Quotient completion for preordered fibrations

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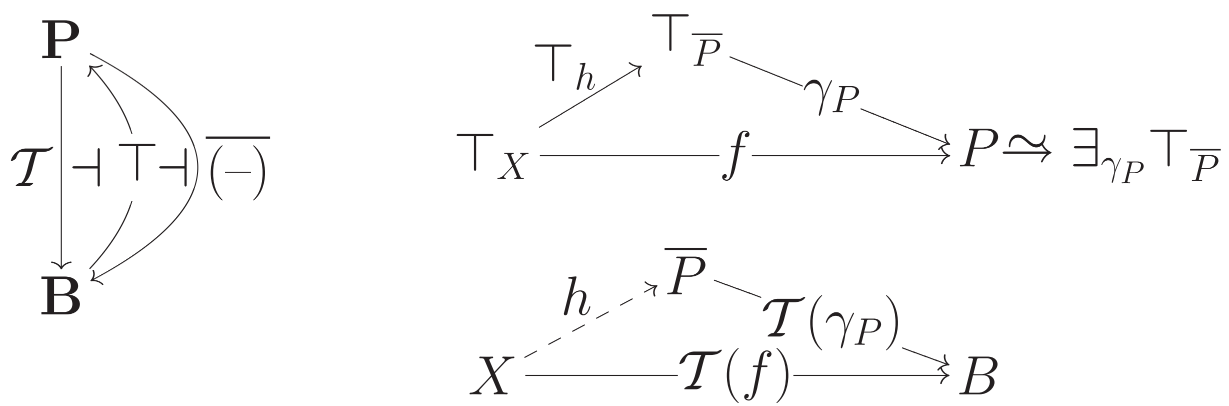
A regular category from a fibration

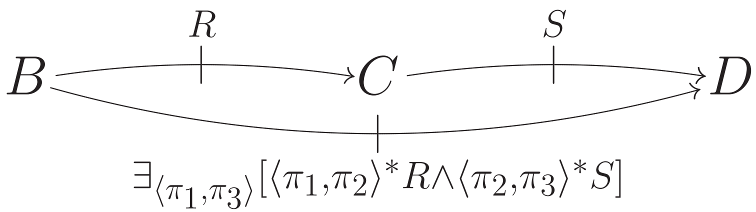
A fibration $\mathcal{T} : \mathbf{P} \to \mathbf{B}$ with finite products in $\mathbf{B}$, preordered fibers, and fibred finite limits $\wedge, \top$

left adjoints $\exists_f \dashv f^*$ satisfying Beck-Chevalley and Frobenius

full comprehension



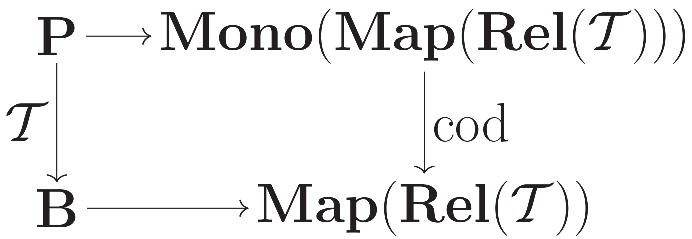
$\mathbf{Rel}(\mathcal{T})$:



with $\mathcal{T}(R) = B \times C$, $\mathcal{T}(S) = C \times D$.

$\mathbf{Rel}(\mathcal{T})$ is a preordered cartesian bicategory. All objects are discrete and Frobenius.

Moreover $\mathbf{Map}(\mathbf{Rel}(\mathcal{T}))$ is regular and



is a change of base.

R.F.C. Walters, R.J. Wood, *Frobenius objects in cartesian bicategories*, T.A.C. 20 (2008)

The only ~~An~~ example that comes to mind

$\mathbf{B}$ with finite limits and a stable factorization $(\mathcal{E}, \mathcal{M})$

Take

$$\begin{array}{c} \mathbf{P} = \mathcal{M}^{\rightarrow} \\ \Big\downarrow \mathcal{T} = \mathrm{cod} \\ \mathbf{B} \end{array}$$

Then $\mathbf{Rel}(\mathrm{cod})$ is $\mathbf{Rel}(\mathbf{B}, \mathcal{E}, \mathcal{M})$ as in

G.M. Kelly, *A note on relations relative to a factorization system*, CT'90

In particular,

$$\mathbf{B} \longrightarrow \mathbf{Map}(\mathbf{Rel}(\mathbf{B}, \mathcal{E}, \mathcal{M}))$$

is the reflection from the 2-category of categories with a stable factorization system to that of regular categories, inverting precisely the monos in $\mathcal{E}$.

