_{D,D}$, $t'_{D,D} = c^{-1}_{D,D}$, i.e. the weak Yang-Baxter operator is the braiding of the braided category $\mathcal{C}$, and $\eta_D \otimes \eta_D = \delta_D \circ \eta_D$, D is a Hopf algebra in a braided category (**Majid**).
- If $\mathcal{C}$ is braided, $t_{D,D} = c_{D,D}$, $t'_{D,D} = c^{-1}_{D,D}$, we introduce the definition of weak Hopf algebra in a braided category.

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- If $\mathcal{C}$ is braided, $t_{D,D} = c_{D,D}$, $t'_{D,D} = c^{-1}_{D,D}$, we introduce the definition of weak Hopf algebra in a braided category.
  Recently, **Pastro and Street** (**2008**), called these objects weak Hopf monoids.

Proposition. Let H be a weak Hopf algebra in a strict symmetric monoidal category $\mathcal{C}$ with split idempotents and such that λ_H is an isomorphism. Let $(D, u_D, m_D, e_D, \Delta_D, \lambda_D)$ be a Hopf algebra in ${}^H_H\mathcal{YD}$ with action φ_D and coaction ϱ_D. Let $t_{D,D} = (\varphi_D \otimes D) \circ (H \otimes c_{D,D}) \circ (\varrho_D \otimes D)$ be the weak Yang-Baxter operator and $\nabla_{D\otimes D} = i_{D,D} \circ p_{D,D}$ the associated idempotent. Then

$D = (D, \eta_D = u_D \circ p_L \circ \eta_H, \mu_D = m_D \circ p_{D,D}, \varepsilon_D = \varepsilon_H \circ i_L \circ e_D, \delta_D = i_{D,D} \circ \Delta_D, \lambda_D)$

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- If $\lambda_D \wedge id_D = \varepsilon_D \otimes \eta_D$ we have $u_D \circ e_D = \eta_D \circ \varepsilon_D$ and then

$$id_{H_L} = p_L \circ \eta_H \circ \varepsilon_H \circ i_L.$$

Therefore, $\Pi_H^L = \varepsilon_H \otimes \eta_H$ and we obtain that H also is a Hopf algebra.

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  $$id_{H_L} = p_L \circ \eta_H \circ \varepsilon_H \circ i_L.$$

  Therefore, $\Pi_H^L = \varepsilon_H \otimes \eta_H$ and we obtain that H also is a Hopf algebra.
- Finally, D is not a weak Hopf algebra since the condition

  $$\delta_D \circ \mu_D = (\mu_D \otimes \mu_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D)$$

  does not imply $\delta_D \circ \mu_D = (\mu_D \otimes \mu_D) \circ (D \otimes c_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D)$ where $c_{D,D}$ is the symmetric braiding of $\mathcal{C}$.

WBHA and Weak Entwining Structures

Definition. (**Caenepeel and De Groot, Cont. Math., 2000**)
A right-right weak entwining structure on a strict monoidal category $\mathcal{C}$ with split idempotents, consists of a triple (A, C, ψ_{RR}), where A is an algebra, C a coalgebra, and $\psi_{RR} : C \otimes A \to A \otimes C$ a morphism satisfying the relations

(1) $\psi_{RR} \circ (C \otimes \mu_A) = (\mu_A \otimes C) \circ (A \otimes \psi_{RR}) \circ (\psi_{RR} \otimes A)$,

(2) $(A \otimes \delta_C) \circ \psi_{RR} = (\psi_{RR} \otimes C) \circ (C \otimes \psi_{RR}) \circ (\delta_C \otimes A)$,

(3) $\psi_{RR} \circ (C \otimes \eta_A) = (e_{RR} \otimes C) \circ \delta_C$,

(4) $(A \otimes \varepsilon_C) \circ \psi_{RR} = \mu_A \circ (e_{RR} \otimes A)$,

where $e_{RR} : C \to A$ is the morphism defined by $e_{RR} = (A \otimes \varepsilon_C) \circ \psi_{RR} \circ (C \otimes \eta_A)$. The morphism ψ_{RR} is known as entwining morphism.

