This is a joint work with Boris Chorny.

Definition. A category $\mathcal{K}$ is called λ-*class-accessible*, where λ is a regular cardinal, provided that

(1) $\mathcal{K}$ has λ-filtered colimits,

(2) $\mathcal{K}$ has a class $\mathcal{A}$ of λ-presentable objects such that every object of $\mathcal{K}$ is a λ-filtered colimit of objects from $\mathcal{A}$.

B. Banaschewski and H. Herrlich introduced these categories in 1976 – it means long ago the appearence of Makkai and Paré.

A category is *class-accessible* if it is λ-class-accessible for some regular cardinal λ. A complete and cocomplete λ-class-accessible category is called *locally* λ-*class-presentable*. A category is *locally class-presentable* is it is locally λ-class-presentable for some regular cardinal λ.

Each λ-accessible category is λ-class-accessible. Since each locally presentable category is complete, each locally λ-presentable category is locally λ-class-presentable.

Given a category $\mathcal{A}$, $\mathcal{P}(\mathcal{A})$ will denote the category of *small presheaves* on $\mathcal{A}$, i.e. functors $F : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ which are small colimits of hom-functors. Of course, for a small category $\mathcal{A}$, each F is small.

The category $\mathcal{P}(\mathcal{A})$ is always finitely class-accessible because each small presheaf is a small filtered colimit of finite colimits of hom-functors and the latter are finitely presentable. $\mathcal{P}(\mathcal{A})$ is always cocomplete but not necessarily complete. For instance, it does not have a terminal object in the case when $\mathcal{A}$ is a large discrete category (it means that it has a proper class of objects and the only morphisms are the identities). It explains why we added completeness into the definition of a locally class-presentable category.

Given a category $\mathcal{A}$, let $\mathbf{Ind}_\lambda(\mathcal{A})$ be the full subcategory of $\mathcal{P}(\mathcal{A})$ consisting of small λ-filtered colimits of hom-functors. $\mathbf{Ind}_\lambda(\mathcal{A})$ is always λ-class-accessible. In fact, each λ-class-accessible category $\mathcal{K}$ is equivalent to $\mathbf{Ind}_\lambda(\mathcal{A})$ for $\mathcal{A}$ being the full subcategory of $\mathcal{K}$ consisting of λ-presentable objects.

A functor $F : \mathcal{K} \to \mathcal{L}$ is called λ-*class-accessible* (where λ is a regular cardinal) if $\mathcal{K}$ and $\mathcal{L}$ are λ-class-accessible categories and F preserves λ-filtered colimits. A λ-class-accessible functor preserving λ-presentable objects is called *strongly* λ-*class-accessible*. F is called *(strongly) class-accessible* if it is (strongly) λ-class-accessible for some regular cardinal λ.

Each accessible functor is strongly accessible but this does not generalize to class-accessible functors.

Proposition. Let λ be a regular cardinal and $F : \mathcal{K} \to \mathcal{M}$ and $G : \mathcal{L} \to \mathcal{M}$ (strongly) λ-class-accessible functors. Then their pseudopullback

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\bar{F}"] \arrow[d, "\bar{G}"'] & \mathcal{L} \arrow[d, "G"] \\
\mathcal{K} \arrow[r, "F"'] & \mathcal{M}
\end{tikzcd}$$

is a λ^+-class-accessible category and the functors $\bar{F}, \bar{G}$ are (strongly) λ^+-class accessible.

The analogous result holds for equifiers and I believe that class-accessible categories are closed under lax limits.

Definition. A class $\mathcal{C}$ of morphisms of a category $\mathcal{K}$ is called *locally small* if, for each morphism f in $\mathcal{K}$, there is a subset $\mathcal{S}$ of $\mathcal{C}$ such that each morphism $g \to f$ in $\mathcal{K}^{\to}$ with $g \in \mathcal{C}$ factorizes as

$$g \to h \to f$$

with $h \in \mathcal{S}$.

Each set $\mathcal{C}$ of morphisms is locally small.

Theorem. Let $\mathcal{K}$ be a locally class-presentable category, $\mathcal{C}$ a locally small class of morphisms of $\mathcal{K}$ and assume that there is a regular cardinal λ such that each morphism from $\mathcal{C}$ has a λ-presentable domain. Then $(\mathrm{cof}(\mathcal{C}), \mathcal{C}^{\square})$ is a weak factorization system in $\mathcal{K}$.