Manufacturing other examples

fibration with fibred fin.lim's

$$\begin{array}{ccc} \mathbf{E} & \longrightarrow & \mathbf{Vert}(p) \\ {\scriptstyle p}\big\downarrow & & \mathrm{c}(p)\big\downarrow \dashv \mathrm{id} \dashv \mathrm{dom} \\ \mathbf{A} & \longrightarrow & \mathbf{E} \end{array}$$

fibration of vertical arrows

comprehension is full

is the left biadjoint to forgetting comprehension in fibrations with fibred finite limits.

Moreover

- if $\mathbf{A}$ has finite products, then $\mathbf{E}$ has finite products
- if $\begin{array}{c} \mathbf{E} \\ {\scriptstyle p}\big\downarrow \\ \mathbf{A} \end{array}$ has left adjoints to reindexing with BCC and FR, then so does $\begin{array}{c} \mathbf{Vert}(p) \\ {\scriptstyle \mathrm{c}(p)}\big\downarrow \\ \mathbf{E} \end{array}$
- if $\begin{array}{c} \mathbf{E} \\ {\scriptstyle p}\big\downarrow \\ \mathbf{A} \end{array}$ is preordered, then $\begin{array}{c} \mathbf{Vert}(p) \\ {\scriptstyle \mathrm{c}(p)}\big\downarrow \\ \mathbf{E} \end{array}$ is preordered

Manufacturing other examples, 2

- a tripos $\begin{array}{c}\mathbf{T}\\ p\downarrow\\ \mathbf{Set}\end{array}$ as in

  J. Hyland, P. Johnstone, A. Pitts, *Tripos Theory*, Math.Proc.Camb.Phil.Soc. 88 (1980)

- for a geometric theory T, the fibration of formulas $\begin{array}{c}\mathbf{WFF}_T\\ p\downarrow\\ \mathbf{Sort}_T\end{array}$ as in

  M. Makkai, G. Reyes, *First Order Categorical Logic*, LNM 611, 1977

- for a category $\mathbf{A}$ with fin.products and weak equalizers, the fibred preordered reflection $\begin{array}{c}(\mathbf{A}^{\rightarrow})_{\mathsf{po}}\\ \mathrm{cod}\downarrow\\ \mathbf{A}\end{array}$

essentially as in

  A. Carboni, E. Vitale, *Regular and exact completions*, J.Pure Appl.Alg. 125 (1998)

- for a left exact functor $F\colon\mathbf{A}\to\mathbf{C}$, a suitable fibred preordered reflection $\begin{array}{c}(\mathbf{pb}_{\mathbf{C}}(F))_{\mathsf{po}}\\ \mathrm{cod}\downarrow\\ \mathbf{A}\end{array}$ as in

  P. Hofstra, *Relative completions*, J.Pure Appl.Alg. 192 (2004)

Quotient completion

$$\begin{array}{ccccccc}
\mathbf{E} & \longrightarrow & \mathbf{Vert}(p) & \longrightarrow & \mathbf{Mono}(\mathbf{Map}(\mathbf{Rel}(\mathrm{c}(p)))) & \longrightarrow & \mathbf{Mono}(\mathbf{Map}(\mathbf{Rel}(\mathrm{c}(p)))_{\mathrm{ex/reg}}) \\
{\scriptstyle p}\big\downarrow & & {\scriptstyle \mathrm{c}(p)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{cod}} & & \big\downarrow{\scriptstyle \mathrm{cod}} \\
\mathbf{A} & \longrightarrow & \mathbf{E} & \longrightarrow & \mathbf{Map}(\mathbf{Rel}(\mathrm{c}(p))) & \longrightarrow & \mathbf{Map}(\mathbf{Rel}(\mathrm{c}(p)))_{\mathrm{ex/reg}} \\
 & & & & & & \wr\Vert \\
 & & & & & & \mathbf{Map}(\mathbf{Split}_{\mathrm{equiv}}(\mathbf{Rel}(\mathrm{c}(p))))
\end{array}$$

For $\mathbf{A}$ with finite products and weak equalizers

$$\begin{array}{ccccccc}
(\mathbf{A}^{\to})_{\mathsf{po}} & \longrightarrow & \mathcal{M}(\mathbf{Fr}(\mathbf{A})) & \longrightarrow & \mathbf{Mono}(\mathbf{A}_{\mathrm{reg}}) & \longrightarrow & \mathbf{Mono}(\mathbf{A}_{\mathrm{ex}}) \\
{\scriptstyle \mathrm{cod}}\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathbf{A} & \longrightarrow & \mathbf{Fr}(\mathbf{A}) & \longrightarrow & \mathbf{A}_{\mathrm{reg}} & \longrightarrow & \mathbf{A}_{\mathrm{ex}}
\end{array}$$