WBHA and Weak Entwining Structures

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(1) $\psi_{RR} \circ (C \otimes \mu_A) = (\mu_A \otimes C) \circ (A \otimes \psi_{RR}) \circ (\psi_{RR} \otimes A)$,

(2) $(A \otimes \delta_C) \circ \psi_{RR} = (\psi_{RR} \otimes C) \circ (C \otimes \psi_{RR}) \circ (\delta_C \otimes A)$,

(3) $\psi_{RR} \circ (C \otimes \eta_A) = (e_{RR} \otimes C) \circ \delta_C$,

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By Δ_{RR} we denote the morphism

$$\Delta_{RR} = (\mu_A \otimes C) \circ (A \otimes \psi_{RR}) \circ (A \otimes C \otimes \eta_A) : A \otimes C \to A \otimes C$$

WBHA and Weak Entwining Structures

Definition. A left-left weak entwining structure on a strict monoidal category $\mathcal{C}$ with split idempotents, consists of a triple (A, C, ψ_{LL}), where A is an algebra, C a coalgebra, and $\psi_{LL} : A \otimes C \to C \otimes A$ a morphism satisfying the relations

(1) $\psi_{LL} \circ (\mu_A \otimes C) = (C \otimes \mu_A) \circ (\psi_{LL} \otimes A) \circ (A \otimes \psi_{RR})$,

(2) $(\delta_C \otimes A) \circ \psi_{LL} = (C \otimes \psi_{LL}) \circ (\psi_{LL} \otimes C) \circ (A \otimes \delta_C)$,

(3) $\psi_{LL} \circ (\eta_A \otimes C) = (C \otimes e_{LL}) \circ \delta_C$,

(4) $(\varepsilon_C \otimes A) \circ \psi_{LL} = \mu_A \circ (A \otimes e_{LL})$,

where $e_{LL} : C \to A$ is the morphism defined by $e_{LL} = (\varepsilon_C \otimes A) \circ \psi_{LL} \circ (\eta_A \otimes C)$.

Definition. A left-left weak entwining structure on a strict monoidal category $\mathcal{C}$ with split idempotents, consists of a triple (A, C, ψ_{LL}), where A is an algebra, C a coalgebra, and $\psi_{LL} : A \otimes C \to C \otimes A$ a morphism satisfying the relations

(1) $\psi_{LL} \circ (\mu_A \otimes C) = (C \otimes \mu_A) \circ (\psi_{LL} \otimes A) \circ (A \otimes \psi_{RR})$,

(2) $(\delta_C \otimes A) \circ \psi_{LL} = (C \otimes \psi_{LL}) \circ (\psi_{LL} \otimes C) \circ (A \otimes \delta_C)$,

(3) $\psi_{LL} \circ (\eta_A \otimes C) = (C \otimes e_{LL}) \circ \delta_C$,

(4) $(\varepsilon_C \otimes A) \circ \psi_{LL} = \mu_A \circ (A \otimes e_{LL})$,

where $e_{LL} : C \to A$ is the morphism defined by $e_{LL} = (\varepsilon_C \otimes A) \circ \psi_{LL} \circ (\eta_A \otimes C)$.

By Δ_{LL} we denote the morphism

$$\Delta_{LL} = (C \otimes \mu_A) \circ (\psi_{LL} \otimes A) \circ (\eta_A \otimes C \otimes A) : C \otimes A \to C \otimes A$$

WBHA and Weak Entwining Structures

Theorem. Let D be an algebra-coalgebra in an strict monoidal category $\mathcal{C}$ with split idempotents. Let $t_{D,D} : D \otimes D \to D \otimes D$ be a morphism in $\mathcal{C}$. The following assertions are equivalent.

a) We have the following:

(a1) The triple $(D, D, t_{D,D})$ is a right-right weak entwining structure.

(a2) The triple $(D, D, t_{D,D})$ is a left-left weak entwining structure.

(a3) $\Delta_{RR} = \Delta_{LL}$.

(a4) $e_{RR} = e_{LL}$.

(a5) $e_{RR} \wedge id_D = id_D \wedge e_{LL} = id_D$.