Given a weak factorization system $(\mathcal{L}, \mathcal{R})$, then the class $\mathcal{L}$ is always locally small.

Corollary. Let $\mathcal{K}$ be a locally class-presentable category, $\mathcal{C}$ a locally small class of morphisms of $\mathcal{K}$. Let λ be a regular cardinal such that each morphism from $\mathcal{C}$ has a λ-presentable domain. Then $\mathbf{Inj}(\mathcal{C})$ is weakly reflective and closed under λ-filtered colimits in $\mathcal{K}$.

But $\mathbf{Inj}(\mathcal{C})$ does not need to be class-accessible.

Theorem. Let $\mathcal{K}$ be a locally class-presentable category, $\mathcal{C}$ a locally small class of morphisms of $\mathcal{K}$ and assume that there is a regular cardinal λ such that each morphism from $\mathcal{C}$ has a λ-presentable domain and codomain. Then $(\mathrm{colim}(\mathcal{C}), \mathcal{C}^{\perp})$ is a factorization system in $\mathcal{K}$.

Corollary. Let $\mathcal{K}$ be a locally λ-class-presentable category, $\mathcal{C}$ a locally small class of morphisms of $\mathcal{K}$ such that each morphism from $\mathcal{C}$ has a λ-presentable domain and codomain. Then $\mathbf{Ort}(\mathcal{C})$ is reflective and closed under λ-filtered colimits in $\mathcal{K}$. Moreover, $\mathbf{Ort}(\mathcal{C})$ is locally λ-class-presentable.

Definition Let $\mathcal{K}$ be a locally class-presentable category. A weak factorization system $(\mathcal{L}, \mathcal{R})$ in $\mathcal{K}$ is called *cofibrantly class-generated* if $\mathcal{L} = \mathrm{cof}(\mathcal{C})$ for a locally small class $\mathcal{C}$ of morphisms having λ-presentable domains and codomains (for some regular cardinal λ).

Definition. A model category is called *class-combinatorial* if its underlying category $\mathcal{K}$ is locally class-presentable and both (cofibrations, trivial fibrations) and (trivial cofibrations, fibrations) are cofibrantly class-generated weak factorization systems.

Each combinatorial model category is class-combinatorial.

Given a simplicial category $\mathcal{A}$, then the category of small simplicial presheaves $\mathcal{A}^{\mathrm{op}} \to \mathbf{SSet}$ with the projective model category structure is class-combinatorial.

Theorem. Let $\mathcal{K}$ be a class-combinatorial model category and $\mathcal{W}$ its class of weak equivalences. Then the inclusion of $\mathcal{W}$ in $\mathcal{K}^{\to}$ is a class-accessible functor. In particular, $\mathcal{W}$ is a class-accessible category.

For combinatorial model categories, this result was claimed by J. H. Smith. Last year, I gave a proof based on the fact that homotopy equivalences are the full image of an acccessible functor and using a pseudopullback

$$\begin{tikzcd}
\mathcal{K}^{\to} \arrow[r, "R"] & \mathcal{K}^{\to} \\
\mathcal{W} \arrow[u] \arrow[r] & \mathcal{H} \arrow[u]
\end{tikzcd}$$

where R is the replacement functor.

A. Stanculescu informed me that J. Lurie gave a simpler proof based on a pseudopullback

$$\begin{tikzcd}
\mathcal{K}^{\to} \arrow[r, "F"] & \mathcal{K}^{\to} \\
\mathcal{W} \arrow[u] \arrow[r] & \mathcal{F} \arrow[u]
\end{tikzcd}$$

where F gives the fibration part in the (trivial cofibration, fibration) factorization and $\mathcal{F}$ denotes trivial fibrations.

Since $\mathcal{F}$ is not necessarily class-accessible in a class-combinatorial setting, I have to combine both mine and Lurie's proof here. Mine yields (2) and Lurie's (1) in the definition of a class-accessible category.

Our goal has been to develop the theory of left Bousfield localizations in the class-combinatorial setting. We have some particular result for simplicial class-combinatorial model categories.