(b) There exists an unique idempotent morphism $\nabla_{D\otimes D} : D \otimes D \to D \otimes D$ such that $(D, t_{D,D}, \nabla_{D\otimes D})$ satisfies the identities (1) to (3) of the definition of WBHA.

WBHA and Weak Entwining Structures

(1)(1-1) $\mu_D \circ \nabla_{D\otimes D} = \mu_D,$

(1-2) $\nabla_{D\otimes D} \circ (\mu_D \otimes D) = (\mu_D \otimes D) \circ (D \otimes \nabla_{D\otimes D}),$

(1-3) $\nabla_{D\otimes D} \circ (D \otimes \mu_D) = (D \otimes \mu_D) \circ (\nabla_{D\otimes D} \otimes D).$

(2)(2-1) $\nabla_{D\otimes D} \circ \delta_D = \delta_D,$

(2-2) $(\delta_D \otimes D) \circ \nabla_{D\otimes D} = (D \otimes \nabla_{D\otimes D}) \circ (\delta_D \otimes D),$

(2-3) $(D \otimes \delta_D) \circ \nabla_{D\otimes D} = (\nabla_{D\otimes D} \otimes D) \circ (D \otimes \delta_D).$

(3)(3-1) $t_{D,D} \circ (\eta_D \otimes D) = \nabla_{D\otimes D} \circ (D \otimes \eta_D),$

(3-2) $t_{D,D} \circ (D \otimes \eta_D) = \nabla_{D\otimes D} \circ (\eta_D \otimes D),$

(3-3) $t_{D,D} \circ (\mu_D \otimes D) = (D \otimes \mu_D) \circ (t_{D,D} \otimes D) \circ (D \otimes t_{D,D}),$

(3-4) $t_{D,D} \circ (D \otimes \mu_D) = (\mu_D \otimes D) \circ (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D),$

(3-5) $(\varepsilon_D \otimes D) \circ t_{D,D} = (D \otimes \varepsilon_D) \circ \nabla_{D\otimes D},$

(3-6) $(D \otimes \varepsilon_D) \circ t_{D,D} = (\varepsilon_D \otimes D) \circ \nabla_{D\otimes D},$

(3-7) $(\delta_D \otimes D) \circ t_{D,D} = (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D) \circ (D \otimes \delta_D),$

(3-8) $(D \otimes \delta_D) \circ t_{D,D} = (t_{D,D} \otimes D) \circ (D \otimes t_{D,D}) \circ (\delta_D \otimes D).$

WBHA and Weak Entwining Structures

(1)(1-1) $\mu_D \circ \nabla_{D\otimes D} = \mu_D,$

(1-2) $\nabla_{D\otimes D} \circ (\mu_D \otimes D) = (\mu_D \otimes D) \circ (D \otimes \nabla_{D\otimes D}),$

(1-3) $\nabla_{D\otimes D} \circ (D \otimes \mu_D) = (D \otimes \mu_D) \circ (\nabla_{D\otimes D} \otimes D).$

(2)(2-1) $\nabla_{D\otimes D} \circ \delta_D = \delta_D,$

(2-2) $(\delta_D \otimes D) \circ \nabla_{D\otimes D} = (D \otimes \nabla_{D\otimes D}) \circ (\delta_D \otimes D),$

(2-3) $(D \otimes \delta_D) \circ \nabla_{D\otimes D} = (\nabla_{D\otimes D} \otimes D) \circ (D \otimes \delta_D).$

(3)(3-1) $t_{D,D} \circ (\eta_D \otimes D) = \nabla_{D\otimes D} \circ (D \otimes \eta_D),$

(3-2) $t_{D,D} \circ (D \otimes \eta_D) = \nabla_{D\otimes D} \circ (\eta_D \otimes D),$

(3-3) $t_{D,D} \circ (\mu_D \otimes D) = (D \otimes \mu_D) \circ (t_{D,D} \otimes D) \circ (D \otimes t_{D,D}),$

(3-4) $t_{D,D} \circ (D \otimes \mu_D) = (\mu_D \otimes D) \circ (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D),$

(3-5) $(\varepsilon_D \otimes D) \circ t_{D,D} = (D \otimes \varepsilon_D) \circ \nabla_{D\otimes D},$

(3-6) $(D \otimes \varepsilon_D) \circ t_{D,D} = (\varepsilon_D \otimes D) \circ \nabla_{D\otimes D},$

(3-7) $(\delta_D \otimes D) \circ t_{D,D} = (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D) \circ (D \otimes \delta_D),$

(3-8) $(D \otimes \delta_D) \circ t_{D,D} = (t_{D,D} \otimes D) \circ (D \otimes t_{D,D}) \circ (\delta_D \otimes D).$

WBHA and Weak Entwining Structures

(1)(1-1) $\mu_D \circ \nabla_{D\otimes D} = \mu_D,$

(1-2) $\nabla_{D\otimes D} \circ (\mu_D \otimes D) = (\mu_D \otimes D) \circ (D \otimes \nabla_{D\otimes D}),$

(1-3) $\nabla_{D\otimes D} \circ (D \otimes \mu_D) = (D \otimes \mu_D) \circ (\nabla_{D\otimes D} \otimes D).$

(2)(2-1) $\nabla_{D\otimes D} \circ \delta_D = \delta_D,$

(2-2) $(\delta_D \otimes D) \circ \nabla_{D\otimes D} = (D \otimes \nabla_{D\otimes D}) \circ (\delta_D \otimes D),$

(2-3) $(D \otimes \delta_D) \circ \nabla_{D\otimes D} = (\nabla_{D\otimes D} \otimes D) \circ (D \otimes \delta_D).$

(3)(3-1) $t_{D,D} \circ (\eta_D \otimes D) = \nabla_{D\otimes D} \circ (D \otimes \eta_D),$

(3-2) $t_{D,D} \circ (D \otimes \eta_D) = \nabla_{D\otimes D} \circ (\eta_D \otimes D),$

(3-3) $t_{D,D} \circ (\mu_D \otimes D) = (D \otimes \mu_D) \circ (t_{D,D} \otimes D) \circ (D \otimes t_{D,D}),$

(3-4) $t_{D,D} \circ (D \otimes \mu_D) = (\mu_D \otimes D) \circ (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D),$

(3-5) $(\varepsilon_D \otimes D) \circ t_{D,D} = (D \otimes \varepsilon_D) \circ \nabla_{D\otimes D},$

(3-6) $(D \otimes \varepsilon_D) \circ t_{D,D} = (\varepsilon_D \otimes D) \circ \nabla_{D\otimes D},$

(3-7) $(\delta_D \otimes D) \circ t_{D,D} = (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D) \circ (D \otimes \delta_D),$

(3-8) $(D \otimes \delta_D) \circ t_{D,D} = (t_{D,D} \otimes D) \circ (D \otimes t_{D,D}) \circ (\delta_D \otimes D).$

WBHA and Weak Entwining Structures

Definition. A right-left weak entwining structure on a strict monoidal category $\mathcal{C}$ with split idempotents, consists of a triple (A, C, ψ_{RL}), where A is an algebra, C a coalgebra, and $\psi_{RL} : A \otimes C \to A \otimes C$ a morphism such that there exist morphisms $\tau_{A,C} : A \otimes C \to C \otimes A$, $\sigma_{C,A} : C \otimes A \to A \otimes C$, $r_{C,C}, s_{C,C} : C \otimes C \to C \otimes C$, satisfying the relations

(1) $(\mu_A \otimes C) \circ (A \otimes \sigma_{C,A}) \circ (\psi_{RL} \otimes A) \circ (A \otimes (\tau_{A,C} \circ \psi_{RL})) = \psi_{RL} \circ (\mu_A \otimes C)$

(2) $(\psi_{RL} \otimes C) \circ (A \otimes s_{C,C}) \circ (\psi_{RL} \otimes C) \circ (A \otimes (r_{C,C} \circ \delta_C)) = (A \otimes \delta_C) \circ \psi_{RL}$

(3) $\psi_{RL} \circ (\eta_A \otimes C) = (e_{RL} \otimes C) \circ \delta_C,$

(4) $(A \otimes \varepsilon_C) \circ \psi_{RL} = \mu_A \circ (A \otimes e_{RL}),$

where $e_{RL} : C \to A$ is the morphism defined by $e_{RL} = (A \otimes \varepsilon_C) \circ \psi_{RL} \circ (\eta_A \otimes C)$.

WBHA and Weak Entwining Structures

Definition. A left-right weak entwining structure on a strict monoidal category $\mathcal{C}$ with split idempotents, consists of a triple (A, C, ψ_{LR}), where A is an algebra, C a coalgebra, and $\psi_{LR} : C \otimes A \to C \otimes A$ a morphism such that there exist morphisms $\tau_{A,C} : A \otimes C \to C \otimes A$, $\sigma_{C,A} : C \otimes A \to A \otimes C$, $r_{A,A}, s_{A,A} : A \otimes A \to A \otimes A$, satisfying the relations

(1) $(C \otimes (\mu_A \circ r_{A,A})) \circ (\psi_{LR} \otimes A) \circ (C \otimes s_{A,A}) \circ (\psi_{LR} \otimes A)) = \psi_{LR} \circ (A \otimes \mu_A)$

(2) $(C \otimes (\psi_{LR} \circ \tau_{A,C})) \circ (\psi_{LR} \otimes C) \circ (C \otimes \sigma_{C,A}) \circ (\delta_C \otimes A) = (\delta_C \otimes A) \circ \psi_{LR}$

(3) $\psi_{LR} \circ (C \otimes \eta_A) = (C \otimes e_{LR}) \circ \delta_C$,

(4) $(\varepsilon_C \otimes A) \circ \psi_{LR} = \mu_A \circ (e_{LR} \otimes A)$,

where $e_{LR} : C \to A$ is the morphism defined by $e_{LR} = (\varepsilon_C \otimes A) \circ \psi_{LR} \circ (C \otimes \eta_A)$.

WBHA and Weak Entwining Structures

Proposition. Let D be a WBHA in $\mathcal{C}$ with weak Yang-Baxter operator $t_{D,D}$ and associated idempotent $\nabla_{D\otimes D}$. Then,

(1) $(D, D, \psi_1 = (D \otimes \mu_D) \circ (t_{D,D} \otimes D) \circ (D \otimes \delta_D))$ is a right-right weak entwining structure.

(2) $(D, D, \psi_2 = (\mu_D \otimes D) \circ (D \otimes t_{D,D}) \circ (\delta_D \otimes D))$ is a left-left weak entwining structure.

(3) $(D, D, \psi_3 = (D \otimes \mu_D) \circ (D \otimes \delta_D))$ is a right-left weak entwining structure where $\tau_{D,D} = r_{D,D} = t'_{D,D}$ and $\sigma_{D,D} = s_{D,D} = t_{D,D}$.

(4) $(D, D, \psi_4 = (\mu_D \otimes D) \circ (\delta_D \otimes D))$ is a left-right weak entwining structure where $\tau_{D,D} = r_{D,D} = t'_{D,D}$ and $\sigma_{D,D} = s_{D,D} = t_{D,D}$.

WBHA and Weak Entwining Structures

Theorem. Let D be an algebra-coalgebra in $\mathcal{C}$. Let $t_{D,D}$ be a weak Yang-Baxter operator with associated idempotent $\nabla_{D\otimes D}$ such that $(D, D, t_{D,D})$ is a right-right and a left-left weak entwining structure, $\nabla_{D\otimes D} = \Delta_{RR} = \Delta_{LL}$, $e_{RR} = e_{LL}$, and $e_{RR} \wedge id_D = id_D \wedge e_{LL} = id_D$. The following assertions are equivalent:

(1) D is a weak braided bialgebra.

(2) (D, D, ψ_1) is a right-right weak entwining structure and (D, D, ψ_3) is a right-left weak entwining structure.

(3) (D, D, ψ_2) is a left-left weak entwining structure and (D, D, ψ_4) is a left-right weak entwining structure.

(4) (D, D, ψ_2) is a left-left weak entwining structure and (D, D, ψ_3) is a right-left weak entwining structure.

(5) (D, D, ψ_1) is a right-right weak entwining structure and (D, D, ψ_4) is a left-right weak entwining structure.

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